PH. D. THESIS

# The cyclic homology of truncated quiver algebras and the Hochschild homology dimension
# (切頂箙多元環の巡回ホモロジーとホッホシルトホモロジー次元)

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# Contents

# Chapter 1

# Introduction

This thesis is based on [**13**] and [**14**].

Let $\Delta$ be a finite quiver and $K$ a field. We fix a positive integer $m \geq 2$. The truncated quiver algebra is defined by $K\Delta/R_\Delta^m$ where $R_\Delta$ is the arrow ideal of $K\Delta$.

In [**19**], Sköldberg computed the Hochschild homology of a truncated quiver algebra $A$ over a commutative ring using an explicit description of the minimal left $A^e$-projective resolution $\boldsymbol{P}$ of $A$. He also computed the Hochschild homology of quadratic monomial algebras. On the other hand, Cibils gave a useful projective resolution $\boldsymbol{Q}$ for more general algebras in [**6**]. If $A$ is a $K$-algebra with a decomposition $A = E \oplus r$, where $E$ is a separable subalgebra of $A$ and $r$ a two-sided ideal of $A$, then the $E$-normalized mixed complex is given by Cibils in [**7**]. Sköldberg [**20**] gave chain maps between the left $A^e$-projective resolution given in [**19**] and $\boldsymbol{Q}$ above for a quadratic monomial algebra $A$, and he obtained the cyclic homology by computing $E^2$-terms of a spectral sequence determined by the above mixed complex due to Cibils. By the similar way, the author computes the cyclic homology of an algebra associated with a cyclic quiver and a monic polynomial in [**12**].

In this thesis, by means of chain maps between the left $A^e$-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra $A$ over an arbitrary field, we compute the cyclic homology following the Sköldberg's way. On the other hand, by means of [**17**, Theorem 4.1.13], Taillefer [**21**] gave a dimension formula for the cyclic homology of truncated quiver algebras over a field of characteristic zero. Our result generalizes the formula into the case of fields of any characteristic. Recently, Volčič [**22**] generalizes our result into the case of commutative rings.

Moreover, by means of the module structure of the Hochschild homology of truncated quiver algebras, we show the following assertion: Let $K$ be a field, $\Delta$ a finite quiver, $R_\Delta$ the arrow ideal of $K\Delta$ and $m \geq 2$ a positive integer. If an algebra $K\Delta/I$ with an ideal $I \subset K\Delta$ contained in $R_\Delta^m$ has an $m$-truncated cycle, then $K\Delta/I$ has infinitely many nonzero Hochschild homology groups (Theorem 4.9). Consequently, in the case $I$ is an admissible ideal of $K\Delta$ which is contained in $R_\Delta^m$, then $K\Delta/I$ satisfies an $m$-truncated cycles version of the "no loops conjecture". That is, if $K\Delta/I$ has finite global dimension, then it contains no $m$-truncated cycles (Corollary 4.10). This result generalizes the result [**5**, Corollary 3.3].

In [**9**], Happel asked that if all higher Hochschild cohomology groups of a finite dimensional algebra vanish, then is the algebra of finite global dimension? This is called "Happel's question". It is shown in [**4**] that this does not hold in general. On the other hand, in [**8**], Han conjectured the homology version of Happel's question, that is, "if all higher Hochschild homology groups of a finite dimensional algebra vanish, then the algebra is of finite global dimension". Moreover, he showed that the counter example of Happel's question in [**4**] satisfies Han's conjecture in [**8**].

In [**5**], Han's conjecture is approached with focusing on the combinatorics of quivers of algebras. In particular, it is shown that all algebras having 2-truncated cycles in which the product of two consecutive arrows is always zero, have infinitely many nonzero Hochschild homology groups. Consequently, 2-truncated cycles version of the well-known "no loops conjecture" holds, that is, algebras of finite global dimension have no 2-truncated cycles. In addition, for arbitrary integer $m \geq 2$, an $m$-truncated cycles version of the "no loops conjecture" is conjectured. In particular, it is shown that monomial algebras satisfy an $m$-truncated cycles version of the "no loops conjecture". However, in [**10**], it was shown that the original no loops conjecture is true for finite dimensional elementary algebras, and this can be obtained by an earlier result of Lenzing in [**16**] (cf. [**11**]).

This thesis is organized as follows: In Chapter 2, we describe the definitions and the notation for the Hochschild homology, cyclic homology and truncated quiver algebras, etc. In Chapter 3, we recall chain maps between Sköldberg's projective resolution $\boldsymbol{P}$ and Cibils' projective resolution $\boldsymbol{Q}$ which are given by Ames, Cagliero and Tirao [**1**]. By means of these chain maps, we consider the spectral sequence determined by the mixed complex due to Cibils. That is, using these chain maps, we investigate the $E^1$-page consisted of the Hochschild homologies which are computed by Sköldberg, and we compute $E^2$-terms. Then we obtain the dimension formula of the cyclic homology of truncated quiver algebras over an arbitrary field (Theorem 3.8). Moreover, we apply the result to the case of cyclic quivers (Example 3.10).

In Chapter 4, we give the elements which correspond to nonzero homology classes for the Hochschild homology of a truncated quiver algebra $A$ in [**19**] (Lemma 4.4 and Lemma 4.5). In addition, we describe Cibils' projective resolution $\boldsymbol{Q}$ and some chain maps which induce quasi-isomorphisms in order to show the main result in this chapter. Furthermore, we describe the definition of an $m$-truncated cycle, and we prove our main theorem (Theorem 4.9) by means of the composition map of quasi-isomorphisms. Moreover, we obtain a corollary (Corollary 4.10), that extends [**5**, Corollary 3.3] for arbitrary $m$, which is an $m$-truncated cycles version of the "no loops conjecture". On the other hand, as we point out in the Remark 4.11, it is known that if a bound quiver algebra $KQ/I$ has a cyclically free oriented cycle then $KQ/I$ has infinite global dimension by Igusa, Liu and Paquette [**11**, Corollary 2.4]. So we show an example

(Example 4.12) of an algebra with infinite global dimension which has a 3-truncated cycle and has no cyclically free oriented cycles.

For general facts on quivers we refer to [**2**], and for the cyclic homology we refer to [**17**] and [**18**].

# Chapter 2

# Preliminaries

In this chapter, we recall the definitions and the notation for the Hochschild homology and cyclic homology and truncated quiver algebras. Let $K$ be a commutative ring and $A$ a unital $K$-algebra. For each $n \geq 1$, we denote the $n$-fold tensor product $A \otimes_K \cdots \otimes_K A$ of $A$ by $A^{\otimes n}$ and an enveloping algebra by $A^e$.

## 2.1. Hochschild homology groups

**Definition 2.1** ([**17**])**.** The Hochschild complex is the following complex:

$$\cdots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A^{\otimes 2} \xrightarrow{b} M \otimes A \xrightarrow{b} M,$$

where $M$ is a bimodule over $A$, the module $M \otimes A^{\otimes n}$ is in degree $n$ and the map $b : M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1}$ given by the formula

$$\begin{aligned} b(x \otimes a_1 \otimes \cdots \otimes a_n) :=& xa_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i (x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

The $n$-th Hochschild homology group of the unital $K$-algebra with coefficients in the $A$-bimodule $M$ is the $n$-th homology group of the Hcohschild complex, and called the $n$-th Hochschild homology group of $A$ simply if $M = A$. The $n$-th Hochschild homology group of $A$ is denoted by $HH_n(A)$.

In [**5**], the Hochschild homology dimension of the algebra $A$ is defined by

$$\operatorname{HHdim} A = \sup\{n \in \mathbb{Z} \mid HH_n(A) \neq 0\}.$$

It is well known that if the unital $K$-algebra $A$ is a projective $K$-module, then the $n$-th Hochschild homology group of $A$ is given by $\operatorname{Tor}_n^{A^e}(A, A)$. Now we recall the definition of the bar resolution of $A$.

**Definition 2.2** ([**17**]). Let $A$ be a unital $K$-algebra. The following resolution of the left $A^e$-module $A$ denoted by $C^{\mathrm{bar}}$ is called the *bar resolution*:

$$C^{\mathrm{bar}} :\longrightarrow A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \longrightarrow \cdots \longrightarrow A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0,$$

where $\mu$ is multiplication and $b'$ is defined by

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n).$$

Let $A$ and $B$ be two $K$-algebras and suppose that $A \xrightarrow{f} B$ is a $K$-algebra homomorphism. Then $f$ is a homomorphism of rings, and the composition map of $f$ and the map $K \to A$ giving the $K$-algebra structure of $A$ is equal to the map $K \to B$ giving the $K$-algebra structure of $B$. This implies that $bf^{\otimes(n+1)} = f^{\otimes n}b$, therefore $\{f^{\otimes n}\}_{n\in\mathbb{N}}$ is a chain map between the Hochschild complex of $A$ and the one of $B$. For each $n \geq 0$, this map of Hochschild complexes induces a map $f^{\otimes(n+1)} : HH_n(A) \to HH_n(B)$ of Hochschild homology groups. The following fact is the key of the main theorem in [**5**]: if we can show that the image of $HH_n(A) \to HH_n(B)$ is nonzero, then this forces $HH_n(A)$ to be nonzero.

## 2.2. Cyclic homology groups

Following Loday [**17**], we recall the definition of the cyclic homology groups of $A$. The cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ action on the module $A^{\otimes n+1}$ is given by letting its generator $t = t_n$ act by

$$t_n(a_0, \cdots, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1})$$

on the generators of $A^{\otimes n+1}$. It is then extended to $A^{\otimes n+1}$ by linearity; it is called the *cyclic operator*. Let $N = 1 + t + \cdots + t^n$ denote the corresponding *norm operator* on $A^{\otimes n+1}$. The *cyclic bicomplex* is defined by the following first quadrant bicomplex denoted $CC(A)$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & \cdots
\end{array}$$

By convention the module $A$, which is in the left-hand corner, is of bidegree $(0,0)$, so $CC_{p,q}(A) = C_q(A) = A^{\otimes q+1}$. The *cyclic homology groups* $HC_n(A), n \geq 0$, of the associative $k$-algebra $A$ are the homology groups of the total complex $\operatorname{Tot} CC(A)$:

$$HC_n(A) := H_n(\operatorname{Tot} CC(A)).$$

Let us recall Kassel's concept of mixed complexes.

**Definition 2.3** ([**15**])**.** A mixed complex $(C, b, B)$ is a family $C = \{C_n\}_{n\geq 0}$ of $K$-modules together with endomorphism $b$ of degree $-1$ and $B$ of degree $+1$ satisfying the equations

$$b^2 = B^2 = bB + Bb = 0.$$

The mixed complex $(C, b, B)$ is considered as the double complex $\mathcal{B}C$ such that its $(p, q)$-term is the module $C_{q-p}$ if $q \geq p$ and $0$ if $q < p$, the horizontal differential is $B$ and the vertical differential is $b$. If we filter the total complex $\operatorname{Tot} \mathcal{B}C$ by the column filtration, then, since $\mathcal{B}C$ is a first quadrant double complex, we have a convergent spectral sequence

$$E^2_{p,q} \underset{p}{\Rightarrow} H_{p+q}(\operatorname{Tot} \mathcal{B}C).$$

It is well known that the cyclic homology of a $K$-algebra $A$ coincides with the cyclic homology of the mixed complex $(C, b, B)$ with the modules $C_n = A^{\otimes n+1}$ and the differentials $b$, $B$ given by

$$\begin{aligned}
b_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &:= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
&\qquad\qquad + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \\
B_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &:= \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \\
&\quad - \sum_{i=0}^{n} (-1)^{ni} a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.
\end{aligned}$$

Therefore, we have a convergent spectral sequence

$$E^2_{p,q} \underset{p}{\Rightarrow} HC_{p+q}(A).$$

In particular, the $E^0$-page is given by $E^0_{p,q} = C_{q-p}$ and the differential is $d^0_{p,q} = b_{q-p}$. Thus the $E^1$-term of this spectral sequence is $E^1_{p,q} = HH_{q-p}(A)$.

