Doctoral thesis

Effective field theory of modified gravity on cosmological and spherically symmetric backgrounds

（宇宙論的及び球対称的な背景時空での修正重力理論に関する有効場の理論）

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Abstract

In this thesis we study the effective field theory of modified gravity on cosmological and spherically symmetric backgrounds.

First, we study the effective field theory of modified gravity on the cosmological background. Modification of gravity usually generates at least one scalar propagating degree of freedom which is responsible for dark energy. We take into account multiple scalar fields χ_I $(I = 1, 2 \cdots, N-1)$ characterized by the Lagrangians $P^{(I)}(Y_I)$ with $Y_I = \partial_\mu \chi_I \partial^\mu \chi_I$ in addition to the dark energy field ϕ. These additional scalar fields χ_I can model the perfect fluids of radiation and non-relativistic matter. The Lagrangian involves three dimensional geometric quantities appearing in the 3+1 decomposition of space-time. We expand a general action up to second order in the perturbations of geometric scalars and additional scalar fields. In doing so we derive propagation speeds of scalar and tensor perturbations as well as conditions for the absence of ghosts. Our analysis covers a wide range of gravitational theories– not only Horndeski theory but also its recent generalizations. The theories beyond Horndeski induce non-trivial modifications to all the propagation speeds of N scalar fields, but the modifications to those for the matter fields χ_I are generally suppressed relative to that for the dark energy field ϕ. We apply our general results to the covariantized Galileon with an Einstein-Hilbert term in which partial derivatives of the Minkowski Galileon are replaced by covariant derivatives. Unlike the covariant Galileon with second-order equations of motion in general space-time, the scalar propagation speed squared c_{s1}^2 associated with the field ϕ becomes negative during the matter era for late-time tracking solutions, so the two Galileon theories can be clearly distinguished at the level of linear cosmological perturbations.

Secondly, on the static and spherically symmetric background, we consider perturbations of such a background endowed with a metric tensor and a scalar field in the framework of the effective field theory of modified gravity. We employ the previously developed 2+1+1 canonical formalism of a double Arnowitt-Deser-Misner (ADM) decomposition of space-time, which singles out both time and radial directions. Our building block is a general gravitational action that depends on scalar quantities constructed from the 2+1+1 canonical variables and the lapse. Variation of the action up to first-order in perturbations gives rise to three independent background equations of motion, as expected from spherical symmetry. The dynamical equations of linear perturbations follow from the second-order Lagrangian after a suitable gauge fixing. We derive conditions for the avoidance of ghosts and Laplacian instabilities for the odd-type perturbations. We show that our results not only incorporate those derived in the Horndeski theories but they can be applied to more generic theories beyond Horndeski as in the case of the effective field theory of modified gravity on the cosmological background.

Acknowledgement

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Contents

Chapter 1

Introduction

In 1998, the late-time cosmic acceleration was discovered from the observations of type Ia supernovae (SN Ia) in high redshifts [1, 2] and the source for this acceleration was dubbed "dark energy". In Ref. [2], Perlmutter *et al.* showed that the dark energy exists at the 99% confidence level by using 42 high-redshift and 18 low-redshift SN Ia data (see figure 1.1). The latest observations of the cosmic microwave background (CMB) have shown that about 68% of the present energy of the Universe is dominated by dark energy [3]. The simplest origin for this present-day acceleration is the cosmological constant, but the vacuum energy appearing in particle physics is vastly larger than the observed energy scale of dark energy [4]. In detail, summing up zero-point energies of all normal modes of some field and taking the cut-off scale of the momentum at the Planck scale, the vacuum energy density is theoretically estimated to be $\rho_{\rm vac} \sim 10^{74}$ GeV^4 while the observed value of dark energy is $\rho_{\rm obs} \sim 10^{-47}$ GeV^4.

If the origin of dark energy is not the cosmological constant, there is a possibility that the accelerated expansion of the Universe is driven by a scalar field [5, 6] or some modification of gravity [7, 8, 9, 10, 11, 12, 13, 14, 15]. The former is called modified matter theories, and the latter, modified gravity theories.

Interestingly, the recent combined analysis based on the observations of SN Ia, CMB, and Baryon Acoustic Oscillations (BAO) showed that the cosmological constant is in mild tension with the data [3, 16] as shown in figure 1.2. Especially, CMB measurement by Planck [3] combined with data of the WMAP polarization [16] and the SN Ia (from SNLS [17]) showed that the dark energy equation of state, which characterizes the dark energy w, is

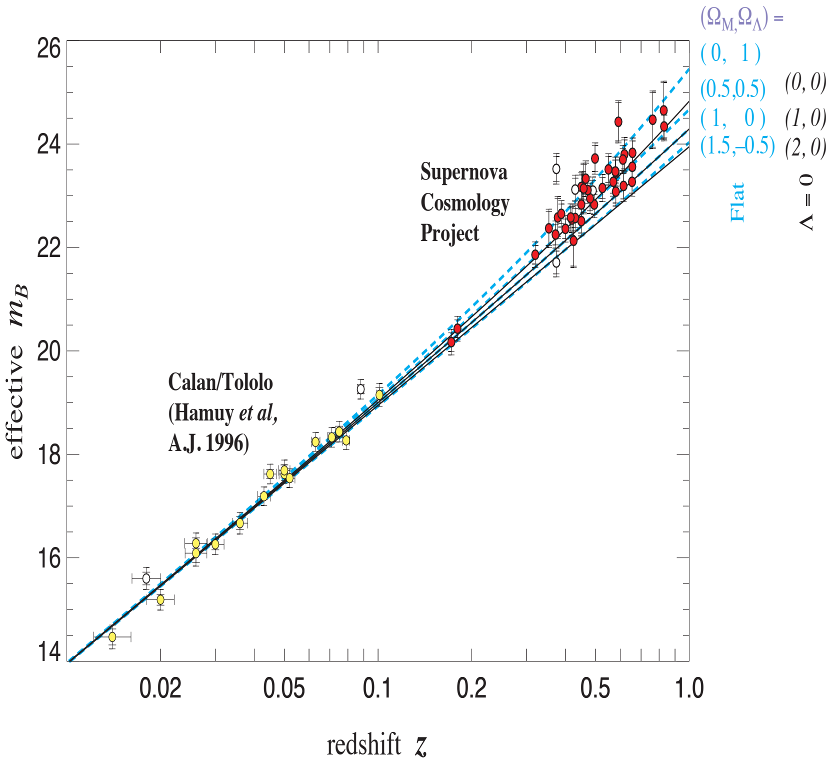

Figure 1.1: The effective apparent luminosity m_B versus the redshift z for several SN Ia data. The dashed curves are the theoretical prediction for m_B when the cosmological constant Λ exists while the solid curves correspond to that without the cosmological constant. Perlmutter *et al.* found that the observational data in the high redshift regime favor the existence of the dark energy. This figure is taken from Ref. [2].

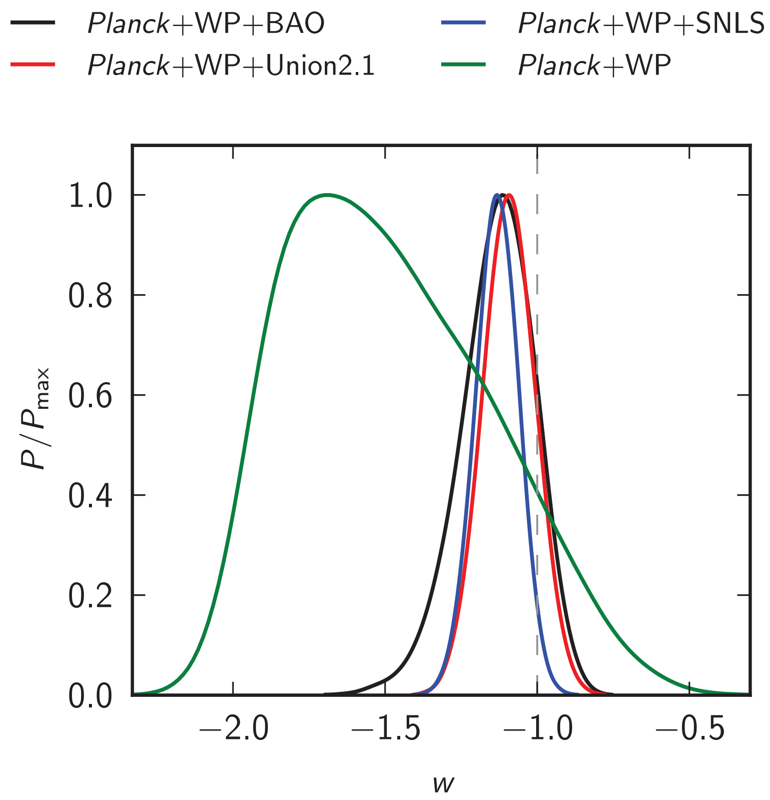

Figure 1.2: Observational constraints on the constant dark energy equation of state w for several combinations of observational data, i.e. CMB, SN Ia, BAO and WMAP polarization. This figure is taken from Ref. [3].

constrained to be $w = -1.13^{+0.13}_{-0.14}$ (95 % CL) for constant w. In GR, including the cosmological constant and modified matter theories, it is generally difficult to explain $w < -1$ unless a ghost mode is introduced, but the modification of gravity allows a possibility of realizing such an equation of state while avoiding ghosts and instabilities [18].

1.1 Horndeski theories and their generalization

The modification of gravity is usually associated with the propagation of a scalar degree of freedom coupled to non-relativistic matter (see Refs. [18] for reviews). Most of the dark energy models based on modified gravity–such as $f(R)$ gravity [8, 9, 10], Brans-Dicke theory [12, 19], and Galileons [13, 14, 20, 21]– belong to the category of the Horndeski theory [22, 23, 24], i.e., the most general scalar-tensor theory with second-order equations of motion. These models realize the late-time cosmic acceleration through the modification of gravity at large distances, i.e. cosmological scales. In the Horndeski theory, the conditions for avoiding ghosts and Laplacian instabilities of scalar and tensor perturbations have been derived in Refs. [25, 26, 27] on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological background in the absence/presence of matter. Imposing these conditions and studying the background dynamics as well as the growth of density perturbations [28], we can test for theoretical consistent models of dark energy with numerous observational data.

On the other hand, the dark energy models based on modified gravity are required to recover Newtonian gravity at short distances for the consistency with local gravity tests in the Solar System. There are several ways to suppress the propagation of the fifth force induced by a scalar degree of freedom ϕ. One of them is the Vainshtein mechanism [29], under which non-linear scalar-field self interactions appearing e.g., in Galileon gravity, lead to the decoupling of the scalar field from baryons inside the radius much larger than the solar system [30]. Another is the chameleon mechanism [31] applicable to $f(R)$ gravity [32] and Brans-Dicke theory [12], under which the fifth force outside a spherically symmetric body is suppressed by the formation of a thin shell inside the body with a large effective mass of the scalar field. For the purpose of understanding the screening mechanism of the fifth force in general, the equations of motion in the Horndeski theory were derived on the spherical symmetric background [33, 34, 35]. The stability of static and spherically symmetric vacuum solutions in the same theory was also studied in Ref. [36] by considering the odd-parity mode of perturbations (associated with tensor perturbations). The analysis of the even-parity perturbations, which is much more involved due to a non-trivial coupling between the scalar field and gravity, was recently performed in Ref. [37]. The spherically symmetric background solutions of viable modified gravity models need to accommodate the screening mechanism of the fifth force, while satisfying the stability conditions against perturbations.

If the derivatives higher than second order appear in the equations of motion, the corresponding theory is usually prone to a ghost-like (Ostrogradski) instability [38] related with the Hamiltonian unbounded from below. The Horndeski Lagrangian was constructed to be manifestly free from the Ostrogradski instability. Recently, Gleyzes, Langlois, Piazza, and Vernizzi (GLPV) proposed the generalization of Horndeski theories in which the higher order derivatives could appear in general space-time [39]. Interestingly, they showed that, on the flat FLRW background, the perturbation equations of motion in the generalized version of Horndeski theories are also of second order with one scalar propagating degree of freedom [39]. This second-order property also holds for the odd-type perturbations on the spherically symmetric and static background [40]. In GLPV theories, the presence of symmetries in space-time allows for the absence of derivatives higher than quadratic order.

A concrete example classified into GLPV theories is the covariantized version of the original Galileon–whose Lagrangian is derived by replacing partial derivatives of the Minkowski Galileon [13] with covariant derivatives– belongs to a class of GLPV theories [39]. This is different from the covariant Galileon [14] in which gravitational counter terms are added to eliminate derivatives higher than second order in general space-time. In other words, the covariant Galileon falls in a class of Horndeski theories, while the covariantized Galileon does not.

For the unified description of modified gravitational theories, there is another approach based on the effective field theory (EFT) which can deal with modified gravity models even beyond Horndeski theories.

1.2 The EFT approach on the cosmological background

The EFT of cosmological perturbations is a powerful framework to deal with the low-energy degree of freedom of dark energy in a systematic and unified way [41]-[65]. This approach is based on the expansion of a general four-dimensional action about the flat FLRW background in terms of the perturbations of three-dimensional geometric scalar quantities appearing in the 3+1 Arnowitt-Deser-Misner (ADM) [66] decomposition of space-time. Such geometric scalars involve the traces and squares of the extrinsic curvature $K_{\mu\nu}$ and the three-dimensional intrinsic curvature $\mathcal{R}_{\mu\nu} \equiv {}^{(3)}R_{\mu\nu}$ as well as the lapse function N. The Lagrangian generally depends on a scalar field ϕ, but such dependence can be absorbed into the lapse dependence by choosing the so-called unitary gauge in which the field perturbation $\delta\phi$ vanishes.

The EFT formalism can incorporate a wide variety of modified gravitational theories known in the literature, e.g. Horndeski theories.[1] In the EFT approach time derivatives are of second order by construction, but there exist spatial derivatives higher than second order in general [50, 51]. In Ref. [52], the conditions for the absence of such higher-order spatial derivatives have been derived by expanding the action up to second order in the perturbations of geometric scalars. In fact the Horndeski theory satisfies such conditions, so the resulting second-order Lagrangian is simplify expressed by the sum of time and spatial derivatives $\dot{\zeta}^2$ and $(\partial\zeta)^2$ of curvature perturbations ζ with time-dependent coefficients [52]. This feature also holds in GLPV theories [39].

In order to study the cosmological dynamics based on GLPV theories, we need to take into account matter fields (such as non-relativistic matter and radiation) other than the scalar field ϕ responsible for dark energy. In the presence of an additional scalar field χ with a kinetic energy $Y = \partial_\mu\chi\partial^\mu\chi$, the conditions for eliminating derivatives higher than second order have been derived in Ref. [68] for the action depending on χ and Y as well as on other

[1] While the Horndeski theory is Lorentz-invariant, the EFT approach can also cover Lorentz-violating theories such as Hořava-Lifshitz gravity [67] where spatial derivatives higher than second order appear in the action. This is first shown in Ref. [60], but the contributions of those terms to the second-order action of cosmological perturbations were not explicitly computed for scalar perturbations. In Ref. [64], we studied the second-order cosmological perturbation in the EFT approach of modified gravity including those higher-order spatial derivatives.

ADM scalar quantities. In Ref. [68] the authors also obtained conditions for the avoidance of ghosts and Laplacian instabilities associated with scalar and tensor perturbations. In GLPV theories it was recognized that the matter propagation speed c_m is affected by the scalar degree of freedom χ [39], but this is not the case for Horndeski theories [27, 68].

In Chapter 2 of this thesis we study the effective field theory of modified gravity in the presence of multiple matter fields on the flat FLRW background. In addition to the dark energy field ϕ, we take into account scalar fields χ_I $(I = 1, 2, \cdots, N-1)$ with the Lagrangians $P^{(I)}(X_I)$ depending on $Y_I = \partial_\mu \chi_I \partial^\mu \chi_I$. This prescription can accommodate the perfect fluids of radiation and non-relativistic matter [69, 70]. Expanding the action up to second order in the perturbations of geometric scalars and multiple matter fields, we obtain propagation speeds of scalar and tensor perturbations as well as no-ghost conditions. We apply our general results to Horndeski and GLPV theories. We obtain an algebraic equation for the propagation speeds of multiple scalar fields and estimate to what extent the difference arises by going beyond Horndeski theories.

We then apply our results to two different theories– covariantized Galileon and covariant Galileon. Although the background equations of motion for the covariantized Galileon are exactly the same as those for the covariant Galileon. At the level of perturbations, however, these two theories can be clearly distinguished from each other. For the covariantized Galileon the propagation speed squared c_{s1}^2 of the field ϕ becomes negative in the deep matter era for late-time tracking solutions, whereas in the covariant Galileon it remains positive. We also show that the matter sound speeds squared of the fields χ_I for the covariantized Galileon are similar to those for the covariant Galileon.

1.3 The EFT approach on the spherically symmetric background

The EFT of modified gravity on the isotropic cosmological background allows a possibility of dealing with the theories beyond Horndeski in a systematic and unified way. If we try to apply a similar formalism to the spherical symmetric background, there is another spatial direction singled out by the ADM decomposition besides the temporal direction. The EFT of modified gravity with the singled-out radial direction is first worked out in Ref. [40].

There are several ways to deal with the perturbations of spherically symmetric and static space-times. Some of the approaches monitor the metric perturbations and they heavily rely both on the decomposition of the perturbations into even and odd modes (under parity transformations on the sphere) and on a full gauge fixing. This line of research includes the pioneering work of Regge, Wheeler and Zerilli [71, 72], leading to the Regge-Wheeler equation for the odd modes and the Zerilli equation for the even modes of general relativistic black hole perturbations. The discussion of perturbations in the Horndeski class of theories presented in Refs. [36, 37] falls into this class.

We employ yet another formalism based on the s+1+1 decomposition, where s is an arbitrary positive integer [73, 74] (developed with the application to braneworld models in mind). This ADM inspired formalism, based on a double foliation of space-time, relies on a canonical rather than a covariant approach. The clear advantage of this procedure is a much lower number of variables. In comparison with the metric perturbation formalism (for $s = 2$) the number of variables is the same, nevertheless the variables in the 2+1+1 ADM formalism carry canonical interpretation, which is a clear virtue when it comes to the EFT approach.

In Chapter 3, we study the EFT of modified gravity on a static and spherically symmetric background by employing the 2+1+1 ADM formalism. In the gravitational action, we take into account all the possible scalar combinations constructed from geometric quantities. We show that the Horndeski and GLPV theory can be accommodated in our general framework, by explicitly rewriting the corresponding Lagrangians in terms of the 2+1+1 covariant variables as in the case of the EFT on cosmological background. The three independent background equations of motion are derived in simple forms, which will be useful for the study of the screening mechanism in general modified gravitational theories.

We also obtain the second-order Lagrangian for odd-type perturbations in the EFT framework to discuss the stability of spherically symmetric and static vacuum solutions. We derive conditions for avoiding ghosts/Laplacian instabilities and apply our results to both Horndeski and GLPV theories (including covariantized Galileon and covariant Galileon). We defer the study of the even-type perturbations to a follow-up work due to its non-triviality and complexity.

1.4 Contents of this thesis

This thesis is organized as follows.

Chapter 2 is devoted for the EFT of modified gravity on the cosmological background in the presence of multiple matter fields.

- In Sec. 2.1 we provide the basic tools to understand the late-time cosmic acceleration on the cosmological background. We derive the condition for the equation of state of the dark energy to realize such an acceleration and show the evolution of the equation of state in several dark energy models.

- In Sec. 2.2 we briefly review the EFT approach of modified gravity on the cosmological background. Introducing geometric scalar quantities appearing in the ADM formalism we provide a general action that depends on such scalars. In the presence of the matter we expand the action up to linear order in cosmological perturbations and derive equations of motion for the background.

- In Sec. 2.3 we show how the EFT action accommodates Horndeski and GLPV theories on the flat FLRW space-time. We apply the general results in the previous section to these theories and derive the background equations of motion.

- In Sec. 2.4 we introduce multiple scalar fields associated with matter components. We first expand the action up to second order in cosmological tensor perturbations and derive conditions to avoid ghost and Laplacian instability for the tensor perturbations. Next we expand the action up to second order in scalar perturbations and derive conditions for the absence of spatial derivatives higher than second order. These conditions are automatically satisfied in Horndeski and GLPV theories. Under these conditions, we derive no-ghost conditions and an N-th order algebraic equation for the scalar propagation speed squared c_s^2.

- In Sec. 2.5 we study the cosmology based on the two Galileon theories (covariantized and covariant Galileons). We discuss how these theories can be distinguished from each other, paying particular attention to the evolution of the scalar propagation speeds.

- In Sec. 2.6 we conclude this chapter.

In Chapter 3 the EFT of modified gravity on the spherically symmetric background is discussed.

- In Sec. 3.1 we briefly review how the screening mechanism suppress the modification of gravity at short distances taking the Vainshtein mechanism as an example.

- In Sec. 3.2 the basic elements of the 2+1+1 ADM decomposition will be reviewed as a brief summary of the formalism developed in Refs. [73, 74].

- In Sec. 3.3 we present a variational principle for a general action in unitary gauge expressed in terms of scalars constructed from the geometric quantities arising in the 2+1+1 decomposition. Varying the action up to first order in perturbations allows us to derive three equations of motion for the background.

- In Sec. 3.4 we express the Lagrangians of both Horndeski and GLPV theories in terms of the variables appearing in the 2+1+1 formalism and show that they belong to the sub-class of our general framework.

- In Sec. 3.5 we explore the diffeomorphism gauge freedom in dealing with perturbations on the static and spherically symmetric background. After choosing the unitary gauge $\delta\phi = 0$, there is still a remaining gauge degree of freedom associated with the time component of a coordinate transformation vector ξ^μ. We show that this residual gauge degree of freedom does not affect the odd-type perturbations studied in this chapter.

- In Sec. 3.6 we derive the second-order perturbed Lagrangian density for the odd-mode perturbations expressed in terms of a dynamical scalar variable and its derivatives.

- In Sec. 3.7 we discuss conditions for the absence of ghosts and Laplacian instabilities and apply the results to both Horndeski and GLPV theories. We also specialize our results for two covariant Galileon models.

- In Sec. 3.8 we conclude this chapter.

Finally, Chapter 4 gives a summary of this thesis.

Throughout this thesis, we use the metric signature $(-,+,+,+)$ and units $c = \hbar = k_B = 1$. $M_{\rm pl}$ is the reduced Planck mass defined as $M_{\rm pl} \equiv 1/(8\pi G)$ where G is the gravitational constant. We denote time derivatives by a dot and the derivatives along the singled-out spatial direction by a prime. All quantities defined on the background will carry an overbar.

In Chapter 2 Greek and Latin indices denote components in space-time and in a three-dimensional space-adapted basis, respectively.

In Chapter 3 we use the abstract index notation, hence tensors defined on the full space-time and on the 2-dimensional surface carry the same set of Latin indices, but the latter obey certain projection conditions.

Chapter 2

Effective field theory of modified gravity on the cosmological background

In this Chapter we investigate the EFT of modified gravity on the cosmological background in the presence of matter fields [61].

2.1 The basis of the late-time cosmic acceleration

First of all we briefly review the basics tools to study the late-time cosmic acceleration and the representative examples of the dark energy models in this section.

2.1.1 Basic tools to study the late-time cosmic acceleration

The cosmological principle, which states that the Universe is homogeneous and isotropic at large distances, is supported by several observations such as CMB. The line-element that describes a 4-dimensional homogeneous, isotropic and flat space-time is called the FLRW space-time and is given by

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \,, \tag{2.1}$$

where a is a scale factor representing a size of the Universe. Using this metric and deriving the Einstein equations we obtain the following Friedmann equations:

$$3M_{\rm pl}^2 H^2 = \rho \,, \tag{2.2}$$

$$2M_{\rm pl}^2 \dot{H} = -(\rho + P) \,, \tag{2.3}$$

where $H \equiv \dot{a}/a$ is a Hubble parameter. Here ρ and P respectively correspond to the energy density and the pressure of the matter component described by the perfect fluid with the equation of state

$$w \equiv \frac{P}{\rho} \,, \tag{2.4}$$

which is 0 for non-relativistic matter and 1/3 for radiation.

Let us consider the case where the Universe is dominated by a single component with a constant equation of state. Solving the equations (2.2) and (2.3) we obtain the following solution

$$a = (t - t_i)^{2/[3(1+w)]} \,, \tag{2.5}$$

where t_i is an integration constant. In the matter era when non-relativistic matter ($w = 0$) gives the dominant contribution in the universe, Eq. (2.5) reduces to $a \propto t^{2/3}$ and the expansion of the Universe slows down during the matter era. In order to realize the cosmic acceleration we require $\ddot{a} < 0$ which reduces to the condition

$$w < -\frac{1}{3} \,. \tag{2.6}$$

Note that the above condition cannot be satisfied for the ordinary matter such as non-relativistic matter or radiation since their equations of state are positive. Although the cosmological constant ($w = -1$) satisfies the above condition, this model is plagued by the energy scale problem as we mentioned in Sec. 1. In the rest of this section we briefly review the viable alternative models based on modified matter theories and modified gravity theories.

2.1.2 Quintessence

As an example of the modified matter theories, let us consider the so-called "quintessence" model [5, 75]. This model contains a canonical scalar field

ϕ with a potential $V(\phi)$. Unlike the cosmological constant the equation of state of quintessence varies with time. In order to realize the late-time cosmic acceleration the effective mass of the field must be very light such as $m_\phi \sim H_0 \sim 10^{-33}$eV where H_0 is today's Hubble parameter.

The quintessence model is described by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S^M\,, \qquad (2.7)$$

where g is a determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar and S^M is the matter action. Here the first term is the so-called Einstein-Hilbert term which gives the Einstein tensor after varying with respect to the metric. The second and the third terms correspond to the kinetic and the potential terms of the scalar field, respectively. Varying the Lagrangian (2.7) with respect to the metric we obtain the Friedmann equations (2.2) and (2.3) with $\rho = \dot{\phi}^2/2+V$ and $P = \dot{\phi}^2/2 - V$. Then the equation of state (2.4) reduces to

$$w_{\rm DE} = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V}\,. \qquad (2.8)$$

As long as the potential V varies slowly and the evolution of the field is sufficiently slow so that $|\dot{\phi}^2| \ll |V|$, the condition (2.6) is satisfied and the cosmic acceleration arises. The evolution of $w_{\rm DE}$ is obtained by using Eq. (2.8) with (2.2) and (2.3) after specifying the form of the potential.

The evolution of $w_{\rm DE}$ depends on the shape of the potential V. Broadly speaking, they are classified into (i) "freezing" models and (ii) "thawing" models [76].

(i) Freezing models

An example of the representative potential that belongs to this class is the inverse power-low potential $V = M^{4+n}\phi^{-n}$. In this class the field rolls along the potential in the past and slows down once the system enters the cosmic acceleration regime.

(ii) Thawing models

An example of the representative potential that belongs to this class is $V = M^4\cos^2(\phi/f)$ which appears in the context of a dynamical supersymmetry breaking. In this class the field is nearly frozen around the potential maximum, but once the Hubble parameter become as small as the effective mass of the field it starts to evolve.

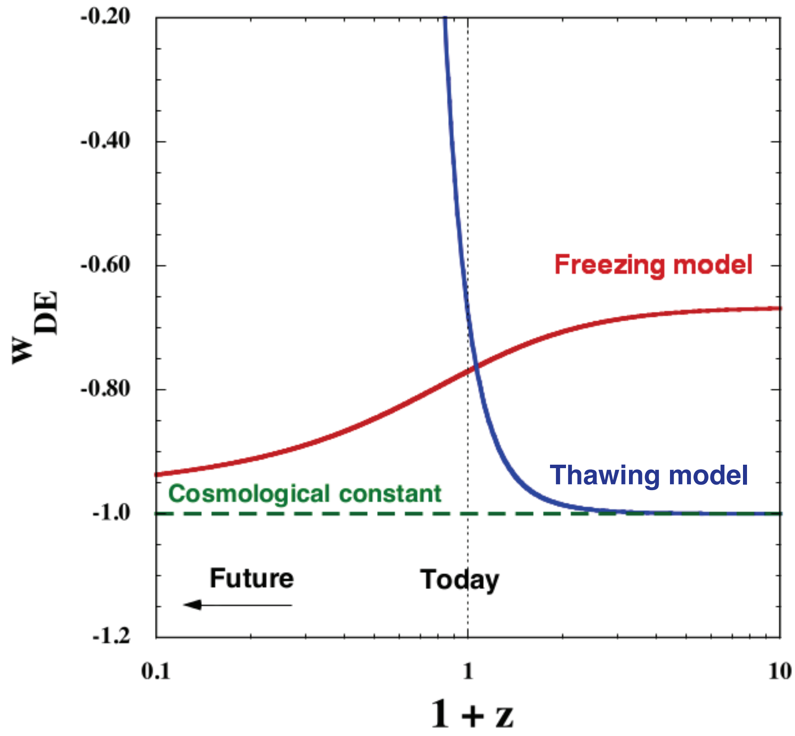

Figure 2.1: Evolution of $w_{\rm DE}$ for the freezing and the thawing models versus the redshift z. The green dashed line represents the cosmological constant.

In figure 2.1 we show an example of the evolution of $w_{\rm DE}$ for the freezing and the thawing models. Although the recent observation tends to favor $w_{\rm DE} < -1$, the equation of state for quintessence is restricted to be $w_{\rm DE} > -1$ as is obvious from Eq. (2.8). One possibility to realize $w_{\rm DE} < -1$ in the context of quintessence is introducing a ghost mode [77]. However in this case the energy of the system is not bounded below and the so-called ghost instability occurs.

2.1.3 Modified gravity

As examples of dark energy models based on modified gravity, we briefly review the $f(R)$ gravity and the covariant Galileon in the following. Both models allow a possibility of realizing $w_{\rm DE} < -1$ around today while avoiding ghost instabilities.

