SUT Journal of Mathematics
Vol. 50, No. 2 (2014), 353–382

Harmonic maps in almost contact geometry

Jun-ichi Inoguchi

(Received August 4, 2014; Revised January 20, 2015)

Abstract. We study harmonicity and pluriharmonicity of holomorphic maps in almost contact geometry.

AMS 2010 *Mathematics Subject Classification.* 58E20, 53C25, 53C43.

Key words and phrases. Harmonic map; holomorphic map; pluriharmonic map; almost contact manifolds.

§1. Introduction

Lichnerowicz showed that every holomorphic map between compact almost Kähler manifolds (*i.e.*, compact symplectic manifolds with compatible almost Hermitian structures) minimizes the energy in its homotopy class, hence holomorphic maps are *stable harmonic maps* [30].

Since the harmonicity of smooth maps from Riemannian 2-manifolds into Riemannian manifolds is invariant under conformal transformations of domain manifolds, harmonicity makes sense for maps from *Riemann surfaces* into Riemannian manifolds.

Let M be a complex manifold. A smooth map $\varphi : M \to N$ into a Riemannian manifold is said to be *pluriharmonic* if its $(0,1)$-exterior derivative of the $(1,0)$-derivative $\partial\varphi$ of φ vanishes. In particular, when M is of complex dimension 1, pluriharmonicity is equivalent to harmonicity. Thus pluriharmonic map theory is a higher dimensional generalization of theory of harmonic maps from Riemann surfaces.

For maps between Kähler manifolds, pluriharmonicity is weaker than holomorphicity and stronger than harmonicity. One can see many interesting examples of non-holomrphic pluriharmonic maps between Kähler manifolds. Moreover, pluriharmonic maps into Riemannian symmetric spaces admit natural deformation family. Thus pluriharmonic immersions can be regarded as a higher dimensional generalization of minimal surfaces in Euclidean 3-space.

The existence of natural deformation family characterizes pluriharmonicity (Eschenburg-Tribuzy [17]). The so-called *loop group methods* can be applied to pluriharmonic maps into Riemannian symmetric spaces (Dorfmeister-Eschenburg [11]). Pluriharmonic maps play important roles in Kähler geometry. See *e.g.*, Udagawa [52].

On the other hand, in odd-dimensional geometry, more precisely, contact geometry (or *CR*-geometry), one can study holomorphic maps (or *CR*-maps) and harmonic maps between contact Riemannian manifolds.

In contrast to Kähler geometry, although holomorphic maps between compact strongly pseudo convex *CR*-manifolds are harmonic, but not necessarily, energy-minimizing. In fact, the identity map of odd-dimensional sphere $\mathbb{S}^{2n+1}$ is unstable ([58]).

From the viewpoint of G-structures, both Kähler manifolds and strongly pseudo convex *CR*-manifolds can be treated as special examples of so-called Riemannian *f-manifolds* or manifolds with $\mathrm{U}(n) \times \mathrm{O}(k)$–*structure*.

An f-structure on a manifold M is an endomorphism field F such that $F^3 + F = 0$. This notion was introduced by Yano [59].

Rawnsley studied harmonicity of holomorphic curves in f-manifolds [41]. Lichnerowicz's theorem for holomorphic maps can be generalized to more general Riemannian f-manifolds under the condition (A) in the sense of Rawnsley [41] (see Definition 5).

On the other hand, inspired by Lichnerowicz' result, Urakawa [54] initiated the study of harmonic maps between strongly pseudo convex *CR*-manifolds.

Strongly pseudo convex *CR*-manifolds can be characterized as contact Riemannian manifolds satisfying certain integrability condition (see [51]). Strongly pseudo convex property is weaker than Sasakian condition for contact Riemannian manifolds.

However, the condition (A) does not fit with contact or *CR*-structure. In fact, Sasakian manifolds (normal strongly pseudo convex *CR*-manifolds) never satisfy condition (A). Ianus and Pastore have shown that every holomorphic map between contact Riemannian manifolds is harmonic [21]. This is a generalization of Urakawa's result [54].

Several kinds of harmonic maps from or into contact manifolds are studied by Gherghe, Ianus and Pastore [18], [21]. In [43], Saotome studied holomorphic maps from Sasakian 3-manifolds into strongly pseudo convex *CR*-manifolds.

On the other hand, study of harmonic maps into non-contact almost contact manifolds are very few.

In this paper we study harmonicity of holomorphic maps into quasi-Sasakian manifolds and Kenmotsu manifolds.

In stead of harmonicity, Barletta, Dragomir and Urakawa [1] introduced the notion of *pseudo-harmonic map* for maps from nondegenerate pseudo-Hermitian *CR*-manifolds into manifolds with linear connection. They gave a variational characterization of pseudo-harmonic maps and studied stability of those maps.

In this paper we discuss several candidates of "pluriharmonic maps" in *CR*-geometry.

The author would like to thank professor Seiichi Udagawa for his careful reading of the manuscript and his information on Iwamatsu's work [22]. He also would like to thank professor Marian Ioan Munteanu and the referee for their useful comments.

§2. Harmonic maps

Let (M, g) and (N, h) be Riemannian manifolds and $\varphi : M \to N$ a smooth map. Then φ induces a vector bundle φ^*TN over N by

$$\varphi^*TN = \{(p, X) \in M \times TN \mid \varphi(p) = \pi(X)\}.$$

Here TN is the tangent bundle of N with natural projection $\pi : TN \to N$ (see [53, p. 124]). The space of all smooth sections of TN and φ^*TN are denoted by $\Gamma(TN)$ and $\Gamma(\varphi^*TN)$, respectively.

The Levi-Civita connection ${}^h\nabla$ of N induces a connection ${}^h\nabla^\varphi$ on φ^*TN which satisfies the condition

$${}^h\nabla^\varphi_X(V \circ \varphi) = ({}^h\nabla_{\varphi_* X}V) \circ \varphi,$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(TN)$ (see [14, p. 4] or [53, p. 126]).

The *second fundamental form* ${}^h\nabla \mathrm{d}\varphi$ of φ is defined by

$$({}^h\nabla \mathrm{d}\varphi)(Y; X) = {}^h\nabla^\varphi_X \mathrm{d}\varphi(Y) - \mathrm{d}\varphi({}^g\nabla_X Y), \quad X, Y \in \Gamma(TM).$$

Here ${}^g\nabla$ is the Levi-Civita connection of (M, g).

The *energy density* $e(\varphi)$ of φ is a smooth function on M defined by

$$e(\varphi) := \frac{1}{2}\,\mathrm{tr}_g(\varphi^* h).$$

The *energy* of φ over a compact region $\mathcal{D}$ is defined by

$$E(\varphi; \mathcal{D}) = \int_{\mathcal{D}} e(\varphi)\mathrm{d}v_g.$$

Here $\mathrm{d}v_g$ is the volume element of (M, g). A smooth map φ is said to be *harmonic* if it is a critical point of the energy over any compact region of M.