## 2.3. Gradings on truncated quiver algebras by Sköldberg and period of cycles

Let $K$ be a commutative ring, $\Delta$ a finite quiver, $m(\geq 2)$ a positive integer and $R_\Delta$ the arrow ideal of $K\Delta$. An algebra formed $K\Delta/R_\Delta^m$ is called truncated quiver algebra.

For $\alpha \in \Delta_1$, its source and target are denoted by $s(\alpha)$ and $t(\alpha)$, respectively. A path in $\Delta$ is a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$. The set of all paths of length $n$ is denoted by $\Delta_n$.

By adjoining the element $\perp$, we will consider the following set (cf. [**19**], [**20**]):

$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i.$$

This set is a semigroup with the multiplication defined by

$$\delta \cdot \gamma = \begin{cases} \delta\gamma & \text{if} \quad t(\delta) = s(\gamma), \\ \perp & \text{otherwise}, \end{cases} \qquad \delta, \gamma \in \bigcup_{i=0}^{\infty} \Delta_i,$$

and

$$\perp \cdot \gamma = \gamma \cdot \perp = \perp, \quad \gamma \in \hat{\Delta}.$$

Hence, $K\hat{\Delta}$ is a semigroup algebra and the path algebra $K\Delta$ is isomorphic to $K\hat{\Delta}/(\perp)$. So, $K\Delta$ is a $\hat{\Delta}$-graded algebra with a basis consisting of the paths in $\Delta$. Moreover, $K\Delta$ is $\mathbb{N}$-graded, that is, $K\Delta = \bigoplus_{i=0}^{\infty} K\Delta_i$. In particular, $R_{\Delta}^m$ is $\hat{\Delta}$-graded and $\mathbb{N}$-graded, thus the truncated quiver algebra $A = K\Delta/R_{\Delta}^m$ is a $\hat{\Delta}$-graded and $\mathbb{N}$-graded algebra.

For an $\mathbb{N}$-graded vector space $V = \bigoplus_{i \geq 0} V_i$, $V_+$ is defined by $V_+ = \bigoplus_{i \geq 1} V_i$.

Let $\Delta$ be a finite quiver. For a path $\gamma$, $|\gamma|$ denotes the length of $\gamma$. A path $\gamma$ is said to be a cycle if $|\gamma| \geq 1$ and its source and target coincide. The period of a cycle $\gamma$ is defined by the smallest integer $i$ such that $\gamma = \delta^j$ $(j \geq 1)$ for a cycle $\delta$ of length $i$, which is denoted by $\operatorname{per}\gamma$. A cycle is said to be a basic cycle if the length of the cycle coincides with its period. It is also called a proper cycle [**8**]. Denote by $\Delta_n^{\mathrm{c}}$ (respectively $\Delta_n^{\mathrm{b}}$) the set of cycles (respectively basic cycles) of length $n$. Let $G_n = \langle g \rangle$ be the cyclic group of order $n$ and the path $\alpha_1 \cdots \alpha_{n-1}\alpha_n$ a cycle where $\alpha_i$ is an arrow in $\Delta$. Then we define the action of $G_n$ on $\Delta_n^{\mathrm{c}}$ by $g \cdot (\alpha_1 \cdots \alpha_{n-1}\alpha_n) := \alpha_n\alpha_1 \cdots \alpha_{n-1}$, and $\Delta_n^{\mathrm{c}}/G_n$ denotes the set of all $G_n$-orbits on $\Delta_n^{\mathrm{c}}$. Similarly, $G_n$ acts on $\Delta_n^{\mathrm{b}}$, and $\Delta_n^{\mathrm{b}}/G_n$ denotes the set of all $G_n$-orbits on $\Delta_n^{\mathrm{b}}$. For $\bar{\gamma} \in \Delta_n^{\mathrm{c}}/G_n$, we denote by $\operatorname{per}\bar{\gamma}$ the period of $\gamma$, that is $\operatorname{per}\bar{\gamma} := \operatorname{per}\gamma$. For convenience we use the notation $\Delta_0^{\mathrm{c}}/G_0$ for the set of vertices $\Delta_0$.

# Chapter 3

# The dimension formula of the cyclic homology of truncated quiver algebras over a field of positive characteristic

In this chapter, by means of chain maps between the left $A^e$-projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ of a truncated quiver algebra $A$ over an arbitrary field, we compute the cyclic homology following the Sköldberg's way.

On the other hand, by means of [**17**, Theorem 4.1.13], Taillefer [**21**] gave a dimension formula for the cyclic homology of truncated quiver algebras over a field of characteristic zero. Our result generalizes the formula into the case of the field of any characteristic.

## 3.1. Chain maps between the projective resolutions constructed by Sköldberg and one by Cibils

In this section, we recall the projective resolutions of a truncated quiver algebra $\boldsymbol{P}$ and $\boldsymbol{Q}$ constructed by Sköldberg and Cibils, respectively. Moreover, we describe chain maps between $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Sköldberg gave the following projective resolution of a truncated quiver algebra $A$ as a left $A^e$-module.

**Theorem 3.1** ([**19**, Theorem 1]). *The following is a projective $\hat{\Delta}$-graded resolution of $A$ as a left $A^e$-module:*

$$\boldsymbol{P} : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0.$$

*Here the modules are defined by*

$$P_i = A \otimes_{K\Delta_0} K\Gamma^{(i)} \otimes_{K\Delta_0} A,$$

*where* $\Gamma^{(i)}$ *is given by*

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \textit{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \textit{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

*and the differentials are defined by*

$$\begin{aligned} & d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta) \\ & \quad = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta, \end{aligned}$$

*and*

$$d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta) = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta.$$

*The augmentation* $\varepsilon \colon A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A \longrightarrow A$ *is defined by*

$$\varepsilon(\alpha \otimes \beta) = \alpha\beta.$$

On the other hand, Cibils gave the following another projective resolution of $A$ as a left $A^e$-module.

**Lemma 3.2** ([**6**, Lemma 1.1])**.** *Let* $\Delta$ *be a finite quiver,* $I$ *an admissible ideal,* $K\Delta_0$ *the subalgebra of* $A = K\Delta/I$ *generated by* $\Delta_0$ *and* $r$ *the Jacobson radical of* $A$*. The following is a projective resolution of* $A$ *as a left* $A^e$*-module*:

$$\begin{aligned} \boldsymbol{Q} : \cdots \longrightarrow A \otimes_{K\Delta_0} r^{\otimes^i_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{d_i} A \otimes_{K\Delta_0} r^{\otimes^{i-1}_{K\Delta_0}} \otimes_{K\Delta_0} A \longrightarrow \cdots \\ \longrightarrow A \otimes_{K\Delta_0} r \otimes_{K\Delta_0} A \xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0, \end{aligned}$$

*where*

$$\begin{aligned} & d_0(\lambda[\quad]\mu) = \lambda\mu, \\ & d_i(\lambda[x_1|\cdots|x_i]\mu) = \lambda x_1[x_2|\cdots|x_i]\mu + \sum_{j=1}^{i-1}(-1)^i\lambda[x_1|\cdots|x_jx_{j+1}|\cdots|x_i]\mu \\ & \qquad\qquad + (-1)^i\lambda[x_1|\cdots|x_{i-1}]x_i\mu \quad \textit{for } i \geq 1, \end{aligned}$$

*and we use the bar notation* $\lambda[x_1|\cdots|x_i]\mu$ *for* $\lambda \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes \mu$*.*

Chain maps between the projective resolutions $\boldsymbol{P}$ and $\boldsymbol{Q}$ are given in [**1**]. We describe a map $\iota : \boldsymbol{P} \to \boldsymbol{Q}$ in Proposition 3.3, and we give an alternative proof that $\iota$ is a chain map, which is direct and shorter than the proof in [**1**].

**Proposition 3.3** ([**1**]). *Define the map $\iota : \boldsymbol{P} \to \boldsymbol{Q}$ as follows*:

$$
\begin{aligned}
&\iota_0(\alpha\otimes\beta)=\alpha[\quad]\beta,\ \iota_1(\alpha\otimes\alpha_1\otimes\beta)=\alpha[\alpha_1]\beta,\\
&\iota_{2c}(\alpha\otimes\alpha_1\cdots\alpha_{cm}\otimes\beta)\\
&\quad=\sum_{0\le j_1,\dots,j_c\le m-2}\alpha[\alpha_1\cdots\alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1}\cdots\alpha_{3+j_1+j_2}|\alpha_{4+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c-1+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{cm}\beta,\\
&\iota_{2c+1}(\alpha\otimes\alpha_1\cdots\alpha_{cm+1}\otimes\beta)\\
&\quad=\sum_{0\le j_1,\dots,j_c\le m-2}\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta,
\end{aligned}
$$

*where $\alpha_1,\alpha_2,\ldots\in\Delta_1$ and $\alpha,\beta\in A$. Then, $\iota$ is a chain map.*

*Proof.* $\mathrm{id}_A\varepsilon=\varepsilon\iota_0$ and $\iota_0 d_1=d_1\iota_1$ are clear. We will check, for $c>0$, $\iota_{2c}d_{2c+1}=d_{2c+1}\iota_{2c+1}$. Similarly, we have $\iota_{2c-1}d_{2c}=d_{2c}\iota_{2c}$. Now we have