$f(R)$ gravity

This model is the simplest extension of general relativity in which the Einstein-Hilbert term is generalized as a function of the Ricci scalar:

$$S = \frac{M_{\rm pl}^2}{2} \int d^4x \sqrt{-g} f(R) + S^M \,. \tag{2.9}$$

As long as $f_{,RR} \equiv d^2 f/dR^2 \neq 0$ holds there is a gravitational scalar degree of freedom. In order to understand that let us consider the equivalent action

$$S = \frac{M_{\rm pl}^2}{2} \int d^4x \sqrt{-g} \left[f(\chi) + f_{,\chi}(\chi)(R - \chi) \right] + S^M \,. \tag{2.10}$$

Varying this action with respect to χ we obtain

$$f_{,\chi\chi}(R - \chi) = 0 \,, \tag{2.11}$$

which leads $\chi = R$ as long as $f_{,\chi\chi} \neq 0$. Inserting back this solution into Eq. (2.10) we recover the original action (2.9). Redefining a scalar field $\phi = f_{,\chi}(\chi)$ Eq. (2.10) can be written as the following

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} \phi R - V(\phi) \right] + S^M \,, \tag{2.12}$$

where $V \equiv (M_{\rm pl}^2/2)[\phi\chi(\phi) - f(\chi(\phi))]$ corresponds to the potential term. Here the first term represents a coupling between the scalar field and the metric field. This type of coupling is called a non-minimal coupling. Note that the non-minimal coupling give rise to an interaction between the scalar field and the baryon at short distances where the curvature R becomes large. However this interaction should be suppressed for the consistency with local gravity tests in the Solar System. In $f(R)$ gravity the so-called chameleon mechanism can be at work in the high density local region [31, 32].

The action (2.12) needs to satisfy the conditions $\phi = f_{,R}(R)$ and $m_\phi^2 \equiv d^2V/d\phi^2 > 0$, to avoid a ghost instability for the tensor perturbation and a tachyonic instability, respectively. An example of viable models satisfying above conditions is

$$f(R) = R - \lambda R_0 \frac{(R/R_0)^{2n}}{(R/R_0)^{2n} + 1} \,, \tag{2.13}$$

where n, λ and R_0 are positive constants [9]. If R_0 is of the order of H_0, Eq. (2.13) approximately behaves as the cosmological constant, $f(R) \simeq R - \lambda R_0$ at the early stage of the cosmological history ($R \ll R_0$). Once R becomes as small as R_0 around the present epoch Eq. (2.13) differs from the cosmological constant. In the asymptotic future the system approaches the de Sitter point characterized by $f_{,R}(R) = 2f(R)$ where the scale factor grows exponentially.

Varying the action (2.9) with respect to the metric we obtain the modified Friedmann equations on the cosmological background. Moving the extra terms appearing due to the modification of gravity into the right-hand sides of the Friedmann equations, one can write these equations in the form of Eqs. (2.2) and (2.3) and define the energy density and the pressure of the effective dark energy. Using Eq. (2.4) with (2.2) and (2.3) we can calculate the evolution of $w_{\rm DE}$. In figure 2.2 we show the evolution of w_{DE} in the model (2.13). As we mentioned in Chapter 1, $w_{\rm DE} < -1$ is realized around the present epoch while the conditions to avoid the ghost and the tachyonic instabilities are kept to be satisfied.

Covariant Galileon

In the Dvali-Gabadadze-Porrati (DGP) braneworld scenario [7], in which the cosmic acceleration is realized by a gravitational leakage to the extra dimension, the field self-interaction of the form $(\nabla\phi)^2\Box\phi$ appears. This non-linear term has nice feature to recover GR in a local region through the so-called Vainshtein mechanism[1] while this model exhibits a ghost mode [78].

The self-interacting Lagrangian $(\nabla\phi)^2\Box\phi$ appearing in the DGP model satisfies the Galilean symmetry $\partial_\mu\phi \to \partial_\mu\phi + b_\mu$ in the Minkowski background. Imposing the Galilean symmetry in the flat space-time one can show that the field Lagrangian consists of five terms $L_1, \cdots, L_5$, where the term $(\nabla\phi)^2\Box\phi$ corresponds to L_3 [13]. In Refs. [14] these terms were extended to covariant forms in the curved space-time as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + \sum_{i=1}^{5} c_i L_i\right] + S^M\,, \tag{2.14}$$

[1] We will briefly review this mechanism in Sec. 3.1

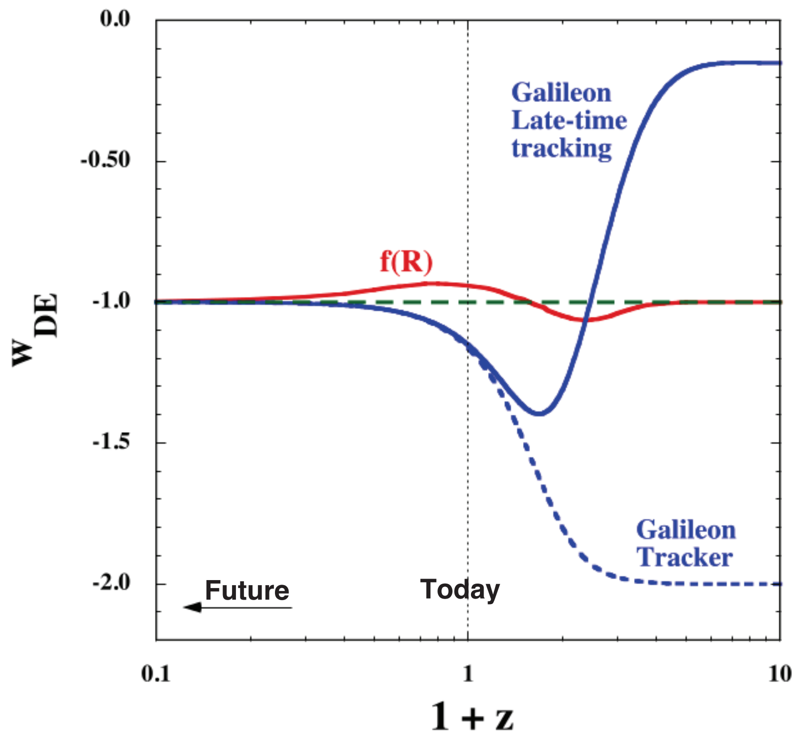

Figure 2.2: Evolution of w_{DE} for the $f(R)$ gravity and the covariant Galileon model versus the redshift z. The green dashed line represents the cosmological constant.

with

$$
\begin{aligned}
& L_1 = \phi\,, \qquad L_2 = (\nabla\phi)^2 \qquad L_3 = (\nabla\phi)^2\Box\phi\,, \\
& L_4 = (\nabla\phi)^2\left[2(\Box\phi)^2 - 2\phi_{;\mu\nu}\phi^{;\mu\nu} - R(\nabla\phi)^2/2\right]\,, \\
& L_5 = (\nabla\phi)^2[(\Box\phi)^3 - 3(\Box\phi)\,\phi_{;\mu\nu}\phi^{;\mu\nu} \\
& \qquad\qquad +2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\rho}\phi_{;\rho}{}^{\mu} - 6\phi_{;\mu}\phi^{;\mu\nu}\phi^{;\rho}G_{\nu\rho}]\,,
\end{aligned}
\tag{2.15}
$$

where c_i are model parameters, a semicolon represents a covariant derivative and $G_{\nu\rho}$ is the Einstein tensor. The cosmological dynamics of covariant Galileon theory except for the term L_1 are studied in Refs. [20, 21]. There exist de Sitter (dS) solutions responsible for dark energy driven by the field kinetic energy. Unlike in the case of the DGP model there exist the viable space of model parameter c_i in which the appearance of ghosts and instabilities of scalar and tensor perturbations can be avoided.

In figure 2.2 the evolution of $w_{\rm DE}$ in the covariant Galileon model is shown. In the past there exist a tracker solution characterized by $w_{\rm DE} = -2$. The epoch at which the solutions approach the tracker depends on the initial conditions. The system finally approaches the de Sitter attractor ($w_{\rm DE} = -1$) characterized by $\dot{\phi} =$ constant. In Ref. [86] it was shown that the observations favor the late-time tracking solution which approaches the tracker only around the present epoch.

As we have seen in this section there exist several viable models which realize $w_{\rm DE} < -1$ without exhibiting ghost instabilities in modified gravitational theories. In the following chapter we investigate the EFT of modified gravity which can deal with almost all the viable models in a systematic and unified way.

2.2 The general EFT action of modified gravity

In the following we briefly review the EFT of modified gravity on the cosmological background.

We first employ the $3+1$ decomposition in the ADM formalism described by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.16}$$

where N is the lapse, N^i is the shift vector, and h_{ij} is the three-dimensional metric. A unit normal vector orthogonal to constant time hypersurfaces Σ_t is given by $n_\mu = (-N, 0, 0, 0)$ with the normalization $n_\mu n^\mu = -1$. The extrinsic curvature of Σ_t is defined by

$$K_{\mu\nu} \equiv h^\lambda_\mu h^\sigma_\nu \, n_{\sigma;\lambda} = n_{\nu;\mu} + n_\mu n^\lambda n_{\nu;\lambda} \,. \tag{2.17}$$

In the second equality of Eq. (2.17) we have used the fact that the three-dimensional metric $h_{\mu\nu}$ can be expressed as $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. The extrinsic curvature can be expressed in terms of a temporal derivative of the induced metric h_{ij} and spatial derivative of the shift vector as

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i\right) , \tag{2.18}$$

where $\nabla^{(3)}_i$ represents a three-dimensional covariant derivative associated with the induced metric h_{ij}.

The internal geometry of the hypersurfaces is characterized by the three-dimensional Ricci tensor $\mathcal{R}_{\mu\nu} \equiv {}^{(3)}R_{\mu\nu}$. Similar to the definition of the four-dimensional Ricci curvature, $\mathcal{R}_{ij}$ is defined by the three-dimensional Christoffel symbol ${}^{(3)}\Gamma^k_{ij} = h^{kl}(\nabla^{(3)}_i h_{lj} + \nabla^{(3)}_j h_{li} - \nabla^{(3)}_l h_{ij})/2$. The three dimensional Ricci scalar $\mathcal{R}$ is related with the four dimensional Ricci scalar R via the Gauss-Coddazi relation as

$$R = \mathcal{R} - K^2 + \mathcal{S} + 2(Kn^\mu - a^\mu)_{;\mu} \,, \tag{2.19}$$

where a_μ is the so-called acceleration defined as $a_\mu \equiv n^\nu n_{\mu;\nu} = h^\nu_\mu N_{;\nu}/N$.

The EFT of cosmological perturbations advocated in Refs. [41]-[52] is based on the combination of geometric scalar quantities:

$$\begin{aligned} &K \equiv K^\mu{}_\mu \,, \qquad \mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu} \,, \\ &\mathcal{R} \equiv \mathcal{R}^\mu{}_\mu \,, \qquad \mathcal{Z} \equiv \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \,, \qquad \mathcal{U} \equiv \mathcal{R}_{\mu\nu}K^{\mu\nu} \,, \end{aligned} \tag{2.20}$$

as well as the lapse N.

The action of general modified gravitational theories that depends on the above mentioned scalar quantities is given by

$$S = \int d^4x\sqrt{-g}\, L\,(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}; t) + S^M \,, \tag{2.21}$$

where L is a Lagrangian. S^M is the matter action with energy density ρ_M and pressure P_M. The dependence of the lapse N and the time t is included since we are in mind considering the existence of a scalar degree of freedom ϕ associated with the modification of gravity. Under the unitary gauge in which the constant field hypersurfaces coincide with the constant time hypersurfaces, the scalar field reduces simply to the function of time $\phi = \phi(t)$ and its kinetic term can be expressed in terms of the lapse as $X = -N^{-2}\dot{\phi}(t)^2$. Thus the ϕ and X dependence in the Lagrangian L can be interpreted as the N and t dependence. We will give a more detailed explanation in Sec. 2.3.

On the flat FLRW background, the equations of motion can be derived by expanding the action (2.21) up to first order in perturbations. The background line element (2.1) corresponds to $\bar{N} = 1$, $\bar{N}_i = 0$ and $\bar{h}_{ij} = \delta_{ij}$. Substituting this background metric into Eq. (2.18) and the definition of $\mathcal{R}_{\mu\nu}$ we obtain

$$\bar{K}_{\mu\nu} = 3H\bar{h}_{\mu\nu}\,, \qquad \bar{\mathcal{R}}_{\mu\nu} = 0\,. \tag{2.22}$$

Thus the background value of ADM variables defined in Eq. (2.20) leads

$$\bar{K} = 3H\,, \qquad \bar{\mathcal{S}} = 3H^2\,, \qquad \bar{\mathcal{R}} = \bar{\mathcal{U}} = \bar{\mathcal{Z}} = 0\,, \tag{2.23}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Then we first define the following perturbations for the scalars composed of the extrinsic curvature

$$\delta K = K - 3H\,, \qquad \delta\mathcal{S} = \mathcal{S} - 3H^2 = 2H\delta K + \delta K^\mu_\nu \delta K^\nu_\mu\,. \tag{2.24}$$

where $\delta K_{\mu\nu} = K_{\mu\nu} - Hh_{\mu\nu}$. Secondly, the scalars composed of the three-dimensional Ricci tensor appear only as perturbations:

$$\delta\mathcal{R} = \delta_1\mathcal{R} + \delta_2\mathcal{R}\,, \qquad \delta\mathcal{Z} = \delta\mathcal{R}^\mu_\nu \delta\mathcal{R}^\nu_\mu\,, \tag{2.25}$$

where $\delta_1\mathcal{R}$ and $\delta_2\mathcal{R}$ represent the first oder and the second order perturbations in $\delta\mathcal{R}$, respectively. Finally the mixture term $\mathcal{U}$ of the extrinsic and the intrinsic curvature can be expressed as

$$\delta\mathcal{U} = H\mathcal{R} + \mathcal{R}^\mu_\nu \delta K^\nu_\mu\,. \tag{2.26}$$

We now expand the Lagrangian L in Eq. (2.21) up to first order in perturbations as

$$L = \bar{L} + L_N\delta N + L_K\delta K + L_\mathcal{S}\delta\mathcal{S} + L_\mathcal{R}\delta\mathcal{R} + L_\mathcal{Z}\delta\mathcal{Z} + L_\mathcal{U}\delta\mathcal{U} + \mathcal{O}(2)\,, \tag{2.27}$$

where a lower index of L denotes the partial derivatives with respect to the scalar quantities evaluated at the background, e.g., $L_N = \overline{\partial L/\partial N}$. In order to derive the first-order Lagrangian, we first compute the combination $L_K \delta K + L_{\mathcal{S}} \delta \mathcal{S}$ in Eq. (2.27). Making use of the second and third relations of Eq. (2.24) and defining the quantity

$$\mathcal{F} \equiv L_K + 2HL_{\mathcal{S}} \,, \tag{2.28}$$

it follows that

$$L_K \delta K + L_{\mathcal{S}} \delta \mathcal{S} = \mathcal{F}(K - 3H) + O(2) \,, \tag{2.29}$$

up to first order. Since $K = n^{\mu}{}_{;\mu}$ from Eq. (2.17), the term $\mathcal{F}K$ is partially integrated to give

$$\int d^4x \sqrt{-g}\, \mathcal{F}K = -\int d^4x \sqrt{-g}\, \mathcal{F}_{;\mu} n^{\mu} = -\int d^4x \sqrt{-g} \frac{\dot{\mathcal{F}}}{N} \,, \tag{2.30}$$

up to a boundary term. Expanding the term $N^{-1} = (1+\delta N)^{-1}$ up to second order, Eq. (2.29) reduces to

$$L_K \delta K + L_{\mathcal{S}} \delta \mathcal{S} = -\dot{\mathcal{F}} - 3H\mathcal{F} + \dot{\mathcal{F}} \delta N + O(2) \,. \tag{2.31}$$

Substituting Eq. (2.31) into (2.27) and integrating by parts the linear order Lagrangian leads

$$L = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_{,N}) \delta N \,, \tag{2.32}$$

where we ignored the last three terms in Eq. (2.27) since they reduce to total derivatives at the linear order in perturbations.

In summary, the first-order action is given by $S = \int d^4x \sqrt{-g}\, L + S^M$ with the Lagrangian

$$L = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_{,N}) \delta N \,. \tag{2.33}$$

We define the Lagrangian density as $\mathcal{L} = \sqrt{-g} L = N\sqrt{h}\, L$ where h is the determinant of the three-dimensional metric h_{ij}. Then, the zeroth-order and first-order Lagrangian densities read

$$\mathcal{L}_0 = a^3(\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F}) \,, \tag{2.34}$$

$$\mathcal{L}_1 = a^3(\bar{L} + L_{,N} - 3H\mathcal{F}) \delta N + (\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F}) \delta \sqrt{h} \,. \tag{2.35}$$

Varying the first-order Lagrangian density (2.35) in terms of δN and $\delta\sqrt{h}$, we obtain the following equations of motion

$$\bar{L} + L_N - 3H\mathcal{F} = \rho_M \,, \tag{2.36}$$
$$\bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} = -P_M \,. \tag{2.37}$$

Once we specify the form of the Lagrangian and substituting it into these general results, we can easily derive the equations of motion on the cosmological background.

Expanding the action (2.21) up to second order in cosmological perturbations about the flat FLRW background, we obtain the equations of motion for the background and linear perturbations. Before doing so, we shall review the theories that belong to the action (2.21).

2.3 Horndeski and GLPV theories in the EFT language

In this section we show that Horndeski and GLPV theories can be fully expressed in terms of ADM variables introduced in the action (2.21).

Let us consider four-dimensional Horndeski theories characterized by the Lagrangian [22, 24, 23, 25]

$$L = \sum_{i=2}^{5} L_i^{\rm H} \,, \tag{2.38}$$

with

$$L_2^{\rm H} = G_2(\phi, X), \tag{2.39}$$
$$L_3^{\rm H} = G_3(\phi, X)\Box\phi, \tag{2.40}$$
$$L_4^{\rm H} = G_4(\phi, X)\, R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right] , \tag{2.41}$$
$$\begin{aligned} L_5^{\rm H} = {} & G_5(\phi, X) G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X) \\ & \times \left[(\Box\phi)^3 - 3(\Box\phi)\,\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma}\right] , \end{aligned} \tag{2.42}$$

where $\Box\phi \equiv (g^{\mu\nu}\phi_{;\nu})_{;\mu}$, and G_i ($i = 2, 3, 4, 5$) are functions in terms of a scalar field ϕ and its kinetic energy $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ with the partial derivatives $G_{iX} \equiv \partial G_i/\partial X$ and $G_{i\phi} \equiv \partial G_i/\partial\phi$. R and $G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor in four dimensions, respectively.

Horndeski theory covers a wide variety of gravitational theories with a single scalar degree of freedom. First of all, the k-essence scalar field [6, 79] in the framework of GR is described by the functions $G_2 = P(\phi, X), G_3 = 0, G_4 = M_{\rm pl}^2/2, G_5 = 0$, where $M_{\rm pl}$ is the reduced Planck mass. The canonical scalar field with a potential $V(\phi)$ corresponds to a particular function $G_2 = -X/2 - V(\phi)$.

Brans-Dicke (BD) theory [19] with a potential $V(\phi)$ is characterized by the functions $G_2 = -M_{\rm pl}\omega_{\rm BD}X/(2\phi) - V(\phi)$, $G_3 = 0$, $G_4 = M_{\rm pl}\phi/2$, $G_5 = 0$, where $\omega_{\rm BD}$ is the BD parameter. The metric $f(R)$ gravity [8, 10] and dilaton gravity [80] correspond to the particular cases of BD theory with $\omega_{\rm BD} = 0$ and $\omega_{\rm BD} = -1$, respectively.

The covariant Galileon [14] corresponds to the functions $G_2 = \beta_2 X$, $G_3 = \beta_3 X$, $G_4 = M_{\rm pl}^2/2 + \beta_4 X^2$, $G_5 = \beta_5 X^2$, where β_i $(i = 2, 3, 4, 5)$ are constants. A scalar field whose derivatives couple to the Einstein tensor in the form $G_{\mu\nu}\partial^\mu\phi\partial^\nu\phi$ [81, 82] can be accommodated by the functions $G_2 = -X/2 - V(\phi)$, $G_3 = 0$, $G_4 = 0$, $G_5 = c\phi$, where c is a constant and $V(\phi)$ is a field potential.

In what follows, we translate the Horndeski Lagrangian series $L_{2-5}^{\rm H}$ into the EFT language. In unitary gauge the constant time hypersurfaces correspond to the constant ϕ hypersurfaces on the cosmological background. Thus unit normal vector orthogonal to the constant time hypersurfaces can be written as

$$n_\mu = -\gamma\phi_{;\mu}\,, \qquad \gamma = \frac{1}{\sqrt{-X}}\,. \tag{2.43}$$

Taking the covariant derivative of n_μ and using the expression (2.17), it follows that

$$\phi_{;\mu\nu} = -\frac{1}{\gamma}\left(K_{\mu\nu} - n_\mu a_\nu - n_\nu a_\mu\right) + \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}n_\mu n_\nu\,, \tag{2.44}$$

Then the term $\Box\phi$ is expressed as

$$\Box\phi = -\frac{1}{\gamma}K + \frac{\phi^{;\lambda}X_{;\lambda}}{2X}\,. \tag{2.45}$$

Remembering that the scalar field corresponds to time in unitary gauge and its kinetic term can be written in terms of the lapse as $X = -\dot{\phi}^2/N^2$, the Lagrangian $L_2^{\rm H}$ is simply written as

$$L_2^{\rm H} = G_2(t, N)\,. \tag{2.46}$$

Using Eq. (2.45) the Lagrangian $L_3^{\rm H}$ reduces to

$$L_3^{\rm H} = 2(-X)^{3/2} F_{3X} - X F_{3\phi} \,, \tag{2.47}$$

where we introduced an auxiliary function $F_3(\phi, X)$ which satisfies $G_3 = F_3 + 2XF_{3X}$. Note that equations of motion can be written in terms of G_3 and its derivatives without using F_3 as we will see later.

Substituting Eqs. (2.19), (2.44) and (2.45) into Eq. (2.41) the Lagrangian $L_4^{\rm H}$ can be written in terms of ADM variables as

$$L_4^{\rm H} = G_4 \mathcal{R} + (2XG_{4X} - G_4)(K^2 - \mathcal{S}) - 2\sqrt{-X} G_{4\phi} K \,. \tag{2.48}$$

In a similar way the Lagrangian $L_5^{\rm H}$ can be expressed as [52]

$$\begin{aligned} L_5^{\rm H} &= \frac{1}{2} X G_{5\phi} (K^2 - \mathcal{S}) - \frac{1}{3}(-X)^{3/2} G_{5X} K_3 \\ &\quad + \frac{1}{2} X (G_{5\phi} - F_{4\phi}) \mathcal{R} - \sqrt{-X} F_5 \left(\mathcal{U} - \frac{1}{2} K \mathcal{R} \right) \,, \end{aligned} \tag{2.49}$$

where $F_5(\phi, X)$ is an auxiliary function satisfying $G_{5X} = F_5/(2X) + F_{5X}$ and

$$K_3 \equiv K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{il}K^j{}_l \,. \tag{2.50}$$

Up to quadratic order in perturbations, the term K_3 is given by

$$K_3 = 3H \left(2H^2 - 2KH + K^2 - \mathcal{S} \right) + O(3) \,. \tag{2.51}$$

As in the case of Eq. (2.47) equations of motion can be written in terms of G_5 and its derivatives in spite of this auxiliary function.

Combining Eqs. (2.46)-(2.49), the Horndeski Lagrangian series can be fully expressed in terms of the ADM scalar quantities as [52]

$$L = A_2 + A_3 K + A_4 (K^2 - \mathcal{S}) + B_4 \mathcal{R} + A_5 K_3 + B_5 \left(\mathcal{U} - K\mathcal{R}/2 \right) \,, \tag{2.52}$$

where

$$\begin{aligned} &A_2 = G_2 - X F_{3\phi} \,, \qquad A_3 = 2(-X)^{3/2} F_{3X} - 2\sqrt{-X} G_{4\phi} \,, \\ &A_4 = 2XG_{4X} - G_4 + XG_{5\phi}/2 \,, \qquad B_4 = G_4 + X(G_{5\phi} - F_{5\phi})/2 \,, \\ &A_5 = -(-X)^{3/2} G_{5X}/3 \,, \qquad B_5 = -\sqrt{-X} F_5 \,. \end{aligned} \tag{2.53}$$

From Eq. (2.53) the coefficients A_4, B_4, A_5, B_5 in Horndeski theories are related with each other as

$$A_4 = 2YB_{4X} - B_4\,, \qquad A_5 = -YB_{5X}/3\,. \tag{2.54}$$

The GLPV theories [39] correspond to the Lagrangian (2.52) in which the conditions (2.54) do not necessarily hold. Thus Horndeski and GLPV theories are subclasses of the EFT of modified gravity. Note that even in the case of GLPV theories derivatives higher than second order do not appear in the quadratic action for cosmological perturbations as we will see in the next section.

Substituting the Lagrangian (2.52) into Eqs. (2.36) and (2.37), the background equations of motion are given as

$$A_2 - 6H^2A_4 - 12H^3A_5 + 2\dot{\phi}^2\left(A_{2X} + 3HA_{3X} + 6H^2A_{4X} + 6H^3A_{5X}\right) = \rho_M\,, \tag{2.55}$$

$$A_2 - 6H^2A_4 - 12H^3A_5 - \dot{A}_3 - 4\dot{H}A_4 - 4H\dot{A}_4 - 12H\dot{H}A_5 - 6H^2\dot{A}_5 = -P_M\,. \tag{2.56}$$

Substituting the functions A_i of Eq. (2.53) into Eqs. (2.55)-(2.56), we reproduce the background equations of motion in Horndeski theories [25, 28] derived by the direct variation of the action (2.38) with (2.39)-(2.42).

Equations (2.55) and (2.56) do not contain the functions B_4 and B_5. This means that, at the background level, the theories with same values of A_2, A_3, A_4, A_5 but with different values of B_4 and B_5 cannot be distinguished from each other. In fact, this happens for the covariantized Galileon and the covariant Galileon mentioned in Introduction. However, it is possible to distinguish between such theories at the level of perturbations. We shall address this issue in Sec. 2.5.

2.4 Cosmological perturbations and propagation speeds of tensor and scalar modes

In this section, we derive no-ghost conditions and scalar propagation speeds for the theory described by the Lagrangian $L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}; t)$ in the presence of multiple scalar fields χ_I ($I = 1, 2, \cdots, N-1$). As we already mentioned, we choose the unitary gauge in which the perturbation of the field ϕ

vanishes ($\delta\phi = 0$). The k-essence Lagrangian $P^{(I)}(Y_I)$ with a kinetic energy $Y_I \equiv g^{\mu\nu}\partial_\mu\chi_I\partial_\nu\chi_I$ can describe the perturbation of a barotropic perfect fluid [69, 70, 68]. Let us then consider the theory with N scalar fields (ϕ and $\chi_1, \cdots, \chi_{N-1}$) given by the action

$$S = \int d^4x\sqrt{-g}\left[L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}; t) + \sum_{I=1}^{N-1} P^{(I)}(Y_I)\right], \tag{2.57}$$

which covers the theory (2.38) with (2.52) as a special case. The energy density $\rho^{(I)}$ and the equation of state w_I of the scalar field χ_I are given, respectively, by

$$\rho^{(I)} = 2Y_I P^{(I)}_{Y_I} - P^{(I)}, \qquad w_I = \frac{P^{(I)}}{2Y_I P^{(I)}_{Y_I} - P^{(I)}}, \tag{2.58}$$

where $P^{(I)}_{Y_I} = \partial P^{(I)}/\partial Y_I$. Then, the total energy density ρ_M and the pressure P_M of the scalar fields $\chi_1, \cdots, \chi_{N-1}$ read

$$\rho_M = \sum_{I=1}^{N-1}\left[2Y_I P^{(I)}_{Y_I} - P^{(I)}\right], \qquad P_M = \sum_{I=1}^{N-1} P^{(I)}. \tag{2.59}$$

Combining Eqs. (2.36) and (2.37), we obtain

$$L_N + \dot{\mathcal{F}} = \sum_{I=1}^{N-1} 2Y_I P^{(I)}_{Y_I}. \tag{2.60}$$

In Sec. 2.5.1 we will show that the above k-essence description can accommodate non-relativistic matter and radiation by choosing specific forms of $P^{(1)}(Y_1)$ and $P^{(2)}(Y_2)$.

In Ref. [68] the conditions for eliminating derivatives higher than quadratic order were derived for the two-field action

$$S = \int d^4x\sqrt{-g}\, L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}, \chi_1, Y_1; t).$$

In this case, the higher-order spatial derivatives do not appear under the certain conditions which we will show later. The mixture of temporal and spatial derivatives higher than second order can be eliminated under the conditions $L_{KY_1} + 2HL_{\mathcal{S}Y_1} = 0$ and $L_{\mathcal{R}Y_1} + HL_{\mathcal{U}Y_1} = 0$ [68]. For the separate Lagrangian

$L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \mathcal{U}; t) + P^{(1)}(Y_1)$, these two conditions are automatically satisfied. This is also the case for the action (2.57) of N scalar fields. In the following we study the second order perturbations for the action (2.57).

We now expand the action (2.57) up to second order in perturbations. In doing so, we express the three-dimensional metric h_{ij} and the shift N_i in the form [83]

$$\begin{aligned} & h_{ij} = a^2(t)e^{2\zeta}\hat{h}_{ij}\,, \qquad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \gamma_{il}\gamma_{lj}/2\,, \qquad \det\hat{h} = 1\,, \\ & N_i = \partial_i\psi \equiv \partial\psi/\partial x^i\,, \end{aligned} \tag{2.61}$$

where ζ and ψ are the scalar perturbations and γ_{ij} is the tensor perturbation satisfying traceless and transverse conditions $\gamma_{ii} = \partial_i\gamma_{ij} = 0$.