The first variation formula of the energy is given by (see [15]):

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E(\varphi_t; \mathcal{D}) = -\int_{\mathcal{D}} h(\tau(\varphi), V)\mathrm{d}v_g, \quad V = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t,$$

where $\varphi_t : M \times (-\varepsilon, \varepsilon) \to N$ is a smooth variation through $\varphi = \varphi_0$ and $\tau(\varphi)$ is a section of $\varphi^* TN$ defined by

$$\tau(\varphi) = \mathrm{tr}_g\, ({}^h\nabla^{\varphi} \mathrm{d}\varphi).$$

The section $\tau(\varphi)$ is called the *tension field* of φ.

The first variation formula implies that, a map φ is harmonic if and only if its tension field vanishes.

Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field of M. Then the tension field is computed as

$$\tau(\varphi) = \sum_{i=1}^{m} \left\{ {}^h\nabla^{\varphi}_{e_i}(\varphi_* e_i) - \varphi_*({}^g\nabla_{e_i} e_i) \right\}. \tag{2.1}$$

Next, we recall the notion of vertical harmonicity introduced by Wood [57].

Let (P, g_P) and (B, g_B) be Riemannian manifolds and assume that there exists a Riemannian submersion $\pi : P \to B$. Denote by $\mathcal{V}$ the *vertical subbundle* of the tangent bundle TP:

$$\mathcal{V} = \bigcup_{u \in P} \mathcal{V}_u, \quad \mathcal{V}_u = \mathrm{Ker}(\,\pi_{*u}), \quad u \in P.$$

The *horizontal distribution* $\mathcal{H}$ is defined by

$$\mathcal{H} = \bigcup_{u \in P} \mathcal{H}_u, \quad \mathcal{H}_u = \mathrm{Ker}(\,\pi_{*u})^{\perp}, \quad u \in P.$$

Now let $\varphi : (M, g) \to (P, g_P)$ be a smooth map. Then the tension field $\tau(\varphi)$ is decomposed as $\tau(\varphi) = \tau^{\mathcal{H}}(\varphi) + \tau^{\mathcal{V}}(\varphi)$ according to the splitting:

$$T_u P = \mathcal{H}_u \oplus \mathcal{V}_u.$$

Then φ is said to be *vertically harmonic* if its *vertical tension field* $\tau^{\mathcal{V}}(\varphi)$ vanishes.

In case $\varphi : M = B \to P$ is a section, then the vertical harmonicity of φ is equivalent to the criticality of the *vertical energy* of φ. See [57].

§3. *CR*-manifolds

Here we recall the notion of *CR*-structure (Cauchy-Riemann structure).

Definition 1. Let M be a manifold. A complex vector subbundle $\mathcal{S}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ is said to be an *almost CR-structure* if $\mathcal{S} \cap \bar{\mathcal{S}} = \{0\}$. A manifold M equipped with an almost *CR*-structure is called an *almost CR-manifold*. An almost *CR*-structure is called *integrable* if the space $\Gamma(\mathcal{S})$ of all smooth sections satisfies the following integrability condition:

$$[\Gamma(\mathcal{S}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S}). \tag{3.1}$$

A manifold M together with an integrable almost *CR*-structure is called a *CR-manifold*. An integrable almost *CR*-structure is called a *CR-structure*.

Let $(M, \mathcal{S})$ be an almost *CR*-manifold. Then there exists a real vector subbundle P of the tangent bundle TM and an endomorphism field $J \in \Gamma(\mathrm{End}P)$ such that

$$P^{\mathbb{C}} = \mathcal{S} \oplus \bar{\mathcal{S}}, \quad J^2 = -I.$$

In fact, J is uniquely defined by

$$J(Z + \bar{Z}) = \sqrt{-1}(Z - \bar{Z}), \quad Z \in \Gamma(\mathcal{S}).$$

The pair (P, J) is called the *real expression* of $\mathcal{S}$.

An almost complex manifold (M, J) is an almost *CR*-manifold. In fact,

$$\mathcal{S} = T^{(1,0)}M = \{X - \sqrt{-1}JX \mid X \in TM\}$$

is an almost *CR*-structure. The resulting almost *CR*-manifold $(M, \mathcal{S})$ is integrable if and only if J is integrable.

Definition 2. Let E be a complex vector bundle over $(M, \mathcal{S})$. Then E is said to be *holomorphic* if there exists a differential operator

$$\bar{\partial} = \bar{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \bar{\mathcal{S}}^*); \ \zeta \longmapsto \bar{\partial}\zeta$$

such that

$$\bar{\partial}_{\bar{Z}}(f\,\zeta) = (\bar{Z}f)\zeta + f\bar{\partial}_{\bar{Z}}\zeta, \tag{3.2}$$

$$\bar{\partial}_{\bar{Z}}(\bar{\partial}_{\overline{W}}\zeta) - \bar{\partial}_{\overline{W}}(\bar{\partial}_{\bar{Z}}\zeta) - \bar{\partial}_{[\bar{Z},\overline{W}]}\zeta = 0 \tag{3.3}$$

for all $Z, W \in \Gamma(\mathcal{S})$, $f \in C^\infty(M, \mathbb{C})$ and $\zeta \in \Gamma(E)$. Here we used the notation $\bar{Z}\,\zeta = \bar{\partial}_{\bar{Z}}\zeta$.

The operator $\bar{\partial}$ is called the *Cauchy-Riemann operator* of E.

As we shall see later the *CR*-structure $\mathcal{S}$ is a holomorphic vector bundle over a normal strongly pseudo convex *CR*-manifold.

§4. Riemannian f-manifolds

In the preceding section we discussed almost *CR*-structures which are regarded as generalizations of almost complex structure. In this section we discuss another generalization of almost complex structure introduced by Yano.

Definition 3. ([59]) Let M be a manifold and F is an endomorphism field. Then F is said to be an *f-structure* if $F^3 + F = 0$.

Stong showed that every f-structure has constant rank [44].

Obviously, the almost complex structure J of an almost complex manifold is a typical example of f-structure. A manifold M equipped with a f-structure is called a *f-manifold*.

A Riemannian metric g on an f-manifold is said to be *compatible* if F is skew-adjoint with respect to it, *i.e.*,

$$g(FX, Y) = -g(X, FY), \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. An f-manifold together with a compatible metric is called a *Riemannian f-manifold*. Almost Hermitian manifolds are typical examples of Riemannian f-manifolds.