$$
\begin{aligned}
&\iota_{2c}d_{2c+1}(\alpha\otimes\alpha_1\cdots\alpha_{cm+1}\otimes\beta)\\
&\quad=\iota_{2c}(\alpha\alpha_1\otimes\alpha_2\cdots\alpha_{cm+1}\otimes\beta-\alpha\otimes\alpha_1\cdots\alpha_{cm}\otimes\alpha_{cm+1}\beta)\\
&\quad=\sum_{0\le j_1,\dots,j_c\le m-2}(\alpha\alpha_1[\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&\qquad-\alpha[\alpha_1\cdots\alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1\cdots\alpha_{3+j_1+j_2}}|\alpha_{4+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c-1+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&d_{2c+1}\iota_{2c+1}(\alpha\otimes\alpha_1\cdots\alpha_{cm+1}\otimes\beta)\\
&\quad=\sum_{0\le j_1,\dots,j_c\le m-2}d_{2c+1}(\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta)\\
&\quad=\sum_{0\le j_1,\dots,j_c\le m-2}\Big(\alpha\alpha_1[\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&\qquad-\alpha[\alpha_1\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{c-1}\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\cdots\\
&\quad|\alpha_{2k-2+j_1+\cdots+j_{k-2}}\cdots\alpha_{2k-2+j_1\cdots+j_{k-1}}|\alpha_{2k-1+j_1+\cdots+j_{k-1}}\\
&\qquad|\alpha_{2k+j_1+\cdots+j_{k-1}}\cdots\alpha_{2k+1+j_1+\cdots+j_k}|\alpha_{2k+2+j_1+\cdots+j_k}\cdots\alpha_{2k+2+j_1+\cdots+j_{k+1}}\\
&\qquad\quad|\alpha_{2k+3+j_1+\cdots+j_{k+1}}|\alpha_{2k+4+j_1+\cdots+j_{k+1}}\cdots\alpha_{2k+4+j_1+\cdots+j_{k+2}}|\alpha_{2k+5+j_1+\cdots+j_{j_k+2}}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&-\sum_{k=1}^{c-1}\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\quad|\alpha_{2k-2+j_1+\cdots+j_{k-2}}\cdots\alpha_{2k-2+j_1+\cdots+j_{k-1}}|\alpha_{2k-1+j_1+\cdots+j_{k-1}}\\
&\qquad|\alpha_{2k+j_1+\cdots+j_{k-1}}\cdots\alpha_{2k+j_1+\cdots+j_k}|\alpha_{2k+1+j_1+\cdots+j_k}\cdots\alpha_{2k+2+j_1+\cdots+j_{k+1}}\\
&\qquad\quad|\alpha_{2k+3+j_1+\cdots+j_{k+1}}|\alpha_{2k+4+j_1\cdots+j_{k+1}}\cdots\alpha_{2k+4+j_1+\cdots+j_{k+2}}|\alpha_{2k+5+j_1+\cdots+j_{k+2}}|\cdots\\
&\qquad\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&+\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c-2+j_1+\cdots+j_{c-2}}\cdots\alpha_{2c-2+j_1+\cdots+j_{c-1}}|\alpha_{2c-1+j_1+\cdots+j_{c-1}}\\
&\qquad\qquad\quad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&-\alpha[\alpha_1|\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\cdots\\
&\qquad\qquad|\alpha_{2c-1+j_1+\cdots+j_{c-1}}|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}]\\
&\qquad\qquad\qquad\qquad\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\Big)\\
=&\sum_{0\le j_1,\ldots,j_c\le m-2}(\alpha\alpha_1[\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\qquad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta)\\
&-\sum_{j_1=1}^{m-2}\sum_{0\le j_2,\ldots,j_c\le m-2}(\alpha[\alpha_1\cdots\alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1}\cdots\alpha_{3+j_1+j_2}|\alpha_{4+j_1+j_2}|\cdots\\
&\quad|\alpha_{2c-1+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta)\\
&-\alpha[\alpha_1|\alpha_2|\alpha_3\cdots\alpha_{m+1}|\alpha_{m+2}|\cdots|\alpha_{(c-2)m+3}\cdots\alpha_{(c-1)m+1}|\alpha_{(c-1)m+2}]\\
&\qquad\qquad\qquad\qquad\alpha_{(c-1)m+3}\cdots\alpha_{cm+1}\beta\\
=&\sum_{0\le j_1,\ldots,j_c\le m-2}(\alpha\alpha_1[\alpha_2\cdots\alpha_{2+j_1}|\alpha_{3+j_1}|\alpha_{4+j_1}\cdots\alpha_{4+j_1+j_2}|\alpha_{5+j_1+j_2}|\cdots\\
&\quad|\alpha_{2c+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c+j_1+\cdots+j_c}|\alpha_{2c+1+j_1+\cdots+j_c}]\alpha_{2c+2+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta\\
&\qquad-\alpha[\alpha_1\cdots\alpha_{1+j_1}|\alpha_{2+j_1}|\alpha_{3+j_1}\cdots\alpha_{3+j_1+j_2}|\alpha_{4+j_1+j_2}|\cdots\\
&\qquad\quad|\alpha_{2c-1+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c-1+j_1+\cdots+j_c}|\alpha_{2c+j_1+\cdots+j_c}]\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{cm+1}\beta).
\end{aligned}
$$

□

**Proposition 3.4** ([**1**]). *Let $x_1, x_2, \ldots$ be paths in $\Delta$ and $m_1, m_2, \ldots$ the lengths of*

*$x_1, x_2, \ldots$, respectively. We set $x_1 = \alpha_1\alpha_2\cdots\alpha_{m_1}, x_2 = \alpha_{m_1+1}\alpha_{m_1+2}\cdots\alpha_{m_1+m_2}, \ldots$, where $\alpha_1, \alpha_2, \ldots \in \Delta_1$. Then there exists a chain map $\pi : \boldsymbol{Q} \to \boldsymbol{P}$ defined by the following equations*:

$$\pi_0(\alpha[\quad]\beta) = \alpha \otimes \beta,$$

$$\pi_1(\alpha[x_1]\beta) = \sum_{j=1}^{m_1} \alpha\alpha_1\cdots\alpha_{j-1} \otimes \alpha_j \otimes \alpha_{j+1}\cdots\alpha_{m_1}\beta,$$

$$\pi_{2c}(\alpha[x_1|x_2|\cdots|x_{2c}]\beta) = \begin{cases} \alpha \otimes \alpha_1\cdots\alpha_{cm} \otimes \alpha_{cm+1}\cdots\alpha_{m_1+\cdots+m_{2c}}\beta \\ \qquad\qquad \textit{if} \quad m_{2i-1} + m_{2i} \geq m \ (1 \leq i \leq c), \\ 0 \qquad\qquad \textit{otherwise}, \end{cases}$$

$$\pi_{2c+1}(\alpha[x_1|x_2|\cdots|x_{2c+1}]\beta) = \begin{cases} \sum_{j=1}^{m_1} \alpha\alpha_1\cdots\alpha_{j-1} \otimes \alpha_j\cdots\alpha_{j+cm} \otimes \\ \qquad\qquad \alpha_{j+cm+1}\cdots\alpha_{m_1+\cdots+m_{2c+1}}\beta \\ \qquad\qquad \textit{if} \quad m_{2i} + m_{2i+1} \geq m \ (1 \leq i \leq c), \\ 0 \qquad\qquad \textit{otherwise}. \end{cases}$$

These chain maps induce quasi-isomorphisms $\bar{\iota} = \mathrm{id}_A \otimes \iota$ and $\bar{\pi} = \mathrm{id}_A \otimes \pi$ between the complexes $A \otimes_{A^e} \boldsymbol{P}$ and $A \otimes_{A^e} \boldsymbol{Q}$. These complexes are $\Delta_*^{\mathrm{c}}/G_*$-graded([**19**], [**20**]), and the chain maps $\bar{\iota}$ and $\bar{\pi}$ respect this grading. That is, we have the following isomorphisms:

$$\begin{array}{llll} \varphi: & A \otimes_{A^e} P_n \xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Gamma^{(n)} \xrightarrow{\sim} & \displaystyle\bigoplus_{\bar{\gamma} \in \Delta_*^c/G_*} (A \otimes_{K\Delta_0^e} K\Gamma^{(n)})_{\bar{\gamma}} \\ & \bar{\iota}\big\downarrow \ \big\uparrow\bar{\pi} & \\ \psi: & A \otimes_{A^e} Q_n \xrightarrow{\sim} A \otimes_{K\Delta_0^e} A_+^{\otimes_{K\Delta_0}^n} \xrightarrow{\sim} & \displaystyle\bigoplus_{\bar{\gamma} \in \Delta_*^c/G_*} (A \otimes_{K\Delta_0^e} A_+^{\otimes_{K\Delta_0}^n})_{\bar{\gamma}}, \end{array}$$

where each basis for the $\bar{\gamma}$ component are given by the elements $\alpha \otimes \beta \in (A \otimes_{K\Delta_0^e} K\Gamma^{(n)})_{\bar{\gamma}}$ such that $\alpha\beta = g^i\gamma$ for some $i$ and $x_0[x_1|\cdots|x_n] \in (A \otimes_{K\Delta_0^e} A_+^{\otimes_{K\Delta_0}^n})_{\bar{\gamma}}$ such that $x_1\cdots x_n = g^j\gamma$ for some $j$.

## 3.2. Cibils' mixed complex and a spectral sequence

In this section, we introduce the mixed complex constructed by Cibils [**7**] and we will compute the map $B$ induced by this mixed complex and the $E^2$-term, by means of the chain maps $\iota$ and $\pi$.

### 3.2.1. Cibils' mixed complex and Hochschild homology groups

Cibils constructed the following mixed complex.

**Theorem 3.5** ([**7**], [**20**])**.** *Let $\Delta$ be a finite quiver, $K$ a field, and $A = K\Delta/I$ for $I$ a homogeneous ideal. Define the mixed complex $(C_{K\Delta_0}(A), b, B)$ by*

$$C_{K\Delta_0}(A)_n = A \otimes_{K\Delta_0^e} A_+^{\otimes_{K\Delta_0}^n},$$

*and*

$$\begin{aligned} b(x_0[x_1|\cdots|x_n]) &= x_0x_1[x_2|\cdots|x_n] \\ &\quad + \sum_{i=1}^{n-1}(-1)^i x_0[x_1|\cdots|x_ix_{i+1}|\cdots|x_n] \\ &\quad + (-1)^n x_nx_0[x_1|\cdots|x_{n-1}], \\ B(x_0[x_1|\cdots|x_n]) &= \sum_{i=0}^{n}(-1)^{in}[x_i|\cdots|x_n|x_0|\cdots|x_{i-1}]. \end{aligned}$$

*Then $HH_n(C_{K\Delta_0(A)}) = HH_n(A)$ and $HC_n(C_{K\Delta_0}(A)) = HC_n(A)$.*

In particular, if $A$ is a truncated quiver algebra $K\Delta/R_\Delta^m (m \geq 2)$, then the map $B$ in $(C_{K\Delta_0}(A), b, B)$ respects the $\Delta_*^{\mathrm{c}}/G_*$-grading (cf. [**20**]). Furthermore if we consider the double complex $\mathcal{B}C$ associated to this mixed complex and filter the total complex $\mathrm{Tot}\,\mathcal{B}C$ by the column filtration, then the resulting spectral sequence is $\Delta_*^{\mathrm{c}}/G_*$-graded. Thus $HC_n(A)$ is $\Delta_*^{\mathrm{c}}/G_*$-graded. Moreover, for $\bar{\gamma} \in \Delta_*^{\mathrm{c}}/G_*$ the degree $\bar{\gamma}$ part of the $E^1$-term of this spectral sequence is $E^1_{p,q,\bar{\gamma}} = HH_{q-p,\bar{\gamma}}(A)$.

In [**19**], the Hochschild homology of truncated quiver algebras is computed:

**Theorem 3.6** ([**19**, Theorem 2])**.** *Let $K$ be a commutative ring and $A$ a truncated quiver algebra $K\Delta/R_\Delta^m$ and $q = cm + e$ for $0 \leq e \leq m-1$. Then the degree $q$ part of the pth Hochschild homology $HH_p(A)$ is given by*

$$HH_{p,q}(A) = \begin{cases} K^{a_q} & \text{if } 1 \leq e \leq m-1 \text{ and } 2c \leq p \leq 2c+1, \\[2ex] \displaystyle\bigoplus_{r|q}\left(K^{\gcd(m,r)-1} \bigoplus \mathrm{Ker}\left(\cdot\frac{m}{\gcd(m,r)} : K \longrightarrow K\right)\right)^{b_r} & \\ & \text{if } e = 0 \text{ and } 0 < 2c-1 = p, \\[2ex] \displaystyle\bigoplus_{r|q}\left(K^{\gcd(m,r)-1} \bigoplus \mathrm{Coker}\left(\cdot\frac{m}{\gcd(m,r)} : K \longrightarrow K\right)\right)^{b_r} & \\ & \text{if } e = 0 \text{ and } 0 < 2c = p, \\[2ex] K^{\#\Delta_0} & \text{if } p = q = 0, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

*Here we set* $a_q := \#(\Delta_q^{\mathrm{c}}/G_q)$ *and* $b_r := \#(\Delta_r^{\mathrm{b}}/G_r)$.

### 3.2.2. Computation of the $E^2$-term

In this subsection, we consider a truncated quiver algebra $A = K\Delta/R_\Delta^m$ over a field $K$. By investigating the basis of the Hochschild homology and the map $B : HH_{p,\bar{\gamma}}(A) \to HH_{p+1,\bar{\gamma}}(A)$ induced by the differential of the Cibils' mixed complex, for $\bar{\gamma} \in \Delta_t^{\mathrm{c}}/G_t$ we have the degree $\bar{\gamma}$ part of the $E^2$-term of the spectral sequence associated with the Cibils' mixed complex.