The second-order action for the tensor mode is the same as that derived in Refs. [52, 68]:

$$S_h^{(2)} = \int d^4x \frac{a^3}{4} L_{\mathcal{S}} \left[\dot{\gamma}_{ij}^2 - c_t^2 \frac{(\partial_k \gamma_{ij})^2}{a^2}\right]\,, \tag{2.62}$$

where the propagation speed c_t is given by

$$c_t^2 = \frac{\mathcal{E}}{L_{\mathcal{S}}}\,, \qquad \mathcal{E} \equiv L_{\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{U}} + \frac{3}{2}HL_{\mathcal{U}}\,. \tag{2.63}$$

The tensor ghosts and Laplacian instabilities are absent under the conditions

$$L_{\mathcal{S}} > 0\,, \tag{2.64}$$

$$\mathcal{E} > 0\,. \tag{2.65}$$

For the scalar perturbations the second-order action can be written in the form $S_s^{(2)} = \int d^4x\, \mathcal{L}_2$, with the Lagrangian density

$$\begin{aligned} \mathcal{L}_2 = {} & \delta\sqrt{h}[(\dot{\mathcal{F}} + L_N)\delta N + \mathcal{E}\delta_1\mathcal{R}] + a^3[(L_N + L_{NN}/2)\delta N^2 + \mathcal{E}\delta_2\mathcal{R} \\ & + \mathcal{A}\delta K^2/2 + \mathcal{B}\delta K\delta N + \mathcal{C}\delta K\delta_1\mathcal{R} + (\mathcal{D} + \mathcal{E})\delta N\delta_1\mathcal{R} \\ & + \mathcal{G}\delta_1\mathcal{R}^2/2 + L_{\mathcal{S}}\delta K^\mu_\nu\delta K^\nu_\mu + L_{\mathcal{Z}}\delta\mathcal{R}^\mu_\nu\delta\mathcal{R}^\nu_\mu] + \mathcal{L}_2^M\,, \end{aligned} \tag{2.66}$$

where

$$\begin{aligned} & \mathcal{A} \equiv L_{KK} + 4HL_{\mathcal{S}K} + 4H^2L_{\mathcal{S}\mathcal{S}}\,, \qquad \mathcal{B} \equiv L_{KN} + 2HL_{\mathcal{S}N}\,, \\ & \mathcal{C} \equiv L_{K\mathcal{R}} + 2HL_{\mathcal{S}\mathcal{R}} + L_{\mathcal{U}}/2 + HL_{K\mathcal{U}} + 2H^2L_{\mathcal{S}\mathcal{U}}\,, \\ & \mathcal{D} \equiv L_{N\mathcal{R}} - \dot{L}_{\mathcal{U}}/2 + HL_{N\mathcal{U}}\,, \qquad \mathcal{G} \equiv L_{\mathcal{R}\mathcal{R}} + 2HL_{\mathcal{R}\mathcal{U}} + H^2L_{\mathcal{U}\mathcal{U}}\,. \end{aligned} \tag{2.67}$$

Substituting Eq. (2.61) into (2.18) and the definition of the intrinsic curvature it follows that

$$
\begin{aligned}
&\delta\mathcal{R}_{ij} = -(\delta_{ij}\partial^2\zeta + \partial_i\partial_j\zeta)\,,\\
&\delta_1\mathcal{R} = -4a^{-2}\partial^2\zeta\,,\\
&\delta_2\mathcal{R} = -2a^{-2}[(\partial\zeta)^2 - 4\zeta\partial^2\zeta]\,,\\
&\delta K^i_j = (\dot\zeta - H\delta N)\delta^i_j - \delta^{ik}(\partial_k N_j + \partial_j N_k)/(2a^2)\,,\\
&\delta K = 3(\dot\zeta - H\delta N) - \partial^2\psi/a^2\,. \qquad (2.68)
\end{aligned}
$$

Using these relations and Eqs. (2.24)-(2.26) the second order Lagrangian density can be expressed in terms of the variables in the perturbed metric as

$$
\begin{aligned}
\mathcal{L}_2 = a^3 \Bigg[&\frac{1}{2}(2L_N + L_{NN} + 9\mathcal{A}H^2 - 6\mathcal{B}H + 6L_{\mathcal{S}}H^2)\delta N^2 \\
&+(\mathcal{B} - 3\mathcal{A}H - 2L_{\mathcal{S}}H)\left(3\dot\zeta - \frac{\partial^2\psi}{a^2}\right)\delta N + 4(3\mathcal{C}H - \mathcal{D} - \mathcal{E})\frac{\partial^2\psi}{a^2}\delta N \\
&-(3\mathcal{A} + 2L_{\mathcal{S}})\dot\zeta\frac{\partial^2\psi}{a^2} - 12\mathcal{C}\dot\zeta\frac{\partial^2\zeta}{a^2} + \left(\frac{9}{2}\mathcal{A} + 3L_{\mathcal{S}}\right)\dot\zeta^2 + 2\mathcal{E}\frac{(\partial\zeta)^2}{a^2} \\
&+\frac{1}{2}(\mathcal{A} + 2L_{\mathcal{S}})\frac{(\partial^2\psi)^2}{a^4} + 4\mathcal{C}\frac{(\partial^2\psi)(\partial^2\zeta)}{a^4} + 2(4\mathcal{G} + 3L_{\mathcal{Z}})\frac{(\partial^2\zeta)^2}{a^4}\Bigg] + \mathcal{L}_2^M\,,
\end{aligned}
\qquad (2.69)
$$

with the notation $\partial^2\zeta \equiv \partial_j\partial_j\zeta$ (the quantities with the same lower index j are summed). Here the last three terms in the square brackets exhibit spatial derivatives higher than second order. These higher order derivatives vanish under the conditions [52]

$$
\mathcal{A} + 2L_{\mathcal{S}} = 0\,, \qquad \mathcal{C} = 0\,, \qquad 4\mathcal{G} + 3L_{\mathcal{Z}} = 0\,, \qquad (2.70)
$$

or explicitly,

$$
L_{KK} + 4HL_{\mathcal{S}K} + 4H^2L_{\mathcal{S}\mathcal{S}} + 2L_{\mathcal{S}} = 0\,, \qquad (2.71)
$$

$$
L_{K\mathcal{R}} + 2HL_{\mathcal{S}\mathcal{R}} + \frac{1}{2}L_{\mathcal{U}} + HL_{K\mathcal{U}} + 2H^2L_{\mathcal{S}\mathcal{U}} = 0\,, \qquad (2.72)
$$

$$
4\left(L_{\mathcal{R}\mathcal{R}} + 2HL_{\mathcal{R}\mathcal{U}} + H^2L_{\mathcal{U}\mathcal{U}}\right) + 3L_{\mathcal{Z}} = 0\,. \qquad (2.73)
$$

It is clear that the Lagrangian (2.52) satisfies these three conditions (2.71)-(2.73) even without the restriction (2.54), so the linear perturbation equations of motion on the FLRW background do not contain derivatives higher than second order both in Horndeski and GLPV theories. Hereafter we focus on the subclass of the EFT in which Eqs. (2.71)-(2.73) are satisfied.

The Lagrangian density $\mathcal{L}_2^M$ corresponds to the contribution coming from the matter fields χ_I:

$$\mathcal{L}_2^M \equiv \sum_{I=1}^{N-1} \left[P_{Y_I}^{(I)} \delta\sqrt{h}\, \delta_1 Y_I + a^3 \left(P_{Y_I}^{(I)} \delta_2 Y_I + P_{Y_I Y_I}^{(I)} \delta_1 Y_I^2/2 + P_{Y_I}^{(I)} \delta N \delta_1 Y_I \right) \right] , \tag{2.74}$$

where the first-order and second-order contributions to Y_I are given, respectively, by

$$\delta_1 Y_I = 2\dot{\chi}_I^2 \delta N - 2\dot{\chi}_I \dot{\delta\chi}_I , \tag{2.75}$$

$$\delta_2 Y_I = -\dot{\delta\chi}_I^2 - 3\dot{\chi}_I^2 \delta N^2 + 4\dot{\chi}_I \dot{\delta\chi}_I \delta N + \frac{2\dot{\chi}_I}{a^2} \partial_j \psi \partial_j \delta\chi_I + \frac{1}{a^2} (\partial \delta\chi_I)^2 , \tag{2.76}$$

with $(\partial\delta\chi_I)^2 \equiv \partial_j \delta\chi_I \partial_j \delta\chi_I$. On using Eq. (2.60), one can eliminate some of the terms involving ζ. On using Eq. (2.68) and the relation $\delta\sqrt{h} = 3a^3\zeta$, the Lagrangian density (2.69) can be expressed as

$$\begin{aligned} \mathcal{L}_2 \;=\; a^3 \Bigg[& \frac{1}{2}(2L_N + L_{NN} - 6H\mathcal{W} + 12H^2 L_{\mathcal{S}}) \delta N^2 + \mathcal{W} \left(3\dot{\zeta} - \frac{\partial^2 \psi}{a^2} \right) \delta N \\ & - 4(\mathcal{D} + \mathcal{E}) \frac{\partial^2 \zeta}{a^2} \delta N + 4 L_{\mathcal{S}} \dot{\zeta} \frac{\partial^2 \psi}{a^2} - 6 L_{\mathcal{S}} \dot{\zeta}^2 + 2\mathcal{E} \frac{(\partial\zeta)^2}{a^2} \\ & + \sum_{I=1}^{N-1} \Bigg\{ (2\dot{\chi}_I^2 P_{Y_I Y_I}^{(I)} - P_{Y_I}^{(I)})(\dot{\chi}_I^2 \delta N^2 - 2\dot{\chi}_I \dot{\delta\chi}_I \delta N + \dot{\delta\chi}_I^2) \\ & - 6\dot{\chi}_I P_{Y_I}^{(I)} \zeta \dot{\delta\chi}_I - 2\dot{\chi}_I P_{Y_I}^{(I)} \delta\chi_I \frac{\partial^2 \psi}{a^2} + P_{Y_I}^{(I)} \frac{(\partial\delta\chi_I)^2}{a^2} \Bigg\} \Bigg] , \end{aligned} \tag{2.77}$$

where

$$\mathcal{W} \equiv L_{KN} + 2H L_{\mathcal{S}N} + 4H L_{\mathcal{S}} . \tag{2.78}$$

Varying the Lagrangian density (2.77) with respect to δN and $\partial^2\psi$, we obtain the Hamiltonian and momentum constraints

$$(2L_N + L_{NN} - 6H\mathcal{W} + 12H^2 L_{\mathcal{S}})\delta N + \mathcal{W}\left(3\dot{\zeta} - \frac{\partial^2\psi}{a^2}\right) - 4(\mathcal{D}+\mathcal{E})\frac{\partial^2\zeta}{a^2}$$
$$+\sum_{I=1}^{N-1} 2\dot{\chi}_I(P^{(I)}_{Y_I} - 2\dot{\chi}_I^2 P^{(I)}_{Y_I Y_I})(\dot{\delta\chi_I} - \dot{\chi}_I\delta N) = 0\,, \tag{2.79}$$

$$\mathcal{W}\delta N - 4L_{\mathcal{S}}\dot{\zeta} + \sum_{I=1}^{N-1} 2\dot{\chi}_I P^{(I)}_{Y_I}\delta\chi_I = 0\,. \tag{2.80}$$

Solving Eqs. (2.79)-(2.80) for δN, $\partial^2\psi$ and substituting the resulting relations into Eq. (2.77), the second-order Lagrangian density can be written in the form

$$\mathcal{L}_2 = a^3\left(\dot{\vec{\mathcal{X}}}^t \boldsymbol{K}\dot{\vec{\mathcal{X}}} - \frac{1}{a^2}\partial_j\vec{\mathcal{X}}^t\boldsymbol{G}\partial_j\vec{\mathcal{X}} - \vec{\mathcal{X}}^t\boldsymbol{B}\dot{\vec{\mathcal{X}}} - \vec{\mathcal{X}}^t\boldsymbol{M}\vec{\mathcal{X}}\right)\,, \tag{2.81}$$

where $\boldsymbol{K}$, $\boldsymbol{G}$, $\boldsymbol{B}$, $\boldsymbol{M}$ are $N\times N$ matrices, and the vector $\vec{\mathcal{X}}$ is composed from the dimensionless multiple fields, as

$$\vec{\mathcal{X}}^t = (\zeta, \delta\chi_1/M_{\rm pl}, \cdots, \delta\chi_{N-1}/M_{\rm pl})\,. \tag{2.82}$$

Here $M_{\rm pl}$ is the reduced Planck mass.

The two matrices $\boldsymbol{K}$ and $\boldsymbol{G}$ determine no-ghost conditions and the scalar propagation speeds. Their components are given by

$$K_{11} = \frac{2L_{\mathcal{S}}}{\mathcal{W}^2}\left(g_2 + \frac{8L_{\mathcal{S}}}{M_{\rm pl}^2}\sum_{I=2}^{N}\dot{\chi}_{I-1}^2 K_{II}\right)\,,$$
$$K_{II} = \left[2\dot{\chi}_{I-1}^2 P^{(I-1)}_{Y_{I-1}Y_{I-1}} - P^{(I-1)}_{Y_{I-1}}\right]M_{\rm pl}^2\,, \quad K_{1I} = K_{I1} = -\frac{4L_{\mathcal{S}}\dot{\chi}_{I-1}}{M_{\rm pl}\mathcal{W}}K_{II}\,, \tag{2.83}$$
$$G_{11} = -\frac{1}{2}\left(\dot{\mathcal{C}}_3 + H\mathcal{C}_3 + 4\mathcal{E}\right)\,,$$
$$G_{II} = -P^{(I-1)}_{Y_{I-1}}M_{\rm pl}^2\,, \qquad G_{1I} = G_{I1} = \frac{\mathcal{C}_3\dot{\chi}_{I-1}}{4L_{\mathcal{S}}M_{\rm pl}}G_{II}\,, \tag{2.84}$$

where $2 \le I \le N$ and other components are 0. The functions g_2 and $\mathcal{C}_3$ are defined by

$$g_2 \equiv 4L_{\mathcal{S}}(2L_N + L_{NN}) + 3(L_{KN} + 2HL_{\mathcal{S}N})^2\,, \tag{2.85}$$
$$\mathcal{C}_3 \equiv -\frac{16L_{\mathcal{S}}(\mathcal{D}+\mathcal{E})}{\mathcal{W}}\,. \tag{2.86}$$

For the derivation of G_{11} we have used the property that

$$\int d^4x\, a\mathcal{C}_3\dot{\zeta}\partial^2\zeta = \int d^4x\, (a/2)(\dot{\mathcal{C}}_3 + H\mathcal{C}_3)(\partial\zeta)^2\,, \tag{2.87}$$

up to a boundary term.

If the symmetric matrix $\boldsymbol{K}$ is positive definite, the scalar ghosts are absent. The necessary and sufficient conditions for the positivity of $\boldsymbol{K}$ are that the determinants of principal submatrices of $\boldsymbol{K}$ are positive, i.e.,

$$\frac{2L_{\mathcal{S}}}{\mathcal{W}^2}\prod_{I=2}^{\ell} K_{II}\left(g_2 + \frac{8L_{\mathcal{S}}}{M_{\rm pl}^2}\sum_{J=\ell+1}^{N}\dot{\chi}_{J-1}^2 K_{JJ}\right) > 0 \qquad (\ell = 1, 2, \cdots, N)\,, \tag{2.88}$$

where $\prod_{I=2}^{\ell} K_{II} = 1$ for $\ell = 1$ and $\sum_{J=\ell+1}^{N}\dot{\chi}_{J-1}^2 K_{JJ} = 0$ for $\ell = N$. Under the tensor no-ghost condition (2.64), all the N conditions (2.88) hold for $g_2 > 0$ and $K_{II} > 0$ ($I = 2, 3, \cdots, N$). Hence the scalar ghost is absent for

$$g_2 = 4L_{\mathcal{S}}(2L_N + L_{NN}) + 3(L_{KN} + 2HL_{\mathcal{S}N})^2 > 0\,, \tag{2.89}$$

$$2\dot{\chi}_I^2 P_{Y_IY_I}^{(I)} - P_{Y_I}^{(I)} > 0 \qquad (I = 1, 2, \cdots, N-1). \tag{2.90}$$

The dispersion relation following from the Lagrangian (2.81) in the limit of a large wave number k with a frequency ω is given by

$$\det\left(\omega^2\boldsymbol{K} - k^2\boldsymbol{G}/a^2\right) = 0\,. \tag{2.91}$$

Introducing the scalar sound speed c_s as $\omega^2 = c_s^2 k^2/a^2$, Eq. (2.91) reduces to

$$\prod_{I=1}^{N}\left(c_s^2 K_{II} - G_{II}\right) - \sum_{I=2}^{N}\left[\left(c_s^2 K_{1I} - G_{1I}\right)^2 \prod_{J\neq I, J\geq 2}^{N}\left(c_s^2 K_{JJ} - G_{JJ}\right)\right] = 0\,. \tag{2.92}$$

For the theory described by the Lagrangian (2.52), it follows that

$$\mathcal{D} + \mathcal{E} = B_4 + B_{4N} - \frac{1}{2}HB_{5N}\,, \qquad L_{\mathcal{S}} = -A_4 - 3HA_5\,. \tag{2.93}$$

We recall that in Horndeski theories the relation (2.54) holds, and hence $\mathcal{D} + \mathcal{E} = L_{\mathcal{S}}$. Then, the term $\mathcal{C}_3$ in Eq. (2.86) reads

$$\mathcal{C}_{3\rm H} = -\frac{16L_{\mathcal{S}}^2}{\mathcal{W}}\,, \tag{2.94}$$

where the lower index "H" represents the values in Horndeski theories. Substituting the relation (2.94) into Eq. (2.84) and using Eq. (2.83), the propagation speed $c_{s\mathrm{H}}$ in Horndeski theories satisfies

$$c_{s\mathrm{H}}^2 K_{1I} - G_{1I} = -\frac{4L_{\mathcal{S}}\dot{\chi}_{I-1}}{M_{\mathrm{pl}}\mathcal{W}}\left(c_{s\mathrm{H}}^2 K_{II} - G_{II}\right) . \tag{2.95}$$

Plugging Eq. (2.95) into Eq. (2.92), we obtain the following algebraic equation

$$\left[c_{s\mathrm{H}}^2 K_{11} - G_{11} - \left(\frac{4L_{\mathcal{S}}}{M_{\mathrm{pl}}\mathcal{W}}\right)^2 \sum_{I=2}^{N} \dot{\chi}_{I-1}^2 \left(c_{s\mathrm{H}}^2 K_{II} - G_{II}\right)\right] \times \prod_{I=2}^{N}\left(c_{s\mathrm{H}}^2 K_{II} - G_{II}\right) = 0 \,, \tag{2.96}$$

whose solutions are given by

$$\begin{aligned} c_{s\mathrm{H}1}^2 &= \frac{G_{11} - [4L_{\mathcal{S}}/(M_{\mathrm{pl}}\mathcal{W})]^2 \sum_{I=2}^{N} \dot{\chi}_{I-1}^2 G_{II}}{K_{11} - [4L_{\mathcal{S}}/(M_{\mathrm{pl}}\mathcal{W})]^2 \sum_{I=2}^{N} \dot{\chi}_{I-1}^2 K_{II}} \\ &= \frac{\mathcal{W}^2}{2L_{\mathcal{S}}g_2}\left[G_{11} + \frac{16L_{\mathcal{S}}^2}{\mathcal{W}^2}\sum_{I=2}^{N}\dot{\chi}_{I-1}^2 P_{Y_{I-1}}^{(I-1)}\right] , \end{aligned} \tag{2.97}$$

$$c_{s\mathrm{H}I}^2 = \frac{G_{II}}{K_{II}} = \frac{P_{Y_{I-1}}^{(I-1)}}{P_{Y_{I-1}}^{(I-1)} - 2\dot{\chi}_{I-1}^2 P_{Y_{I-1}Y_{I-1}}^{(I-1)}} \qquad (I = 2, 3, \cdots, N) \,. \tag{2.98}$$

The matter sound speed squared (2.98) coincides with that derived in Ref. [84] in the context of single-field k-inflation. In Horndeski theories, each $c_{s\mathrm{H}I}$ ($I \geq 2$) is not affected by other scalar fields. The presence of the matter fields χ_I gives rise to modifications to the first propagation speed $c_{s\mathrm{H}1}$, which was already derived in Ref. [68] for $N = 2$.

In GLPV theories where the conditions (2.54) are not satisfied, we cannot write Eq. (2.92) in the separate form like Eq. (2.96). On using the propagation speeds (2.97) and (2.98), Eq. (2.92) can be written in the following form:

$$\prod_{I=1}^{N}\left(c_s^2 - c_{s\mathrm{H}I}^2\right) = -\frac{8L_{\mathcal{S}}}{g_2}\left(\frac{\mathcal{C}_3\mathcal{W}}{16L_{\mathcal{S}}^2}+1\right)\sum_{I=2}^{N}\left[\dot{\chi}_{I-1}^2 P_{Y_{I-1}}^{(I-1)} \times\left\{2c_s^2 + c_{s\mathrm{H}I}^2\left(\frac{\mathcal{C}_3\mathcal{W}}{16L_{\mathcal{S}}^2}-1\right)\right\}\prod_{J\neq I, J\geq 2}^{N}\left(c_s^2 - c_{s\mathrm{H}J}^2\right)\right] , \tag{2.99}$$

where, for $N=2$, $\prod_{J\neq I,J\geq 2}^{N}(c_s^2-c_{sHJ}^2)=1$. Since $\mathcal{C}_3\neq -16L_{\mathcal{S}}^2/\mathcal{W}$ in GLPV theories, the right hand side of Eq. (2.99) does not vanish. Hence c_s^2 differs from the value c_{sHI}^2. This means that not only the propagation speed c_{sH1} but also the matter sound speeds c_{sHI} $(I\geq 2)$ are affected by the presence of other scalar fields. When $N=2$, Eq. (2.99) reduces to

$$\left(c_s^2-c_{sH1}^2\right)\left(c_s^2-c_{sH2}^2\right) = -\frac{8L_{\mathcal{S}}}{g_2}\left(\frac{\mathcal{C}_3\mathcal{W}}{16L_{\mathcal{S}}^2}+1\right)\dot{\chi}_1^2P_{Y_1}^{(1)} \times\left[2c_s^2+c_{sH2}^2\left(\frac{\mathcal{C}_3\mathcal{W}}{16L_{\mathcal{S}}^2}-1\right)\right]\,, \tag{2.100}$$

which agrees with Eq. (22) of Ref. [39].[2]

Let us consider the case in which the deviation of $\mathcal{C}_3$ from the value $-16L_{\mathcal{S}}^2/\mathcal{W}$ is small, i.e.,

$$\mathcal{C}_3=-\frac{16L_{\mathcal{S}}^2}{\mathcal{W}}\left(1+\delta\mathcal{C}_3\right)\,,\qquad |\delta\mathcal{C}_3|\ll 1\,. \tag{2.101}$$

Under this approximation we write the two solutions for c_s^2 in Eq. (2.99), as

$$c_{s1}^2 = c_{sH1}^2+\delta c_{s1}^2\,, \tag{2.102}$$
$$c_{sI}^2 = c_{sHI}^2+\delta c_{sI}^2 \qquad (I=2,3,\cdots,N)\,. \tag{2.103}$$

Substituting Eq. (2.102) into Eq. (2.99), we obtain

$$\delta c_{s1}^2\simeq\sum_{I=2}^{N}\xi_{I-1}\,\delta\mathcal{C}_3\,, \tag{2.104}$$

where

$$\xi_I\equiv\frac{16L_{\mathcal{S}}\dot{\chi}_I^2P_{Y_I}^{(I)}}{g_2}\,. \tag{2.105}$$

[2] In Ref. [39], replacing $\phi\leftrightarrow\chi$, the definition of the first propagation speed squared is given by

$$\tilde{c}_{sH1}^2=\frac{\mathcal{W}^2}{2L_{\mathcal{S}}g_2}\left[G_{11}+\left(\frac{\mathcal{C}_3}{4L_{\mathcal{S}}}\right)^2\dot{\chi}_1^2P_{Y_1}^{(1)}\right]\,,$$

whereas the definition of c_{sH2}^2 is the same as ours. In the Horndeski limit $\mathcal{C}_3\to -16L_{\mathcal{S}}^2/\mathcal{W}$, c_{sH1}^2 is identical to $\tilde{c}_{sH1}^2$.

When we substitute Eq. (2.103) into Eq. (2.99), we employ the approximation $|\delta c_{sI}^2| \ll c_{sHI}^2$, whose validity should be checked after deriving the solution of δc_{sI}^2. It then follows that

$$\delta c_{sI}^2 \simeq -\frac{c_{sHI}^2}{2(c_{sHI}^2 - c_{sH1}^2 - \delta c_{s1}^2)} \xi_{I-1} \delta \mathcal{C}_3^2 \qquad (I = 2, 3, \cdots, N). \tag{2.106}$$

If the quantities $|\xi_{I-1}|$ $(I \geq 2)$ are much larger than 1, it is possible to have $|\delta c_{s1}^2|$ of the order of 1 even for $|\delta \mathcal{C}_3| \ll 1$. In fact, this happens for the cosmology of the covariantized Galileon studied in Sec. 2.5. On the other hand, δc_{sI}^2 $(I \geq 2)$ contains an additional suppression factor $\delta \mathcal{C}_3$. In the cosmological epoch where the field χ_{I-1} $(I \geq 2)$ dominates the energy density of the Universe, we have $\delta c_{s1}^2 \simeq \xi_{I-1} \, \delta \mathcal{C}_3$ from Eq. (2.104). Provided that the terms $|\delta c_{s1}^2|$ and $|c_{sHI}^2 - c_{sH1}^2|$ are at most of the order of 1, it follows that $|\delta c_{sI}^2| \ll c_{sHI}^2$. This discussion implies that the deviation from Horndeski theories may potentially lead to a considerable modification to c_{sH1}^2, but the modification to c_{sHI}^2 $(I \geq 2)$ should be suppressed. In Sec. 2.5.4 we shall study this issue for concrete models of dark energy.

2.5 Application to Galileon theories

The covariant Galileon advocated in Ref. [14] belongs to a class of the Horndeski Lagrangian (2.38) with the functions

$$G_2 = \frac{c_2}{2} X \,, \qquad G_3 = \frac{c_3}{2M^3} X \,, \qquad G_4 = \frac{M_{\rm pl}^2}{2} - \frac{c_4}{4M^6} X^2 \,, \qquad G_5 = \frac{3c_5}{4M^9} X^2 \,, \tag{2.107}$$

where $c_{2,3,4,5}$ are dimensionless constants and M is a constant having a dimension of mass. In this case the auxiliary functions F_3 and F_5 can be chosen as $F_3 = c_3 Y/(6M^3)$ and $F_5 = 3c_5 Y^2/(5M^9)$, respectively. Then, the covariant Galileon [dubbed Model (A)] corresponds to the Lagrangian (2.52) with the functions

$$\begin{aligned} \text{Model (A)}: \quad & A_2 = \frac{c_2}{2} X \,, \qquad A_3 = \frac{c_3}{3M^3} (-X)^{3/2} \,, \\ & A_4 = -\frac{M_{\rm pl}^2}{2} - \frac{3c_4}{4M^6} X^2 \,, \qquad A_5 = \frac{c_5}{2M^9} (-X)^{5/2} \,, \\ & B_4 = \frac{M_{\rm pl}^2}{2} - \frac{c_4}{4M^6} X^2 \,, \qquad B_5 = -\frac{3c_5}{5M^9} (-X)^{5/2} \,. \end{aligned} \tag{2.108}$$

The Lagrangian of the original Galileon [13] was constructed such that the field equations of motion satisfy the Galilean symmetry $\partial_\mu\phi \to \partial_\mu\phi + b_\mu$ in Minkowski space-time. In curved space-time, the covariantized version of the Minkowski Galileon follows by replacing partial derivatives of the field with covariant derivatives. Although this process generally gives rise to derivatives higher than second order, the equations of motion for the covariantized Galileon remain of second order on the isotropic cosmological background. Taking into account the Einstein-Hilbert term $(M_{\rm pl}^2/2)R$, the Lagrangian of the covariantized Galileon [dubbed Model (B)] is given by

$$
\begin{aligned}
\text{Model (B)}: \quad & A_2 = \frac{c_2}{2}X\,, \qquad A_3 = \frac{c_3}{3M^3}(-X)^{3/2}\,, \\
& A_4 = -\frac{M_{\rm pl}^2}{2} - \frac{3c_4}{4M^6}X^2\,, \qquad A_5 = \frac{c_5}{2M^9}(-X)^{5/2}\,, \\
& B_4 = \frac{M_{\rm pl}^2}{2}\,, \qquad B_5 = 0\,.
\end{aligned} \tag{2.109}
$$

The additional terms $-c_4X^2/(4M^6)$ and $-3c_5(-X)^{5/2}/(5M^9)$ appearing in the terms B_4 and B_5 of the covariant Galileon Lagrangian (2.108) correspond to the gravitational counter terms that eliminate derivatives higher than second order in general space-time.

2.5.1 Background cosmology

Even though the coefficients B_4 and B_5 in Model (B) are different from those in Model (A), the coefficients A_i $(i = 2, 3, 4, 5)$ are the same in both cases. Hence the background cosmological dynamics in Model (B) are exactly the same as those in Model (A). Substituting the functions A_i of Eqs. (2.108) and (2.109) into Eqs. (2.55) and (2.56), we obtain the equations of motion

$$3M_{\rm pl}^2H^2 = \rho_{\rm DE} + \rho_M\,, \tag{2.110}$$

$$3M_{\rm pl}^2H^2 + 2M_{\rm pl}^2\dot{H} = -P_{\rm DE} - P_M\,, \tag{2.111}$$

where the energy density $\rho_{\rm DE}$ and the pressure $P_{\rm DE}$ of the "dark" component are given by

$$\rho_{\rm DE} = -\frac{1}{2}c_2\dot{\phi}^2 + \frac{3c_3H\dot{\phi}^3}{M^3} - \frac{45c_4H^2\dot{\phi}^4}{2M^6} + \frac{21c_5H^3\dot{\phi}^5}{M^9}\,, \tag{2.112}$$

$$P_{\rm DE} = -\frac{1}{2}c_2\dot{\phi}^2 - \frac{c_3\dot{\phi}^2\ddot{\phi}}{M^3} + \frac{3c_4\dot{\phi}^3}{2M^6}\left[8H\ddot{\phi} + (3H^2 + 2\dot{H})\dot{\phi}\right]$$

$$-\frac{3c_5 H\dot{\phi}^4}{M^9}\left[5H\ddot{\phi}+2(H^2+\dot{H})\dot{\phi}\right]\,. \tag{2.113}$$

Equations (2.110) and (2.111) coincide with those derived in Refs. [20, 21] for the covariant Galileon. The dark energy equation of state is defined by $w_{\rm DE} \equiv P_{\rm DE}/\rho_{\rm DE}$.