The *fundamental* 2*-form* $\omega = \omega_F$ of a Riemannian f-manifold is defined by

$$\omega(X, Y) = g(X, FY), \quad X, Y \in \mathfrak{X}(M).$$

Definition 4. Let (M, F) and $(N, \tilde{F})$ be f-manifolds. Then a smooth map $\varphi : M \to N$ is said to be an *f-holomorphic map* (or *holomorphic map*, in short) provided that

$$\mathrm{d}\varphi \circ F = \tilde{F} \circ \mathrm{d}\varphi.$$

Anti f-holomorphic maps are defined in a similar manner. More precisely, a smooth map $\varphi : (M, F) \to (N, \tilde{F})$ between f-manifolds is said to be *anti f-holomorphic* if $\mathrm{d}\varphi \circ F = -\tilde{F} \circ \mathrm{d}\varphi$. We write these alternatives together as *$\pm f$-holomorphic* (or *$\pm$ holomorphic*, in short).

Example 1. Let us denote by $\mathbb{H}^3$ the hyperbolic 3-space of constant curvature -1. Then its unit tangent sphere bundle $\mathrm{U}\mathbb{H}^3$ admits two standard f-structures F_1 and F_2. Note that $\mathrm{U}\mathbb{H}^3$ equipped with the *CR*-structure determined by F_1 is the *twistor CR-manifold* of $\mathbb{H}^3$ in the sense of LeBrun [29].

For a conformally immersed surface $M \subset \mathbb{H}^3$, its *Gauss map* $\varphi : M \to \mathrm{U}\mathbb{H}^3$ is F_1-holomorphic if and only if M is totally umbilical. On the other hand φ is F_2-holomorphic if and only if M is minimal (Salamon [42]). A loop group method for constructing F_2-holomorphic maps into $\mathrm{U}\mathbb{H}^3$ is established in [12].

Lichnerowicz showed that every holomorphic map between almost Kähler manifolds is harmonic, especially, it is an energy-minimizing map in its homotopy class. The following generalization of Lichnerowicz theorem is known.

Proposition 1. ([3],[41]) *Let (M, g, J) be an almost Hermitian manifold, (N, h, F) a Riemannian f-manifold and $\varphi : M \to N$ an f-holomorphic map. Assume that M and N satisfy the following conditions*:

(1) *ω is coclosed, i.e.,* $\mathrm{d}(*\omega) = 0$,

(2) $(\mathrm{d}^\nabla F)^{(1,1)} = 0$.

Then φ is harmonic. Here ω is the fundamental 2-form of M, $*$ is the Hodge star operator and $\mathrm{d}^\nabla F$ is the covariant-exterior derivative of F defined by

$$(\mathrm{d}^\nabla F)(X, Y) = ({}^h\nabla_X F)Y - ({}^h\nabla_Y F)X, \quad X, Y \in \Gamma(TN).$$

On a Riemannian f-manifold (M, F, g), the complexified tangent bundle $T^{\mathbb{C}}M$ splits into the direct sum:

$$T^{\mathbb{C}}M = \mathcal{S} \oplus \overline{\mathcal{S}} \oplus \mathcal{F}.$$

Here S, $\bar{S}$ and $\mathcal{F}$ are eigen-subbundles of $T^{\mathbb{C}}M$ with respect to F corresponding to the eigenvalue $\sqrt{-1}, -\sqrt{-1}$ and 0, respectively. The subbundle $\mathcal{S}$ is an almost *CR*-structure on M. We denote the projection $T^{\mathbb{C}}M$ onto $\mathcal{F}$ by Π_0. The projections onto $\mathcal{S}$ and $\overline{\mathcal{S}}$ are denoted by Π_+ and Π_-, respectively.

Let $\varphi : (M, F) \to N$ be a smooth map into a manifold N, its differential $\mathrm{d}\varphi$ is decomposed as

$$\mathrm{d}\varphi = \mathrm{d}^+\varphi + \mathrm{d}^-\varphi + \eth\varphi,$$

with

$$\mathrm{d}^+\varphi = \mathrm{d}\varphi \circ \Pi_+, \quad \mathrm{d}^-\varphi = \mathrm{d}\varphi \circ \Pi_-, \quad \eth\varphi = \mathrm{d}\varphi \circ \Pi_0.$$

Proposition 2. *Let (M, F) and $(N, \tilde{F})$ be f-manifolds and $\varphi : M \to N$ a smooth map. Denote by $\mathcal{S}_M$ and $\mathcal{S}_N$ the associated almost CR-structure. Then φ is f-holomorphic if and only if*

$$\mathrm{d}\varphi(\mathcal{S}_M) \subset \mathcal{S}_N, \quad \mathrm{d}\varphi(\bar{\mathcal{S}}_M) \subset \bar{\mathcal{S}}_N, \quad \mathrm{d}\varphi(\mathcal{F}_M) \subset \mathcal{F}_N.$$

Lemma 3. *Let (M, F, g) be a Riemannian f-manifold. Then M satisfies $(\mathrm{d}^\nabla F)^{(1,1)} = 0$ if and only if*

$${}^g\nabla_{\bar{Z}}\Gamma(\mathcal{S}) \subset \Gamma(\mathcal{F} \oplus \mathcal{S}), \quad Z \in \Gamma(\mathcal{S}),$$

$$\Pi_0({}^g\nabla_Z \overline{W} + {}^g\nabla_{\overline{W}} Z) = 0, \quad Z, W \in \Gamma(\mathcal{S}).$$

Definition 5 ([41])**.** We say a Riemanian f-manifold (M, F, g) satisfies the *Rawnsley's condition* (A) if

$$^{g}\nabla_{\bar{Z}}\, \Gamma(\mathcal{S}) \subset \Gamma(\mathcal{S}), \quad Z \in \Gamma(\mathcal{S}).$$

In particular, if (M, F, g) is an almost Hermitian manifold, then the condition (A) is equivalent to $(d\omega)^{1,2} = 0$. An almost Hermitian manifold satisfying this condition is said to be $(1,2)$-*symplectic*. The nearly Kähler 6-sphere is a typical example of $(1,2)$-symplectic almost Hermitian manifold. (see [42, Proposition 1.4]). Note that $(1,2)$-symplectic property is stronger than cosymplectic (coclosed fundamental 2-form) property. Hermitian $(1,2)$-symplectic manifolds are Kähler.