**Proposition 3.7.** *The* $E^2$*-term of the spectral sequence associated with the Cibils' mixed complex is given as follows*:

$$E^2_{p,q,\varepsilon} = \begin{cases} K & \text{\it if} \quad p = q \geq 0, \\ 0 & \text{\it otherwise}, \end{cases} \quad \text{for } \varepsilon \in \Delta_0,$$

$$E^2_{p,q,\bar{\gamma}} = \begin{cases} K & \text{\it if} \quad q = 2c,\ p = 0, \\ K & \text{\it if} \quad 2c \leq q - p \leq 2c+1, \\ & \qquad q \geq 2c+1 \text{ \it and } \operatorname{char} K | z, \\ 0 & \text{\it otherwise}, \end{cases}$$
$$\text{for } \bar{\gamma} \in \Delta^{\mathrm{c}}_{cm+e}/G_{cm+e}\ (1 \leq e \leq m-1 \text{ and } c \geq 0),$$

*and*

$$E^2_{p,q,\bar{\gamma}} = \begin{cases} K & \text{\it if} \quad d \cdot \operatorname{char} K | m, \\ 0 & \text{\it otherwise}, \end{cases} + \begin{cases} K^{d-1} & \text{\it if} \quad p = 0,\ q = 2c+1 \\ & \qquad \text{\it or} \quad \operatorname{per}\bar{\gamma} \cdot \operatorname{char} K | cd, \\ 0 & \text{\it otherwise}, \end{cases}$$
$$\text{for } \bar{\gamma} \in \Delta^{\mathrm{c}}_{cm}/G_{cm} (c \geq 0),$$

*where* $\operatorname{per}\bar{\gamma}$ *is the period of* $\bar{\gamma} \in \Delta^{\mathrm{c}}_*/G_*$ *(see Preliminaries for its definition), and we set* $z = |\bar{\gamma}|/\operatorname{per}\bar{\gamma}$ *and* $d = \gcd(m, \operatorname{per}\bar{\gamma})$.

*Proof.* We will investigate the basis of the Hochschild homology and the map $B : HH_{p,\bar{\gamma}}(A) \to HH_{p+1,\bar{\gamma}}(A)$ induced by the differential of the Cibils' mixed complex dividing into the following three cases (cf. [**19**]):

*Case* 1: $\Delta_0^{\mathrm{c}}/G_0 (= \Delta_0)$-degrees,

*Case* 2: $\Delta_t^{\mathrm{c}}/G_t$-degrees for $t \equiv 1, \ldots, m-1 \pmod{m}$ $(t > 0)$,

*Case* 3: $\Delta_t^{\mathrm{c}}/G_t$-degrees for $t \equiv 0 \pmod{m}$ $(t > 0)$.

*Case* 1: We consider the $\Delta_0^{\mathrm{c}}/G_0$-degrees: Suppose that $\varepsilon \in \Delta_0$. Then by Theorem 3.6 we get

$$E^2_{p,q,\varepsilon} = \begin{cases} K & \text{if} \quad p = q \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Case* 2: In this case, we set $t = cm + e$, for $1 \leq e \leq m-1$. We suppose that $\bar{\gamma} \in \Delta_t^c/G_t$ with $\gamma = \alpha_1 \cdots \alpha_t$. We set $z = |\bar{\gamma}|/\mathrm{per}\,\bar{\gamma}$. Then we get

$$HH_{p,\bar{\gamma}}(A) = \begin{cases} K & \text{if} \quad p = 2c,\ 2c+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the basis of $HH_{2c,\bar{\gamma}}(A)$ and $HH_{2c+1,\bar{\gamma}}(A)$ are $\alpha_{cm+1} \cdots \alpha_{cm+e} \otimes \alpha_1 \cdots \alpha_{cm}$ and $\sum_{i=1}^{\mathrm{per}\,\bar{\gamma}} \alpha_{cm+i+1} \cdots \alpha_{cm+e} \alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \cdots \alpha_{cm+i}$, respectively. Hence we have

$$E^1_{p,q,\bar{\gamma}} = \begin{cases} K & \text{if} \quad 2c \leq q-p \leq 2c+1 \text{ and } q \geq 2c, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we obtain

$$\begin{aligned}
&\alpha_{cm+1} \cdots \alpha_{cm+e} \otimes \alpha_1 \cdots \alpha_{cm} \\
&\overset{\psi\bar{\iota}\varphi^{-1}}{\longmapsto} \sum_{0 \leq j_1, \dots, j_c \leq m-2} \alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm+e} \\
&\qquad [\alpha_1 \cdots \alpha_{1+j_1} | \alpha_{2+j_1} | \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} | \alpha_{4+j_1+j_2} | \cdots \\
&\qquad\quad | \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} | \alpha_{2c+j_1+\cdots+j_c}] \\
&\overset{B}{\longmapsto} \sum_{0 \leq j_1, \dots, j_c \leq m-2} \Big( [\alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm+e} \\
&\qquad | \alpha_1 \cdots \alpha_{1+j_1} | \alpha_{2+j_1} | \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} | \alpha_{4+j_1+j_2} | \cdots \\
&\qquad\quad | \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} | \alpha_{2c+j_1+\cdots+j_c}] \\
&\quad + \sum_{i'=1}^{c} (-1)^{2i' \cdot 2c} [\alpha_{2i'+j_1+\cdots+j_{i'}} | \alpha_{2i'+1+j_1+\cdots+j_{i'}} \cdots \alpha_{2i'+1+j_1+\cdots+j_{i'+1}} | \alpha_{2i'+2+j_1+\cdots+j_{i'+1}} \\
&\qquad | \alpha_{2i'+3+j_1+\cdots+j_{i'+1}} \cdots \alpha_{2i'+3+j_1+\cdots+j_{i'+2}} | \alpha_{2i'+4+j_1+\cdots+j_{i'+2}} | \cdots \\
&\qquad | \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} | \alpha_{2c+j_1+\cdots+j_c} \\
&\qquad | \alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm+e} | \alpha_1 \cdots \alpha_{1+j_1} \\
&\qquad | \alpha_{2+j_1} | \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} | \alpha_{4+j_1+j_2} | \alpha_{5+j_1+j_2} \cdots \alpha_{5+j_1+j_2+j_3} | \cdots \\
&\qquad | \alpha_{2i'-1+j_1+j_{i'-1}} | \alpha_{2i'-1+j_1+\cdots+j_{i'-1}} \cdots \alpha_{2i'-1+j_1+\cdots+j_{i'}}] \\
&\quad + \sum_{i'=1}^{c} (-1)^{(2i'-1) \cdot 2c} [\alpha_{2i'-1+j_1+\cdots+j_{i'-1}} \cdots \alpha_{2i'-1+j_1+\cdots+j_{i'}} | \alpha_{2i'+j_1+\cdots+j_{i'}} \\
&\qquad | \alpha_{2i'+1+j_1+\cdots+j_{i'}} \cdots \alpha_{2i'+1+j_1+\cdots+j_{i'+1}} | \alpha_{2i'+2+j_1+\cdots+j_{i'+1}} | \cdots \\
&\qquad | \alpha_{2c-1+j_1+\cdots+j_{c-1}} \cdots \alpha_{2c-1+j_1+\cdots+j_c} | \alpha_{2c+j_1+\cdots+j_c} \\
&\qquad | \alpha_{2c+1+j_1+\cdots+j_c} \cdots \alpha_{cm+e} | \alpha_1 \cdots \alpha_{1+j_1} | \alpha_{2+j_1} \\
&\qquad | \alpha_{3+j_1} \cdots \alpha_{3+j_1+j_2} | \alpha_{4+j_1+j_2} | \cdots \\
&\qquad | \alpha_{2i'-3+j_1+\cdots+j_{i'-2}} \cdots \alpha_{2i'-3+j_1+\cdots+j_{i'-1}} | \alpha_{2i'-2+j_1+\cdots+j_{i'-1}}] \Big)
\end{aligned}$$

$$\overset{\varphi\bar{\pi}\psi^{-1}}{\longmapsto} \sum_{j=1}^{e} \alpha_{cm-e+j+1}\cdots\alpha_{cm+j-1}\otimes\alpha_{cm+j}\cdots\alpha_{cm+e}\alpha_1\cdots\alpha_{cm+e-j}$$
$$+\sum_{i'=1}^{c}\sum_{j=e-1}^{n-2}\alpha_{(i'-1)m-e+j+3}\cdots\alpha_{(i'-1)m+1+j}$$
$$\otimes\,\alpha_{(i'-1)m+2+j}\cdots\alpha_{cm+e}\alpha_1\cdots\alpha_{(i'-1)m-e+j+2}$$
$$+\sum_{i'=1}^{c}\sum_{j=1}^{e-1}\alpha_{(i'-1)m-e+j+1}\cdots\alpha_{(i'-1)m+j-1}$$
$$\otimes\,\alpha_{(i'-1)m+j}\cdots\alpha_{cm+e}\alpha_1\cdots\alpha_{(i'-1)m-e+j}$$
$$=\sum_{i=1}^{cm+e}\alpha_{cm+i+1}\cdots\alpha_{cm+e}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{cm+i}$$
$$=z\sum_{i=1}^{\operatorname{per}\bar{\gamma}}\alpha_{cm+i+1}\cdots\alpha_{cm+e}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{cm+i}.$$

Therefore we conclude that the $E^1$-page in degree $\bar{\gamma}$ is drawn as follows:

$$\begin{array}{ccccccc}
0 & & 0 & & K & \overset{\cdot z}{\leftarrow} & K \\
0 & & K & \overset{\cdot z}{\leftarrow} & K & & 0 \\
K & \overset{\cdot z}{\leftarrow} & K & & 0 & & 0 \\
K & & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the bottom left $K$ lies in bidegree $(0, 2c)$. Therefore we have $E^2$-term as follows:

$$E^2_{p,q,\bar{\gamma}}=\begin{cases} K & \text{if} \quad q=2c,\ p=0,\\ K & \text{if} \quad 2c\le q-p\le 2c+1,\ q\ge 2c+1 \text{ and } \operatorname{char}K|z,\\ 0 & \text{otherwise.}\end{cases}$$

*Case* 3: In this case, we set $t = cm$ and suppose that $\bar{\gamma}\in\Delta_t^c/G_t$ with $\gamma=\alpha_1\cdots\alpha_t$, $z=|\bar{\gamma}|/\operatorname{per}\bar{\gamma}$ and $d=\gcd(m,\operatorname{per}\bar{\gamma})$. Then we have

$$HH_{2c-1,\bar{\gamma}}(A)=\begin{cases} K^d & \text{if} \quad d\cdot\operatorname{char}K|m,\\ K^{d-1} & \text{otherwise,}\end{cases}$$

and

$$HH_{2c,\bar{\gamma}}(A)=\begin{cases} K^d & \text{if} \quad d\cdot\operatorname{char}K|m,\\ K^{d-1} & \text{otherwise,}\end{cases}$$

where the basis of $HH_{2c-1,\bar{\gamma}}(A)$ and $HH_{2c,\bar{\gamma}}(A)$ are given by

$$\begin{cases} \alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i} \ (1\le i\le d-1) \\ \qquad \text{and} \displaystyle\sum_{i=1}^{d}\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i} \text{ if } d\cdot\operatorname{char}K|m, \\ \alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i} \ (1\le i\le d-1) \quad \text{otherwise,} \end{cases}$$

and

$$\begin{cases} \displaystyle\sum_{j=0}^{\operatorname{per}\bar{\gamma}/d-1}(1\otimes\alpha_{i+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+jd-1}-1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{1+jd-1}) \\ \hfill (2\le i\le d) \\ \qquad \text{and} \displaystyle\sum_{j=0}^{\operatorname{per}\bar{\gamma}/d-1}1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{1+jd-1} \qquad \text{if} \quad d\cdot\operatorname{char}K|m, \\ \displaystyle\sum_{j=0}^{\operatorname{per}\bar{\gamma}/d-1}(1\otimes\alpha_{i+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+jd-1}-1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{1+jd-1}) \\ \hfill (2\le i\le d) \qquad \text{otherwise,} \end{cases}$$

respectively.