For the matter component labelled by the lower index "M" in Eqs. (2.110)-(2.111), we take into account radiation and non-relativistic matter. The perfect fluids of radiation and non-relativistic matter can be modeled by two scalar fields χ_1 and χ_2, respectively, with the Lagrangians

$$P^{(1)}(Y_1) = b_1 Y_1^2\,, \tag{2.114}$$
$$P^{(2)}(Y_2) = b_2 (Y_2 - Y_0)^2\,, \tag{2.115}$$

where b_1, b_2, and Y_0 are constants. If we add a constant term Λ to the Lagrangian (2.115), this corresponds to the unified model of dark matter and dark energy proposed by Scherrer [85].

From Eq. (2.58) the energy density and the equation of state of radiation are given, respectively, by $\rho_r = 3b_1 Y_1^2$ and $w_r = 1/3$. The no-ghost condition (2.90) is satisfied for $b_1 > 0$. In Horndeski theories including Model (A), Eq. (2.98) shows that the sound speed squared of radiation is given by

$$c_{s\rm H2}^2 = \frac{1}{3}\,. \tag{2.116}$$

In Model (B), the radiation sound speed squared c_{s2}^2 deviates from $c_{s\rm H2}^2$ with the difference estimated by Eq. (2.106).

The energy density and the equation of state of non-relativistic matter following from Eq. (2.115) are given, respectively, by

$$\rho_m = b_2 (Y_2 - Y_0)(3Y_2 + Y_0)\,, \qquad w_m = \frac{Y_2 - Y_0}{3Y_2 + Y_0}\,. \tag{2.117}$$

Provided that Y_2 is close to Y_0, the field χ_2 behaves as non-relativistic matter with $w_m \simeq 0$. Then, the no-ghost condition (2.90) is satisfied for $b_2 > 0$. From the continuity equation $\dot{\rho}_m + 3H(1+w_m)\rho_m = 0$, we obtain the dependence $\rho_m \propto a^{-3}$ for $\epsilon \equiv (Y_2 - Y_0)/Y_0 \ll 1$ and hence $\epsilon \propto w_m \propto a^{-3}$. The matter energy density can be expressed as $\rho_m = 16 b_2 Y_0^2 w_m/(1-3w_m)^2$. In Horndeski theories, the sound speed squared of non-relativistic matter reads

$$c_{s\rm H3}^2 = \frac{Y_2 - Y_0}{3Y_2 - Y_0} = \frac{2w_m}{1+3w_m}\,, \tag{2.118}$$

which is much smaller than 1 for $w_m \ll 1$. The matter sound speed squared c_{s3}^2 in Model (B) is subject to change relative to c_{sH3}^2 given above. The presence of an additional pressure affects the gravitational growth of matter perturbations. For the successful structure formation, we require that $c_{s3}^2 \ll 1$ during the matter-dominated epoch.

The background cosmology based on Eqs. (2.110) and (2.111) has been studied in detail in Refs. [20, 21]. In what follows we shall briefly review the background dynamics and then study how the two Galileon theories can be distinguished from each other at the level of perturbations. A de Sitter solution ($H = H_{\rm dS} =$ constant) responsible for the late-time cosmic acceleration can be realized for a constant field velocity $\dot{\phi}_{\rm dS}$. Normalizing the mass M as $M^3 = M_{\rm pl} H_{\rm dS}^2$ and defining the dimensionless variable $x_{\rm dS} \equiv \dot{\phi}_{\rm dS}/(H_{\rm dS} M_{\rm pl})$, the coefficients c_2 and c_3 are related with the quantities $\alpha \equiv c_4 x_{\rm dS}^4$ and $\beta \equiv c_5 x_{\rm dS}^5$, as

$$c_2 x_{\rm dS}^2 = 6 + 9\alpha - 12\beta\,, \qquad c_3 x_{\rm dS}^3 = 2 + 9\alpha - 9\beta\,. \tag{2.119}$$

We also introduce the following dimensionless variables:

$$r_1 \equiv \frac{\dot{\phi}_{\rm dS} H_{\rm dS}}{\dot{\phi} H}\,, \qquad r_2 \equiv \frac{H}{H_{\rm dS}} \left(\frac{\dot{\phi}}{\dot{\phi}_{\rm dS}} \right)^5\,, \tag{2.120}$$

which are normalized as $r_1 = r_2 = 1$ at the de Sitter fixed point. The Friedmann equation (2.110) can be written in the form

$$\Omega_m = 1 - \Omega_r - \Omega_{\rm DE}\,, \tag{2.121}$$

where $\Omega_m \equiv \rho_m/(3M_{\rm pl}^2 H^2)$, $\Omega_r \equiv \rho_r/(3M_{\rm pl}^2 H^2)$, and

$$\Omega_{\rm DE} \equiv \frac{\rho_{\rm DE}}{3M_{\rm pl}^2 H^2} = -\frac{1}{2}(2+3\alpha-4\beta) r_1^3 r_2 + (2+9\alpha-9\beta) r_1^2 r_2 - \frac{15}{2}\alpha r_1 r_2 + 7\beta r_2\,. \tag{2.122}$$

The autonomous equations of motion for the variables r_1, r_2, and Ω_r are presented in Appendix A (see also Ref. [21] for detail). The variation of the Hubble parameter is known by $H'/H = -5r_1'/(4r_1) - r_2'/(4r_2)$, where a prime represents the derivative with respect to $\ln a$.

There exists a so-called tracker solution characterized by $r_1 = 1$, along which the field velocity evolves as $\dot{\phi} \propto H^{-1}$ [20, 21]. Along the tracker, the variable r_2 grows as $r_2' = 2r_2(3 - 3r_2 + \Omega_r)/(1 + r_2)$ from the regime $r_2 \ll 1$

to the de Sitter fixed point characterized by $r_2 = 1$ and $\Omega_r = 0$. The dark energy equation of state on the tracker is given by

$$w_{\rm DE} = -\frac{\Omega_r + 6}{3(1 + r_2)}\,, \tag{2.123}$$

which evolves as $w_{\rm DE} = -7/3 \to -2 \to -1$ during the cosmological sequence of radiation ($\Omega_r \simeq 1$, $r_2 \ll 1$), matter ($\Omega_r \ll 1$, $r_2 \ll 1$), and de Sitter ($\Omega_r \ll 1$, $r_2 = 1$) epochs. However, the tracker equation of state (2.123) is in tension with the joint data analysis of Sn Ia, CMB, and BAO because of the large deviation of $w_{\rm DE}$ from -1 during the matter era [86].

The solutions that approach the tracker at late times can be consistent with the observational data. In this case, the variable r_1 is much smaller than 1 during the early stage of the cosmological evolution. In the regime $r_1 \ll 1$, the variables r_1 and r_2 approximately obey the differential equations

$$r_1' \simeq \frac{9 + \Omega_r + 21\beta r_2}{8 + 21\beta r_2} r_1\,, \qquad r_2' \simeq \frac{3 + 11\Omega_r - 21\beta r_2}{8 + 21\beta r_2} r_2\,. \tag{2.124}$$

Provided that $|\beta r_2| \ll 1$, we obtain the solutions $r_1 \propto a^{5/4}$, $r_2 \propto a^{7/4}$ during the radiation era and $r_1 \propto a^{9/8}$, $r_2 \propto a^{3/8}$ during the matter era. When $r_1 \ll 1$, the dark energy equation of state is given by

$$w_{\rm DE} \simeq -\frac{1 + \Omega_r}{8 + 21\beta r_2}\,. \tag{2.125}$$

In the regime $|\beta r_2| \ll 1$, $w_{\rm DE}$ evolves from the value $-1/4$ (radiation era) to the value $-1/8$ (matter era). Once r_1 approaches 1, the solutions enter the tracking regime characterized by the equation of state (2.123). Provided that the approach to the tracker occurs at low redshifts, $w_{\rm DE}$ takes a minimum value larger than -1.3 and then it approaches the de Sitter value -1 in the asymptotic future. Such late-time tracking solutions are consistent with the combined data analysis of Sn Ia, CMB, and BAO [86].

2.5.2 No-ghost conditions

Let us discuss no-ghost conditions for tensor and scalar perturbations in Models (A) and (B). From Eq. (2.64) the tensor ghost is absent for $L_{\mathcal{S}} = -A_4 - 3HA_5 > 0$, whose condition is the same in both Models (A) and (B). This property also holds for no-ghost conditions of the scalar mode, because

Eq. (2.88) does not involve the functions B_4, B_5 and their derivatives. We recall that, for the matter fields characterized by the Lagrangians (2.114) and (2.115), the conditions (2.90) are satisfied for $b_1 > 0$ and $b_2 > 0$, respectively.

In the following, let us consider the case in which the sign of $\dot{\phi}$ does not change during the cosmic expansion history, i.e., $r_1 > 0$ and $r_2 > 0$. On using the variables r_1 and r_2, the no-ghost conditions (2.64) and (2.89) for tensor and scalar modes are given, respectively, by

$$
\begin{aligned}
L_{\mathcal{S}} &= [2 + 3r_2(\alpha r_1 - 2\beta)]M_{\rm pl}^2/4 > 0\,, &(2.126)\\
g_2 &= 3M_{\rm pl}^4 H_{\rm dS}^2 \sqrt{r_2/r_1^5} \\
&\quad \times\{(72\alpha^2 + 81\beta^2 - 150\alpha\beta + 30\alpha - 36\beta + 4)r_1^4 r_2 \\
&\qquad + [8\beta - 6\alpha - 4 - (162\alpha^2 + 24\beta^2 - 180\alpha\beta + 36\alpha - 12\beta)r_2]r_1^3 \\
&\qquad + [36\alpha - 36\beta + 8 + (90\alpha^2 - 162\beta^2 + 162\alpha\beta + 36\beta)r_2]r_1^2 \\
&\qquad - 12\alpha(3 + 16\beta r_2)r_1 + 105\beta^2 r_2 + 40\beta\} > 0\,. &(2.127)
\end{aligned}
$$

In the regime $r_1 \ll 1$ and $r_2 \ll 1$ the condition (2.126) is satisfied, while another condition (2.127) translates to

$$\beta > 0\,. \tag{2.128}$$

In order to satisfy Eqs. (2.126) and (2.127) in the tracking regime ($r_1 = 1$ and $0 < r_2 \leq 1$), we require that

$$-2 < 3(\alpha - 2\beta) < 2\,. \tag{2.129}$$

The conditions (2.128) and (2.129) need to obey for avoiding tensor and scalar ghosts.

2.5.3 Tensor propagation speeds

Since $\mathcal{E} = B_4 + \dot{B}_5/2$ for the Lagrangian (2.52), the tensor propagation speed squared $c_t^2 = \mathcal{E}/L_{\mathcal{S}}$ is different between the two Galileon theories. For Model (A) it is given by [20, 21]

$$c_t^2 = \frac{2r_1(2 - \alpha r_1 r_2) - 3\beta(r_1 r_2' + r_2 r_1')}{2r_1[2 + 3r_2(\alpha r_1 - 2\beta)]} \qquad \text{[Model (A)]}, \tag{2.130}$$

which is close to 1 for $r_2 \ll 1$. At the de Sitter fixed point we have $c_t^2 = (2-\alpha)/(2 + 3\alpha - 6\beta)$, so we require $\alpha < 2$ to avoid the Laplacian instability of tensor perturbations under the condition (2.129). During the transition from the regime $r_1 = 1, r_2 \ll 1$ to the regime $r_1 = 1, r_2 = 1$, it happens that c_t^2 has a minimum. Imposing that $c_t^2 > 0$ at the minimum, it follows that $\alpha < 12\sqrt{\beta} - 9\beta - 2$ [20, 21].

For Model (B) we have $\mathcal{E} = M_{\rm pl}^2/2$, so the tensor propagation speed squared is simply given by

$$c_t^2 = \frac{2}{2 + 3r_2(\alpha r_1 - 2\beta)} \qquad [\text{Model (B)}] \,. \tag{2.131}$$

Under the no-ghost condition (2.126), c_t^2 is positive. As long as the tensor perturbation is concerned, the viable parameter space of Model (B) is not restrictive compared to that of Model (A).

2.5.4 Scalar propagation speeds

Model (A)

The covariant Galileon model (A) belongs to a class of Horndeski theories, so the three scalar propagation speeds squared c_s^2 follow from Eqs. (2.97)-(2.98) with $I = 2, 3$. Among them the sound speed squared c_{s2}^2 and c_{s3}^2 of radiation and non-relativistic matter are given, respectively, by Eqs. (2.116) and (2.118). On using the relation

$$\dot{\chi}_I^2 P_{Y_I}^{(I)} = -\frac{1}{2}(\rho^{(I)} + P^{(I)}) = -\frac{3}{2} M_{\rm pl}^2 H^2 (1 + w_{(I)})\Omega_{(I)} \,, \tag{2.132}$$

where $w_{(I)}$ and $\Omega_{(I)}$ are the equation of state and the density parameter of the field χ_I, the first propagation speed squared c_{s1}^2 follows from Eq. (2.97), as

$$\begin{aligned} c_{s1}^2 \;=\; & -\frac{\mathcal{W}^2 M_{\rm pl}^2}{4 L_{\mathcal{S}} g_2} \Bigg[\left(1 - \frac{H'}{H}\right) \tilde{C}_{3\rm H} + \tilde{C}_{3\rm H}' + \frac{4\mathcal{E}_{\rm H}}{M_{\rm pl}^2} \\ & + \frac{48 L_{\mathcal{S}}^2 H^2}{\mathcal{W}^2} \left\{ \Omega_r (1 + w_r) + \Omega_m (1 + w_m) \right\} \Bigg] \,, \end{aligned} \tag{2.133}$$

where $\tilde{C}_{3\rm H} = H \mathcal{C}_{3\rm H}/M_{\rm pl}^2$, $\mathcal{E}_{\rm H} = [2r_1(2 - \alpha r_1 r_2) - 3\beta(r_1 r_2' + r_2 r_1')] M_{\rm pl}^2/(8r_1)$, and

$$\mathcal{W} = M_{\rm pl}^2 H_{\rm dS} (r_1^5 r_2)^{-1/4} \left[2 - 21\beta r_2 + 15\alpha r_1 r_2 - (2 + 9\alpha - 9\beta) r_1^2 r_2\right] \,. \tag{2.134}$$

Note that the matter density parameter Ω_m can be eliminated by using the relation (2.121). The evolution of c_{s1}^2 in three asymptotic regimes is given by [20, 21]:

$$c_{s1}^2 = \begin{cases} \frac{1}{40}(\Omega_r + 1) & [\text{(i) } r_1 \ll 1,\ r_2 \ll 1]\,, \\ \frac{8 + 10\alpha - 9\beta + \Omega_r(2 + 3\alpha - 3\beta)}{3(2 - 3\alpha + 6\beta)} & [\text{(ii) } r_1 = 1,\ r_2 \ll 1]\,, \\ \frac{(\alpha - 2\beta)(4 + 15\alpha^2 - 48\alpha\beta + 36\beta^2)}{2(2 + 3\alpha - 6\beta)(2 - 3\alpha + 6\beta)} & [\text{(iii) } r_1 = 1,\ r_2 = 1]\,. \end{cases} \tag{2.135}$$

In the regime (i) we have $c_{s1}^2 = 1/20$ and $1/40$ during the radiation and matter eras, respectively, so there is no Laplacian instability. If the solutions enter the tracking regime (ii) during the radiation era, we require the stability condition $10 + 13\alpha - 12\beta > 0$. Taking into account the condition (2.129), the de Sitter fixed point (iii) is stable for $\alpha > 2\beta$. The theoretically viable parameter space is shown in figure 1 of Ref. [20]. The evolution of matter density perturbations and observational constraints on the covariant Galileon from large-scale structures have been studied in Ref. [87].

Model (B)

In the case of Model (B), we need to solve the coupled equation (2.92) for $N = 3$, i.e.,

$$\left(c_s^2 K_{11} - G_{11}\right)\left(c_s^2 K_{22} - G_{22}\right)\left(c_s^2 K_{33} - G_{33}\right) - \left(c_s^2 K_{12} - G_{12}\right)^2\left(c_s^2 K_{33} - G_{33}\right) \\ - \left(c_s^2 K_{13} - G_{13}\right)^2\left(c_s^2 K_{22} - G_{22}\right) = 0\,. \tag{2.136}$$

The solutions to this third-order equation for c_s^2 are given by

$$c_s^2 = -\frac{a_2}{3a_1} + u_+ + u_-\,, \quad -\frac{a_2}{3a_1} + u_+\omega + u_-\omega^2\,, \quad -\frac{a_2}{3a_1} + u_+\omega^2 + u_-\omega\,, \tag{2.137}$$

where $\omega = -(1 + \sqrt{3}\,i)/2$, $u_\pm = [(-q \pm \sqrt{q^2 + 4p^3/27})/2]^{1/3}$, $p = a_3/a_1 - a_2^2/(3a_1^2)$, $q = 2a_2^3/(27a_1^3) - a_2 a_3/(3a_1^2) + a_4/a_1$, and

$$a_1 = K_{11}K_{22}K_{33} - K_{12}^2 K_{33} - K_{13}^2 K_{22}\,, \tag{2.138}$$

$$a_2 = K_{12}^2 G_{33} + K_{13}^2 G_{22} - K_{11}K_{22}G_{33} - K_{11}G_{22}K_{33} - G_{11}K_{22}K_{33} \\ + 2K_{12}G_{12}K_{33} + 2K_{13}G_{13}K_{22}\,, \tag{2.139}$$

$$
\begin{aligned}
a_3 &= K_{11}G_{22}G_{33} + G_{11}K_{22}G_{33} + G_{11}G_{22}K_{33} - G_{12}^2K_{33} - G_{13}^2K_{22} \\
&\quad -2K_{12}G_{12}G_{33} - 2K_{13}G_{13}G_{22}\,, \qquad (2.140) \\
a_4 &= G_{12}^2G_{33} + G_{13}^2G_{22} - G_{11}G_{22}G_{33}\,. \qquad (2.141)
\end{aligned}
$$

When $q^2 + 4p^3/27 < 0$, all the solutions (2.137) are real.

One of the solutions c_{s1}^2 in Eq. (2.137) is associated with the propagation speed squared of the dark energy field ϕ. In three asymptotic regimes it is given by

$$
c_{s1}^2 = \begin{cases} \dfrac{1}{40}(3\Omega_r - 1) & [\text{(i)}\ r_1 \ll 1,\ r_2 \ll 1]\,, \\ \dfrac{16 - 15(\alpha - 2\beta) + \Omega_r(4 - 3\alpha + 6\beta)}{6(2 - 3\alpha + 6\beta)} & [\text{(ii)}\ r_1 = 1,\ r_2 \ll 1]\,, \\ \dfrac{\alpha - 2\beta}{2 + 3\alpha - 6\beta} & [\text{(iii)}\ r_1 = 1,\ r_2 = 1]\,. \end{cases} \qquad (2.142)
$$

Under the no-ghost condition (2.129), the propagation speed squared c_{s1}^2 in the regime (ii) is positive. The de Sitter fixed point (iii) is stable for

$$
\alpha > 2\beta\,. \qquad (2.143)
$$

In the regime (i) we have $c_{s1}^2 = 1/20$ for $\Omega_r = 1$, but $c_{s1}^2 = -1/40$ for $\Omega_r = 0$. This means that the perturbations are plagued by short-scale Laplacian instabilities during the matter era for late-time tracking solutions. As long as the solutions approach the tracker by the end of the radiation era, it is possible to avoid the Laplacian instability of scalar perturbations. We recall however that only the background trajectories approaching the tracker around the end of the matter era are consistent with the joint data analysis of Sn Ia, CMB, and BAO [86]. Then the solutions need to be in the regime (i) during most of the matter era, in which case the Laplacian instability cannot be avoided.

In the regime (i), the quantity $c_{s\mathrm{H}1}$ defined by Eq. (2.97) evolves as

$$
c_{s\mathrm{H}1}^2 = \frac{1}{40}\left(7\Omega_r + 11\right)\,, \qquad (2.144)
$$

which is positive. The difference between (2.144) and $c_s^2 = (3\Omega_r - 1)/40$ in Eq. (2.142) should be induced from the term $\delta\mathcal{C}_3$ in Eq. (2.101). This term can be expressed as

$$
\delta\mathcal{C}_3 = -\frac{3r_2(\alpha r_1 - 2\beta)}{2 + 3r_2(\alpha r_1 - 2\beta)}\,, \qquad (2.145)
$$

which means that $|\delta\mathcal{C}_3| \ll 1$ in the regimes (i) and (ii). For radiation, the quantity ξ_1 defined by Eq. (2.105) evolves as $\xi_1 = -2/(15\beta r_2)$ in the regime (i) and hence $|\xi_1| \gg 1$. From Eq. (2.104) we then have $\delta c_{s1}^2 = -2/5$ during the radiation era, so that $c_{s1}^2 = c_{s\mathrm{H}1}^2 + \delta c_{s1}^2 = 1/20$. For non-relativistic matter, the evolution of the quantity ξ_2 is given by $\xi_2 = -1/(10\beta r_2)$ and hence $\delta c_{s1}^2 = -3/10$ during the regime (i) of the matter era. Hence we obtain the negative propagation speed squared $c_{s1}^2 = c_{s\mathrm{H}1}^2 + \delta c_{s1}^2 = -1/40$. Interestingly, even if the difference between $\mathcal{C}_3$ and $\mathcal{C}_{3\mathrm{H}}$ is small in the regime (i), the modification to $c_{s\mathrm{H}1}^2$ cannot be negligible. From Eq. (2.106) the corrections δc_{sI}^2 $(I = 2, 3)$ to $c_{s\mathrm{H}2}^2$ and $c_{s\mathrm{H}3}^2$ of radiation and non-relativistic matter are suppressed relative to δc_{s1}^2 by the additional factor $\delta\mathcal{C}_3$, so the deviations of c_{s2}^2 and c_{s3}^2 from the values (2.116) and (2.118) are very small in the regime (i).

Evaluating the term ξ_2 for non-relativistic matter along the tracker ($r_1 = 1$), the correction δc_{s1}^2 to $c_{s\mathrm{H}1}^2$ after the onset of the matter-dominated epoch is given by

$$\delta c_{s1}^2 \simeq \frac{3(\alpha - 2\beta)(1 - r_2)}{(2 - 3\alpha + 6\beta)(1 + r_2)} \,. \tag{2.146}$$

In the regime (ii) we have $\delta c_{s1}^2 \simeq 3(\alpha - 2\beta)/(2 - 3\alpha + 6\beta)$, while δc_{s1}^2 vanishes at the de Sitter point (iii). From Eq. (2.106) the correction δc_{s3}^2 to the matter sound speed squared $c_{s\mathrm{H}3}^2$ $(= O(w_m) \ll 1)$ on the tracker can be estimated as

$$\delta c_{s3}^2 \simeq \frac{c_{s\mathrm{H}3}^2}{c_{s\mathrm{H}3}^2 - c_{s\mathrm{H}1}^2 - \delta c_{s1}^2} \frac{9(\alpha - 2\beta)^2 r_2 \Omega_m}{2[2 + 3r_2(\alpha - 2\beta)](1 + r_2)(2 - 3\alpha + 6\beta)} \,, \tag{2.147}$$

which is suppressed both in the regimes (ii) and (iii). The correction δc_{s3}^2 can provide some contribution to $c_{s\mathrm{H}3}^2$ around $r_2 = O(0.1)$, but δc_{s3}^2 is still much smaller than 1 due to the multiplication of the small term $c_{s\mathrm{H}3}^2$ in Eq. (2.147).

In Fig. 2.3 we plot the evolution of the scalar propagation speeds squared c_{s1}^2, c_{s2}^2, and c_{s3}^3 for the initial conditions $r_1 \ll 1$ and $r_2 \ll 1$ in the deep radiation-dominated epoch. In this case, the dark energy equation of state w_{DE} starts to evolve from $-1/4$ (radiation era) to $-1/8$ (matter era) and then it reaches a minimum -1.1 around the redshift $z = 0.3$. This late-time tracking behavior is consistent with the observational data of Sn Ia, CMB, and BAO at the background level [86].

From Fig. 2.3 we find that the first propagation speed squared c_{s1}^2 evolves from the value $1/20$ in the radiation era, which is followed by the decrease to the value close to $-1/40$ in the matter era. The solution stays in the

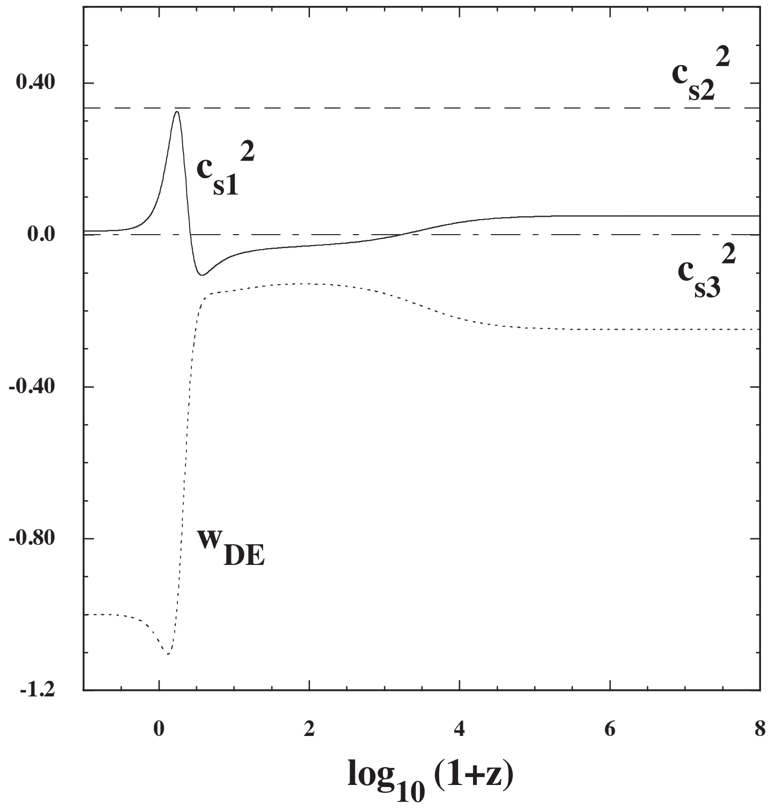

Figure 2.3: Evolution of the scalar propagation speeds squared $c_{s1}^2, c_{s2}^2, c_{s3}^2$ and the dark energy equation of state $w_{\rm DE}$ versus the redshift $z = 1/a - 1$ in Model (B). We choose the model parameters $\alpha = 0.3$ and $\beta = 0.14$ with the initial conditions $r_1 = 5 \times 10^{-11}$, $r_2 = 8 \times 10^{-12}$, $\Omega_r = 0.999995$, and $w_m = 10^{-3}$ at $z = 6.0 \times 10^8$. This case corresponds to the late-time tracking solution that approaches the tracker ($r_1 = 1$) at low redshifts. During most of the radiation and matter eras the solution is in the regime $r_1 \ll 1$ and $r_2 \ll 1$, in which case the first propagation speed squared is given by $c_{s1}^2 \simeq (3\Omega_r - 1)/40$.

regime (i) during most of the matter-dominated epoch. The period during which the solution is in the regime (ii) is short, so c_{s1}^2 soon approaches the value 9.7×10^{-3} at the de Sitter fixed point (iii) after its temporal variation around $z \lesssim O(1)$. Figure 2.3 shows that the sound speeds squared c_{s2}^2 and c_{s3}^3 of radiation and non-relativistic matter are close to the values 1/3 and 0, respectively. This result is consistent with the analytic estimation given above. For the parameters used in the numerical simulations of Fig. 2.3, we find that the deviation of c_{s2}^2 from the value 1/3 is less than the order of 10^{-4} in the matter era.

2.6 Conclusions

We have studied the cosmology of the recently proposed generalized Horndeski theories on the flat FLRW background. The Lagrangian of these theories is simply expressed in terms of three-dimensional scalar quantities constructed in the 3+1 ADM decomposition of space-time. In Horndeski theories there are particular relations (2.54) between the coefficients A_i and B_i, but GLPV theories are not subject to this restriction. On the isotropic cosmological background, the perturbation equations of motion in GLPV theories are of second order with one scalar degree of freedom.

In the presence of multiple scalar fields ϕ_I ($I = 1, 2, \cdots, N-1$) described by the Lagrangians $P^{(I)}(X_I)$, we have expanded the action (2.57) up to quadratic order in perturbations of the ADM scalar quantities. We have in mind the application to dark energy with additional perfect fluids of radiation and non-relativistic matter. The second-order action for tensor perturbations is given by Eq. (2.62) with the propagation speed squared $c_t^2 = \mathcal{E}/L_{\mathcal{S}}$, so the tensor ghosts and Laplacian instabilities are absent for $L_{\mathcal{S}} > 0$ and $\mathcal{E} > 0$. We have derived the second-order Lagrangian density for scalar perturbations of the form (2.81), which explicitly shows the absence of derivatives higher than second order.

The positivity of the $N \times N$ matrix $\boldsymbol{K}$ implies that the scalar ghosts do not appear under the conditions (2.88). The scalar propagation speeds c_s obey the algebraic equation (2.92). In Horndeski theories this equation can be written as the separate form (2.96), so the solutions to c_s^2 are simply given by Eqs. (2.97)-(2.98). In GLPV theories the propagation speeds squared are coupled each other in the form (2.99), whose right hand side vanishes in the Horndeski limit. Under the condition that the deviation of the term $\mathcal{C}_3$ from the Horndeski value $-16L_S^2/\mathcal{W}$ is small, we have estimated the propagation speeds in Eqs. (2.102)-(2.106). Compared to the modification δc_{s1}^2 to the first sound speed squared $c_{s\mathrm{H}1}^2$ associated with the dark energy field χ, the corrections δc_{sI}^2 $(I = 2, 3, \cdots, N)$ to the matter sound speeds squared $c_{s\mathrm{H}I}^2$ are generally suppressed.