Proposition 4. ([41, Proposition 2.6]) *A Riemannian f-manifold (M, F, g) satisfies condition* (A) *if and only if*

$$({}^{g}\nabla_{\overline{Z}} F)W = 0, \quad Z, W \in \Gamma(\mathcal{S}).$$

Bejan and Benyounes [2] considered the following condition for Riemannian f-manifolds:

Definition 6. We say that a Riemannian f-manifold (M, F, g) satisfies condition $(\tilde{\mathrm{A}})$ if (M, F, g) satisfies Rawnsley's condition (A) and in addition M satisfies

$$^{g}\nabla_{U} \Gamma(\mathcal{F}) \subset \Gamma(\mathcal{F}), \quad U \in \Gamma(\mathcal{F}).$$

Obviously, if $\mathcal{F}$ is parallel with respect to the Levi-Civita connection $^{g}\nabla$, condition $(\tilde{\mathrm{A}})$ is equivalent to condition (A). As we will see later, the condition (A) is a very strong restriction for (almost) contact Riemannian structures.

§5. Almost contact manifolds

Let M be a manifold of odd dimension $m = 2n + 1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$ (*cf.* Ogiue [31, 32] and Ogiue-Okumura [33]). This is equivalent to existence of an endomorphism field F, a vector field ξ and a 1-form η satisfying

$$F^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

From these conditions one can deduce that

$$F\xi = 0, \quad \eta \circ F = 0.$$

Obviously F is an f-structure on M. Moreover, since $\mathrm{U}(n)\times\{1\}\subset\mathrm{SO}(2n+1)$, M admits a Riemannian metric g satisfying

$$g(FX,FY)=g(X,Y)-\eta(X)\eta(Y)$$

for all X, $Y\in\mathfrak{X}(M)$. Such a metric is called an *associated metric* of the almost contact manifold $M=(M,F,\xi,\eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X,\xi)=\eta(X)$$

for all $X\in\mathfrak{X}(M)$. A structure (F,ξ,η,g) on M is called an *almost contact Riemannian structure*, and a manifold M equipped with an almost contact Riemannian structure is said to be an *almost contact Riemannian manifold*. Thus every almost contact Riemannian manifold is a Riemannian f-manifold.

The fundamental 2-form of $(M;F,\xi,\eta,g)$ is usually denoted by Φ, *i.e.*,

$$\Phi(X,Y)=g(X,FY),\quad X,Y\in\mathfrak{X}(M).$$

On an almost contact Riemannian manifold M, we define a distribution $\mathcal{D}$ by

$$\mathcal{D}=\{X\in TM\mid\eta(X)=0\}.$$

Then one can see that

$$\mathcal{S}=\{X-\sqrt{-1}JX\mid X\in\mathcal{D}\},\quad J:=F|_{\mathcal{D}}$$

is an almost *CR*-structure on M with real expression $(\mathcal{D},J)$. This almost *CR*-structure is called the *standard almost CR-structure* on M.

An almost contact Riemannian manifold $(M;F,\xi,\eta,g)$ is said to be *normal* if it satisfies

$$[F,F](X,Y)+2\mathrm{d}\eta(X,Y)\xi=0,\quad X,Y\in\mathfrak{X}(M),$$

where $[F,F]$ is the *Nijenhuis torsion* of F defined by

$$[F,F](X,Y)=[FX,FY]+F^2[X,Y]-F[FX,Y]-F[X,FY]$$

for any $X,Y\in\mathfrak{X}(M)$.

Ianus [20] showed that the standard almost *CR*-structures of normal almost contact Riemannian manifolds are integrable.

Definition 7. An almost contact Riemannian manifold M is said to be a *contact Riemannian manifold* if $\Phi=\mathrm{d}\eta$.

Remark 1. A 1-form η on a manifold of dimension $m = 2n + 1$ is called a *contact form* if $(\mathrm{d}\eta)^n \wedge \eta \neq 0$. A manifold M together with a contact form is called a *contact manifold* (in the strict sense). The unique vector field ξ on a contact manifold (M, η) satisfying $\eta(\xi) = 1$ and $\mathrm{d}\eta(\xi, \cdot) = 0$ is called the *Reeb vector field* of a contact manifold (M, η). One can see that on a contact Riemannian manifold $(M; F, \xi, \eta, g)$, the 1-form η is a *contact form* with Reeb vector field ξ.

Let M be a contact Riemannian manifold. Then its standard almost *CR*-structure is integrable if and only if its *Tanno tensor field* $\mathcal{Q}$ vanishes:

$$\mathcal{Q}(X, Y) = ({}^g\nabla_Y F)X + \{({}^g\nabla_Y \eta)FX\}\xi + \eta(X)F({}^g\nabla_Y \xi).$$

A contact Riemannian manifold M is said to be *integrable* if $\mathcal{Q} = 0$. An integrable contact Riemannian manifold is regarded as a *strongly pseudo convex CR-manifold*. This follows from the following

Remark 2. Let $(M, \mathcal{S})$ be an almost *CR*-manifold of dimension $m = 2n + 1$ with real expression (P, J). Assume that there exists a contact form η such that P is defined by the Pfaff equation $\eta = 0$. Then the *Levi-form* L is defined by

$$L(X, Y) = -\mathrm{d}\eta(X, JY), \quad X, Y \in \Gamma(P).$$

If L is J-invariant, then (M, η, L) is said to be a *non-degenerate pseudo-Hermitian manifold*. In particular, if L is positive definite, (M, η, L) is called a *strongly pseudo convex CR-manifold*. On a strongly pseudo convex *CR* manifold (M, η, L), we can extend L to the Riemannian metric g (called the *Webster metric*) on M by the formula $g = L + \eta \otimes \eta$. Next, take the Reeb vector field ξ of η and extend J to the $F \in \Gamma(\mathrm{End}\, TM)$ by $F\xi = 0$, then $(M; F, \xi, \eta, g)$ is a contact Riemannian manifold satisfying $\mathcal{Q} = 0$.

Conversely, let $(M; F, \xi, \eta, g)$ be an integrable contact Riemannian manifold and denote by L and J the restrictions of g and F to $\mathcal{D}$, respectively. Then (M, η, L) is a strongly pseudo convex *CR*-manifold ([51], see also Blair and Dragomir [7] for different conventions).

Definition 8. Let $(M; F, \xi, \eta, g)$ be a contact Riemannian manifold. Then M is said to be a *K-contact manifold* if ξ is a Killing vector field with respect to g.

Proposition 5. *On a contact Riemannian manifold M, the Reeb vector field is a Killing vector field if and only if* ${}^g\nabla\xi = -F$.

A smooth map $\varphi : (M; F, \xi, \eta, g) \to (N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ between almost contact Riemannian manifolds is said to be a *holomorphic map* if it is f-holomorphic with respect to F and $\tilde{F}$, *i.e.*, $\tilde{F} \circ \mathrm{d}\varphi = \mathrm{d}\varphi \circ F$. Analogously, a smooth map

$\varphi : (M; F, \xi, \eta, g) \to (N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ is said to be an *anti holomorphic map* if it is anti f-holomorphic with respecto to F and $\tilde{F}$.