We will compute the map $B$ induced by the differential of the mixed complex in detail. For $1\le i\le d$, we have

$$\begin{aligned}
&\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i} \\
&\overset{\psi\bar{\iota}\varphi^{-1}}{\longmapsto} \alpha_{i+(c-1)m+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1} \\
&\qquad\qquad [\alpha_i|\alpha_{i+1}\cdots\alpha_{i+m-1}|\alpha_{i+m}|\alpha_{i+m+1}\cdots\alpha_{i+2m-1}|\alpha_{i+2m}|\cdots \\
&\qquad\qquad\qquad |\alpha_{i+(c-2)m+1}\cdots\alpha_{i+(c-1)m-1}|\alpha_{i+(c-1)m}] \\
&\overset{B}{\longmapsto} [\alpha_{i+(c-1)m+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}|\alpha_i \\
&\qquad\qquad |\alpha_{i+1}\cdots\alpha_{i+m-1}|\alpha_{i+m}|\alpha_{i+m+1}\cdots\alpha_{i+2m-1}|\alpha_{i+2m}|\cdots \\
&\qquad\qquad\qquad |\alpha_{i+(c-2)m+1}\cdots\alpha_{i+(c-1)m-1}|\alpha_{i+(c-1)m}] \\
&\qquad +\sum_{i'=1}^{c}(-1)^{(2i'-1)\cdot(2c-1)}[\alpha_{i+(i'-1)m}|\alpha_{i+(i'-1)m+1}\cdots\alpha_{i+i'm-1} \\
&\qquad\qquad |\alpha_{i+i'm}|\alpha_{i+i'm+1}\cdots\alpha_{i+(i'+1)m-1}|\cdots \\
&\qquad\qquad\quad |\alpha_{i+(c-2)m}|\alpha_{i+(c-2)m+1}\cdots\alpha_{i+(c-1)m-1} \\
&\qquad\qquad\qquad |\alpha_{i+(c-1)m}|\alpha_{i+(c-1)m+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1} \\
&\qquad\qquad\qquad\quad |\alpha_i|\alpha_{i+1}\cdots\alpha_{i+m-1}|\alpha_{i+m}|\alpha_{i+m+1}\cdots\alpha_{i+2m-1}|\cdots \\
&\qquad\qquad\qquad\qquad |\alpha_{i+(i'-2)m}|\alpha_{i+(i'-2)m+1}\cdots\alpha_{i+(i'-1)m-1}]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i'=1}^{c-1}(-1)^{2i'\cdot(2c-1)}[\alpha_{i+(i'-1)m+1}\cdots\alpha_{i+i'm-1}|\alpha_{i+i'm}\\
&\qquad|\alpha_{i+i'm+1}\cdots\alpha_{i+(i'+1)m-1}|\alpha_{i+(i'+1)m}|\cdots\\
&\qquad\quad|\alpha_{i+(c-2)m+1}\cdots\alpha_{i+(c-1)m-1}|\alpha_{i+(c-1)m}\\
&\qquad\qquad|\alpha_{i+(c-1)m+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}|\alpha_i\\
&\qquad\qquad\quad|\alpha_{i+1}\cdots\alpha_{i+m-1}|\alpha_{i+m}|\alpha_{i+m+1}\cdots\alpha_{i+2m-1}|\alpha_{i+2m}|\cdots\\
&\qquad\qquad\qquad|\alpha_{i+(i'-2)m+1}\cdots\alpha_{i+(i'-1)m-1}|\alpha_{i+(i'-1)m}]\\
\overset{\varphi\bar\pi\psi^{-1}}{\longmapsto}&\sum_{i'=1}^{c}(1\otimes\alpha_{i+(i'-1)m+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+(i'-1)m}\\
&\qquad\qquad-1\otimes\alpha_{i+(i'-1)m}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+(i'-1)m-1})\\
=&\left(\frac{cd}{\operatorname{per}\bar\gamma}\right)\Bigg(\sum_{j=0}^{\operatorname{per}\bar\gamma/d-1}(1\otimes\alpha_{i+1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+jd}\\
&\qquad\qquad\qquad-1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{jd})\\
&-\sum_{j=0}^{\operatorname{per}\bar\gamma/d-1}(1\otimes\alpha_{i+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1+jd}-1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{jd})\Bigg).
\end{aligned}
$$

Thus, we have

$$
\varphi\bar\pi\psi B\psi\bar\iota\varphi^{-1}\left(\sum_{i=1}^{d}\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i}\right)=0.
$$

Hence, we set that

$$
\begin{aligned}
u_i&=\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i}\quad(1\le i\le d-1),\\
u_d&=\sum_{i=1}^{d}\alpha_{(c-1)m+i+1}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i-1}\otimes\alpha_i\cdots\alpha_{(c-1)m+i}\qquad\text{if}\quad d\cdot\operatorname{char}K|m,
\end{aligned}
$$

and

$$
\begin{aligned}
v_i&=\sum_{j=0}^{\operatorname{per}\bar\gamma/d-1}(1\otimes\alpha_{i+1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{i+1+jd-1}\\
&\qquad\qquad-1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{1+jd-1})\qquad(1\le i\le d-1),\\
v_d&=\sum_{j=0}^{\operatorname{per}\bar\gamma/d-1}1\otimes\alpha_{1+jd}\cdots\alpha_{cm}\alpha_1\cdots\alpha_{1+jd-1}\qquad\text{if}\quad d\cdot\operatorname{char}K|m.
\end{aligned}
$$

If $d{\cdot}\mathrm{char}\,K|m$, then the $E^1$-page in degree $\bar{\gamma}$ is drawn as follows:

$$\begin{array}{ccccccc}
0 & & 0 & & K^d & \xleftarrow{B} & K^d \\
0 & & K^d & \xleftarrow{B} & K^d & & 0 \\
K^d & \xleftarrow{B} & K^d & & 0 & & 0 \\
K^d & & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the bottom left $K^d$ lies in bidegree $(0, 2c-1)$ and $B$ is given by

$$B(u_1, \ldots, u_d) = (v_1, \ldots, v_d) \left( \frac{cd}{\mathrm{per}\,\bar{\gamma}} \right) \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & -1 & 0 \\ & & & \ddots & 1 & 0 \\ 0 & & \cdots & & 0 & 0 \end{pmatrix}.$$

If $d{\cdot}$ $\mathrm{char}\,K \nmid m$, then the $E^1$-page in degree $\bar{\gamma}$ is drawn as follows:

$$\begin{array}{ccccccc}
0 & & 0 & & K^{d-1} & \xleftarrow{B} & K^{d-1} \\
0 & & K^{d-1} & \xleftarrow{B} & K^{d-1} & & 0 \\
K^{d-1} & \xleftarrow{B} & K^{d-1} & & 0 & & 0 \\
K^{d-1} & & 0 & & 0 & & 0 \\
0 & & 0 & & 0 & & 0 \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the bottom left $K^{d-1}$ lies in bidegree $(0, 2c-1)$ and $B$ is given by

$$B(u_1, \ldots, u_{d-1}) = (v_1, \ldots, v_{d-1}) \left( \frac{cd}{\mathrm{per}\,\bar{\gamma}} \right) \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Thus we have $E^2$-term as follows:

$$E^2_{p,q,\bar{\gamma}} = \begin{cases} K & \text{if} \quad d \cdot \mathrm{char}\,K|m, \\ 0 & \text{otherwise,} \end{cases} \quad + \begin{cases} K^{d-1} & \text{if} \quad p = 0,\ q = 2c+1 \\ & \quad \text{or} \quad \mathrm{per}\,\bar{\gamma} \cdot \mathrm{char}\,K|cd, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

## 3.3. The dimension formula of the cyclic homology of truncated quiver algebras

In this section, let $K$ be a field of characteristic $\zeta$ including the case $\zeta = 0$. By the result of Proposition 3.7, we can determine the dimension formula of the cyclic homology $HC_n(A)$. We obtain the following main theorem.

**Theorem 3.8.** *Suppose that $m \geq 2$ and $A = K\Delta/R_\Delta^m$. Then the dimension formula of the cyclic homology of $A$ is given by*

$$\dim_K HC_0(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e},$$

$$\dim_K HC_1(A) = \sum_{\substack{r>0\\ \text{s.t. } r|m}} (\gcd(m,r)-1)b_r + \sum_{e=1}^{m-1} \sum_{\substack{r>0\\ \text{s.t. } r\zeta|e}} b_r + \sum_{\substack{r>0\\ \text{s.t. } r|m,\\ \gcd(m,r)\zeta|m}} b_r,$$

$$\begin{aligned}\dim_K HC_{2c}(A) = \#\Delta_0 &+ \sum_{e=1}^{m-1} a_{cm+e} + \sum_{c'=0}^{c-1}\sum_{e=1}^{m-1} \sum_{\substack{r>0\\ \text{s.t. } r\zeta|c'm+e}} b_r \\ &+ \sum_{c'=1}^{c} \sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0\\ \text{s.t.} r\zeta|\gcd(m,r)c'}} (\gcd(m,r)-1)b_r,\end{aligned}$$

*for $c \geq 1$,*

*and*

$$\begin{aligned}\dim_K HC_{2c+1}(A) = &\sum_{\substack{r>0\\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r + \sum_{c'=0}^{c}\sum_{e=1}^{m-1} \sum_{\substack{r>0\\ \text{s.t. } r\zeta|c'm+e}} b_r \\ &+ \sum_{c'=1}^{c+1} \sum_{\substack{r>0\\ \text{s.t. } r|c'm,\\ \gcd(m,r)\zeta|m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0\\ \text{s.t. } r\zeta|\gcd(m,r)c'}} (\gcd(m,r)-1)b_r,\end{aligned}$$

*for $c \geq 1$.*

*Proof.* By the $E^2$-term of the spectral sequence, we have

$$\begin{cases} \dim_K HC_{2c,c'm+e}(A) = 0 & \text{if} \quad 0 \leq e \leq m-1 \text{ and } c' > c, \\ \dim_K HC_{2c+1,c'm+e}(A) = 0 & \text{if} \quad 1 \leq e \leq m-1 \text{ and } c' > c, \\ \dim_K HC_{2c+1,c'm}(A) = 0 & \text{if} \quad c' > c+1 \text{ and } c' = 0. \end{cases}$$

Suppose that $c \geq 1$. By the above, we have

$$\begin{aligned}
\dim_K HC_{2c}(A) &= \sum_t \dim_K HC_{2c,t}(A) \\
&= \sum_t \sum_{\substack{p+q=2c \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta_t^{\mathrm{c}}/G_t} \dim_K E^2_{p,q,\bar{\gamma}} \\
&= \sum_{\substack{p+q=2c \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta_0} \dim_K E^2_{p,q,\bar{\gamma}} + \sum_{e=1}^{m-1} \sum_{\substack{p+q=2c \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{cm+e}/G_{cm+e}} \dim_K E^2_{p,q,\bar{\gamma}} \\
&\quad + \sum_{c'=0}^{c-1}\sum_{e=1}^{m-1} \sum_{\substack{p+q=2c \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{c'm+e}/G_{c'm+e}} \dim_K E^2_{p,q,\bar{\gamma}} \\
&\quad + \sum_{c'=1}^{c} \sum_{\substack{p+q=2c \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{c'm}/G_{c'm}} \dim_K E^2_{p,q,\bar{\gamma}} \\
&= \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e} + \sum_{c'=0}^{c-1}\sum_{e=1}^{m-1} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta \mid c'm+e}} b_r \\
&\quad + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\mid c'm, \\ \gcd(m,r)\zeta\mid m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta\mid \gcd(m,r)c'}} (\gcd(m,r)-1)b_r.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\dim_K HC_{2c+1}(A) &= \sum_t \dim_K HC_{2c,t}(A) \\
&= \sum_t \sum_{\substack{p+q=2c+1 \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta_t^{\mathrm{c}}/G_t} \dim_K E^2_{p,q,\bar{\gamma}} \\
&= \sum_{c'=0}^{c}\sum_{e=1}^{m-1} \sum_{\substack{p+q=2c+1 \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{c'm+e}/G_{c'm+e}} \dim_K E^2_{p,q,\bar{\gamma}} \\
&\quad + \sum_{c'=1}^{c} \sum_{\substack{p+q=2c+1 \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{c'm}/G_{c'm}} \dim_K E^2_{p,q,\bar{\gamma}} \\
&\quad + \sum_{\substack{p+q=2c+1 \\ p,q\geq 0}} \sum_{\bar{\gamma}\in\Delta^{\mathrm{c}}_{(c+1)m}/G_{(c+1)m}} \dim_K E^2_{p,q,\bar{\gamma}}
\end{aligned}$$

$$= \sum_{\substack{r>0 \\ \text{s.t. } r|(c+1)m}} (\gcd(m,r)-1)b_r + \sum_{e=1}^{m-1}\sum_{c'=0}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta|c'm+e}} b_r$$
$$+ \sum_{c'=1}^{c+1} \sum_{\substack{r>0 \\ \text{s.t. } r|c'm, \\ \gcd(m,r)\zeta|m}} b_r + \sum_{c'=1}^{c} \sum_{\substack{r>0 \\ \text{s.t. } r\zeta|\gcd(m,r)c'}} (\gcd(m,r)-1)b_r.$$