We have applied our results in Sec. 2.4 to the cosmology based on the covariantized Galileon (a class of GLPV theories) and the covariant Galileon (a class of Horndeski theories) in the presence of perfect fluids of radiation and non-relativistic matter. These two theories give rise to the background equations of motion exactly the same as each other, so we cannot distinguish them at the background level.

At the level of perturbations, however, different choices of the functions B_4, B_5 give rise to different values of $\mathcal{E}$, $\mathcal{C}_3$ defined respectively by Eqs. (2.63) and (2.86). As a consequence, the first scalar propagation speed squared $c_{s\mathrm{H}1}^2$ in Eq. (2.97) differ in these two theories. Moreover, in GLPV theories, there is a correction term δc_{s1}^2 to $c_{s\mathrm{H}1}^2$ estimated approximately by Eq. (2.104). Indeed the first scalar propagation speed squared c_{s1}^2 in the covariantized Galileon becomes negative $(-1/40)$ in the deep matter era for late-time tracking solutions, while in the covariant Galileon $c_{s1}^2 = c_{s\mathrm{H}1}^2 = 1/40$. Hence the former is plagued by the small-scale instability problem of dark energy perturbations, while the latter has a theoretically consistent parameter space. The matter sound speeds squared of radiation and non-relativistic matter for the covariantized Galileon are close to the values (2.98) in the Horndeski limit.

We have thus provided a general scheme for studying the evolution of background and perturbations in dark energy models based on GLPV theories. These results will be useful in both placing model-independent constraints on the properties of dark energy/modified gravity and in imposing bounds on individual models. For the latter, it may be of interest to search for theoretically and observationally allowed parameter spaces in the covariantized version of the extended Galileon scenario advocated in Refs. [88, 89].

Chapter 3

Effective field theory of modified gravity on the spherically symmetric background

In the previous chapter we studied the EFT of modified gravity on the cosmological background. That framework is useful to study the cosmological dynamics of the dark energy model based on modified gravity in addition to their stability conditions in a systematic and unified way.

Although the modification of gravity realizes the late-time cosmic acceleration, there must be a mechanism which suppresses the modification of gravity at short distances since local gravity tests in the Solar System agree with general relativity in high precision. In order to understand this mechanism and confront several models with the Solar System constraint, one need to study modified gravity on the spherically symmetric background. The EFT of modified gravity on the spherically symmetric background [40] can be a powerful framework to study these screening mechanism in a systematic and unified way.

3.1 The basis of the screening mechanism

Before investigating the EFT of modified gravity on the spherically symmetric background, we briefly review the screening mechanism in this section.

We start with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} e^{-2Q\phi/M_{\rm pl}} R + \frac{f(\phi)}{2} X - \frac{g(\phi)}{2M^3} X \Box \phi \right] + S^M \,, \qquad (3.1)$$

where the exponential coupling in the first term is related with a dilaton field appearing in the low-energy effective string theory [90], for instance, and Q is a coupling constant of the order of unity. The third term in Eq. (3.1) is the so-called Galileon term appearing in the covariant Galileon as L_3. Due to the existence of this non-linear term the Vainshtein mechanism can be at work and the modification of gravity can be suppressed inside the Vainshtein radius r_V. We assume that$f(\phi)$ and $g(\phi)$ are slowly varying dimensionless functions of the order of unity, such that

$$|M_{\rm pl} f_{,\phi}/f| \lesssim 1 \,, \qquad |M_{\rm pl} g_{,\phi}/g| \lesssim 1 \,. \qquad (3.2)$$

On the spherically symmetric background the line element is given as

$$ds^2 = -e^{2\Psi(r)} dt^2 + e^{2\Phi(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \,, \qquad (3.3)$$

where $\Psi(r)$ and $\Phi(r)$ are functions with respect to the distance r from the center of symmetry. Varying the action (3.1) with respect to the metric one can derive equations of motion on the spherically symmetric background. On the weak gravitational background characterized by the conditions $|\Phi| \ll 1$ and $|\Psi| \ll 1$, and assuming that the terms appearing from the modification of gravity should be suppressed at short distances, the equation of motion for the scalar field reduce to

$$\begin{aligned} \frac{d}{dr}(r^2 \phi') = & \frac{r[2QF/M_{\rm pl} + (f - 2M^{-3} g_{,\phi} X)\phi' r - 5M^{-3} g X]}{2F[(f - 2M^{-3} g_{,\phi} X) r - 4M^{-3} g \phi']} \rho_m r^2 \\ & - \frac{f_{,\phi} X r^2 + 4M^{-3} g_{,\phi} X \phi' r + [(f_{,\phi} - M^{-3} g_{,\phi\phi} X/2) r^2 + 6M^{-3} g]\phi'^2}{(f - 2M^{-3} g_{,\phi} X) r - 4M^{-3} g \phi'} r \,. \end{aligned} \qquad (3.4)$$

The qualitative behavior of solutions in Eq. (3.4) is different depending on the radius r. The behavior of solutions changes at the radius r_V characterized by

$$\left| f - 4M^{-3} g_{,\phi} X(r_V) \right| r_V = 4M^{-3} \left| g\phi'(r_V) \right| \,, \qquad (3.5)$$

Hence there should be thwo different regimes: (a) $r \gg r_V$, (b) $r \ll r_V$. In the following we shall derive the solutions to Eq. (3.4) in each regime.

(a) $r \gg r_V$

In this regime linear terms coming from the first and the second term in Eq. (3.1) give dominant contributions to Eq. (3.4). Then it reduces as

$$\frac{d}{dr}(r^2\phi') \simeq \frac{Q}{M_{\rm pl} f}\rho_m r^2 \,. \tag{3.6}$$

As long as the function f is nearly constant Eq. (3.6) is integrated to give

$$\phi'(r) \simeq \frac{QM_{\rm pl}}{f}\frac{r_g}{r^2} \,, \tag{3.7}$$

where we have introduced the Schwarzschild radius r_g of the source, as

$$r_g \equiv \frac{1}{M_{\rm pl}^2}\int_0^r \rho_m \tilde{r}^2 d\tilde{r} \,. \tag{3.8}$$

Substituting the solution (3.7) into the equations of motion for the metric field, i.e. modified Einstein equations, one can show that the effective gravitational coupling can be represented as $G_{\rm eff} \simeq G_N(1 + 2Q^2/f)$ where G_N is Newton's gravitational constant [61]. This correction term is of the order of unity as long as Q and f are of $\mathcal{O}(1)$ so that the gravitational coupling is strongly modified relative to GR in this regime.

Under the approximation that the solution (3.7) is valid at $r = r_V$, it follows that

$$r_V \simeq \left|\frac{4gQM_{\rm pl}r_g}{f^2M^3}\right|^{1/3} \,. \tag{3.9}$$

If the model (3.1) is responsible for the late-time cosmic acceleration the mass M is related to the today's Hubble radius $r_c = H_0^{-1} \approx 10^{28}\,{\rm cm}$ via $M \approx (M_{\rm pl}^{-1}r_c^2)^{-1/3}$. Using this relation and assuming f and g are of the order of unity Eq. (3.9) reduces to $r_V \approx (|Q|r_g^2r_c^2)^{1/3}$. For the Sun ($r_g \approx 10^5\,{\rm cm}$) one has $r_V \approx 10^{20}\,{\rm cm}$ for $|Q| = \mathcal{O}(1)$ which is much larger than the Solar System scale.

(b) $r \ll r_V$

In this regime the non-linear self interaction term, i.e. the Galileon term, gives the dominant contribution to Eq. (3.4), and it simply reduces as

$$\frac{d}{dr}(r^2\phi') \simeq \frac{3}{2}r\phi' \,. \tag{3.10}$$

Integrating Eq. (3.10) and matching with the exterior solution (3.7) at $r = r_V$ we obtain

$$\phi'(r) \simeq \frac{QM_{\rm pl} r_g}{f r_V^2} \left(\frac{r}{r_V}\right)^{-1/2} . \tag{3.11}$$

Compared to the solution (3.7) the field derivative varies more slowly in the regime $r \ll r_V$. This is the region in which the Vainshtein mechanism is at work.

Substituting the interior solution (3.11) into the modified Einstein equations and solving them for $\Phi(r)$ and $\Psi(r)$, one can estimate the modifications to the Newtonian gravitational potentials. In Ref. [61] we obtained

$$\Phi \simeq \frac{r_g}{2r}\left[1 - \frac{2Q^2}{f}\left(\frac{r}{r_V}\right)^{3/2}\right] , \tag{3.12}$$

$$\Psi \simeq -\frac{r_g}{2r}\left[1 - \frac{4Q^2}{f}\left(\frac{r}{r_V}\right)^{3/2}\right] . \tag{3.13}$$

Clearly the second terms on the r.h.s. of the square brackets of Eqs. (3.12) and (3.13) are much smaller than unity in the regime $r \ll r_V$, so that the fifth force is suppressed.

We define the post-Newtonian parameter γ, as

$$\gamma \equiv -\Phi/\Psi . \tag{3.14}$$

The present tightest experimental bound on γ is $|\gamma - 1| < 2.3 \times 10^{-5}$ [91]. Using the solutions (3.12) and (3.13) this constraint translates into

$$\frac{2Q^2}{|f|}\left(\frac{r}{r_V}\right)^{3/2} < 2.3 \times 10^{-5} . \tag{3.15}$$

For r much less than r_V the bound (3.15) can be satisfied even for $|Q| = \mathcal{O}(1)$.

In the following sections we investigate the EFT of modified gravity on the spherically symmetric background. This framework can be used, for instance, to study the screening mechanism such as the Vainshtein mechanism we briefly reviewed for the model (3.1) in this section.

3.2 The 2+1+1 formalism

We assume that the 4-dimensional space-time allows for a double foliation in the 2+1+1 formalism, e.g., it can be foliated both by constant time hypersurfaces Σ_t and by constant spatial coordinate hypersurfaces Σ_r. The time-like unit congruence n^a (satisfying $n^a n_a = -1$) is orthogonal to Σ_t, while the unit vector l^a of the singled-out spatial direction (satisfying $l^a l_a = 1$) is orthogonal to Σ_r. For convenience we choose them mutually orthogonal ($n^a l_a = 0$). The 2-surface orthogonal to both congruences is labeled as Σ_{tr}.

In terms of the 4-dimensional metric g_{ab} and the unit vectors mentioned above, the induced metric h_{ab} on the 2-dimensional space is given by

$$h_{ab} = g_{ab} + n_a n_b - l_a l_b \,, \tag{3.16}$$

which satisfies the orthogonal relations $n^a h_{ab} = 0$ and $l^a h_{ab} = 0$.

Time evolution proceeds along the integral lines of the congruence

$$\left(\frac{\partial}{\partial t}\right)^a = N n^a + N^a \,, \tag{3.17}$$

where N^a and N are the shift vector and lapse function related to the foliation Σ_t. The singled-out spatial evolution proceeds along

$$\left(\frac{\partial}{\partial r}\right)^a = M l^a + M^a \,, \tag{3.18}$$

where M^a and M represent the shift vector and lapse function associated with the singled-out spatial direction. In contrast with N and N^a, the scalar M and vector M^a represent true gravitational degrees of freedom, contributing to the spatial 3-metric.

The shift vectors are restricted as $M^a n_a = M^a l_a = N^a n_a = N^a l_a = 0$, so they have only two independent components each. The gravitational sector is described by $\{h_{ab}, M^a, M, N^a, N\}$, a set of one variable less than the number of variables contained in g_{ab}. This is because mutually perpendicular foliations are chosen through the condition $n^a l_a = 0$.

In order to accommodate the possibility that the perturbations may affect the perpendicularity of the foliations, we consider the 4-dimensional metric in the system adapted to the coordinates (t, r, x^a) (here x^a being coordinates adapted to Σ_{tr}) in full generality [73]:

$$\begin{aligned} ds^2 &= (N_a N^a + \mathcal{N}^2 - N^2) dt^2 + 2 N_a dt dx^a + 2\,(N_a M^a + \mathcal{N} M)\, dt dr \\ &\quad + h_{ab} dx^a dx^b + 2 M_a dx^a dr + (M_a M^a + M^2) dr^2 \,. \end{aligned} \tag{3.19}$$

The requirement of a double foliation (of both vector fields n and l being vorticity-free), as shown in the Appendix of Ref. [73], imposes a proportionality of the metric functions M and $\mathcal{N}$. The easiest way to obey this constraint is to chose $\bar{\mathcal{N}} = 0$, equivalent to the perpendicularity of the foliations on the background. The latter condition can be fulfilled even in the presence of the perturbations by a suitable gauge fixing:

$$\mathcal{N} = 0\,. \tag{3.20}$$

The embedding of the co-dimension 2 surfaces is characterized by two types of extrinsic curvatures, related to each of the normal vector fields n^a and l^a:

$$K_{ab} = h^c{}_a h^d{}_b \nabla_c n_d\ , \qquad L_{ab} = h^c{}_a h^d{}_b \nabla_c l_d\ , \tag{3.21}$$

where ∇ denotes the g-metric compatible connection.

There are also two normal fundamental forms

$$\mathcal{K}_a = h^b{}_a l^c \nabla_c n_b\ , \qquad \mathcal{L}_a = -h^b{}_a n^c \nabla_c l_b\ , \tag{3.22}$$

and two normal fundamental scalars

$$\mathcal{K} = l^a l^b \nabla_a n_b\,, \qquad \mathcal{L} = n^a n^b \nabla_a l_b \tag{3.23}$$

to consider[1]. To summarize, the covariant derivatives of the normal vectors can be expressed as

$$\begin{aligned} \nabla_a n_b &= K_{ab} + l_a \mathcal{K}_b + l_b \mathcal{K}_a + l_a l_b \mathcal{K} + n_a \alpha_b\,, &(3.24)\\ \nabla_a l_b &= L_{ab} + n_a \mathcal{L}_b + n_b \mathcal{L}_a + n_a n_b \mathcal{L} + l_a \beta_b\,, &(3.25) \end{aligned}$$

where α^a and β^a are the curvatures of the congruences n^a and l^a, defined by

$$\begin{aligned} \alpha^b &= n^c \nabla_c n^b = D^b (\ln N) - \mathcal{L} l^b\ , &(3.26)\\ \beta^b &= l^c \nabla_c l^b = -D^b (\ln M) + \mathcal{K} n^b\,. &(3.27) \end{aligned}$$

Occasionally, both α^a and β^a will be referred as accelerations.

From the symmetric property of the extrinsic curvatures and the relation $n^a l_a = 0$, it has been shown in Ref. [73] that $\mathcal{K}_a = \mathcal{L}_a$ holds. The quantities L_{ab} and $\mathcal{L}$ are expressed in terms of r-derivatives and the covariant derivatives D_a associated with h_{ab} as [73]

$$\begin{aligned} L_{ab} &= \frac{1}{2M}\left(\frac{\partial h_{ab}}{\partial r} - 2D_{(a}M_{b)}\right)\ , &(3.28)\\ \mathcal{L} &= -\frac{1}{MN}\left(\frac{\partial N}{\partial r} - M^a D_a N\right)\ . &(3.29) \end{aligned}$$

[1]The sets (K_{ab}, $\mathcal{K}_a$, $\mathcal{K}$) and (L_{ab}, $\mathcal{L}_a$, $\mathcal{L}$) can also be interpreted as the tensorial, vectorial and scalar contributions in the 2+1 split of the extrinsic curvatures of the hypersurfaces perpendicular to the congruences n^a and l^a, respectively.

Hence they are just convenient abbreviations for spatial derivatives.

By contrast, the quantities K_{ab}, $\mathcal{K}^a$ and $\mathcal{K}$ give the time evolution of h_{ab}, M^a and M, respectively [73]:

$$
\begin{aligned}
K_{ab} &= \frac{1}{2N}\left(\frac{\partial h_{ab}}{\partial t} - 2D_{(a}N_{b)}\right) , && (3.30a)\\
\mathcal{K}^a &= \frac{1}{2MN}\left(\frac{\partial M^a}{\partial t} - \frac{\partial N^a}{\partial r} + M^b D_b N^a - N^b D_b M^a\right) , && (3.30b)\\
\mathcal{K} &= \frac{1}{MN}\left(\frac{\partial M}{\partial t} - N^a D_a M\right) , && (3.30c)
\end{aligned}
$$

so that they are velocity-type variables.

Thus the coordinates in the velocity phase-space are

$$
\{h_{ab}, M^a, M; K_{ab}, \mathcal{K}^a, \mathcal{K}\} . \qquad (3.31)
$$

This is a feature that any 2+1+1 covariant Lagrangian description of modified gravity should take into account.

Note that the time and spatial derivatives along the singled-out directions of any tensor $T^{a_1...a_r}_{b_1...b_s}$ which has vanishing contraction with both n^a and l^a in all indices are defined as projected Lie-derivatives [73, 74]:

$$
\begin{aligned}
\frac{\partial}{\partial t} T^{a_1...a_r}_{b_1...b_s} &\equiv h^{a_1}_{c_1}...h^{a_r}_{c_r} h^{d_1}_{b_1}...h^{d_s}_{b_s}\, {}^{(4)}\mathcal{L}_{\frac{\partial}{\partial t}} T^{c_1...c_r}_{d_1...d_s} \equiv \mathcal{L}_{\frac{\partial}{\partial t}} T^{a_1...a_r}_{b_1...b_s} \\
&= N \mathcal{L}_{\boldsymbol{n}} T^{a_1...a_r}_{b_1...b_s} + \mathcal{L}_{\boldsymbol{N}} T^{a_1...a_r}_{b_1...b_s} , && (3.32)\\
\frac{\partial}{\partial r} T^{a_1...a_r}_{b_1...b_s} &\equiv h^{a_1}_{c_1}...h^{a_r}_{c_r} h^{d_1}_{b_1}...h^{d_s}_{b_s}\, {}^{(4)}\mathcal{L}_{\frac{\partial}{\partial r}} T^{c_1...c_r}_{d_1...d_s} \equiv \mathcal{L}_{\frac{\partial}{\partial r}} T^{a_1...a_r}_{b_1...b_s} \\
&= M \mathcal{L}_{\boldsymbol{l}} T^{a_1...a_r}_{b_1...b_s} + \mathcal{L}_{\boldsymbol{M}} T^{a_1...a_r}_{b_1...b_s} , && (3.33)
\end{aligned}
$$

where ${}^{(4)}\mathcal{L}_{\boldsymbol{V}}$ and $\mathcal{L}_{\boldsymbol{V}}$ hold for the 4-dimensional and 2-dimensional Lie-derivatives along any vector congruence $\boldsymbol{V}$. For a scalar quantity S, one has

$$
\begin{aligned}
\frac{\partial}{\partial t} S &= N\left(n^a \nabla_a\right) S + N^a D_a S , && (3.34)\\
\frac{\partial}{\partial r} S &= M\left(l^a \nabla_a\right) S + M^a D_a S . && (3.35)
\end{aligned}
$$

From the above expressions, it is immediate to see that the time and spatial derivatives along the singled-out direction of scalars which are constant on Σ_{tr} (such that the last terms in Eqs. (3.34) and (3.35) vanish) are also expressible as projected covariant derivatives, a property we will employ in what follows.

3.3 Equations of motion on the spherically symmetric background

We consider general gravitational theories with a single scalar degree of freedom ϕ. On the background the scalar field has only radial dependence. As will be discussed in detail in Sec. 3.5, we choose a radial unitary gauge $\phi = \phi(r)$. Then the kinetic term of the scalar field can be expressed in terms of the radial lapse M and the radial derivative of the field. Hence we render the scalar field into the gravitational sector (the radial lapse) and into the explicit radial dependence of the action. We will therefore consider an action principle with the Lagrangian depending on variables constructed from the metric alone, however with explicit radial dependence allowed.

3.3.1 Action principle

We elaborate the variational principle developed for a cosmological setup [52] in a way that it applies to a spherically symmetric background. For this purpose we employ scalar quantities related to the velocity phase-space variables (3.31) emerging in the 2+1+1 decomposition. We introduce the gravitational action

$$S^{\mathrm{EFT}} = \int d^4x\sqrt{-g}\, L^{\mathrm{EFT}}\left(N, \mathcal{L}; M, \mathcal{K}; \mathfrak{M}, \mathfrak{K}; \mathcal{R}, K, \varkappa, L, \lambda; r\right) , \qquad (3.36)$$

where we have denoted the gravitational Lagrangian by L^{EFT} and

$$\begin{aligned} \mathcal{R} &\equiv {}^{(2)}R^{a}{}_{a}\,, \qquad \mathfrak{M} \equiv M_a M^a\,, \qquad \mathfrak{K} \equiv \mathcal{K}_a\mathcal{K}^a = \mathcal{L}_a\mathcal{L}^a\,, \\ K &\equiv K^{a}{}_{a}\,, \qquad \varkappa \equiv K^{a}{}_{b}K^{b}{}_{a}\,, \qquad L \equiv L^{a}{}_{a}\,, \qquad \lambda \equiv L^{a}{}_{b}L^{b}{}_{a}\,. \end{aligned} \qquad (3.37)$$

Here ${}^{(2)}R_{ab}$ is the 2-dimensional Ricci tensor.

The action (3.36) depends on the lapse and the velocity phase-space variables (3.31) discussed in the previous section. Symmetry allows us to use fewer variables. While the scalar sector $\{M; \mathcal{K}\}$ is fully included, the vectorial sector $\{M^a; \mathcal{K}^a\}$ appears through the quantities $\{\mathfrak{M}; \mathfrak{K}\}$. The tensorial sector $\{h_{ab}; K_{ab}\}$ also appears through $\mathcal{R}$ (which in two dimensions is the only independent component of the Riemann curvature tensor constructed from h_{ab}) and the quantities $K, \varkappa$. Besides, the scalars $\{\mathcal{L}, L, \lambda\}$ formed from spatial derivatives of N and h_{ab} are also introduced.

In comparison to the corresponding action of the cosmological setup [52], the action (3.36) does not depend on the variable $\mathcal{Z} \equiv {}^{(2)}R_{ab}{}^{(2)}R^{ab}$, as in two dimensions the intrinsic curvature has only one degree of freedom. In particular, the relation ${}^{(2)}R_{ab} = (\mathcal{R}/2)\, h_{ab}$ holds, so that $\mathcal{Z} = \mathcal{R}^2/2$. By contrast, the extrinsic curvatures K_{ab} and L_{ab} of the 2-dimensional surface have two independent components respectively (related to the two sectional curvatures), hence we keep $\varkappa$ (denoted $\mathcal{S}$ in Ref. [52]) and the new variable λ.

In summary, we have taken into account scalars equivalent to the variables of the velocity phase-space. As the action contains a Lagrangian density $\sqrt{-g}\, L^{\mathrm{EFT}}$, scalars representing spatial derivatives have been also included. Instead of the induced 2-metric h_{ab}, we have included the 2-dimensional scalar curvature (in two dimensions the curvature generated by the metric is equivalent to the scalar curvature). Finally, we included the scalars $\varkappa$, $\mathfrak{K}$ and λ for later convenience, as they also appear in the 2+1+1 version of the twice contracted Gauss equation [73]:

$$\begin{aligned} \mathcal{R} \; = \; & {}^{(4)}R + K^2 - \varkappa - 2\mathfrak{K} + 2\mathcal{K}K - L^2 + \lambda + 2\mathcal{L}L + 2\alpha^b\beta_b \\ & + 2\nabla_a(\alpha^a - \beta^a - Kn^a + Ll^a) \; . \end{aligned} \tag{3.38}$$

Note that Eq. (3.38) contains a 4-dimensional covariant derivative and is not completely written in 2-dimensional language, but it is adequate for our work. The expression for the Ricci scalar fully translated into 2-dimensional language can be found in Ref. [74][2].

3.3.2 Background equations of motion

In what follows, we will proceed in deriving the equations of motion by taking variations of the action on a spherically symmetric and static background. Under the assumption of spherical symmetry, the line element (3.19) contains only two free functions of $(t,\, r)$ and it simplifies to

$$d\bar{s}^2 = -\bar{N}^2 dt^2 + \bar{M}^2 dr^2 + r^2 d\Omega^2 \,, \tag{3.39}$$

where $d\Omega^2 = d\theta^2 + (\sin^2\theta) d\varphi^2$ is the surface element of the unit sphere. Since $\bar{N}^a = \bar{M}^a = 0$, it follows that $\bar{\mathfrak{M}} = \bar{\mathfrak{K}} = 0$. The one-forms n_a and l_a are given

[2] After the change in notation, $(\mathcal{R},\, {}^{(4)}R,\, h_{ab}) \leftrightarrow (R\,,\, \tilde{R}\,,\, g_{ab})$, one can show the equivalence between Eq. (3.38) in this paper and Eq. (A1) in Ref. [74].

by $n_a = (-N, 0, 0, 0)$ and $l_a = (0, M, 0, 0)$, respectively. Note that these expressions of n_a and l_a stay valid to first order in the perturbations. The extrinsic curvatures obey $\bar{K}_{ab} = \bar{K}\bar{h}_{ab}/2$ and $\bar{L}_{ab} = \bar{L}\bar{h}_{ab}/2$, hence $\bar{\varkappa} = \bar{K}^2/2$ and $\bar{\lambda} = \bar{L}^2/2$. If the background is further time-independent, then the relations $\bar{K} = \bar{\varkappa} = \bar{\mathcal{K}} = 0$ hold. Other non-vanishing geometric quantities are given by

$$\bar{\mathcal{R}} = \frac{2}{r^2}\,, \qquad \bar{\mathcal{L}} = -\frac{\bar{N}'}{\bar{N}\bar{M}}\,, \qquad \bar{L} = \frac{2}{\bar{M}r}\,, \qquad \bar{\lambda} = \frac{2}{\bar{M}^2 r^2}\,. \tag{3.40}$$

We expand the action (3.36) up to second order in perturbations of the geometric scalar quantities. In doing so, we define the variation of the velocity phase-space variables in the action as the difference between the background and perturbed variables. In particular, we have

$$\delta\mathcal{R} \equiv \mathcal{R} - \frac{2}{r^2}\,, \qquad \delta\mathcal{L} \equiv \mathcal{L} + \frac{\bar{N}'}{\bar{N}\bar{M}}\,, \qquad \delta L \equiv L - \frac{2}{\bar{M}r}\,, \qquad \delta\lambda \equiv \lambda - \frac{2}{\bar{M}^2 r^2}\,, \tag{3.41}$$

and

$$\delta\mathfrak{K} \equiv \mathfrak{K}\,, \qquad \delta\varkappa \equiv \varkappa\,, \qquad \delta\mathfrak{M} \equiv \mathfrak{M}\,, \qquad \delta K \equiv K\,, \qquad \delta\mathcal{K} \equiv \mathcal{K}\,. \tag{3.42}$$

Alternatively, from the definitions of the variables, we obtain the following explicit expressions

$$\begin{aligned}
\delta\lambda &= L^a{}_b L^b{}_a - \bar{L}^a{}_b \bar{L}^b{}_a = \frac{2}{\bar{M}r}\delta L + \delta L^a{}_b \delta L^b{}_a\,, \\
\delta\mathfrak{M} &= M^a M_a - \bar{M}^a \bar{M}_a = \delta M_a \delta M^a\,, \\
\delta\mathfrak{K} &= \mathcal{K}^a \mathcal{K}_a - \bar{\mathcal{K}}^a \bar{\mathcal{K}}_a = \delta\mathcal{K}_a \delta\mathcal{K}^a\,, \\
\delta\varkappa &= K^a{}_b K^b{}_a - \bar{K}^a{}_b \bar{K}^b{}_a = \delta K^a{}_b \delta K^b{}_a\,.
\end{aligned} \tag{3.43}$$

Hence the variables $\mathfrak{M}$, $\mathfrak{K}$ and $\varkappa$ (which vanish on the background) are second order, while λ (non-vanishing on the background) is changed by the perturbations at both first and second order. We also see that the scalar variables λ and L are not independent at first-order accuracy.