The $\pm$ holomorphicity equation $\tilde{F} \circ \mathrm{d}\varphi = \pm \mathrm{d}\varphi \circ F$ implies the following result.

Lemma 6. *Let* $\varphi : (M; F, \xi, \eta, g) \to (N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ *be a* $\pm$ *holomorphic map between almost contact Riemannian manifolds. Then there exits a smooth function* λ *on* M *such that* $\varphi_* \xi = \lambda \tilde{\xi}$.

A holomorphic diffeomorphism $\varphi : M \to N$ between almost contact Riemannian manifolds is called an *almost contact isomorphism* if

$$\varphi^* \tilde{\eta} = \eta, \ \varphi^* h = g.$$

An *almost contact automorphism* φ is an almost contact isomorphism $\varphi : M \to M$ on an almost contact Riemannian manifold.

Remark 3. A diffeomorphism on a contact manifold (M, η) is called a *contact transformation* if $\varphi^* \eta = \lambda \eta$ for some non-vanishing function λ. In particular, a diffeomorphism φ is said to be a *strict contact transformation* if $\varphi^* \eta = \eta$. On a compact contact manifold, the group of all contact transformations admits a structure of infinite dimensional Lie group (more precisely, ILH-Lie group structure and Fréchet Lie group structure). See Omori [38, 39].

The set $\mathrm{Aut}(M)$ of all almost contact automorphisms is a finite dimensional Lie group, since it is a subgroup of the isometry group of M ([48]).

As an odd-dimensional analogue of Kähler manifold, the notion of Sasakian manifold is introduced in the following way:

Definition 9. Let $(M; F, \xi, \eta, g)$ be a contact Riemannian manifold. Then M is called a *Sasakian manifold* if it is normal.

Proposition 7. *An almost contact Riemannian manifold* $(M; F, \xi, \eta, g)$ *is a Sasakian manifold if and only if*

$$({}^g\nabla_X F)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathfrak{X}(M).$$

On a Sasakian manifold, the Reeb vector field is a Killing vector field. Note that the fundamental 2-form of a Sasakian manifold is exact. The above covariant derivative formula implies that Sasakian manifolds cannot satisfy Rawnsley's condition (A).

Sasakian manifolds cannot be of negative curvature. In fact, the sectional curvature of planes tangent to the Reeb vector field are constant 1.

Blair introduced the notion of quasi-Sasakian manifold.

Definition 10 ([4])**.** Let M be an almost contact Riemannian manifold. Then M is said to be a *quasi-Sasakian manifold* if M is normal and the fundamental 2-form is closed.

Remark 4. Quasi-Sasakian manifolds are *CR-Kähler manifolds* in the sense of Burstall [9] and Eells-Lemaire [15].

More precisely, let $\mathcal{D}$ be a real 1-codimensional subbundle of TM of a manifold M together with a bundle endomorphism J on $\mathcal{D}$ such that $J^2 = -I$. Thus $(M, \mathcal{D}, J)$ becomes an almost *CR*-manifold. The almost *CR*-manifold $(M, \mathcal{D}, J)$ equipped with a Riemannian metric g is called an *almost CR-Hermitian manifold* if g is J-invariant over $\mathcal{D}$. On an almost *CR*-Hermitian manifold, we define a 2-form ω by $\omega(X, Y) = g(X, JY)$, $X, Y \in \Gamma(\mathcal{D})$ and $\omega(V, \cdot) = 0$, $V \perp \mathcal{D}$. Then an almost *CR*-Hermitian manifold is said to be *almost CR-Kähler* if ω is closed. An almost *CR*-Kähler manifold is called a *CR*-Kähler manifold if its almost *CR*-structure is integrable.

According to this definition, quasi-Sasakian manifolds are *CR*-Kähler.

Let (M, J, g) be a Hermitian manifold and $(N, \widetilde{\mathcal{D}}, \tilde{J}, h)$ an almost *CR*-Hermitian manifold. Then a smooth map $\phi : M \to N$ is called a *CR-map* if $\mathrm{d}\varphi(T^{(1,0)}M) \subset \widetilde{\mathcal{D}}$ and $\mathrm{d}\varphi \circ J = \tilde{J} \circ \mathrm{d}\varphi$.

Proposition 8 ([9])**.** *If two CR-maps φ_1, $\varphi_2 : M \to N$ agree on an open subset of M, then $\varphi_1 = \varphi_2$.*

Proposition 9. ([9], [41]) *If (M, g, J) is an almost Hermitin cosymplectic manifold and (N, h, J) an almost CR-Kähler, then any CR-map $\varphi : M \to N$ is an energy-minimizing harmonic map.*

Quasi-Sasakian maifolds are characterized by the rank of $\mathrm{d}\eta$ (see [4], [50]). Sasakian manifolds are quasi-Sasakian manifolds whose $\mathrm{d}\eta$ is of full rank. In general, quasi-Sasakian manifolds are non-contact.

Typical examples of quasi-Sasakian manifolds are homogeneous real hypersurfaces of type A_2 in complex projective space (Okumura [35], Olszak [37], see also [10]).

On the other hand, quasi-Sasakian manifolds with vanishing $\mathrm{d}\eta$ are called *coKähler manifolds* ([23]).

Proposition 10. *Let M be an almost contact Riemannian manifold. Then M is coKähler if and only if ${}^g\nabla F = 0$. In this case, M is locally isomorphic to a direct product of a Kähler manifold and the real line.*

Proposition 11 ([37])**.** *Let (M, F, ξ, η, g) be a quasi-Sasakian manifold. Then*

1. *ξ is Killing,*
2. ${}^g\nabla_{FX}\xi = F({}^g\nabla_X \xi)$, $X \in \mathfrak{X}(M)$,

3. $({}^{g}\nabla_X F)Y = -g({}^{g}\nabla_X \xi, FY)\xi - \eta(Y)F({}^{g}\nabla_X \xi)$.

Odd-dimensional unit sphere $\mathbb{S}^{2n+1}$ is a typical example of Sasakian manifold. On the other hand, odd-dimensional unit hyperbolic space $\mathbb{H}^{2n+1}$ admits non-contact normal almost contact structure compatible to the metric.

Definition 11 ([28])**.** An almost contact Riemannian manifold $(M; F, \xi, \eta, g)$ is called a *Kenmotsu manifold* if

$$({}^{g}\nabla_X F)Y = -g(X, FY)\xi - \eta(Y)FX, \quad X, Y \in \mathfrak{X}(M).$$

From this formula we have

$${}^{g}\nabla_X \xi = X - \eta(X)\xi, \ X \in \mathfrak{X}(M).$$

Proposition 12. *A Kenmotsu manifold* (M, F, ξ, η, g) *has the following properties*:

1. *it is noncompact,*
2. ξ *is not Killing,*
3. *it is normal but not quasi-Sasakian,*
4. *it is locally isometric to the warped product* $\mathbb{R}(t) \times_{ce^t} \overline{M}$ *whose fibre is a Kähler manifold* $\overline{M}$*. Here c is a real constant.*

On a Kenmotsu manifold M, $\mathcal{D}$ defines a foliation $\mathcal{F}$ of codimension 1. One can check that $\mathcal{F}$ is Riemannian and tangentially Kähler.