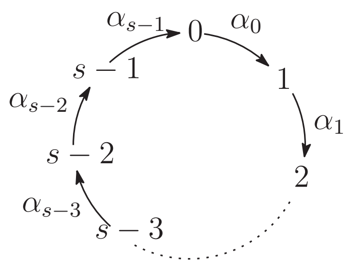
For the dimension formula of $HC_0(A)$ and $HC_1(A)$, we easily have that as a special case of the above computation. $\square$

**Corollary 3.9.** *If the characteristic of $K$ is zero, then the dimension of the cyclic homology of the truncated quiver algebra $A = K\Delta/R_\Delta^m$ is given by*

$$\dim_K HC_{2c}(A) = \#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e},$$
$$\dim_K HC_{2c+1}(A) = \sum_{r|(c+1)m} (\gcd(r,m)-1)b_r.$$

The dimension formula above corrects a result of [**21**, Proposition 4.9].

**Example 3.10.** Let $K$ be a field of characteristic $\zeta$ and $\Delta$ the following quiver:




Suppose $m \geq 2$ and $A = K\Delta/R_\Delta^m$, which is called a truncated cycle algebra in [**3**].

(1) The case $\zeta > 0$. We have

$$\dim_K HC_0(A) = s + \left[\frac{m-1}{s}\right],$$
$$\dim_K HC_1(A) = (\gcd(m,s)-1)\left(\left[\frac{m}{s}\right] - \left[\frac{m-1}{s}\right]\right) + \left[\frac{m-1}{s\zeta}\right]$$
$$+ \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right)\left(\left[\frac{m}{s}\right] - \left[\frac{m-1}{s}\right]\right).$$

On the other hand, since

$$a_r = \begin{cases} 1 & \text{if } \ s|r, \\ 0 & \text{otherwise}, \end{cases} \qquad b_r = \begin{cases} 1 & \text{if } \ s=r, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned}
\dim_K HC_{2c}(A) = s + & \sum_{\substack{1 \leq e \leq m-1 \\ \text{s.t. } s \mid cm+e}} 1 + \sum_{c'=0}^{c-1} \sum_{\substack{1 \leq e \leq m-1 \\ \text{s.t. } s\zeta \mid c'm+e}} 1 \\
& + \sum_{\substack{1 \leq c' \leq c \\ \text{s.t. } s \mid c'm, \\ \gcd(m,s)\zeta \mid m}} 1 + \sum_{\substack{1 \leq c' \leq c \\ \text{s.t. } s\zeta \mid \gcd(m,s)c'}} (\gcd(m,s)-1) \\
& = s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right] + \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right) \\
& + \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right) \sum_{c'=1}^{c}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right) \\
& + (\gcd(m,s)-1)\left[\frac{\gcd(m,s)c}{s\zeta}\right],
\end{aligned}$$

and

$$\begin{aligned}
\dim_K HC_{2c+1}(A) = & \begin{cases} \gcd(m,s)-1 & \text{if} \quad s \mid (c+1)m, \\ 0 & \text{otherwise,} \end{cases} \\
& + \sum_{c'=0}^{c} \sum_{\substack{1 \leq e \leq m-1 \\ \text{s.t. } s\zeta \mid c'm+e}} 1 + \sum_{\substack{1 \leq c' \leq c+1 \\ \text{s.t. } s \mid c'm, \\ \gcd(m,s)\zeta \mid m}} 1 \\
& + \sum_{\substack{1 \leq c' \leq c \\ \text{s.t. } s\zeta \mid \gcd(m,s)c'}} (\gcd(m,s)-1) \\
& = (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right] + \left[\frac{\gcd(m,s)c}{s\zeta}\right]\right) \\
& + \left(\left[\frac{m}{\gcd(m,s)\zeta}\right] - \left[\frac{m-1}{\gcd(m,s)\zeta}\right]\right) \sum_{c'=1}^{c+1}\left(\left[\frac{c'm}{s}\right] - \left[\frac{c'm-1}{s}\right]\right) \\
& + \sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right).
\end{aligned}$$

In particular, we consider the following three cases.

(i) In the case $\gcd(m,s)=1$, we have

$$\begin{aligned}
\dim_K HC_0(A) =& s + \left[\frac{m-1}{s}\right],\\
\dim_K HC_1(A) =& \left[\frac{m-1}{s\zeta}\right],\\
\dim_K HC_{2c}(A) =& s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right]\\
&+ \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right)\\
&+ \left(\left[\frac{m}{\zeta}\right] - \left[\frac{m-1}{\zeta}\right]\right)\left[\frac{c}{s}\right],
\end{aligned}$$

and

$$\begin{aligned}
\dim_K HC_{2c+1}(A) = & \sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right)\\
&+ \left(\left[\frac{m}{\zeta}\right] - \left[\frac{m-1}{\zeta}\right]\right)\left[\frac{c+1}{s}\right].
\end{aligned}$$

(ii) In the case $s|m$, we have

$$\begin{aligned}
\dim_K HC_0(A) &= s + \frac{m}{s} - 1,\\
\dim_K HC_1(A) &= s - 1 + \left[\frac{m}{s\zeta}\right],\\
\dim_K HC_{2c}(A) &= s + \frac{m}{s} - 1 + \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right)\\
&\quad + \left(\left[\frac{m}{s\zeta}\right] - \left[\frac{m-1}{s\zeta}\right]\right)c + (s-1)\left[\frac{c}{\zeta}\right],
\end{aligned}$$

and

$$\begin{aligned}
\dim_K HC_{2c+1}(A) &= s - 1 + \sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right)\\
&\quad + \left(\left[\frac{m}{s\zeta}\right] - \left[\frac{m-1}{s\zeta}\right]\right)(c+1) + (s-1)\left[\frac{c}{\zeta}\right].
\end{aligned}$$

(iii) Suppose $m = ts + u$ $(2 \leq u \leq s-1)$ and $u|s$, then we have

$$\begin{aligned}
\dim_K HC_0(A) &= s + \frac{m-u}{s}, \\
\dim_K HC_1(A) &= \left[\frac{m-1}{s\zeta}\right], \\
\dim_K HC_{2c}(A) &= s + \frac{m-u}{s} + \sum_{c'=0}^{c-1}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right) \\
&\quad + \left(\left[\frac{m}{u\zeta}\right] - \left[\frac{m-1}{u\zeta}\right]\right)\left[\frac{cu}{s}\right] + (u-1)\left[\frac{uc}{s\zeta}\right],
\end{aligned}$$

and

$$\begin{aligned}
\dim_K HC_{2c+1}(A) &= (u-1)\left(\left[\frac{(c+1)u}{s}\right] - \left[\frac{cu}{s}\right]\right) \\
&\quad + \sum_{c'=0}^{c}\left(\left[\frac{(c'+1)m-1}{s\zeta}\right] - \left[\frac{c'm}{s\zeta}\right]\right) \\
&\quad + \left(\left[\frac{m}{u\zeta}\right] - \left[\frac{m-1}{u\zeta}\right]\right)\left[\frac{(c+1)u}{s}\right] + (u-1)\left[\frac{uc}{s\zeta}\right].
\end{aligned}$$

(2) The case $\zeta = 0$. We have

$$\begin{aligned}
\dim_K HC_{2c}(A) &= s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right], \\
\dim_K HC_{2c+1}(A) &= (\gcd(m,s)-1)\left(\left[\frac{(c+1)m}{s}\right] - \left[\frac{(c+1)m-1}{s}\right]\right).
\end{aligned}$$

In particular, we also consider the following three cases as well as (1).

(i) In the case $\gcd(m,s) = 1$, we have

$$\begin{aligned}
\dim_K HC_{2c}(A) &= s + \left[\frac{(c+1)m-1}{s}\right] - \left[\frac{cm}{s}\right] \\
\dim_K HC_{2c+1}(A) &= 0.
\end{aligned}$$

(ii) In the case $s|m$, we have

$$\begin{aligned}
\dim_K HC_{2c}(A) &= s + \frac{m}{s} - 1, \\
\dim_K HC_{2c+1}(A) &= s - 1.
\end{aligned}$$

(iii) Suppose $m = ts + u$ $(2 \leq u \leq s-1)$ and $u|s$, then we have

$$\dim_K HC_{2c}(A) = s + \frac{m-u}{s},$$
$$\dim_K HC_{2c+1}(A) = (u-1)\left(\left\lceil \frac{(c+1)u}{s} \right\rceil - \left\lceil \frac{cu}{s} \right\rceil\right).$$

We remarks that by the similar way, Volčič [**22**] generalizes recently the result of the Theorem 3.8 into the case of the commutative ring. We introduce his result.

**Theorem 3.11** ([**22**, Theorem 4.2])**.** *Let $K$ be a commutative ring and $A = K\Delta/R_\Delta^m$. Then*

$$HC_{2c}(A) \cong K^{\#\Delta_0 + \sum_{e=1}^{m-1} a_{cm+e}} \oplus \bigoplus_{i=0}^{c-1} \bigoplus_{j=1}^{m-1} \mathrm{Ann}_K(im+j)^{a_{im+j}}$$
$$\oplus \bigoplus_{i=0}^{c} \bigoplus_{r|im} \left( \left( K \Big/ \frac{\gcd(r,m)i}{r} K \right)^{\gcd(r,m)-1} \oplus \mathrm{Ann}_K\left(\frac{m}{\gcd(r,m)}\right)\right)^{b_r}$$
$$HC_{2c+1}(A) \cong \bigoplus_{r|(c+1)m} \left( K^{\gcd(r,m)-1} \oplus K \Big/ \frac{m}{\gcd(r,m)} K \right)^{b_r}$$
$$\oplus \bigoplus_{i=0}^{c} \bigoplus_{j=1}^{m-1} \bigoplus_{r|im+j} \left( K \Big/ \frac{im+j}{r} K \right)^{b_r}$$
$$\oplus \bigoplus_{i=1}^{c} \bigoplus_{r|im} \left( \mathrm{Ann}_K\left(\frac{\gcd(r,m)i}{r}\right)^{\gcd(r,m)-1} \oplus K \Big/ \frac{m}{\gcd(r,m)} K \right)^{b_r}$$

*for $c \geq 0$.*

# Chapter 4

# Hochschild homology dimension and truncated cycles

In this chapter, we show the following assertion: Let $K$ be a field, $\Delta$ a finite quiver, $R_\Delta$ the arrow ideal of $K\Delta$ and $m \geq 2$ a positive integer. If an algebra $K\Delta/I$ with an ideal $I \subset K\Delta$ contained in $R_\Delta^m$ has an $m$-truncated cycle, then $K\Delta/I$ has infinitely many nonzero Hochschild homology groups (Theorem 4.9).

## 4.1. The Hochschild homology groups of the truncated quiver algebras

In this section, first of all, we recall the Hochschild homology of truncated quiver algebras following Sköldberg ([**19**]) which is introduced in Chapter 3. We refer to Section 2.3 for the gradings on truncated quiver algebras dealt in this chapter.