Next we expand the Lagrangian in the action (3.36) up to first order in perturbations. In doing so, we keep in mind that $\mathfrak{M}$, $\mathfrak{K}$ and $\varkappa$ are second-order quantities, while at first order $\delta\lambda$ is related to δL. This leaves us with the following Taylor expansion:

$$\begin{aligned}
&L^{\rm EFT}\left(N, \mathcal{L}; M, \mathcal{K}; \mathfrak{M}, \mathfrak{K}; \mathcal{R}, K, \varkappa, L, \lambda; r\right) \\
&= \bar{L}^{\rm EFT} + L_N^{\rm EFT}\delta N + L_{\mathcal{L}}^{\rm EFT}\delta\mathcal{L} + L_M^{\rm EFT}\delta M + L_{\mathcal{K}}^{\rm EFT}\delta\mathcal{K} \\
&\quad + L_{\mathcal{R}}^{\rm EFT}\delta\mathcal{R} + L_K^{\rm EFT}\delta K + \mathcal{F}\delta L\,,
\end{aligned} \tag{3.44}$$

where we introduced the notations $L_G^{\rm EFT} \equiv \overline{\partial L^{\rm EFT}/\partial G}$ for any

$$G = N, \mathcal{L}; M, \mathcal{K}; \mathfrak{M}, \mathfrak{K}; \mathcal{R}, K, \varkappa, L, \lambda\,, \tag{3.45}$$

evaluated on the background, and

$$\mathcal{F} \equiv L_L^{\rm EFT} + \frac{2L_\lambda^{\rm EFT}}{\bar{M}r}\,. \tag{3.46}$$

In what follows, we explore further relations among the scalar variables. On using Eq. (3.16), we have that $L = h^{ab}\nabla_a l_b = \nabla_a l^a + \bar{\mathcal{L}} + \delta\mathcal{L}$. Integrating by parts the term $\sqrt{-g}\,\mathcal{F}\delta L$ in the action and dropping the total covariant divergence term, finally employing Eq. (3.35) and the expression (3.40) of $\bar{\mathcal{L}}$, we obtain

$$\begin{aligned}\int d^4x\sqrt{-g}\,\mathcal{F}\delta L &= -\int d^4x\sqrt{-g}\frac{\mathcal{F}'}{\bar{M}}\left(1-\frac{\delta M}{\bar{M}}\right) \\ &\quad+\int d^4x\sqrt{-g}\,\mathcal{F}\left(-\frac{\bar{N}'}{\bar{N}\bar{M}}+\delta\mathcal{L}-\frac{2}{r\bar{M}}\right)\,,\end{aligned} \tag{3.47}$$

where we have also expanded M^{-1} up to first order. In the same way, using $\delta K = K = \nabla_a n^a - \delta\mathcal{K}$, integrating by parts, dropping the total covariant divergence term and taking into account Eq. (3.34), it follows that

$$\int d^4x\sqrt{-g}\,L_K^{\rm EFT}\delta K = -\int d^4x\sqrt{-g}\,L_K^{\rm EFT}\delta\mathcal{K}\,. \tag{3.48}$$

Then the Lagrangian (3.44) is decomposed as

$$L^{\rm EFT} = \bar{L}_0^{\rm EFT} + \delta L^{\rm EFT}\,, \tag{3.49}$$

where we have denoted

$$\bar{L}_0^{\rm EFT} = \bar{L}^{\rm EFT} - \frac{\mathcal{F}'}{\bar{M}} - \frac{(\bar{N}'r + 2\bar{N})\mathcal{F}}{\bar{N}\bar{M}r}\,, \tag{3.50}$$

and

$$\begin{aligned}\delta L^{\rm EFT} &= L_N^{\rm EFT}\delta N + \left(L_{\mathcal{L}}^{\rm EFT} + \mathcal{F}\right)\delta\mathcal{L} + \left(L_M^{\rm EFT} + \frac{\mathcal{F}'}{\bar{M}^2}\right)\delta M \\ &\quad+ \left(L_{\mathcal{K}}^{\rm EFT} - L_K^{\rm EFT}\right)\delta\mathcal{K} + L_{\mathcal{R}}^{\rm EFT}\delta\mathcal{R}\,.\end{aligned} \tag{3.51}$$

It can be proven that the zeroth-order Lagrangians $\bar{L}_0^{\rm EFT}$ and $\bar{L}^{\rm EFT}$ differ only by a total covariant divergence, which can be dropped.

The Lagrangian density is given by $\mathscr{L} = \sqrt{-g}\, L^{\rm EFT}$, with $\sqrt{-g} = NM\sqrt{h}$ and $\sqrt{h} = r^2 \sin\theta$. It can be decomposed into a background contribution $\bar{\mathscr{L}}_0 = \sqrt{-\bar{g}}\, \bar{L}_0^{\rm EFT}$ and a first-order contribution $\delta\mathscr{L} = \mathscr{L} - \bar{\mathscr{L}}_0$ as follows:

$$\delta\mathscr{L} = \sqrt{-\bar{g}}\, \delta L^{\rm EFT} + \bar{L}_0^{\rm EFT}\, \delta\sqrt{-g}\,. \tag{3.52}$$

Up to first order in perturbations the metric is given by

$$\begin{aligned} ds_1^2 = &-\left(\bar{N}^2 + 2\bar{N}\delta N\right) dt^2 + 2\bar{M}\delta\mathcal{N} dt dr + 2\delta N_a dt dx^a \\ &+ \left(\bar{h}_{ab} + \delta h_{ab}\right) dx^a dx^b + 2\delta M_a dx^a dr + \left(\bar{M}^2 + 2\bar{M}\delta M\right) dr^2\,, \end{aligned} \tag{3.53}$$

and hence

$$\delta\sqrt{-g} = \frac{\sqrt{-\bar{g}}}{2}\bar{g}^{ab}\delta g_{ab} = \sqrt{-\bar{g}}\left(\frac{\delta N}{\bar{N}} + \frac{\delta M}{\bar{M}} + \frac{1}{2}\bar{h}^{ab}\delta h_{ab}\right)\,. \tag{3.54}$$

We assume the form $h_{ab} = e^{2\zeta}\bar{h}_{ab}$, where ζ is the curvature perturbation. This is consistent with allowing only scalar perturbations and suitably fixing the gauge, like in the cosmological case [52, 68], see also the discussion of scalar perturbations in Sec. 3.5. Hence the perturbed and unperturbed metrics are related by a conformal transformation and the respective curvature scalars can be expressed as

$$\mathcal{R} = e^{-2\zeta}\left(\bar{\mathcal{R}} - 2\bar{h}^{ab}\bar{D}_a\bar{D}_b\zeta\right)\,, \tag{3.55}$$

which to linear order gives

$$\delta\mathcal{R} = -2\zeta\bar{\mathcal{R}} - 2\bar{h}^{ab}\bar{D}_a\bar{D}_b\zeta\,. \tag{3.56}$$

In the generalized Stokes theorem, the integral of a differential form ω over the boundary of an oriented manifold $\mathcal{S}$ is equivalent to the integral of the exterior derivative of ω over the manifold $\mathcal{S}$, i.e. $\int_{\mathcal{S}} d\omega = \int_{\partial\mathcal{S}} \omega$. Since there is no boundary of a boundary, the rhs of the generalized Stokes theorem vanishes when $\mathcal{S}$ is some closed surface, e.g., the 2-sphere as in our case. Using this and integrating the second term on the rhs of Eq. (3.56), we obtain

$$\begin{aligned} &\int d^4x \sqrt{-g}\, L_{\mathcal{R}}^{\rm EFT}\left(-2\bar{h}^{ab}\bar{D}_a\bar{D}_b\zeta\right) \\ &= -2\int dt dr \bar{N}\bar{M} r^2 L_{\mathcal{R}}^{\rm EFT}\int d\theta\, d\varphi D_a\left(\sqrt{\bar{h}}\bar{h}^{ab}\bar{D}_b\zeta\right) = 0\,. \end{aligned} \tag{3.57}$$

Hence the variations in the scalar curvature and conformal factor are related by the simple expression

$$\delta\mathcal{R} = -\frac{4\zeta}{r^2}\,. \tag{3.58}$$

Remarkably, the same expression emerges for restricting to spherically symmetric perturbations. Non-spherically symmetric modes in the perturbations do not contribute to the background equations of motion.

Similarly, to linear order in perturbations, we obtain

$$\frac{1}{2}\bar{h}^{ab}\delta h_{ab} = 2\zeta\,, \tag{3.59}$$

which, when employing Eq. (3.58) and the first equation (3.40), becomes

$$\frac{1}{2}\bar{h}^{ab}\delta h_{ab} = -\frac{\delta\mathcal{R}}{\bar{\mathcal{R}}}\,. \tag{3.60}$$

With this, we have completed the program of rewriting the linear-order variation exclusively into terms containing the variation of the scalar variables in the action.

In what follows we further reduce this set at linear order. Substitution of Eqs. (3.54) and (3.60) into the first-order Lagrangian density (3.52) leads to

$$\begin{aligned}\delta\mathscr{L} \;=\; & \sqrt{-\bar{g}}\Bigg[L_N^{\mathrm{EFT}}\delta N + \left(L_{\mathcal{L}}^{\mathrm{EFT}} + \mathcal{F}\right)\delta\mathcal{L} + \left(L_M^{\mathrm{EFT}} + \frac{\mathcal{F}'}{\bar{M}^2}\right)\delta M \\ & + \left(L_{\mathcal{K}}^{\mathrm{EFT}} - L_K^{\mathrm{EFT}}\right)\delta\mathcal{K} + L_{\mathcal{R}}^{\mathrm{EFT}}\delta\mathcal{R}\Bigg] \\ & + \bar{L}_0^{\mathrm{EFT}}\sqrt{-\bar{g}}\left(\frac{\delta N}{\bar{N}} + \frac{\delta M}{\bar{M}} - \frac{\delta\mathcal{R}}{\bar{\mathcal{R}}}\right)\,.\end{aligned} \tag{3.61}$$

By using Eqs. (3.29) and (3.30c), it follows that

$$\delta\mathcal{L} = \frac{\bar{N}'}{\bar{N}\bar{M}}\left(-\frac{\delta N'}{\bar{N}'} + \frac{\delta N}{\bar{N}} + \frac{\delta M}{\bar{M}}\right)\,, \tag{3.62}$$

$$\delta\mathcal{K} = \frac{\delta\dot{M}}{\bar{N}\bar{M}}\,. \tag{3.63}$$

Plugging these expressions into Eq. (3.61) and integrating by parts, we obtain

$$\begin{aligned}\delta\mathscr{L} &= \sqrt{-\bar{g}}\Bigg[\left\{L_N^{\rm EFT}+\frac{\left(L_{\mathcal{L}}^{\rm EFT}+\mathcal{F}\right)'}{\bar{M}\bar{N}}+\frac{(\bar{N}'r+2\bar{N})\left(L_{\mathcal{L}}^{\rm EFT}+\mathcal{F}\right)}{\bar{N}^2\bar{M}r}\right\}\delta N \\ &\quad +\left\{L_M^{\rm EFT}+\frac{\mathcal{F}'}{\bar{M}^2}+\frac{\bar{N}'\left(L_{\mathcal{L}}^{\rm EFT}+\mathcal{F}\right)}{\bar{N}\bar{M}^2}\right\}\delta M+L_{\mathcal{R}}^{\rm EFT}\delta\mathcal{R}\Bigg] \\ &\quad +\bar{L}_0^{\rm EFT}\sqrt{-\bar{g}}\left(\frac{\delta N}{\bar{N}}+\frac{\delta M}{\bar{M}}-\frac{\delta\mathcal{R}}{\bar{\mathcal{R}}}\right). \end{aligned} \tag{3.64}$$

Variation of the three scalars δN, δM, and $\delta\mathcal{R}$ leads, respectively, to

$$\bar{L}^{\rm EFT}+\bar{N}L_N^{\rm EFT}+\frac{(\bar{N}'r+2\bar{N})L_{\mathcal{L}}^{\rm EFT}}{\bar{N}\bar{M}r}+\frac{L_{\mathcal{L}}^{\rm EFT\prime}}{\bar{M}}=0\,, \tag{3.65}$$

$$\bar{L}^{\rm EFT}+\bar{M}L_M^{\rm EFT}-\frac{2\mathcal{F}}{\bar{M}r}+\frac{\bar{N}'L_{\mathcal{L}}^{\rm EFT}}{\bar{M}\bar{N}}=0\,, \tag{3.66}$$

$$\bar{L}^{\rm EFT}-\frac{\mathcal{F}'}{\bar{M}}-\frac{(\bar{N}'r+2\bar{N})\mathcal{F}}{\bar{N}\bar{M}r}-\frac{2L_{\mathcal{R}}^{\rm EFT}}{r^2}=0\,, \tag{3.67}$$

which are the equations of motion on the spherically symmetric and static background. For a given Lagrangian, they can be used for discussing the screening mechanism of the fifth force mediated by the scalar degree of freedom. In Appendix B, we show that, in the Horndeski theory, the background equations of motion following from Eqs. (3.65)-(3.67) coincide with those derived in Refs. [61, 36] by the direct variation of the Horndeski action. In doing so, we need to express the Horndeski action in terms of the variables used in the 2+1+1 decomposition. In the next section we shall address this issue in both Horndeski and GLPV theories.

3.4 2+1+1 decomposition of Horndeski and GLPV theories

In what follows, we prove that, assuming a spherically symmetric and static background, both the Horndeski theory [22] and its recent GLPV [39] generalization are accommodated in the framework of the EFT of modified gravity.

In unitary gauge, the unit normal vector orthogonal to the constant ϕ hypersurfaces (which coincide with the constant r hypersurfaces) can be expressed as

$$l_a = \gamma \nabla_a \phi \,, \qquad \gamma = \frac{1}{\sqrt{X}} \,. \tag{3.68}$$

By virtue of Eq. (3.25), the covariant derivative of $\nabla_a \phi = \gamma^{-1} l_a$ reads

$$\nabla_a \nabla_b \phi = \gamma^{-1} \left(L_{ab} + n_a \mathcal{L}_b + n_b \mathcal{L}_a + n_a n_b \mathcal{L} + l_a \beta_b + l_b \beta_a \right) + \frac{\gamma^2}{2} \nabla^c \phi \nabla_c X l_a l_b \,. \tag{3.69}$$

Finally, the term $\Box \phi = g^{ab} \nabla_a \nabla_b \phi$ becomes

$$\Box \phi = \gamma^{-1} (L - \mathcal{L}) + \frac{\nabla^c \phi \nabla_c X}{2X} \,. \tag{3.70}$$

With the help of these formulas, we will rewrite both the Horndeski and GLPV Lagrangians in terms of the 2+1+1 variables of the action (3.36).

3.4.1 The Horndeski class of theories

The Horndeski theories, the most general scalar-tensor theories with second-order equations of motion [22], can be given as a series of the Lagrangians (2.38).

The analysis of the background gravitational dynamics in the Horndeski theory have been presented in Refs. [33, 35, 61] on the spherically symmetric space-time and specialized for the weak gravity regime, allowing for confrontation with solar-system tests. In the presence of non-linear scalar-field self interactions, the Vainshtein mechanism can be efficient enough to suppress the propagation of the fifth force inside the solar system, provided that the non-minimal derivative coupling to the Einstein tensor is suppressed [33, 35, 61]. At a technical level, this translates into constraining the magnitude of the function G_5 in the $L_5^{\rm H}$ contribution of the Horndeski Lagrangian to be subdominant as compared to the $L_4^{\rm H}$ contribution. For the consistency with solar-system tests, we will consider the subclass of the Horndeski theory with $L_5^{\rm H} = 0$ in the following.

The Lagrangian $L_2^{\rm H}$ depends on the lapse M according to

$$L_2^{\rm H} = G_2(\phi, X(M)) \,, \qquad X(M) = \frac{\phi'^2}{M^2} \,. \tag{3.71}$$

As for the Lagrangian $L_3^{\rm H} = G_3 \Box \phi$, we introduce an auxiliary function $F_3(\phi, X)$ [52] such that

$$G_3 \equiv F_3 + 2X F_{3X} \,. \tag{3.72}$$

Integrating the term $F_3\Box\phi$ by parts, using Eq. (3.70) for the term $2XF_{3X}\Box\phi$, the Lagrangian $L_3^{\rm H}$ reduces to

$$L_3^{\rm H} = 2X^{3/2}F_{3X}(L-\mathcal{L}) - F_{3\phi}X\,. \tag{3.73}$$

By using Eqs. (3.38), (3.69) and (3.70), the Lagrangian $L_4^{\rm H}$ can be expressed as

$$\begin{aligned} L_4^{\rm H} &= G_4\left(\mathcal{R}-K^2+\varkappa\right)+(G_4-2XG_{4X})\left[L^2-\lambda-2L\mathcal{L}+2\mathfrak{K}-2K\mathcal{K}\right. \\ &\quad \left.+2D^a\left(\ln N\right)D_a\left(\ln M\right)\right]+2\sqrt{X}G_{4\phi}(L-\mathcal{L})\,. \end{aligned} \tag{3.74}$$

Thus we have shown that the Horndeski Lagrangians $L_{2,3,4}^{\rm H}$ are fully expressed in terms of 2+1+1 covariant quantities introduced in the action (3.36).

3.4.2 GLPV theories

We proceed to apply our formalism to GLPV theories which we have studied on the cosmological background in Sec. 2. In a manifestly covariant form, the Lagrangian which characterizes GLPV theories is given as

$$L^{\rm GLPV} = \sum_{i=2}^{5} L_i^{\rm GLPV}\,, \tag{3.75}$$

where the series of Lagrangians $L_{2-5}^{\rm GLPV}$ are given by [39]

$$L_2^{\rm GLPV} = A_2(\phi,X)\,, \tag{3.76}$$

$$L_3^{\rm GLPV} = \left[C_3(\phi,X)+2XC_{3X}(\phi,X)\right]\Box\phi + XC_{3\phi}(\phi,X)\,, \tag{3.77}$$

$$\begin{aligned} L_4^{\rm GLPV} &= B_4(\phi,X)R-\frac{B_4(\phi,X)+A_4(\phi,X)}{X}\left[(\Box\phi)^2-\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi\right] \\ &\quad +\frac{2\left[B_4(\phi,X)+A_4(\phi,X)-2XB_{4X}(\phi,X)\right]}{X^2} \\ &\quad \times\left(\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi\Box\phi-\nabla^a\phi\nabla_a\nabla_b\phi\nabla_c\phi\nabla^b\nabla^c\phi\right) \\ &\quad +\left[C_4(\phi,X)+2XC_{4X}(\phi,X)\right]\Box\phi+XC_{4\phi}(\phi,X)\,, \end{aligned} \tag{3.78}$$

$$\begin{aligned} L_5^{\rm GLPV} &= G_5(\phi,X)G_{ab}\nabla^a\nabla^b\phi-|X|^{3/2}A_5(\phi,X)\left[(\Box\phi)^3-3(\Box\phi)\nabla^a\nabla^b\phi\nabla_a\nabla_b\phi\right. \\ &\quad \left.+2\nabla_a\nabla_b\phi\nabla^c\nabla^b\phi\nabla_c\nabla^a\phi\right]+\frac{XB_{5X}(\phi,X)+3A_5(\phi,X)}{|X|^{5/2}} \\ &\quad \times\Big[(\Box\phi)^2\,\nabla_a\phi\nabla^a\nabla^b\phi\nabla_b\phi-2\Box\phi\nabla_a\phi\nabla^a\nabla^b\phi\nabla_b\nabla_c\phi\nabla^c\phi \end{aligned}$$

$$-\nabla_a\nabla_b\phi\nabla^a\nabla^b\phi\nabla_c\phi\nabla^c\nabla^d\phi\nabla_d\phi + 2\nabla_a\phi\nabla^a\nabla^b\phi\nabla_b\nabla_c\phi\nabla^c\nabla^d\phi\nabla_d\phi\Big]$$
$$+C_5(\phi, X)R - 2C_{5X}(\phi, X)\left[(\Box\phi)^2 - \nabla^a\nabla^b\phi\nabla_a\nabla_b\phi\right] , \qquad (3.79)$$

where

$$C_3 = \int dX \frac{A_3}{2|X|^{3/2}} , \qquad C_4 = -\int dX \frac{B_{4\phi}}{|X|} ,$$
$$C_5 = \frac{XG_{5\phi} - |X|^{1/2}B_{5\phi}}{2} , \qquad G_5 = -\int dX \frac{B_{5X}}{|X|^{1/2}} , \qquad (3.80)$$

with $A_{2,3,4,5}$ and $B_{4,5}$ arbitrary functions of a scalar field ϕ and its kinetic term X. The Lagrangians (3.76)-(3.79) arise as an extension of the Horndeski theory by generalizing the Horndeski Lagrangians written in terms of the ADM variables in the isotropic cosmological setup [39].

The Horndeski theory corresponds to

$$A_4 = -B_4 + 2XB_{4X} , \qquad (3.81)$$
$$A_5 = -\frac{XB_{5X}}{3} , \qquad (3.82)$$

under which the terms on the second line of Eq. (3.78) and those in the second and third lines of Eq. (3.79) vanish. Then, the Horndeski Lagrangians (2.39)-(2.42) can be recovered by moving some of the terms (such as $XC_{3\phi}(\phi, X)$) in the Lagrangian $L_i^{\rm GLPV}$ $(i = 3, 4, 5)$ to the previous Lagrangian $L_{i-1}^{\rm GLPV}$.

In comparison to the Horndeski Lagrangians characterized by the functions $G_{2,3,4,5}$, the theories (3.79) have two additional functions included in $A_{2,3,4,5}$ and $B_{4,5}$. Apparently, the equation of motion for the scalar field allows for derivatives higher than second order. In the presence of higher-order derivatives[3], the theory can be plagued by Ostrogradski instabilities associated with the propagation of the extra degrees of freedom [38]. In the GLPV theory, however, a careful counting of the degrees of freedom in the Hamiltonian formulation on the isotropic cosmological background[4] indicates that no additional degrees of freedom would arise.

[3]Although such a higher-order dynamics is non-standard in physics, it has not been unaccounted either. An example for such a dynamics is provided by the (spin-orbit contribution to the) Lagrangian of spinning binary black holes. In this case the Lagrangian depends on the relative acceleration of the black holes, which leads to a third-order Euler-Lagrange equation [92].

[4]In Ref. [39] this has been performed after the scalar degree of freedom is transferred

As in the discussion of the Horndeski theory, we will also drop the contribution of L_5^{GLPV}. The Lagrangians $L_{2,3,4}^{\mathrm{GLPV}}$ can be expressed as

$$\begin{aligned} L_2^{\mathrm{GLPV}} &= A_2\,, &(3.83)\\ L_3^{\mathrm{GLPV}} &= A_3\left(L-\mathcal{L}\right)\,, &(3.84)\\ L_4^{\mathrm{GLPV}} &= B_4\left(\mathcal{R}-K^2+\varkappa\right)-2\left(B_4-2XB_{4X}\right)\left[K\mathcal{K}-D^a\left(\ln N\right)D_a\left(\ln M\right)\right]\\ &\quad -A_4\left(L^2-\lambda-L\mathcal{L}+2\mathfrak{K}\right)\,, &(3.85)\end{aligned}$$

fully rewritten in terms of the 2+1+1 covariant variables of the action (3.36). Hence $L_{2,3,4}^{\mathrm{GLPV}}$ also belong to the class of the EFT of modified gravity. This illustrates that the latter accommodates theories beyond Horndeski.

In Appendix B we show the background equations of motion, as derived from Eqs. (3.65)-(3.67) for the GLPV Lagrangians (3.83)-(3.85). Under the conditions (3.81) and (3.82), the equations of motion coincide[5] with those derived in Refs. [36, 61] in the Horndeski theory. In general, however, they differ from each other.

Thus we have shown that there are theories which at the level of the background are second order and more generic than the Horndeski theory. This seems to contradict the generic claim that the Horndeski theory represents the most generic second-order scalar-tensor dynamics. We have to keep in mind however that we are considering a spherically symmetric and static background. These additional symmetries may render some of the requirements imposed in order to achieve second-order dynamics unnecessarily restrictive.

Further, we comment that, under spherical symmetry and staticity imposed in the generic EFT of modified gravity, the tensorial sector is always governed by second-order dynamics. As we consider a static background, the equations of motion (3.65)-(3.67) represent constraints, containing no time derivatives. Due to the additional spherical symmetry, higher-order derivative terms could emerge only as radial derivatives. This could happen, if the Lagrangian L^{EFT} involves second radial derivatives. Nevertheless, this is forbidden by the very nature of the action. Indeed, the Lagrangian only

into the lapse and coordinate associated with the constant ϕ hypersurfaces. For the spatial hypersurfaces considered there, the usual lapse N and the time t were employed. On the spherically symmetric background the constant ϕ-surfaces have spherical topology, so in this case the scalar degree of freedom is transferred into M and r.

[5] In order to manifestly see this, one has to redefine the functions A_i and B_i. These redefinitions will be discussed in Appendix A in the case when L_5^{GLPV} is dropped.

depends on scalars constructed algebraically from the variables of the 2+1+1 formalism involving the induced metric, extrinsic curvatures, normal fundamental vectors and forms. The latter are related to first temporal and radial derivatives, as Eqs. (3.28)-(3.30c) explicitly show. No second-order derivatives of the metric are included in these variables. Hence the background equations of motion (increasing the differential order of the Lagrangian at most by one) are free from third or higher order radial derivatives of the chosen variables of the action.

Nevertheless, at the level of perturbations, to be discussed in the rest of this chapter, their second-order evolution cannot be guaranteed a priori.

3.5 Gauge transformations and fixing

In this section we discuss the simplifications achieved by suitably employing the available gauge degrees of freedom (diffeomorphism invariance). In doing so, we will adapt the radial coordinate r to the hypersurfaces of constant scalar field even in the perturbed case by requiring

$$\delta\phi = 0\,. \tag{3.86}$$

Next, we will simplify the perturbations of the induced 2-metric to a mere conformal rescaling. Finally we will adopt a gauge which maintains the geometrical interpretation of the variables as arising in the 2+1+1 canonical formalism (e.g., assure $\mathcal{N} = 0$ even in the presence of perturbations).

In a manner analogous to the Helmholtz theorem, any vector $V_a = V_a\left(t, r, \theta, \varphi\right)$ on a sphere can be decomposed by using scalar potentials as follows:

$$V_a = \bar{D}_a V_{\text{rot}} + E^b{}_a \bar{D}_b V_{\text{div}}\,, \tag{3.87}$$

where $V_{\text{rot}} = V_{\text{rot}}\left(t, r, \theta, \varphi\right)$ and $V_{\text{div}} = V_{\text{div}}\left(t, r, \theta, \varphi\right)$ are arbitrary scalars generating a rotation-free part and a divergence-free part, respectively. Here $E_{ab} = \sqrt{\bar{h}}\,\varepsilon_{ab}$ and ε_{ab} stands for the antisymmetric tensor density, defined as $\varepsilon_{\theta\varphi} = 1$ [36]. Similarly, any rank-2 symmetric tensor $T_{ab} = T_{ab}\left(t, r, \theta, \varphi\right)$ on a sphere can be decomposed in terms of a scalar and a vector potential, e.g., T_{scalar} and T_a, as $T_{ab} = \bar{h}_{ab} T_{\text{scalar}} + \left(\bar{D}_a T_b + \bar{D}_b T_a\right)/2$. Applying the decomposition (3.87) to T_a, the tensor T_{ab} is uniquely expressed in terms of the scalar functions T_{scalar}, T_{rot} and T_{div}, as

$$T_{ab} = \bar{h}_{ab} T_{\text{scalar}} + \bar{D}_a \bar{D}_b T_{\text{rot}} + \frac{1}{2}\left(E^c{}_a \bar{D}_c \bar{D}_b + E^c{}_b \bar{D}_c \bar{D}_a\right) T_{\text{div}}\,. \tag{3.88}$$

We apply these decompositions to the metric perturbation (3.53), such that the perturbed quantities can be expressed as

$$\delta N_a = \bar{D}_a P + E^b{}_a \bar{D}_b Q \,, \tag{3.89a}$$
$$\delta M_a = \bar{D}_a V + E^b{}_a \bar{D}_b W \,, \tag{3.89b}$$
$$\delta h_{ab} = \bar{h}_{ab} A + \bar{D}_a \bar{D}_b B + \frac{1}{2}\left(E^c{}_a \bar{D}_c \bar{D}_b + E^c{}_b \bar{D}_c \bar{D}_a\right) C \,. \tag{3.89c}$$

Here the perturbations Q, W and C correspond to either divergence-free terms or to derivatives of such terms (these terms have non-vanishing curls), whereas P, V, A, and B represent either rotation-free terms or derivatives of such terms. As first shown in Ref. [71], after expanding in terms of spherical harmonics, the elements of the first set become odd modes under the parity transformation on the sphere. The quantities of the second set, together with δN, $\delta \mathcal{N}$, and δM of Eq. (3.53), behave as even modes.

In what follows we concentrate on the evolution of these 10 variables, conveniently characterizing the perturbations from the parity point of view. At first, we remark that some of them could be eliminated by making use of the allowed diffeomorphism freedom. In doing so, we consider an infinitesimal coordinate transformation $\tilde{x}^a = x^a + \xi^a$. For the infinitesimal displacement ξ^a we write the time and radial component as ξ^t and ξ^r respectively, while the infinitesimal displacement along the sphere is decomposed as

$$\xi^a = \bar{D}^a \xi + E^{ba} \bar{D}_b \eta \,, \qquad (a = \theta, \varphi) \,. \tag{3.90}$$

Then, the perturbed metric in the new coordinate system becomes $\widetilde{\delta g_{ab}} = \delta g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a$.

The perturbations transform as

$$\widetilde{\delta N} = \delta N - \bar{N}\dot{\xi}^t - \bar{N}'\xi^r \,, \tag{3.91a}$$
$$\widetilde{\delta \mathcal{N}} = \delta \mathcal{N} - \frac{\bar{N}^2}{2\bar{M}}\xi^{t\prime} + \frac{\bar{M}}{2}\dot{\xi}^r \,, \tag{3.91b}$$
$$\widetilde{\delta M} = \delta M + \bar{M}'\xi^r + \bar{M}\xi^{r\prime} \,, \tag{3.91c}$$
$$\widetilde{P} = P - \bar{N}^2\xi^t + \dot{\xi} \,, \tag{3.91d}$$
$$\widetilde{Q} = Q + \dot{\eta} \,, \tag{3.91e}$$
$$\widetilde{V} = V + \bar{M}^2\xi^r + \xi' - \frac{2}{r}\xi \,, \tag{3.91f}$$
$$\widetilde{W} = W + \eta' - \frac{2}{r}\eta \,, \tag{3.91g}$$

$$\widetilde{A} = A + \frac{2}{r}\xi^r , \tag{3.91h}$$
$$\widetilde{B} = B + 2\xi , \tag{3.91i}$$
$$\widetilde{C} = C + 2\eta . \tag{3.91j}$$

Additionally, the linear perturbation $\delta\phi$ of a scalar field $\phi(t,r,\theta,\varphi) = \bar{\phi}(r) + \delta\phi(t,r,\theta,\varphi)$ transforms under an infinitesimal coordinate transformation as

$$\widetilde{\delta\phi} = \delta\phi - \bar{\phi}'\xi^r . \tag{3.92}$$

In the isotropic cosmological setting, the key ingredient in deriving the EFT of modified gravity is the $3+1$ decomposition with the time slicing determined by hypersurfaces of the uniform scalar field [52]. In an analogous way, we consider here the hypersurfaces of constant ϕ as defining the radial slicing with $r =$ const, in a choice which simplifies the EFT of modified gravity on the spherically symmetric background. Therefore, we first fix the gauge ξ^r to obtain $\widetilde{\delta\phi} = 0$. Due to this gauge choice, the action (3.36) does not explicitly include the scalar field as a variable.