For more details on almost contact metric manifolds, we refer to Blair's monographs [5, 6].

§6. Holomorphic maps between quasi-Sasakian manifolds

Ianus and Pastore showed that any holomorphic map between contact Riemannian manifolds is harmonic [21]. In particular, every holomorphic map between Sasakian manifold is harmonic. In this section we compute the tension field of holomorphic maps between quasi-Sasakian manifolds.

Theorem 1. *Let* $(M; F, \xi, \eta, g)$ *and* $(N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ *be quasi-Sasakian manifolds. Then every holomorphic map* $\varphi : M \to N$ *satisfies*

$$\tau(\varphi) = \mathrm{d}\lambda(\xi)\tilde{\xi}$$

Here the function λ *is determined by the equation* $\varphi_*\xi = \lambda\,\xi$.

Thus φ *is harmonic if and only if* $\mathrm{d}\lambda(\xi) = 0$*. In such a case* λ *is constant.*

Proof. Take a local orthonormal frame field of M of the form:

$$u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, \xi, \quad v_j = Fu_j, \ j = 1, 2, \cdots, n.$$

Then for a holomorphic map φ, we have

$$\tilde{\eta}(\varphi_* u_i) = h(\tilde{\xi}, \varphi_* u_i) = h(\tilde{\xi}, -\varphi_* F v_i) = -h(\tilde{\xi}, \tilde{F} \varphi_* v_i) = 0, \quad i = 1, 2, \ldots, n.$$

Next by using the property $\tilde{F} \circ {}^h\nabla \tilde{\xi} = {}^h\nabla \tilde{\xi} \circ \tilde{F}$ and the Killing property of $\tilde{\xi}$, we have

$$\begin{aligned} h({}^h\nabla_{\varphi_* v_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) &= h({}^h\nabla_{\varphi_* F u_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) = h({}^h\nabla_{\tilde{F} \varphi_* u_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) \\ &= h(\tilde{F}({}^h\nabla_{\varphi_* u_i} \tilde{\xi}), \tilde{F} \varphi_* u_i) \\ &= h({}^h\nabla_{\varphi_* u_i} \tilde{\xi}, \varphi_* u_i) - \tilde{\eta}({}^h\nabla_{\varphi_* u_i} \tilde{\xi}) \tilde{\eta}(\varphi_* u_i) \\ &= h({}^h\nabla_{\varphi_* u_i} \tilde{\xi}, \varphi_* u_i) = h(({}^h\nabla \tilde{\xi}) \varphi_* u_i, \varphi_* u_i) = 0. \end{aligned}$$

Now let us compute the tension field of φ. Since φ is holomorphic,

$${}^h\nabla^{\varphi}_{v_i} \varphi_* v_i = {}^h\nabla^{\varphi}_{v_i} \tilde{F}(\varphi_* u_i) = ({}^h\nabla^{\varphi}_{v_i} \tilde{F}) \varphi_* u_i + \tilde{F}({}^h\nabla^{\varphi}_{v_i} \varphi_* u_i).$$

Using the covariant derivative formula of $\tilde{F}$, we get

$$\begin{aligned} ({}^h\nabla^{\varphi}_{v_i} \tilde{F}) \varphi_* u_i &= -h({}^h\nabla_{\varphi_* v_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) \tilde{\xi} - \tilde{\eta}(\varphi_* u_i) \tilde{F}({}^h\nabla_{\varphi_* v_i} \xi) \\ &= -h({}^h\nabla_{\varphi_* v_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) \tilde{\xi} = -h({}^h\nabla_{\varphi_* v_i} \tilde{\xi}, \varphi_* v_i) \tilde{\xi} = 0, \end{aligned}$$

since $\tilde{\xi}$ is Killing. Thus we have ${}^h\nabla^{\varphi}_{v_i} \varphi_* v_i = \tilde{F}({}^h\nabla^{\varphi}_{v_i} \varphi_* u_i)$. The right hand side of this formula is computed as

$$\begin{aligned} \tilde{F}({}^h\nabla^{\varphi}_{v_i} \varphi_* u_i) &= \tilde{F}({}^h\nabla^{\varphi}_{u_i} \varphi_* v_i + \varphi_*[v_i, u_i]) = \tilde{F}({}^h\nabla^{\varphi}_{u_i} \varphi_* F u_i + \varphi_*[v_i, u_i]) \\ &= \tilde{F}({}^h\nabla^{\varphi}_{u_i} \tilde{F} \varphi_* u_i + \varphi_*[v_i, u_i]) \\ &= \tilde{F}(({}^h\nabla^{\varphi}_{u_i} \tilde{F}) \varphi_* u_i + \tilde{F}({}^h\nabla^{\varphi}_{u_i} \varphi_* u_i) + \varphi_*[v_i, u_i]). \end{aligned}$$

Using the covariant derivative formula of $\tilde{F}$ again we get

$$({}^h\nabla^{\varphi}_{u_i} \tilde{F}) \varphi_* u_i = -h({}^h\nabla_{\varphi_* u_i} \tilde{\xi}, \tilde{F} \varphi_* u_i) \tilde{\xi} - \tilde{\eta}(\varphi_* u_i) \tilde{F}({}^h\nabla_{\varphi_* u_i} \tilde{\xi}) = 0.$$

Moreover we have

$$\tilde{F}^2({}^h\nabla^{\varphi}_{u_i} \varphi_* u_i) = -{}^h\nabla^{\varphi}_{u_i} \varphi_* u_i + \tilde{\eta}({}^h\nabla^{\varphi}_{u_i} \varphi_* u_i) \tilde{\xi}.$$

Here we note that

$$\tilde{\eta}({}^h\nabla^{\varphi}_{u_i} \varphi_* u_i) = h(\tilde{\xi}, {}^h\nabla^{\varphi}_{u_i} \varphi_* u_i) = (\varphi_* u_i) \cdot \tilde{\eta}(\varphi_* u_i) - h({}^h\nabla^{\varphi}_{u_i} \tilde{\xi}, \varphi_* u_i) = 0.$$

Thus we obtain

$$^h\nabla^{\varphi}_{v_i}\varphi_* v_i = -{}^h\nabla^{\varphi}_{u_i}\varphi_* u_i + \tilde{F}\varphi_*[v_i, u_i].$$