The minimal projective resolution of a monomial algebra over a field is given by Bardzell [**3**]. For the truncated quiver algebra $A$, we use the following projective resolution given by Sköldberg.

**Theorem 4.1** (= Theorem 3.1 ([**19**, Theorem 1]))**.** *Let $K$ be a commutative ring and $A = K\Delta/R_\Delta^m$. The following is a projective $\hat{\Delta}$-graded resolution of $A$ as a left $A^e$-module*:

$$\boldsymbol{P} : \cdots \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0.$$

*Here $P_i (i \geq 0)$ are defined by*

$$P_i = A \otimes_{K\Delta_0} K\Gamma^{(i)} \otimes_{K\Delta_0} A,$$

*where $\Gamma^{(i)}$ is given by*

$$\Gamma^{(i)} = \begin{cases} \Delta_{cm} & \text{if } i = 2c \ (c \geq 0), \\ \Delta_{cm+1} & \text{if } i = 2c+1 \ (c \geq 0), \end{cases}$$

*and the differentials are defined by*

$$
\begin{aligned}
&d_{2c}(\alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \beta)\\
&\quad = \sum_{j=0}^{m-1} \alpha\alpha_1 \cdots \alpha_j \otimes \alpha_{1+j} \cdots \alpha_{(c-1)m+1+j} \otimes \alpha_{(c-1)m+2+j} \cdots \alpha_{cm}\beta,\\
&d_{2c+1}(\alpha \otimes \alpha_1 \cdots \alpha_{cm+1} \otimes \beta)\\
&\quad = \alpha\alpha_1 \otimes \alpha_2 \cdots \alpha_{cm+1} \otimes \beta - \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1}\beta,
\end{aligned}
$$

*where* $\alpha_1, \dots, \alpha_{cm} \in \Delta_1$. *The augmentation* $\varepsilon \colon A \otimes_{K\Delta_0} K\Delta_0 \otimes_{K\Delta_0} A \cong A \otimes_{K\Delta_0} A \longrightarrow A$ *is defined by* $\varepsilon(\alpha \otimes \beta) = \alpha\beta$.

Following Sköldberg [**19**], $\boldsymbol{K}$ denotes the complex $A \otimes_{K\Delta_0^e} K\Gamma^{(*)}$, which is isomorphic to $A \otimes_{A^e} \boldsymbol{P}$ by the following isomorphism $\varphi$:

$$\varphi : A \otimes_{A^e} \boldsymbol{P} \xrightarrow{\sim} A \otimes_{A^e} A^e \otimes_{K\Delta_0^e} K\Gamma^{(*)} \xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Gamma^{(*)}.$$

Since the complex $\boldsymbol{K}$ is decomposed into the subcomplexes $\boldsymbol{K}_{\bar{\gamma}}$ by $\boldsymbol{K} \cong \bigoplus_q \bigoplus_{\bar{\gamma} \in \Delta_q^{\mathrm{c}}/G_q} \boldsymbol{K}_{\bar{\gamma}}$ in [**19**], the $p$-th Hochschild homology group of $A$ is $\mathbb{N}_0$-graded for $p \geq 0$, that is, the $p$-th Hochschild homology of $A$ is given by $HH_p(A) = \bigoplus_q HH_{p,q}(A)$. Moreover, the degree $q$ part of $HH_p(A)$ is $\Delta_q^{\mathrm{c}}/G_q$-graded, that is, the degree $q$ part of $p$-th Hochschild homology of $A$ is given by $HH_{p,q}(A) = \bigoplus_{\bar{\gamma} \in \Delta_q^{\mathrm{c}}/G_q} (HH_{p,q}(A))_{\bar{\gamma}}$. We denote the $n$-th module of $\boldsymbol{K}_{\bar{\gamma}}$ by $K_{\bar{\gamma},n}$.

The Hochschild homology $HH_{p,q}(A)$ of a truncated quiver algebra $A$ is computed by means of the above projective resolution in [**19**]. The following is introduced in Chapter 3.

**Theorem 4.2** (=Theorem 3.6 ([**19**, Theorem 2]))**.** *Let $K$ be a commutative ring and $A$ a truncated quiver algebra $K\Delta/R_\Delta^m$ and $q = cm + e$ for $0 \leq e \leq m-1$. Then the degree $q$ part of the pth Hochschild homology $HH_p(A)$ is given by*

$$
HH_{p,q}(A) = \begin{cases}
K^{a_q} & \text{if } 1 \leq e \leq m-1 \text{ and } 2c \leq p \leq 2c+1,\\
\bigoplus_{r|q} \left( K^{\ (m,r)-1} \oplus \operatorname{Ker}\left( \cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\
 & \text{if } e = 0 \text{ and } 0 < 2c-1 = p,\\
\bigoplus_{r|q} \left( K^{\gcd(m,r)-1} \oplus \operatorname{Coker}\left( \cdot \frac{m}{\gcd(m,r)} : K \longrightarrow K \right) \right)^{b_r} & \\
 & \text{if } e = 0 \text{ and } 0 < 2c = p,\\
K^{\#\Delta_0} & \text{if } p = q = 0,\\
0 & \text{otherwise.}
\end{cases}
$$

*Here we set* $a_q := \#(\Delta_q^{\mathrm{c}}/G_q)$ *and* $b_r := \#(\Delta_r^{\mathrm{b}}/G_r)$.

*Remark* 4.3. The degree 0 part of the 0th Hochschild homology, $HH_{0,0}(A)$ in [**19**, Theorem 2] should be corrected as $HH_{0,0}(A) = K^{\#\Delta_0}$ with our notation.

For $K\Delta/R_\Delta^m$, $c \geq 1$ and $\bar{\gamma} \in \Delta_*^{\mathrm{c}}/G_{cm}$, we have the following lemma by computing the degree $\bar{\gamma}$ part of $HH_{2c-1,cm}(K\Delta/R_\Delta^m)$ (see Chapter 3).

**Lemma 4.4.** *Let $K$ be a field and $A = K\Delta/R_\Delta^m$ a truncated quiver algebra. For an element $\bar{\gamma} \in \Delta_{cm}^{\mathrm{c}}/G_{cm}$ with $\gamma = \alpha_1 \cdots \alpha_{cm}(\alpha_1, \ldots, \alpha_{cm} \in \Delta_1)$, the following elements correspond to non-zero homology classes:*

$$\alpha_{(c-1)m+i+1} \cdots \alpha_{cm}\alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \cdots \alpha_{(c-1)m+i} \in A \otimes_{K\Delta_0^e} \Gamma^{((c-1)m+1)},$$

*where $d = \gcd(m, \operatorname{per}\bar{\gamma})$ and $i = 1, 2, \ldots, d-1$.*

*Proof.* Let $\gamma = \alpha_1 \cdots \alpha_{cm}(\alpha_1, \ldots, \alpha_{cm} \in \Delta_1)$. We consider the degree $\bar{\gamma}$ part of the degree $cm$ part of the $(2c-1)$-th Hochschild homology group of $A$, $(HH_{2c-1,cm}(A))_{\bar{\gamma}}$. In [**19**], the modules $(A \otimes_{K\Delta_0^e} \Gamma^{(n)})_{\bar{\gamma}} = 0$ for $n \neq 2c, 2c-1$. We set $u_i = e_i \otimes \alpha_i \cdots \alpha_{cm}\alpha_1 \cdots \alpha_{i-1}$ and $v_i = \alpha_{(c-1)m+i+1} \cdots \alpha_{cm}\alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \cdots \alpha_{(c-1)m+i}$, where the subscripts $i$ of $u_i$ and $v_i$ are considered to be modulo $\operatorname{per}\bar{\gamma}$. Then

$$(A \otimes_{K\Delta_0^e} \Gamma^{(2c)})_{\bar{\gamma}} = \bigoplus_{i=1}^{\operatorname{per}\gamma} K u_i,$$

$$(A \otimes_{K\Delta_0^e} \Gamma^{(2c-1)})_{\bar{\gamma}} = \bigoplus_{i=1}^{d-1} K v_i \oplus K(\sum_{i=1}^{d} v_i) \oplus \bigoplus_{i=1}^{d} \bigoplus_{j=1}^{\operatorname{per}\bar{\gamma}/d-1} K(v_{i+jm} - v_i).$$

The differentials $\overline{d}_{n,\bar{\gamma}} : (A \otimes_{K\Delta_0^e} \Gamma^{(n)})_{\bar{\gamma}} \to (A \otimes_{K\Delta_0^e} \Gamma^{(n-1)})_{\bar{\gamma}}$ is given by $\overline{d}_{n,\bar{\gamma}} = 0$ for $n \neq 2c$ and $\overline{d}_{2c,\bar{\gamma}}(u_i) = \sum_{j=0}^{m-1} v_{i+j}$.

Since $\overline{d}_{2c,\bar{\gamma}}(u_{i+1} - u_i) = v_{i+m} - v_i$ holds, we have $\overline{d}_{2c,\bar{\gamma}}(\sum_{l=0}^{j-1}(u_{i+1+lm} - u_{i+lm})) = v_{i+jm} - v_i$. Hence we have

$$\operatorname{Im}\overline{d}_{2c,\bar{\gamma}} = K\Big(\frac{m}{d}\sum_{i=1}^{d} v_i\Big) \oplus \bigoplus_{i=1}^{d} \bigoplus_{j=1}^{\operatorname{per}\bar{\gamma}-1} K(v_{i+jm} - v_i).$$

Therefore we have the result as desired. $\square$

We also have the following lemma.

**Lemma 4.5.** *Let $K$ be a field and $A = K\Delta/R_\Delta^m$ a truncated quiver algebra. For an element $\bar{\gamma} \in \Delta_{cm+e}^{\mathrm{c}}/G_{cm+e}(1 \leq e \leq m-1)$ with $\gamma = \alpha_1 \cdots \alpha_{cm+e}(\alpha_1, \ldots, \alpha_{cm+e} \in \Delta_1)$, the following element corresponds to a non-zero homology class:*

$$\alpha_{cm+1} \cdots \alpha_{cm+e} \otimes \alpha_1 \cdots \alpha_{cm} \in A \otimes_{K\Delta_0^e} \Gamma^{(cm)}.$$

By means of the basis of $HH_\bullet(A)$ including the basis in the above lemma, we computed the cyclic homology $HC_\bullet(A)$ of $A$ in Chapter 3.

We recall Cibils' projective resolution in [**6**] which is introduced in Chapter 3.

**Lemma 4.6** (= Lemma 3.2 ([**6**, Lemma 1.1]))**.** *Let $K$ be a field and $A = K\Delta/R_\Delta^m$ a truncated quiver algebra. We set $r = R_\Delta/R_\Delta^m$ the Jacobson radical of $A$ and $K\Delta_0$ the subalgebra of $A$ generated by $\Delta_0$. The following is a projective resolution of $A$ as a left $A^e$-module*:

$$\begin{aligned}\boldsymbol{Q} : \cdots \longrightarrow A \otimes_{K\Delta_0} r^{\otimes^i_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{d_i} A \otimes_{K\Delta_0} r^{\otimes^{i-1}_{K\Delta_0}} \otimes_{K\Delta_0} A \longrightarrow \cdots \\ \longrightarrow A \otimes_{K\Delta_0} r \otimes_{K\Delta_0} A \xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0,\end{aligned}$$

*where*

$$\begin{aligned} d_0(\lambda[\quad]\mu) &= \lambda\mu, \\ d_i(\lambda[x_1|\cdots|x_i]\mu) &= \lambda x_1[x_2|\cdots|x_i]\mu + \sum_{j=1}^{i-1}(-1)^i\lambda[x_1|\cdots|x_jx_{j+1}|\cdots|x_i]\mu \\ &\qquad + (-1)^i\lambda[x_1|\cdots|x_{i-1}]x_i\mu \end{aligned}$$

*for $i \geq 1$, and we use the bar notation $\lambda[x_1|\cdots|x_i]\mu$ for $\lambda \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_i \otimes \mu$.*

We note that there is the following chain map $C^{\mathrm{bar}} \to \boldsymbol{Q}$ in [**7**], which we denote by $\theta$: $\theta(a_0 \otimes \cdots \otimes a_{n+2}) = a_0[a_1|a_2|\cdots|a_{n+1}]a_{n+2}$. This chain map $\theta$ induces a quasi-isomorphism $\mathrm{id}_A \otimes \theta : A \otimes_{A^e} C^{\mathrm{bar}} \to A \otimes_{A^e} \boldsymbol{Q}$, which we denote by $\theta$ for the sake of simplicity.