Next, we fix the two gauge degrees of freedom ξ and η such that the anisotropic contributions to δh_{ab} disappear, i.e., $\widetilde{B} = \widetilde{C} = 0$. By doing so, the perturbed and unperturbed induced metrics are simply related by a conformal transformation as $h_{ab} = (1+\widetilde{A})\bar{h}_{ab}$. After redefining $\widetilde{A} = e^{2\zeta} - 1$, the perturbed induced metric coincides with the one employed in Sec. 3.3. Finally, we also need to fix the gauge ξ^t to achieve $\widetilde{\delta\mathcal{N}} = 0$ [see Eq. (3.20)][6].

In summary, the gauge fixing is given by

$$\xi^t = \int dr \frac{2\bar{M}}{\bar{N}^2}\left(\delta\mathcal{N} + \frac{\bar{M}}{2}\dot{\xi}^r\right) + F(t,\theta,\varphi) ,$$
$$\xi^r = \frac{\delta\phi}{\bar{\phi}'} , \qquad \xi = -\frac{B}{2} , \qquad \eta = -\frac{C}{2} , \tag{3.93}$$

where $F(t,\theta,\varphi)$ is an integration function, yet to be fixed [7].

[6]Even if we would not choose $\delta\mathcal{N} = 0$, preserving at the level of perturbations the more general linear relation between $\mathcal{N}$ and M, Eq. (C2) of the Appendix C of Ref. [73] would consume this gauge degree of freedom.

[7]In the particular case where P exhibits the radial dependence $P(t,r,\theta,\varphi) = \bar{N}(r)^2 F(t,\theta,\varphi)$, the remaining gauge transformation $\tilde{t} = t + F(t,\theta,\varphi)$ could be employed to eliminate $\widetilde{P}$. In general, however, this is not possible, so another fixing of the function F would be necessary in order to avoid the appearance of any non-physical gauge mode, similar to the one of the synchronous gauge in cosmology.

With the new notation for the conformal factor in the transformation of the induced metric

$$\delta h_{ab} = \left(e^{2\zeta} - 1\right) \bar{h}_{ab} \,, \tag{3.94}$$

the line element up to first-order accuracy can be written as

$$\begin{aligned} ds_1^2 \;\; &= \;\; - \left(\bar{N}^2 + 2\bar{N}\delta N\right) dt^2 + 2\delta N_a dt dx^a + 2\delta M_a dx^a dr \\ &\quad + \left(\bar{M}^2 + 2\bar{M}\delta M\right) dr^2 + e^{2\zeta}\bar{h}_{ab} dx^a dx^b \,, \end{aligned} \tag{3.95}$$

where δN_a and δM_a are given in terms of parity-related scalars through Eqs. (3.89a) and (3.89b). In the above expression we have omitted the tildes for notational simplicity, and we will do so hereafter.

We now discuss how the gauge fixing affects the even and odd modes. First, we stress that the residual gauge freedom in F does not affect the odd-parity perturbations as it does not appear in the transformation of the odd-sector variables (C, Q, W), as seen from Eqs. (3.91). In fact all these variables transform only in terms of η, which has been fixed such that C could be eliminated. Then the other two odd-sector variables stay arbitrary, unaffected by the three other gauge choices.

Finally, we comment on the elimination of the even-sector variable $\delta\mathcal{N}$. By doing so, the interpretation of the Lagrangian variables in terms of the geometric quantities defined in the 2+1+1 formalism continues to hold even in the presence of perturbations. Such a condition is equivalent to imposing hypersurface-orthogonality of the vector field l^a. The last requirement could be relaxed such that the vector l^a acquires vorticity at a perturbative level. However, this would imply to develop a more involved formalism, allowing at least for a new scalar, a new vectorial and a new tensorial degree of freedom (and all the scalars formed from them). Then we can choose another gauge $\widetilde{P} = 0$, as commonly used in past works. Such a generalization of the formalism for the even-parity perturbations is left for a subsequent work.

3.6 Odd-mode perturbation dynamics

We proceed with the analysis of the odd-parity perturbations by expanding the action up to second order to discuss the dynamical evolution of them.

3.6.1 Second-order perturbed Lagrangian

We expand the action (3.36) at second order for the odd-type perturbations in order to derive linear perturbation equations of motion. As the even and odd sectors decouple in the second-order perturbed Lagrangian, at a formal level, we could just switch off all even-type variables as

$$P = V = \delta N = \delta M = \zeta = 0\,. \tag{3.96}$$

Then the second-order contribution to the Lagrangian density for the odd modes is given by

$$\delta_2 \mathscr{L}^{\rm odd} = \bar{L}_0^{\rm EFT}\delta_2\sqrt{-g} + \delta\sqrt{-g}\delta L^{\rm EFT} + \sqrt{-\bar{g}}\,\delta_2 L^{\rm EFT}\,, \tag{3.97}$$

where δ_2 represents second-order variations.

The second-order contribution to the line element reads

$$\begin{aligned}\delta_2\left(ds^2\right) &= (\delta N_a\delta N^a - \delta N^2)dt^2 + 2\delta N_a\delta M^a dtdr \\ &\quad + \left(\delta M_a\delta M^a + \delta M^2\right)dr^2 + 2\zeta^2\bar{h}_{ab}dx^a dx^b\,.\end{aligned} \tag{3.98}$$

By employing Eqs. (3.95) and (3.98), it follows that

$$\delta_2\sqrt{-g} = \frac{\sqrt{-\bar{g}}}{2}\left[\bar{g}^{ab}\delta_2 g_{ab} + \frac{1}{4}\left(\bar{g}^{ab}\bar{g}^{cd} - 2\bar{g}^{ac}\bar{g}^{bd}\right)\delta g_{ab}\delta g_{cd}\right] = 0\,. \tag{3.99}$$

Thus the first term on the rhs of Eq. (3.97) vanishes identically. Similarly the second term on the rhs of Eq. (3.97) vanishes, since by virtue of Eq. (3.54) the first-order variation $\delta\sqrt{-g}$ consists only of even-mode contributions.

Next we expand the Lagrangian up to second order. Before doing so, we note that the linear and quadratic perturbations of L, $\mathcal{L}$, K, $\mathcal{K}$ and $\mathcal{R}$ arise from even modes only [see Eqs. (3.29), (3.30) and (3.55)], so they do not contribute to the odd-mode dynamics. As a result, the second-order Lagrangian for the odd-type perturbations becomes extremely simple (depending on 4 variables only out of 11):

$$\delta_2 L^{\rm EFT} = L_{\mathfrak{M}}^{\rm EFT}\delta_2\mathfrak{M} + L_{\mathfrak{K}}^{\rm EFT}\delta_2\mathfrak{K} + L_{\varkappa}^{\rm EFT}\delta_2\varkappa + L_{\lambda}^{\rm EFT}\delta_2\lambda\,. \tag{3.100}$$

Substituting Eqs. (3.89a) and (3.89b) into Eqs. (3.29) and (3.30), then integrating by parts (employing once again the generalized Stokes theorem for manifolds without boundaries), the second-order factors in $\delta_2 L^{\rm EFT}$ can be explicitly expressed in terms of the odd-type variables:

$$\delta_2\mathfrak{M} = \left(\bar{D}W\right)^2 , \qquad \delta_2\lambda = \frac{1}{2\bar{M}}\left[\left(\bar{D}^2W\right)^2 - \frac{2}{r^2}\left(\bar{D}W\right)^2\right] ,$$
$$\delta_2\varkappa = \frac{1}{2\bar{N}^2}\left[\left(\bar{D}^2Q\right)^2 - \frac{2}{r^2}\left(\bar{D}Q\right)^2\right] ,$$
$$\delta_2\mathfrak{K} = \frac{1}{4\bar{N}^2\bar{M}^2}\left[\left(\bar{D}\dot{W}\right)^2 + \left(\bar{D}Q'\right)^2 - 2\bar{D}^a\dot{W}\bar{D}_aQ' + \frac{4}{r}\left(\bar{D}^a\dot{W}\bar{D}_aQ - \bar{D}^aQ\bar{D}_aQ'\right) + \frac{4}{r^2}\left(\bar{D}Q\right)^2\right] , \quad (3.101)$$

where the notations $\bar{D}^2 \equiv \bar{D}^a\bar{D}_a$ and $\left(\bar{D}f\right)^2 \equiv \bar{D}^af\bar{D}_af$ have been introduced for $f \equiv (Q, W)$.

Substituting Eqs. (3.99)-(3.101) and $\delta\sqrt{-g} = 0$ into the second-order Lagrangian density (3.97) for the odd modes, we finally obtain

$$\delta_2\mathscr{L}^{\rm odd} = \sqrt{-\bar{g}}\left\{a_1\left(\bar{D}\dot{W} - \bar{D}Q' + \frac{2}{r}\bar{D}Q\right)^2 + a_2\left[\left(\bar{D}^2Q\right)^2 - \frac{2}{r^2}\left(\bar{D}Q\right)^2\right] + a_3\left(\bar{D}^2W\right)^2 + a_4\left(\bar{D}W\right)^2\right\} , \quad (3.102)$$

where the coefficients a_i $(i = 1, \cdots, 4)$ are

$$a_1 = \frac{L^{\rm EFT}_{\mathfrak{K}}}{4\bar{N}^2\bar{M}^2} , \qquad a_2 = \frac{L^{\rm EFT}_{\varkappa}}{2\bar{N}^2} , \qquad a_3 = \frac{L^{\rm EFT}_{\lambda}}{2\bar{M}^2} , \qquad a_4 = L^{\rm EFT}_{\mathfrak{M}} - \frac{2}{r^2}a_3 . \quad (3.103)$$

From the second-order Lagrangian density (3.102), we will derive the equations of motion for the odd-sector perturbations in the next subsection. We remark that the Lagrangian density (3.102) is quadratic in the odd-mode perturbations Q and W, so in what follows we will refer to this Lagrangian contribution as quadratic.

3.6.2 Perturbation equations in the harmonics expansion

We rewrite the quadratic action $S_2 = \int d^4x\, \delta_2\mathscr{L}^{\rm odd}$ in the following form

$$\delta_2\mathscr{L}^{\rm odd} = \sqrt{-\bar{g}}\left[-a_1\left(\dot{W} - Q' + \frac{2}{r}Q\right)\bar{D}^2\left(\dot{W} - Q' + \frac{2}{r}Q\right)\right.$$

$$+a_2 Q\bar{D}^2\left(\bar{D}^2+\frac{2}{r^2}\right)Q+W\bar{D}^2\left(a_3\bar{D}^2-a_4\right)W\Big],\qquad(3.104)$$

in which we have dropped covariant total divergence terms. The resulting equations of motion derived by varying W and Q are given, respectively, by

$$\bar{D}^2\Psi^{(1)}=0\,,\qquad \Psi^{(1)}\equiv a_1\frac{\partial}{\partial t}\left(\dot{W}-Q'+\frac{2Q}{r}\right)+\left(a_3\bar{D}^2-a_4\right)W\,,\quad(3.105)$$

and

$$\frac{1}{\sqrt{-\bar{g}}r^2}\frac{\partial}{\partial r}\left[\sqrt{-\bar{g}}a_1 r^2\bar{D}^2\left(\dot{W}-Q'+\frac{2}{r}Q\right)\right]-a_2\bar{D}^2\left(\bar{D}^2+\frac{2}{r^2}\right)Q=0\,.\qquad(3.106)$$

Since $\sqrt{-\bar{g}}=\bar{N}\bar{M}\sqrt{h}=\bar{N}\bar{M}r^2\sin\theta$ and $\bar{D}_a$ is the covariant derivative compatible with the metric h_{ab}, it follows that $\bar{D}_a\sqrt{-\bar{g}}=0$. On using this identity and the fact that $r^2\bar{D}^2$ has no radial dependence (i.e., it commutes with $\partial/\partial r$), Eq. (3.106) reads

$$\bar{D}^2\Psi^{(2)}=0\,,\ \ \Psi^{(2)}\equiv\frac{1}{\sqrt{-\bar{g}}}\frac{\partial}{\partial r}\left[\sqrt{-\bar{g}}a_1\left(\dot{W}-Q'+\frac{2}{r}Q\right)\right]-a_2\left(\bar{D}^2+\frac{2}{r^2}\right)Q\,.\qquad(3.107)$$

Hence Eqs. (3.105) and (3.107) are of the form $\bar{D}^2\Psi^{(i)}=0$ with $i=1,2$. These are fourth-order coupled differential equations, but in the expressions of $\Psi^{(i)}$ they contain time and radial derivatives up to second orders alone.

In the following, we expand the angular part of the odd-mode perturbations $f\equiv(Q,W)$ in terms of spherical harmonics, i.e.,

$$f(t,r,\theta,\varphi)=\sum_{l,m}f_{lm}(t,r)Y_l^m\,.\qquad(3.108)$$

A similar decomposition of the differential expressions $\Psi^{(i)}$ $(i=1,2)$ is given by

$$\Psi^{(i)}(t,r,\theta,\varphi)=\sum_{l,m}\Psi^{(i)}_{lm}(t,r)Y_l^m\,.\qquad(3.109)$$

Each mode obeys the identity

$$r^2\bar{D}^2\left[\Psi^{(i)}_{lm}(t,r)Y_l^m\right]+l\,(l+1)\left[\Psi^{(i)}_{lm}(t,r)Y_l^m\right]=0\,.\qquad(3.110)$$

The differential order of Eqs. (3.105) and (3.107) can be reduced by two, i.e.,

$$\sum_{l,m} l\,(l+1)\,\Psi^{(i)}_{lm}(t,r)Y_l^m = 0\,, \qquad (i=1,2), \tag{3.111}$$

or explicitly

$$\sum_l l\,(l+1)\,\Psi^{(1)}_l = 0\,, \tag{3.112}$$

$$\sum_l l\,(l+1)\,\Psi^{(2)}_l = 0\,, \tag{3.113}$$

with

$$\Psi^{(1)}_l \equiv a_1\frac{\partial}{\partial t}\left(\dot{W}_l - Q'_l + \frac{2}{r}Q_l\right) - \left[a_3\frac{l\,(l+1)}{r^2} + a_4\right]W_l\,, \tag{3.114}$$

$$\Psi^{(2)}_l \equiv \frac{1}{\sqrt{-\bar{g}}}\frac{\partial}{\partial r}\left[\sqrt{-\bar{g}}\,a_1\left(\dot{W}_l - Q'_l + \frac{2}{r}Q_l\right)\right] + a_2\frac{l\,(l+1)-2}{r^2}Q_l\,. \tag{3.115}$$

Note that we have introduced the notations $f_l \equiv \sum_m f_{lm}Y_l^m$. The f_l modes are orthogonal to each other due to the orthogonality of spherical harmonics, so that $\Psi^{(1)}_l$ and $\Psi^{(2)}_l$ vanish for $l \neq 0$. Hence we have derived a sequence of second-order differential equations $\Psi^{(i)}_l = 0$ $(i=1,2)$ holding for each non-zero l.

There exists a second time derivative of W_l in Eq. (3.112), so this corresponds to a dynamical equation of motion for W_l. The variable Q_l appears only algebraically in the second-order Lagrangian density (3.102) and through a first temporal derivative in Eq. (3.112). Since Eq. (3.113) contains only a first time derivative of W_l with no time derivatives of Q_l, this is a constraint equation in the Lagrangian sense. In Sec. 3.6.4 we shall address the issue of a true dynamical degree of freedom for general l by using a method of the Lagrange multiplier. Before doing so, we shall discuss the specific cases of $l = 0, 1$ in the next subsection.

3.6.3 Monopolar and dipolar perturbations

Monoploar mode ($l = 0$)

The monopolar perturbations trivially obey Eqs. (3.112)-(3.113), so they do not contribute to the dynamics. In fact, after integrations by parts, the quadratic odd-mode Lagrangian density (3.102) can be written in a form containing exclusively Laplacian terms:

$$\begin{aligned}\delta_2\mathscr{L}^{\rm odd} &= \sqrt{-\bar{g}}\Bigg[-a_1\left(\dot{W}-Q'+\frac{2}{r}Q\right)\left(\bar{D}^2\dot{W}-\bar{D}^2Q'+\frac{2}{r}\bar{D}^2Q\right)\\ &\quad + a_2\left(\bar{D}^2Q\right)\left(\bar{D}^2+\frac{2}{r^2}\right)Q+\left(\bar{D}^2W\right)\left(a_3\bar{D}^2-a_4\right)W\Bigg]\,,\end{aligned} \tag{3.116}$$

all of which identically vanish for $l=0$. In the following we consider only perturbations without a monopolar contribution.

Dipolar mode ($l = 1$)

For the dipolar perturbations, the last term of Eq. (3.89c), which contains the term C, vanishes due to the identity (3.110). Hence there is no need to eliminate C by gauge fixing, so that the respective gauge degree of freedom can be used up as

$$\eta = -r^2\int dr\frac{W_1}{r^2}+r^2\mathcal{C}_0(t,\theta,\varphi)\,, \tag{3.117}$$

where $\mathcal{C}_0(t,\theta,\varphi)$ is an integration function. With this choice, $\widetilde{W_1}=0$ and $\widetilde{Q}_1 = Q_1 + r^2\dot{\mathcal{C}}_0(t,\theta,\varphi)$. Omitting tildes as before and noting that the last term of Eq. (3.113) also vanishes due to the identity (3.110), Eqs. (3.112)-(3.113) is simplified as

$$\frac{\partial}{\partial t}\left(Q_1'-\frac{2}{r}Q_1\right)=0\,, \tag{3.118}$$

$$\frac{\partial}{\partial r}\left[\sqrt{-\bar{g}}\,a_1\left(Q_1'-\frac{2}{r}Q_1\right)\right]=0\,. \tag{3.119}$$

The dynamical degree of freedom W does not appear in Eqs. (3.118)-(3.119), suggesting that dipolar perturbations are non-dynamical. Indeed, direct integration of Eqs. (3.118)-(3.119) leads to

$$Q_1 = r^2\mathcal{C}_1(\theta,\varphi)\int\frac{dr}{\sqrt{-\bar{g}}a_1r^2}+r^2\mathcal{C}_2(t,\theta,\varphi)\,, \tag{3.120}$$

where $\mathcal{C}_{1,2}$ are integration functions. The remaining gauge degree of freedom can be exploited as $\dot{\mathcal{C}}_0 = -\mathcal{C}_2$, so the time dependence is completely eliminated from the dipolar odd-mode perturbations. As discussed in Ref. [72], the time-independent contribution to Q_1 appearing as the first term on the r.h.s. of Eq. (3.120) is related to the angular momentum induced by the dipolar perturbation.

3.6.4 Dynamical degree of freedom for $l \geq 2$

The Lagrangian density (3.104) possesses first and second derivatives, which appear quadratically. Hence some of the terms would be of fourth order in spatial derivatives by partial integration (while the time derivatives remain of second order). This is why the perturbation Eqs. (3.105) and (3.107) involve fourth-order spatial differentiations. For $l \geq 2$ these equations of motion reduce to the form $\Psi_l^{(1)} = 0$ and $\Psi_l^{(2)} = 0$ under the expansion of spherical harmonics, where $\Psi_l^{(i)}$ $(i = 1, 2)$ are given by Eqs. (3.112) and (3.113).

As we already mentioned in Sec. 3.6.2, the first equation ($\Psi_l^{(1)} = 0$) describes the dynamical evolution of the variable W_l, whereas the second one ($\Psi_l^{(2)} = 0$) corresponds to a constraint involving a second spatial derivative of the field Q_l. Since the latter constraint equation is not directly solved for Q_l, it is difficult to derive a closed-form differential equation for W_l by eliminating the Q_l-dependent terms appearing in the equation $\Psi_l^{(1)} = 0$. This obstacle can be circumvented by using the method of a Lagrange multiplier. In fact, this method was employed to study the linear perturbations on a spherically symmetric background in modified Gauss-Bonnet gravity [93] and it was further applied to Horndeski theory [36].

Introducing the Lagrange multiplier vector Y^a, the Lagrangian density equivalent to Eq. (3.102) is given by

$$\begin{aligned} \delta_2 \mathscr{L}^{\rm odd} &= \sqrt{-\bar{g}} \left\{ a_1 \left[2Y^a \bar{D}_a \left(\dot{W} - Q' + \frac{2}{r} Q \right) - Y^2 \right] \right. \\ & \left. \qquad + a_2 \left[\left(\bar{D}^2 Q \right)^2 - \frac{2}{r^2} \left(\bar{D} Q \right)^2 \right] + a_3 \left(\bar{D}^2 W \right)^2 + a_4 \left(\bar{D} W \right)^2 \right\} , \end{aligned} \tag{3.121}$$

where $Y^2 = Y^a Y_a$. Variation of Eq. (3.121) with respect to Y^a leads to $Y_a = \bar{D}_a[\dot{W} - Q' + (2/r)Q]$. Substituting this relation into Eq. (3.121), we recover the original Lagrangian density (3.102).

Defining the Lagrange multiplier potential Z as $Y^a = \bar{D}^a Z$, the Lagrangian density (3.121) is characterized by two scalar fields W and Q plus the auxiliary scalar field Z. Varying Eq. (3.121) in terms of W and Q, we obtain

$$\bar{D}^2 \left[a_1 \dot{Z} + \left(a_3 \bar{D}^2 - a_4\right) W\right] = 0\,, \tag{3.122}$$

$$\bar{D}^2 \left[\frac{1}{\sqrt{-\bar{g}}} \frac{\partial}{\partial r} \left(\sqrt{-\bar{g}} a_1 Z\right) - a_2 \left(\bar{D}^2 + \frac{2}{r^2}\right) Q\right] = 0\,. \tag{3.123}$$

For $l \geq 2$ the $\bar{D}^2$ operators acting on the square brackets can be formally omitted, so the l-th multipolar components W_l and Q_l obey the following equations:

$$W_l = \frac{a_1 r^2}{a_3 l\,(l+1) + a_4 r^2} \dot{Z}_l\,, \tag{3.124}$$

$$Q_l = -\frac{r^2}{a_2\,(l+2)\,(l-1)\sqrt{-\bar{g}}} \frac{\partial}{\partial r} \left(\sqrt{-\bar{g}}\, a_1 Z_l\right)\,, \tag{3.125}$$

where Z_l is the l-th component of Z.

Equations (3.124) and (3.125) show that both W_l and Q_l are directly known from Z_l. On using the last of Eq. (3.103), we can also write Eq. (3.124) of the form

$$W_l = \frac{r^2}{a_3\,(l+2)\,(l-1)} \left(a_1 \dot{Z}_l - L^{\rm EFT}_{\mathfrak{M}} W_l\right)\,. \tag{3.126}$$

Substituting Eqs. (3.125) and (3.126) into the l-th component of the multipolar decomposition of the Lagrangian density (3.121), using $Y^a = \bar{D}^a Z$, and integrating it by parts, we finally obtain

$$\delta_2 \mathscr{L}^{\rm odd}_l = \frac{l(l+1)\sqrt{-\bar{g}}}{(l+2)(l-1)} \times \left[-\frac{a_1^2}{a_3} \dot{Z}_l^2 - \frac{a_1^2}{a_2} Z_l'^2 - a_1 (\bar{D} Z_l)^2 - U^{\rm H}(r) Z_l^2 + \frac{a_1}{a_3} L^{\rm EFT}_{\mathfrak{M}} W_l \dot{Z}_l\right]\,, \tag{3.127}$$

where the potential $U^{\rm H}(r)$ is given by

$$U^{\rm H}(r) = -a_1 \frac{\partial}{\partial r} \left[\frac{1}{\sqrt{-\bar{g}} a_2} \frac{\partial}{\partial r} \left(\sqrt{-\bar{g}} a_1\right)\right] - \frac{2a_1}{r^2}\,, \tag{3.128}$$

or more explicitly,

$$U^{\rm H}(r) = -\frac{a_1^2}{a_2}\left[\frac{\bar{N}''}{\bar{N}}+\frac{\bar{M}''}{\bar{M}}-\frac{\bar{N}'^2}{\bar{N}^2}-\frac{\bar{M}'^2}{\bar{M}^2}-\frac{2}{r^2}+\frac{a_1''}{a_1}-\frac{a_1'a_2'}{a_1a_2} +\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right)\left(\frac{\bar{N}'}{\bar{N}}+\frac{\bar{M}'}{\bar{M}}+\frac{2}{r}\right)\right]-\frac{2a_1}{r^2}\,. \quad (3.129)$$

The superscript in $U^{\rm H}(r)$ has been introduced to point out that in the Horndeski limit it reduces to the potential (24) of Ref. [36]. Using the relation $\bar{D}^2 Z_l = -l(l+1)Z_l/r^2$, the third term in the square bracket of Eq. (3.127) is equivalent to $-a_1 l(l+1)Z_l^2/r^2$ up to a boundary term.

The last term in the square bracket of Eq. (3.127) gives rise to a contribution $\dot{Z}_l^2$ with a coefficient including the multipolar index l by virtue of Eq. (3.124). In this case the propagation speeds are different for each multipolar mode, so the global interpretation of the perturbation Z_l and its propagation speeds become far from trivial. Hence, in the following, we impose the following condition

$$L_{\mathfrak{M}}^{\rm EFT} = 0\,. \quad (3.130)$$

In fact, this is satisfied both in the Horndeski theory and in the GLPV theory ($i = 2, 3, 4$ for our cases of interest).

Under the condition (3.130) the second-order Lagrangian density is expressed solely by the quantity Z_l and its time and spatial derivatives, in a mode-independent way. As a result, Z_l is a master variable governing the dynamics of the odd-mode perturbations. Comparing Eqs. (3.122) and (3.123) with Eqs. (3.105) and (3.107), respectively, there is the correspondence $Z \to \dot{W} - Q' + 2Q/r$, which also arises by varying the Lagrangian density (3.121) for the Lagrange multiplier potential Z. While Q and W were eliminated from the Lagrangian density by their respective equations of motion, Eqs. (3.124) and (3.125), we stress that this third equation of motion $Z = \dot{W} - Q' + 2Q/r$ was *not* exploited for deriving Eq. (3.127). In fact, after the substitution of Eqs. (3.124) and (3.125), the Lagrangian density (3.127) already contains the dynamics of the third field Z. If we were to make the additional substitution $Z_l \to \dot{W}_l - Q_l' + 2Q_l/r$, the Lagrangian density (3.121) would reduce to a boundary term $\delta_2\mathscr{L}^{\rm odd} = -(\partial/\partial r)(\sqrt{-\bar{g}}a_1 Ql(l+1)Z_l/r^2)$, which is irrelevant to the true dynamics of perturbations.

On using the equations of motion following from the variation of Eq. (3.127) with respect to Z_l, we can discuss the stability of the odd-type perturbations. In the next section we shall address this issue.

3.7 No-ghost conditions and avoidance of Laplacian instabilities

In the previous section we have seen that in an expansion with respect to spherical harmonics there is no monopolar contribution to the odd modes and the dipolar mode is non-dynamical. In the following we proceed with the stability analysis of quadrupolar and higher multipolar contributions to the odd-mode perturbations ($l \geq 2$), governed by the quadratic Lagrangian density (3.127) under the condition (3.130).

3.7.1 Generalized Horndeski class

We categorize theories satisfying the condition (3.130) as the generalized Horndeski class (including the GLPV theory). In this case the quadratic Lagrangian could depend on the odd-mode variable W, which generates the odd-mode contribution to δM_a through Eq. (3.89b). Nevertheless, the Lagrangian for the background dynamics does not depend on the particular combination $\mathfrak{M} \equiv M_a M^a$. Whenever Eq. (3.130) holds, the quadratic Lagrangian density (3.127) leads to a second-order differential equation for the decoupled master variable Z_l and the usual stability conditions can be imposed on this equation.

The condition for avoidance of the scalar ghost (no negative kinetic term) is satisfied for $a_3 < 0$, i.e.,

$$L_\lambda^{\mathrm{EFT}} < 0\,. \tag{3.131}$$

For the modes with the large wave numbers along the radial or tangential directions, many terms of Eq. (3.127) are suppressed. In particular, $U^{\mathrm{H}}(r)$ as well as the third (for radial modes) or second (for tangential modes) terms are sub-dominant in the high-frequency limit. In these two regimes the dispersion relations following from the Lagrangian density (3.127) are given, respectively, by

$$\omega^2 + \frac{a_3}{a_2} k_r^2 = 0\,, \qquad \omega^2 + \frac{a_3}{a_1} k_\Omega^2 = 0\,, \tag{3.132}$$

where ω is the angular frequency, k_r and k_Ω are the wave numbers along the radial and tangential directions respectively. Introducing proper time $\tau = \int \bar{N} dt$ and tortoise coordinate $r_* = \int \bar{M} dr$, the squared sound speeds of fluctuations along the radial and tangential directions read

$$c_r^2 \equiv \frac{\bar{M}^2 k_r^2}{\bar{N}^2 \omega^2} = -\frac{\bar{M}^2 a_3}{\bar{N}^2 a_2} = -\frac{L_\lambda^{\rm EFT}}{L_\varkappa^{\rm EFT}}\,, \qquad c_\Omega^2 \equiv \frac{k_\Omega^2}{\bar{N}^2 \omega^2} = -\frac{a_3}{\bar{N}^2 a_1} = -\frac{2L_\lambda^{\rm EFT}}{L_\mathfrak{K}^{\rm EFT}}\,, \tag{3.133}$$

respectively. Under the no-ghost requirement (3.131), the conditions for the absence of Laplacian instabilities, i.e., $c_r^2 > 0$ and $c_\Omega^2 > 0$, take a remarkably simple form

$$L_\varkappa^{\rm EFT} > 0\,, \qquad L_\mathfrak{K}^{\rm EFT} > 0\,. \tag{3.134}$$

These simple stability conditions acquire geometrical significance, as $\mathfrak{K}$ is the length squared of the normal fundamental vector, while $\varkappa$ and λ are the traces of the squares of the two extrinsic curvature tensors of the spheres. These are quantities appearing in the 2+1+1 decomposition of the covariant derivatives of the two normal vectors to the spheres, Eqs. (3.24)-(3.25). The additional quantities of these decompositions are the normal fundamental scalars and accelerations. They however do not contribute to the stability conditions for the odd modes as the normal fundamental scalars $\mathcal{L}$ and $\mathcal{K}$ are even-mode variables, while the accelerations α^a and β^a appear in the action only through the curvature scalar $\mathcal{R}$ under divergences. Hence their rotation-free part alone survives under the Helmholtz decomposition, which again generates the even modes.