Next we compute $\varphi_*({}^g\nabla_{v_i} v_i)$.

$$\begin{aligned}
\varphi_*({}^g\nabla_{v_i} v_i) &= \varphi_*({}^g\nabla_{v_i} F u_i) = \varphi_*\{({}^g\nabla_{v_i} F)u_i + F({}^g\nabla_{v_i} u_i)\}\\
&= \varphi_*\{-g({}^g\nabla_{v_i}\xi, F u_i)\xi - \eta(u_i)(F{}^g\nabla_{v_i}\xi) + F({}^g\nabla_{v_i} u_i)\}\\
&= \varphi_*\{-g({}^g\nabla_{v_i}\xi, v_i)\xi + F({}^g\nabla_{v_i} u_i)\}\\
&= \varphi_* F({}^g\nabla_{u_i} v_i + [v_i, u_i])\\
&= \varphi_* F({}^g\nabla_{u_i} F u_i + [v_i, u_i])\\
&= \varphi_* F(({}^g\nabla_{u_i} F)u_i + F({}^g\nabla_{u_i} u_i) + [v_i, u_i])\\
&= \varphi_* F(-g({}^g\nabla_{u_i}\xi, F u_i)\xi - \eta(u_i)F{}^g\nabla_{u_i}\xi + F({}^g\nabla_{u_i} u_i) + [v_i, u_i])\\
&= \varphi_* F^2({}^g\nabla_{u_i} u_i) + \varphi_* F[v_i, u_i]\\
&= \varphi_*(-{}^g\nabla_{u_i} u_i + \eta({}^g\nabla_{u_i} u_i)\xi) + \varphi_* F[v_i, u_i]\\
&= -\varphi_*{}^g\nabla_{u_i} u_i + \varphi_* F[v_i, u_i].
\end{aligned}$$

Here we used a fact $\eta({}^g\nabla_{u_i} u_i) = 0$. From these computations we obtain

$$^h\nabla^{\varphi}_{v_i}\varphi_* v_i - \varphi_*({}^g\nabla_{v_i} v_i) = -{}^h\nabla^{\varphi}_{u_i}\varphi_* u_i + \varphi_*{}^g\nabla_{u_i} u_i. \tag{6.1}$$

Hence the tension field of φ is given by

$$\begin{aligned}
\tau(\varphi) &= \sum_{i=1}^{n}\{{}^h\nabla^{\varphi}_{u_i}\varphi_* u_i - \varphi_*({}^g\nabla_{u_i} u_i)\} + \sum_{i=1}^{n}\{{}^h\nabla^{\varphi}_{v_i}\varphi_* v_i - \varphi_*({}^g\nabla_{v_i} v_i)\}\\
&\quad + {}^h\nabla^{\varphi}_{\xi}\varphi_*\xi - \varphi_*({}^g\nabla_{\xi}\xi)\\
&= {}^h\nabla^{\varphi}_{\xi}\varphi_*\xi.
\end{aligned}$$

From Lemma 6, there exists a smooth function λ such that $\varphi_*\xi = \lambda\tilde{\xi}$. Inserting this into the formula of $\tau(\varphi)$, we get $\tau(\varphi) = \mathrm{d}\lambda(\xi)\,\tilde{\xi}$.

Next, since $\varphi_*\xi = \lambda\,\tilde{\xi}$, we get $\varphi^*\tilde{\eta} = \lambda\,\eta$. Taking the exterior derivative of this relation, we get

$$\mathrm{d}(\varphi^*\tilde{\eta}) = \mathrm{d}(\lambda\,\eta) = \mathrm{d}\lambda \wedge \eta + \lambda\,\mathrm{d}\eta.$$

Hence

$$\mathrm{d}(\varphi^*\tilde{\eta})(\xi, X) = \mathrm{d}\lambda(\xi)\,\eta(X) - \mathrm{d}\lambda(X), \quad X \in \mathfrak{X}(M),$$

since $\mathrm{d}\eta(\xi, \cdot) = 0$ on any quasi-Sasakian manifold (see [4]). On the other hand we have

$$\mathrm{d}(\varphi^*\tilde{\eta})(\xi, X) = \varphi^*(\mathrm{d}\tilde{\eta})(\xi, X) = (\mathrm{d}\tilde{\eta})(\varphi_*\xi, \varphi_* X) = \lambda(\mathrm{d}\tilde{\eta})(\tilde{\xi}, \varphi_* X) = 0.$$

Thus we obtain $\mathrm{d}\lambda = \mathrm{d}\lambda(\xi)\eta$. Hence $\mathrm{d}\lambda(\xi) = 0$ if and only if λ is constant. □

Corollary 1. *Let $\varphi : M \to N$ be an isometric immersion from a quasi-Sasakian manifold into quasi-Sasakian manifold. If φ is holomorphic and $\varphi^*\tilde{\eta} = \eta$, then it is minimal.*

Urakawa obtained the following result.

Proposition 13. ([54]) *Let M and N be strongly pseudo convex CR-manifolds. Then every holomorphic map $\varphi : M \to N$ satisfies*

$$\tau(\varphi) = {}^h\nabla^{\varphi}_{\xi}\varphi_*\xi.$$

Here we recall the following Lemma essentially due to Tanno [47]. See also Ianus and Pastore [21].

Lemma 14. ([21],[47]) *Let $\varphi : (M, F, \xi, \eta, g) \to (N, \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ be a holomorphic map (resp. anti holomorphic map) between contact Riemannian manifolds. Then there exists a positive (resp. negative) constant α such that*

$$\varphi_*\xi = \alpha\tilde{\xi},\ \varphi^*\tilde{\eta} = \alpha\eta,\ \varphi^*h = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta.$$

Olszak [36] obtained the following formula for contact Riemannian manifolds:

$$({}^g\nabla_{FX}F)(FY) + ({}^g\nabla_X F)Y = 2g(X, Y)\xi - \eta(Y)\{(I + \mathrm{h})X + \eta(X)\xi\}$$

for all X,$Y \in \mathfrak{X}(M)$. Here the endomorphism field h is defined by $\mathrm{h} = \pounds_{\xi}F/2$.

Computing the tension field of a $\pm$ holomorphic map between contact Riemannian manifold by using Olszak's formula, Ianus and Pastore obtained

Proposition 15 ([21], see also [18])**.** *Any $\pm$ holomorphic map between contact Riemannian manifolds is harmonic.*

Corollary 2. *Any $\pm$ holomorphic map between strongly pseudo convex CR-manifolds is harmonic.*

Corollary 3 ([16])**.** *Any holomorphic immersion of contact Riemannian manifolds into contact Riemannian manifolds is minimal.*

Corollary 4. *Any $\pm$ holomorphic map between Sasakian manifolds is harmonic.*

Corollary 5 ([49, 60])**.** *Any holomorphic isometric immersion of a Sasakian manifold into a Sasakian manifold is minimal.*

Holomorphic isometric immersions of Sasakian manifolds into Sasakian manifolds are called *Sasakian immersions* or *invariant immersions* ([19, 26, 27]).