We recall the chain map $\pi$ from $\boldsymbol{Q}$ to $\boldsymbol{P}$ given in [**1**] which is used in Chapter 3.

**Proposition 4.7** (= Proposition 3.4 ([**1**]))**.** *Let $x_1, x_2, \dots$ be paths in $\Delta$ and $m_1, m_2, \dots$ the lengths of $x_1, x_2, \dots$, respectively. We set $x_1 = \alpha_1 \cdots \alpha_{m_1}$, $x_2 = \alpha_{m_1+1} \cdots \alpha_{m_1+m_2}, \dots$, where $\alpha_1, \alpha_2, \dots \in \Delta_1$. Then there exists a chain map $\pi : \boldsymbol{Q} \longrightarrow \boldsymbol{P}$ defined by the following equations*:

$$\pi_0(\alpha[\quad]\beta) = \alpha \otimes \beta, \quad \pi_1(\alpha[x_1]\beta) = \sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \otimes \alpha_{j+1} \cdots \alpha_{m_1}\beta,$$

$$\pi_{2c}(\alpha[x_1|x_2|\cdots|x_{2c}]\beta) = \begin{cases} \alpha \otimes \alpha_1 \cdots \alpha_{cm} \otimes \alpha_{cm+1} \cdots \alpha_{m_1+\cdots+m_{2c}}\beta \\ \qquad \textit{if} \quad m_{2i-1} + m_{2i} \geq m \ (1 \leq i \leq c), \\ 0 \qquad \textit{otherwise}, \end{cases}$$

$$\begin{aligned}&\pi_{2c+1}(\alpha[x_1|x_2|\cdots|x_{2c+1}]\beta) \\ &\quad = \begin{cases} \displaystyle\sum_{j=1}^{m_1} \alpha\alpha_1 \cdots \alpha_{j-1} \otimes \alpha_j \cdots \alpha_{j+cm} \otimes \alpha_{j+cm+1} \cdots \alpha_{m_1+\cdots+m_{2c+1}}\beta \\ \qquad\qquad\qquad\qquad \textit{if} \quad m_{2i} + m_{2i+1} \geq m \ (1 \leq i \leq c), \\ 0 \qquad\qquad\qquad\qquad \textit{otherwise}. \end{cases}\end{aligned}$$

By means of this chain map, the cyclic homology of truncated quiver algebra is computed in Chapter 3.

This chain map $\pi$ induces a quasi-isomorphism $\bar{\pi} = \mathrm{id}_A \otimes \pi : A \otimes_{A^e} \boldsymbol{Q} \longrightarrow A \otimes_{A^e} \boldsymbol{P}$. We denote the following composition map of chain maps from the Hochschild complex to Sköldberg's complex by $\Phi$;

$$\begin{array}{ccc} A \otimes_{A^e} Q_n \xleftarrow{\theta} A \otimes_{A^e} (C^{\mathrm{bar}})_n = A \otimes_{A^e} A^{\otimes(n+2)} \xleftarrow{\psi} A^{\otimes(n+1)} \\ \big\downarrow \bar{\pi} \\ A \otimes_{A^e} P_n \xrightarrow{\varphi} A \otimes_{K\Delta_0^e} K\Gamma^{(n)} \xrightarrow{\sim} \bigoplus_i \bigoplus_{\bar{\gamma} \in \Delta_i^c / G_i} K_{\bar{\gamma},n}, \end{array}$$

where $\psi$ is given by $\psi(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes_{A^e} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$.

## 4.2. The $m$-truncated cycles version of the "no loops conjecture"

Let $K$ be a field, $\Delta$ a finite quiver, $R_\Delta$ the arrow ideal of $K\Delta$ and $m \geq 2$ a positive integer.

In this section, we show that if an algebra $K\Delta/I$ with $I \subset R_\Delta^m$ has an $m$-truncated cycle (see Definition 4.8), then the algebra has infinite Hochschild homology dimension. Moreover, we show that the algebra satisfies an $m$-truncated cycles version of the "no loops conjecture".

If $I \subset R_\Delta^2$ is an ideal in the path algebra $K\Delta$, then a finite sequence $\alpha_1, \ldots, \alpha_u$ of arrows which satisfies the equations $t(\alpha_i) = s(\alpha_{i+1})$ $(i = 1, \ldots, u-1)$ and $t(\alpha_u) = s(\alpha_1)$ is called *a cycle* in $K\Delta/I$ in [**5**].

**Definition 4.8** ([**5**])**.** A cycle $\alpha_1, \ldots, \alpha_u$ in $K\Delta/I$ is *$m$-truncated* for an integer $m \geq 2$ if

$$\alpha_i \cdots \alpha_{i+m-1} = 0 \quad \text{and} \quad \alpha_i \cdots \alpha_{i+m-2} \neq 0 \quad \text{in } K\Delta/I$$

for all $i$, where the indices are modulo $u$.

**Theorem 4.9.** *Let $K$ be a field, $\Delta$ a finite quiver and $I \subset K\Delta$ an ideal contained in $R_\Delta^m$. Suppose that $K\Delta/I$ contains an $m$-truncated cycle $\alpha_1, \ldots, \alpha_u$. Then the following holds:*

(i) *Assume that* $\gcd(m, \mathrm{per}(\alpha_1 \cdots \alpha_u)) \neq 1$. *For every $n \geq 1$ with $un \equiv 0 \pmod{m}$, the element*

$$\begin{aligned} &\alpha_{(c-1)m+2} \cdots \alpha_{cm} \otimes \alpha_1 \otimes \alpha_2 \cdots \alpha_m \otimes \alpha_{m+1} \\ &\quad \otimes \alpha_{m+2} \cdots \alpha_{2m} \otimes \alpha_{2m+1} \otimes \cdots \otimes \alpha_{(c-2)m+2} \cdots \alpha_{(c-1)m} \otimes \alpha_{(c-1)m+1}, \end{aligned}$$

*where $c = un/m$, represents a nonzero element in $HH_{2c-1}(K\Delta/I)$.*

(ii) *Let $e$ be an integer with $1 \le e \le m-1$. For every $n \ge 1$ with $un \equiv e \pmod m$, the element*

$$\begin{aligned}\sum_{0\le j_1,\dots,j_c\le m-2} &\alpha_{2c+1+j_1+\cdots+j_c}\cdots\alpha_{un}\\ &\otimes \alpha_1\cdots\alpha_{1+j_1}\otimes\alpha_{2+j_1}\otimes\alpha_{3+j_1}\cdots\alpha_{3+j_1+j_2}\otimes\alpha_{4+j_1+j_2}\otimes\cdots\\ &\qquad\otimes\alpha_{2c-1+j_1+\cdots+j_{c-1}}\cdots\alpha_{2c-1+j_1+\cdots+j_c}\otimes\alpha_{2c+j_1+\cdots+j_c},\end{aligned}$$

*where $c = (un-e)/m$, represents a nonzero element in $HH_{2c}(K\Delta/I)$.*

*In particular, the Hochschild homology dimension* HHdim $(K\Delta/I) = \infty$.

*Proof.* We only prove (i). In this proof, we regard the subscripts $i$ of $\alpha_i$ modulo $u$ and we denote the algebra $K\Delta/I$ by $B$. For every positive integer $n$ with $un \equiv 0 \pmod m$, we consider the Hochschild complex

$$\cdots \to B^{\otimes(2c+1)} \xrightarrow{d_{2c}} B^{\otimes(2c)} \xrightarrow{d_{2c-1}} B^{\otimes(2c-1)} \to \cdots,$$

where $c = un/m$. We denote the expression $\alpha_{(c-1)m+2}\cdots\alpha_{cm}\otimes\cdots\otimes\alpha_{(c-1)m+1}$ in (i) by $x$.

Since the sequence $\alpha_1,\dots,\alpha_u$ is an $m$-truncated cycle, the element $x$ belongs to $\operatorname{Ker} d_{2c-1}$. Let $A = K\Delta/R_\Delta^m$. We consider the natural surjective $K$-algebra homomorphism $f : B = K\Delta/I \to K\Delta/R_\Delta^m = A$ which induces a $K$-homomorphism $f^{\otimes 2c}$ : $HH_{2c-1}(B) \to HH_{2c-1}(A)$. We consider the $K$-isomorphism $HH_{2c-1}(A) \to HH_{2c-1}(A)$ induced by $\Phi$ and denote this by $\Phi$ for the sake of simplicity, where $\Phi$ is defined in the last paragraph of Section 3. Then we have

$$\Phi(f^{\otimes 2c}(x)) = \alpha_{(c-1)m+2}\cdots\alpha_{cm}\otimes\alpha_1\cdots\alpha_{(c-1)m+1}.$$

Hence $\Phi(f^{\otimes 2c}(x))$ is a non-zero element in $HH_{2c-1}(A)$ by the Lemma 4.4, so the element $x$ is a non-zero element in $HH_{2c-1}(B)$.

The statement (ii) can also be proved by means of the Lemma 4.5. □

The above theorem includes the result of [**5**, Theorem 3.1] in the case $K$ is a field and $\Delta$ is a finite quiver.

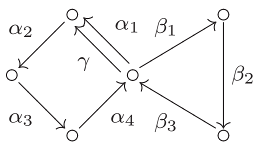
For a basic and connected finite dimensional $K$-algebra $B$ which has finite global dimension, its Hochschild homology is given in [**8**, Proposition 6] and $HH_n(B) = 0$ for $n \ge 1$. Therefore, by the above theorem, we have the following result which generalizes [**5**, Corollary 3.3].

**Corollary 4.10.** *Let $K$ be a field, $\Delta$ a finite quiver and $I$ an admissible ideal in $K\Delta$ with $I \subset R_\Delta^m$. If the algebra $K\Delta/I$ has finite global dimension, then it contains no $m$-truncated cycles.*

*Remark* 4.11. We suppose that a bound quiver algebra $K\Delta/I$ has an $m$-truncated cycle $\alpha_1, \ldots, \alpha_u$ for integers $m$ and $u$ such that $u < m$. Then the oriented cycle $\alpha_1 \cdots \alpha_u$ is cyclically free in $K\Delta/I$ (see [**11**] for the definition of "cyclically free in $K\Delta/I$"). Hence, by [**11**, Corollary 2.4], we obtain that $K\Delta/I$ has infinite global dimension. In the Example 4.12, we show an example of an algebra which has an $m$-truncated cycle and has no cyclically free oriented cycles.

For a monomial algebra, an $m$-truncated cycles version of the "no loops conjecture" is proved in [**5**, Proposition 3.4]. However, the algebra $B$ in the Example 4.12 is not a monomial algebra.

**Example 4.12.** Let $B$ be an algebra given by the quiver with relations:




$$\alpha_i\alpha_{i+1}\alpha_{i+2} = \beta_1\beta_2\beta_3 = \beta_3\gamma\alpha_2 = 0,$$
$$\beta_2\beta_3\alpha_1 = \beta_2\beta_3\gamma,$$

where the indices of $\alpha_i$ are modulo 4 ($1 \leq i \leq 4$). Then $B$ has the 3-truncated cycle $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. By the Theorem 4.9, we have $\operatorname{HHdim} B = \infty$. Therefore, the global dimension of $B$ is infinite.

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