The stability conditions (3.131) and (3.134) can be further specified for the particular case of the Horndeski theory with $L_{2,3,4}^{\rm H}$ and the GLPV theory with $L_{2,3,4}^{\rm GLPV}$ discussed in Sec. 3.4. For this we first remark that, according to Eqs. (3.74) and (3.85), only the contributions $L_4^{\rm H}$ and $L_4^{\rm GLPV}$ depend on the variables λ, $\varkappa$ and $\mathfrak{K}$.

In the Horndeski theory, the stability conditions (3.131) and (3.134) read

$$-L_\lambda^{\rm H} = \frac{1}{2} L_\mathfrak{K}^{\rm H} = G_4 - 2XG_{4X} > 0\,, \qquad L_\varkappa^{\rm H} = G_4 > 0\,. \tag{3.135}$$

The first of these conditions exactly corresponds to Eq. (25) or Eq. (28) of Ref. [36] (these two conditions coincide when $L_5^{\rm H} = 0$). The second is the condition imposed in Ref. [36] for avoiding gradient instabilities, when $L_5^{\rm H} = 0$. Since $X > 0$, the first condition (3.135) gives information beyond the second one only for $G_{4X} > 0$.

In the GLPV theory, the stability conditions reduce to

$$L_\lambda^{\rm GLPV} = -\frac{1}{2} L_\mathfrak{K}^{\rm GLPV} = A_4 < 0\,, \qquad L_\varkappa^{\rm GLPV} = B_4 > 0\,. \tag{3.136}$$

It is easy to see that, in the Horndeski limit characterized by Eq. (3.81), these reduce to Eq. (3.135).

3.7.2 Stability conditions for covariant Galileon models

Covariantized Galileons

The original Galileon model advocated in Ref. [13] is composed of five Lagrangians invariant under the Galilean symmetry $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$ in the Minkowski background. The equations of motion remain of second order by virtue of this symmetry. In the curved background, the original Galileon model can be covariantized by replacing coordinate derivatives with covariant derivatives. This "covariantized Galileon" belongs to a particular case of the GLPV theory given by the Lagrangians (3.76)-(3.78) with the functions

$$A_2 = c_2 X\,, \qquad A_3 = c_3 X^{3/2}\,, \qquad A_4 = -\frac{M_{\rm pl}^2}{2} - c_4 X^2\,, \qquad B_4 = \frac{M_{\rm pl}^2}{2}\,, \tag{3.137}$$

where $c_{2,3,4}$ are constants. Here we have taken into account the Einstein-Hilbert term $M_{\rm pl}^2 R/2$ in the Lagrangian, where $M_{\rm pl}$ is the reduced Planck mass.

In general space-time, the theory described by (3.137) contains derivatives higher than second order. On the flat isotropic cosmological background, however, the equations of motion for the background and linear perturbations are second order without a new propagating degree of freedom [39]. This result was obtained by considering the constant-time hypersurfaces, such that the scalar field plays the role of time. A similar argument may also work for the spherically symmetric background due to the high degree of symmetry, in which case the scalar field takes the role of a radial coordinate r. In fact, substituting Eq. (3.137) into the background equations of motion (B.1)-(B.3), we obtain

$$\frac{M_{\rm pl}^2}{r}\left(\frac{1}{r} - \frac{1}{\bar{M}^2 r} + \frac{2\bar{M}'}{\bar{M}^3}\right) + c_2 X + \frac{3c_3 X}{\bar{M}^2}\left(\frac{\phi' \bar{M}'}{\bar{M}} - \phi''\right)$$
$$-\frac{2c_4 X}{\bar{M}^2 r}\left(\frac{X}{r} - \frac{10X\bar{M}'}{\bar{M}} + \frac{8\phi'\phi''}{\bar{M}^2}\right) = 0\,, \tag{3.138}$$
$$\frac{M_{\rm pl}^2}{r}\left(\frac{1}{r} - \frac{1}{\bar{M}^2 r} - \frac{2\bar{N}'}{\bar{M}^2 \bar{N}}\right) - c_2 X - \frac{3c_3 X\phi'}{\bar{M}^2}\left(\frac{2}{r} + \frac{\bar{N}'}{\bar{N}}\right)$$

$$
-\frac{10c_4X^2}{r\bar{M}^2}\left(\frac{1}{r}+\frac{2\bar{N}'}{\bar{N}}\right)=0\,, \tag{3.139}
$$

$$
\begin{aligned}
&\frac{M_{\rm pl}^2}{\bar{M}^2}\left[\frac{\bar{M}'}{\bar{M}r}-\frac{\bar{N}''}{\bar{N}}-\frac{\bar{N}'}{\bar{N}}\left(\frac{1}{r}-\frac{\bar{M}'}{\bar{M}}\right)\right]+c_2X+\frac{3c_3X}{\bar{M}^2}\left(\frac{\phi'\bar{M}'}{\bar{M}}-\phi''\right)\\
&-\frac{2c_4X}{\bar{M}^2}\left[\frac{X\bar{N}''}{\bar{N}}-\frac{5X\bar{M}'}{\bar{M}r}+\frac{4\phi'\phi''}{\bar{M}^2r}+\frac{\bar{N}'}{\bar{N}}\left(\frac{X}{r}-\frac{5X\bar{M}'}{\bar{M}}+\frac{4\phi'\phi''}{\bar{M}^2}\right)\right]=0\,,
\end{aligned} \tag{3.140}
$$

which are of second order. The equations of motion for the odd-mode perturbations are also of second order. The stability conditions (3.136) translate to

$$
c_4\left(\frac{X}{M_{\rm pl}}\right)^2>-\frac{1}{2}\,. \tag{3.141}
$$

The radial and tangential sound speeds read

$$
c_r^2=1+2c_4\left(\frac{X}{M_{\rm pl}}\right)^2\,,\qquad c_\Omega^2=1\,, \tag{3.142}
$$

respectively.

Covariant Galileons

Higher-order derivatives present for the covariantized Galileon in a general curved space-time can be eliminated by including a non-minimally coupled gravitational contribution to the Lagrangian [14]. The Galileon model with second-order equations of motion is dubbed "covariant Galileon". This is a sub-class of the Horndeski Lagrangians (2.39)-(2.41) with the choice

$$
G_2=\hat{c}_2X\,,\qquad G_3=\hat{c}_3X\,,\qquad G_4=\frac{M_{\rm pl}^2}{2}+\hat{c}_4X^2\,, \tag{3.143}
$$

where $\hat{c}_{2,3,4}$ are constants.

From Eqs. (B.1)-(B.3) the background equations of motion are given by

$$\frac{M_{\rm pl}^2}{r}\left(\frac{1}{r}-\frac{1}{\bar{M}^2 r}+\frac{2\bar{M}'}{\bar{M}^3}\right)+\hat{c}_2 X+\frac{2\hat{c}_3 X}{\bar{M}^2}\left(\frac{\phi'\bar{M}'}{\bar{M}}-\phi''\right)$$
$$+\frac{6\hat{c}_4 X}{\bar{M}^2 r}\left(\frac{\bar{M}^2 X}{3r}+\frac{X}{r}-\frac{10X\bar{M}'}{\bar{M}}+\frac{8\phi'\phi''}{\bar{M}^2}\right)=0\,, \tag{3.144}$$

$$\frac{M_{\rm pl}^2}{r}\left(\frac{1}{r}-\frac{1}{\bar{M}^2 r}-\frac{2\bar{N}'}{\bar{M}^2\bar{N}}\right)-\hat{c}_2 X-\frac{2\hat{c}_3 X\phi'}{\bar{M}^2}\left(\frac{2}{r}+\frac{\bar{N}'}{\bar{N}}\right)$$
$$+\frac{30\hat{c}_4 X^2}{\bar{M}^2 r}\left(-\frac{\bar{M}^2}{5r}+\frac{1}{r}+\frac{2\bar{N}'}{\bar{N}}\right)=0\,, \tag{3.145}$$

$$\frac{M_{\rm pl}^2}{\bar{M}^2}\left[\frac{\bar{M}'}{\bar{M}r}-\frac{\bar{N}''}{\bar{N}}-\frac{\bar{N}'}{\bar{N}}\left(\frac{1}{r}-\frac{\bar{M}'}{\bar{M}}\right)\right]+\hat{c}_2 X+\frac{2\hat{c}_3 X}{\bar{M}^2}\left(\frac{\phi'\bar{M}'}{\bar{M}}-\phi''\right)$$
$$+\frac{6\hat{c}_4 X}{\bar{M}^2}\left[\frac{X\bar{N}''}{\bar{N}}-\frac{5X\bar{M}'}{\bar{M}r}+\frac{4\phi'\phi''}{\bar{M}^2 r}+\frac{\bar{N}'}{\bar{N}}\left(\frac{X}{r}-\frac{5X\bar{M}'}{\bar{M}}+\frac{4\phi'\phi''}{\bar{M}^2}\right)\right]=0\,. \tag{3.146}$$

Compared to the covariantized Galileon, the difference arises from the B_4-dependent terms in Eqs. (B.1) and (B.2). The stability conditions (3.135) translate to

$$-\frac{1}{2}<\hat{c}_4\left(\frac{X}{M_{\rm pl}}\right)^2<\frac{1}{6}\,, \tag{3.147}$$

which is different from Eq. (3.141). The radial and tangential speeds of sound are given, respectively, by

$$c_r^2=\frac{M_{\rm pl}^2-6\hat{c}_4 X^2}{M_{\rm pl}^2+2\hat{c}_4 X^2}\,,\qquad c_\Omega^2=1\,, \tag{3.148}$$

where c_r^2 differs from Eq. (3.142).

We have shown that the background and perturbation equations of motion for both the covariantized Galileon (3.137) and the covariant Galileon (3.143) are of second order on the spherically symmetric background. Their perturbations propagate identically along the spheres, but with different propagation speeds in the radial direction.

3.8 conclutions

In Chapter 3, We have studied the perturbations about a spherically symmetric and static background in the framework of the EFT of modified gravity. Since spherical symmetry selects a preferred radial direction besides the time direction, we employed a more intricate 2+1+1 decomposition which is briefly introduced in Sec. 3.2. Due to the double foliation, there are two sets of extrinsic curvatures in the formalism. Some of them are related to temporal derivatives (K_{ab} , $\mathcal{K}_a$, $\mathcal{K}$), the others to radial derivatives (L_{ab} , $\mathcal{L}_a$, $\mathcal{L}$). We have started from a general action that depends on scalars formed from these quantities, the metric variables of the constant time hypersurfaces (h_{ab}, M_a, M) and the lapse N.

We choose the gauge $\mathcal{N} = 0$ to ensure the perpendicularity of the foliations on the spherical symmetric space-time. Then, the dynamics of the radial and temporal components proceeds in a hypersurface-orthogonal manner without vorticities. By this gauge choice, it is possible to avoid an unnecessary increase in the number of variables associated with vorticity-type quantities. A second gauge fixing is the radial unitary gauge $\phi = \phi(r)$, which switches off the perturbations of the scalar field ($\delta\phi = 0$). In this case, the scalar field is absorbed in the gravitational sector (into the radial lapse M) and an explicit radial dependence of the action.

In Sec. 3.3 we started from the gravitational action (3.36) which incorporates a general system of a single scalar degree of freedom. Despite the relatively large number of scalar variables, variation of the action gives rise to three independent equations of motion at the background level. They are derived by the changes in the lapse δN, in the radial lapse δM, and in the scalar curvature on the sphere $\delta\mathcal{R}$, respectively. Equations (3.65)-(3.67) represent the most generic set of equations of motion in modified gravity theories on the spherically symmetric and static background.

In Sec. 3.4 we have expressed the Horndeski and GLPV Lagrangians in terms of the 2+1+1 variables, proving that they belong to the class of the EFT of modified gravity studied in this thesis. We also derived the background equations of motion explicitly for both under spherical symmetry and staticity. Under these symmetries the GLPV background is also second order, as in the case of the Horndeski theory.

In Sec. 3.6 we expanded the action up to second order for the odd mode perturbations and derived the linear perturbation equations of motion, with the even and odd modes decoupled. In this study we focused on the analysis for the odd-parity mode of perturbations. The originally fourth-order differential equations were reduced to second order by employing a multipolar expansion into spherical harmonics. We derived the second-order Lagrangian density for odd-mode perturbations of the form (3.127).

In Sec. 3.7 we derived the stability conditions. Under the condition (3.130), which is satisfied for both Horndeski and GLPV theories, the Lagrangian density is expressed solely by a dynamical scalar variable Z_l and its derivatives. We established extremely simple conditions for avoiding ghosts and Laplacian instabilities. The propagation speed of odd-mode perturbations depends on the direction of propagation. More specifically, the radial sound speed and the sound speed along the spheres are different, generalizing the corresponding result established for the Horndeski theory [36].

As applications of our general stability analysis, we have i) confirmed the corresponding results for the Horndeski theory, ii) obtained the stability conditions for the recently proposed GLPV theory, iii) derived and compared both the tangential and the radial speeds of sound for two types of Galileon theories: "covariantized Galileon" (derived by replacing coordinate derivatives with covariant derivatives in the original Galileon model) and "covariant Galileon" with second-order dynamics in general space-time (obtained by adding a new term to eliminate higher-order derivatives). Although the background equations of motion are similar in the two Galileon theories, the stability conditions associated with the radial propagation speed c_r are different. This can be traced back to the terms B_4 and B_5 appearing in the Lagrangians (3.78) and (3.79) being different in these two theories. In the Horndeski theory B_4 and B_5 are related to the other terms A_4 and A_5 according to Eqs. (3.81) and (3.82), however in general no such restriction appears in the GLPV theory.

Recently, the cosmology based on the two Galileon theories was studied in Ref. [34] on the flat Friedmann-Lemaître-Robertson-Walker background. It was shown that the propagation speeds of the field ϕ for covariant and covariantized Galileons are different due to the different values of B_4 and B_5 in the two theories. On the isotropic cosmological background, the equations of motion for linear perturbations also remain of second order. In spite of the possible presence of derivatives higher than second order on general backgrounds, the GLPV theory remains healthy on both the static spherically symmetric and the isotropic cosmological backgrounds.

It is possible to extend our work to several interesting directions. First, the background equations of motion (3.65)-(3.67) can be generally applied to the discussion of the screening mechanism of the fifth force mediated by the scalar field ϕ. Second, the analysis of even-parity perturbations, which is much more involved than that of odd-parity modes, will be useful to discuss the full stability of the EFT of modified gravity on the spherically symmetric and static background. Third, the construction of theoretically consistent dark energy models in the framework of the GLPV theory will be also intriguing. We leave these issues for future works.

Chapter 4

Sumarry

In this thesis we have investigated the EFT approach to modified gravity on cosmological and spherically symmetric backgrounds. The EFT of modified gravity is a powerful framework to deal with the low-energy degree of freedom of dark energy in a systematic and unified way.

In order to study the cosmological dynamics of modified gravity we need to take into account matter fields, e.g. non-relativistic matter and radiation in addition to the scalar field ϕ associated with the modification of gravity. In Chapter 2 we studied the EFT of modified gravity on the cosmological background in the presence of multiple scalar fields χ_I $(I = 1, 2 \cdots, N-1)$ [34]. These additional scalar fields χ_I can model non-relativistic matter and radiation. Expanding the general action up to second order in the perturbations of geometric scalars and multiple matter fields, we derived propagation speeds of scalar and tensor perturbations as well as no-ghost conditions. Applying our general results to Horndeski and GLPV theories, we obtained an algebraic equation for the propagation speeds of multiple scalar fields and showed that the theories beyond Horndeski induce non-trivial modifications to all the propagation speeds of N scalar fields. This modification to the dark energy field ϕ can be large compared to that for the matter fields χ_I. Then we applied our general results to the two different theories, the covariantized Galileon (a class of GLPV theories) and the covariant Galileon (a class of Horndeski theories), in the presence of non-relativistic matter and radiation. Though these two theories give completely same equations of motion at the level of background, the differences show up at the perturbation level. We estimated the scalar propagation speeds squared in the several epochs of cosmological history in these two theories. For the covariantized Galileon we found that the propagation speed squared c_{s1}^2 of the field ϕ becomes negative in the deep matter era for late-time tracking solutions, while for the covariant Galileon it remains positive. We also showed that the matter sound speeds squared of the fields χ_I for the covariantized Galileon are similar to those for the covariant Galileon.

The modification of gravity realize the late-time cosmic acceleration at large distances, whereas that should be suppressed at short distances since the Solar System agree with general relativity in high precision. In order to understand the latter mechanism and confront several models with the Solar System constraint, one need to study modified gravity on the spherically symmetric background. The EFT of modified gravity on the spherically symmetric background [40] which we investigated in Chapter 3 provide a powerful framework to study the above issue in a unified way. Compared to the case of the cosmological background, there is particular spatial direction singled out by the ADM decomposition besides the temporal direction. Thus we employed the 2+1+1 decomposition to single out both radial and temporal directions from the four-dimensional space-time. Taking the radial unitary gauge, in which the scalar field ϕ associated with the modification of gravity reduces to a function of radius, a contribution of the scalar field is embedded on the constant radius hypersurfaces. Then we constructed the general EFT action of modified gravity in terms of the 2+1+1 ADM geometrical variables. As in the case of the EFT of modified gravity on the cosmological background, we showed that our general action accommodates theories beyond Horndeski. Expanding the action up to linear order for the perturbations we derived the background equations of motion which can be used for discussing the screening mechanism of the fifth force mediated by the scalar degree of freedom. We also expanded the action up to second order for the odd-parity perturbations and derived linear perturbation equations of motion as well as the stability conditions. We applied our general results to both Horndeski and GLPV theories and specialized them for the two distinct theories, covariantized Galileon and the covariant Galileon.

Our general EFT formalism can be applied to a vast range of dark energy models based on modified gravity. We expect that our general results will be useful for the constructions of viable dark energy models that are consistent with several observations and experiments both at large and short distances. We hope that we will be able to approach the origins of dark energy.

Appendix A

The autonomous equations in two Galileon theories

In both Models (A) and (B) described by the functions (2.108) and (2.109), the variables r_1, r_2, and Ω_r obey the following equations of motion

$$\begin{aligned} r_1' &= \frac{1}{\Delta}\left(r_1-1\right)r_1\left[r_1\left(r_1(-3\alpha+4\beta-2)+6\alpha-5\beta\right)-5\beta\right] \\ &\quad\times\Big[2\left(\Omega_r+9\right)+3r_2\Big(r_1^3(-3\alpha+4\beta-2) \\ &\qquad+2r_1^2(9\alpha-9\beta+2)-15r_1\alpha+14\beta\Big)\Big]\,, \qquad \text{(A.1)} \\ r_2' &= -\frac{1}{\Delta}[r_2(6r_1^2(r_2(45\alpha^2-4(9\alpha+2)\beta+36\beta^2)-(\Omega_r-7)(9\alpha-9\beta+2)) \\ &\qquad+r_1^3(-2(\Omega_r+33)(3\alpha-4\beta+2) \\ &\qquad-3r_2(-2(201\alpha+89)\beta+15\alpha(9\alpha+2)+356\beta^2)) \\ &\qquad-3r_1\alpha(-28\Omega_r+123r_2\beta+36)+10\beta(-11\Omega_r+21r_2\beta-3) \\ &\qquad+3r_1^4r_2(9\alpha^2-30\alpha(4\beta+1)+2(2-9\beta)^2)+3r_1^6r_2(3\alpha-4\beta+2)^2 \\ &\qquad+3r_1^5r_2(9\alpha-9\beta+2)(3\alpha-4\beta+2))], \qquad \text{(A.2)} \\ \Omega_r' &= \frac{2}{\Delta}\Omega_r[r_1^2(4(\Omega_r-1)(9\alpha-9\beta+2)+6r_2(-15\alpha^2+36\alpha\beta+4(2-9\beta)\beta)) \\ &\qquad-2r_1^3((\Omega_r-1)(3\alpha-4\beta+2)+9r_2(18(\alpha+1)\beta+\alpha(9\alpha+2)-36\beta^2)) \\ &\qquad+12r_1\alpha(-3\Omega_r+22r_2\beta+3)-10\beta(-4\Omega_r+21r_2\beta+4) \\ &\qquad+r_1^4r_2(549\alpha^2+\alpha(330-840\beta)+2(2-9\beta)^2)+3r_1^6r_2(3\alpha-4\beta+2)^2 \\ &\qquad-12r_1^5r_2(9\alpha-9\beta+2)(3\alpha-4\beta+2)], \\ &\qquad \text{(A.3)} \end{aligned}$$

where

$$\begin{aligned}
\Delta \equiv\ & 2r_1^4 r_2[72\alpha^2 + 30\alpha(1-5\beta) + (2-9\beta)^2] \\
& +4r_1^2[9r_2(5\alpha^2 + 9\alpha\beta + (2-9\beta)\beta) + 2(9\alpha - 9\beta + 2)] \\
& +4r_1^3[-3r_2\left(-2(15\alpha+1)\beta + 3\alpha(9\alpha+2) + 4\beta^2\right) - 3\alpha + 4\beta - 2] \\
& -24r_1\alpha(16r_2\beta + 3) + 10\beta(21r_2\beta + 8). \qquad \text{(A.4)}
\end{aligned}$$

Appendix B

Equations of motion in the Horndeski and GLPV theories on the spherically symmetric and static background

In this Appendix we present the background equations of motion for the spherically symmetric and static GLPV theory (including the Horndeski theory). Substituting the Lagrangians (3.83)-(3.85) into Eqs. (3.65)-(3.67), it follows that

$$A_2 - \frac{\phi' A_{3\phi} + \bar{X}' A_{3X}}{\bar{M}} + \frac{2A_4}{\bar{M}^2 r}\left(\frac{1}{r} - \frac{2\bar{M}'}{\bar{M}}\right) + \frac{4\left(\phi' A_{4\phi} + \bar{X}' A_{4X}\right)}{\bar{M}^2 r} + \frac{2B_4}{r^2} = 0\,, \tag{B.1}$$

$$A_2 - 2\bar{X} A_{2X} - \frac{2\bar{X} A_{3X}}{\bar{M}}\left(\frac{2}{r} + \frac{\bar{N}'}{\bar{N}}\right) + \frac{2\left(A_4 + 2\bar{X} A_{4X}\right)}{\bar{M}^2 r}\left(\frac{1}{r} + \frac{2\bar{N}'}{\bar{N}}\right) + \frac{2\left(B_4 - 2\bar{X} B_{4X}\right)}{r^2} = 0\,, \tag{B.2}$$

$$A_2 - \frac{\phi' A_{3\phi} + \bar{X}' A_{3X}}{\bar{M}} - \frac{2A_4}{\bar{M}^2}\left[\frac{\bar{N}''}{\bar{N}} + \frac{\bar{M}'}{\bar{M} r} - \frac{\bar{N}'}{\bar{N}}\left(\frac{1}{r} - \frac{\bar{M}'}{\bar{M}}\right)\right] + \frac{2\left(\phi' A_{4\phi} + \bar{X}' A_{4X}\right)}{\bar{M}^2}\left(\frac{1}{r} + \frac{\bar{N}'}{\bar{N}}\right) = 0\,, \tag{B.3}$$

where $\bar{X}$ represents the background value of the kinetic term X, i.e., $\bar{X} = \phi'^2/\bar{M}^2$. The last terms on the lhs of Eqs. (B.1)-(B.2), which include B_4 and its derivative with respect to X, originate from the non-vanishing two-dimensional scalar curvature $\mathcal{R}$ on the spherically symmetric and static background. In the Horndeski theory B_4 is entirely determined by A_4 and X, while in the GLPV theory it is not. Hence the equations of motion for the GLPV theory generally differ form those for the Horndeski theory[1].

Under the condition (3.81) and by redefining the functions A_2, A_3 and B_4 in terms of the new functions G_2, F_3 and G_4 as follows

$$A_2 = G_2 - F_{3\phi}X\,, \qquad A_3 = 2X^{3/2}F_{3X} + 2\sqrt{X}G_{4\phi}\,, \qquad B_4 = G_4\,, \quad \text{(B.4)}$$

the sum of the Lagrangians $L^{\rm GLPV}_{2,3,4}$ manifestly reduces to that of $L^{\rm H}_{2,3,4}$. Applying the same condition and redefinitions to the equations of motion (B.1)-(B.3), we obtain those for the Horndeski theory. In order to compare them with the equations of motion derived in Ref. [61] by a method entirely intrinsic to the Horndeski theory, we further need the conversion in the notations ($\bar{N}$, $\bar{M}$, X, G_3)$\to$($e^{\Psi(r)}$, $e^{\Phi(r)}$, $-2X$, $-G_3$), after which a full agreement is reached.

[1] On the flat isotropic cosmological background the scalar curvature of the constant time hypersurfaces identically vanishes. We verified that no B_4 terms appear in the background equations of motion of the GLPV theory, which then coincide with those of the Horndeski theory at the background level.

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List of publications

Peer-reviewed journal

1. "Effective field theory approach to modified gravity including Horndeski theory and Hořava-Lifshitz gravity,"
   Ryotaro Kase and Shinji Tsujikawa,
   International Journal of Modern Physics D, Vol. 23, No. 14 (2014) 1443008

2. "Cosmology in generalized Horndeski theories with second-order equations of motion,"
   Ryotaro Kase and Shinji Tsujikawa,
   Phys. Rev. D **90**, 044073 (2014) [arXiv:1407.0794 [hep-th]].

3. "Effective field theory of modified gravity on the spherically symmetric background: leading order dynamics and the odd-type perturbations,"
   Ryotaro Kase, László Á. Gergely and Shinji Tsujikawa,
   Phys. Rev. D **90**, 124019 (2014) [arXiv:1406.2402 [hep-th]].

4. "Screening the fifth force in the Horndeski's most general scalar-tensor theories,"
   Ryotaro Kase and Shinji Tsujikawa,
   JCAP **1308**, 054 (2013) [arXiv:1306.6401 [gr-qc]].

5. "Vainshtein mechanism in second-order scalar-tensor theories,"
   Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
   Phys. Rev. D **85**, 044059 (2012) [arXiv:1111.5090 [gr-qc]].

6. "Matter perturbations in Galileon cosmology,"
   Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
   Phys. Rev. D **83**, 043515 (2011) [arXiv:1011.6132 [astro-ph.CO]];

Invited talk

1. Ryotaro Kase, László Á. Gergely and Shinji Tsujikawa,
"Effective field theory of modified gravity on the spherically symmetric background," (Oral presentation),
1st APCTP-TUS workshop on Dark Energy,
Jun. 2014, APCTP Headquarter, Pohang, Korea.

International conferences

1. Ryotaro Kase and Shinji Tsujikawa,
"The effective field theory of modified gravity," (oral and poster presentation),
Testing Gravity 2015,
Jan. 2015, SFU Harbour Center, Vancouver, Canada.

2. Ryotaro Kase and Shinji Tsujikawa,
"Effective field theory approach to modified gravity including Horndeski theory and Hořava-Lifshitz gravity," (oral presentation),
The 24th Workshop on General Relativity and Gravitation (JGRG24),
Nov. 2014, Kavli IPMU, Kashiwa, Japan.

3. Ryotaro Kase and Shinji Tsujikawa,
"Cosmology in generalized Horndeski theories with second-order equations of motion," (oral presentation),
Relativistic Cosmology,
Sep. 2014, YITP, Kyoto, Japan.

4. Ryotaro Kase, László Á. Gergely and Shinji Tsujikawa,
"Effective field theory of modified gravity on the spherically symmetric background," (poster presentation),
569. Wilhelm and Else Heraeus Seminar Quantum Cosmology,
Jul. 2014, hysikzentrum in Bad Honnef, Germany.

5. Ryotaro Kase, László Á. Gergely and Shinji Tsujikawa,
"Effective field theory of modified gravity on the spherically symmetric background," (poster and oral presentation),
Non-Linear Structure in the Modified Universe,
Jul. 2014, Lorentz Center, Leiden University, Netherlands.

6. Ryotaro Kase and Shinji Tsujikawa,
"Screening the fifth force in the Horndeski's most general scalar-tensor theories," (oral presentation),
Seventh Aegean Summer School Beyond Einstein's Theory of Gravity,
Sep. 2013, Paros, Greece.

7. Ryotaro Kase and Shinji Tsujikawa,
"Vainshtein mechanism in the most general scalar-tensor theories," (oral presentation),
The 3rd International Workshop on Dark Matter and Dark Energy,
Dec. 2012, NCTS, National Tsing Hua University, Taiwan.

8. Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
"Vainshtein mechanism in second-order scalar-tensor theories," (oral presentation),
Thirteenth Marcel Grossmann Meeting,
Jul. 2012, Stockholm University, Sweden.

9. Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
"Matter perturbations in Galileon cosmology," (oral presentation),
Sixth Aegean Summer School Quantum Gravity and Quantum Cosmology,
Sep. 2011, Naxos, Greece.

Domestic conferences

1. Ryotaro Kase, László Á. Gergely and Shinji Tsujikawa,
"Effective field theory of modified gravity on the spherically symmetric background Part I. leading order dynamics and the odd-type perturbations," (oral presentation),
The 2014 Annual Autumn Meeting of the Physical Society of Japan,
Sep. 2014, Saga University, Japan.

2. Ryotaro Kase and Shinji Tsujikawa,
"Vainshtein mechanism in Horndeski theories," (oral presentation),
The 68th Annual Meeting of the Physical Society of Japan,
Mar. 2013, Hiroshima University, Japan.

3. Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
"Vainshtein mechanism in the second-order scalar-tensor theories," (oral presentation),
The 68th Annual Meeting of the Physical Society of Japan,
Mar. 2012, Kwansei Gakuin University, Japan.

4. Antonio De Felice, Ryotaro Kase and Shinji Tsujikawa,
"Matter perturbations in Galileon cosmology," (oral presentation),
The 68th Annual Meeting of the Physical Society of Japan,
Mar. 2011, Niigata University, Japan.

Award

1. Outstanding presentation award,
The 24th Workshop on General Relativity and Gravitation (JGRG24),
Nov. 2014, Kavli IPMU, Kashiwa, Japan.