Every holomorphic map between (compact) almost Kähler manifolds are stable harmonic maps. On the other hand, harmonic maps from or into Sasakian manifold $\mathbb{S}^{2n+1}$ are unstable (Xin [58]).

Remark 5. In [43], Saotome studied *J-holomorphic map* in strongly pseudo convex *CR*-manifolds.

Let $(M; F, \xi, \eta, g)$ be a Sasakian 3-manifold and $(N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h)$ a strongly pseudo convex *CR*-manifold. Denote by $(\mathcal{D}, J)$ and $(\widetilde{\mathcal{D}}, \tilde{J})$ the real expressions of standard *CR*-structure $\mathcal{S}$ of M and $\widetilde{\mathcal{S}}$ of N, respectively.

A smooth map $\varphi : M \to N$ is said to be *J-holomorphic* (in the sense of [43]) if

- $\mathrm{d}\varphi \circ J = \tilde{J} \circ \mathrm{d}\varphi$ on the contact distribution $\mathcal{D}$ of M.
- $\mathrm{d}\varphi(\xi) = \lambda\tilde{\xi}$ for some positive function λ on M.
- $\varphi^*\tilde{\eta} = \lambda\eta$.

One can see that every J-holomorphic map in Saotome's sense preserves the *CR*-structure, *i.e.*, $\mathrm{d}\varphi(\mathcal{S}) \subset \widetilde{\mathcal{S}}$ and hence it satisfies $\mathrm{d}\varphi \circ F = \tilde{F} \circ \mathrm{d}\varphi$. Thus J-holomorphic maps in the sense of [43] are harmonic maps.

§7. Holomorphic maps into Kenmotsu manifolds

Let (N, F, η, ξ, h) be a Kenmotsu manifold. As we have seen before, N is locally isomorphic to a warped product. More precisely, for each point $p \in N$, there exists a neighbourhood U of p and a positive number ε such that U is isomorphic to the warped product $(-\varepsilon, \varepsilon) \times_f \overline{U}$, where $\overline{U}$ is a Kähler manifold and $f : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is a smooth positive function. The natural projection $\pi_U : U \to (-\varepsilon, \varepsilon)$ is a Riemannian submersion. The vertical distribution $\mathcal{V}$ is

$$\mathcal{V}_q = \{X \in T_qU \mid \eta_q(X) = 0\},\ q \in U.$$

Following the terminology of [57] explained in §2, we call a smooth map $\varphi : M \to N$ from a Riemannian manifold M into a Kenmotsu manifold a *vertically harmonic map* if $\tau(\varphi) - \eta(\tau(\varphi))\xi = 0$.

Now let (M, J, g) be a Kähler manifold and consider a holomorphic map $\varphi : M \to N$. We compute the tension field of φ. Take a local orthonormal frame field of M of the form: $\{e_1, \cdots, e_m, f_1, \cdots, f_m\}$ such that $f_i = Je_i$.

Direct computations show that

$$({}^h\nabla \mathrm{d}\varphi)(e_i; e_i) = -(\varphi^*h)(f_i; f_i)\xi + F^2({}^h\nabla \mathrm{d}\varphi)(e_i; e_i),$$

$$({}^h\nabla \mathrm{d}\varphi)(f_i; f_i) = -(\varphi^*h)(e_i; e_i)\xi + F^2({}^h\nabla \mathrm{d}\varphi)(f_i; f_i).$$

Summing up these we get,

$$\tau(\varphi) = -2e(\varphi)\xi.$$

This formula implies that every holomorphic map satisfies the vertical harmonicity equation. Thus we obtain

Theorem 2. *Holomorphic maps from a Kähler manifold into a Kenmotsu manifold are vertically harmonic.*

Moreover, a holomorphic map $\varphi : M \to N$ is harmonic if and only if its energy density vanishes. Namely we obtain

Corollary 6. *Holomorphic maps from a Kähler manifold into a Kenmotsu manifold are harmonic if and only if they are constants.*

§8. Pluriharmonic maps

For a smooth map φ from a Riemannian 2-manifold (M, g) into a Riemannian manifold (N, h), the energy functional $E(\varphi)$ is invariant under the conformal transformation of M. Thus the harmonicity makes sense for maps from a 2-manifold equipped with a *conformal structure*, *i.e.*, *Riemann surface* into a Riemannian manifold. More generally, harmonicity can be defined for maps from Riemann surfaces (or Lorentz surfaces) into manifolds with linear connection.

In dimension 2, conformal structure coincides with *complex structure*. The notion of harmonic maps from Riemann surfaces into Riemannian manifolds can be generalized to that of *pluriharmonic maps* from *complex manifold* (with or without metric) into Riemannian manifolds. Here we recall fundamental ingredients of pluriharmonic maps.

Let (M, J) be a complex manifold. With respect to the complex structure J, we decompose the complexified tangent bundle $T^{\mathbb{C}}M$ into the direct sum: $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ with $T^{(0,1)}M = \overline{T^{(1,0)}M}$.

Let us denote by $\bar{\partial}$ the *Cauchy-Riemann operator* of M. Namely $\bar{\partial}$ is an operator

$$\bar{\partial} : \Gamma(T^{(0,1)}M) \times \Gamma(T^{(1,0)}M) \to \Gamma(T^{(1,0)}M)$$

such that

$$\bar{\partial}_{\bar{Z}} W = \sum \overline{Z^i} \frac{\partial W_j}{\partial \bar{z}_i} \frac{\partial}{\partial z_j}$$

for

$$Z = \sum Z_i \frac{\partial}{\partial z_i}, \quad W = \sum W_j \frac{\partial}{\partial z_j} \in \Gamma(T^{(1,0)}M).$$

Note that $(T^{(1,0)}M, \bar{\partial})$ is a holomorphic vector bundle over M and called the *holomorphic tangent bundle* of M.

Let (N, h) be a Riemannian manifold with Levi-Civita connection ${}^h\nabla$ and $\varphi : (M, J) \to N$ a smooth map. By restricting the differential $\mathrm{d}\varphi$ to $T^{(1,0)}M$ and $T^{(0,1)}M$, we obtain vector bundle morphisms;

$$\partial\varphi : T^{(1,0)}M \to \varphi^* T^{\mathbb{C}}N, \quad \bar{\partial}\varphi : T^{(0,1)}M \to \varphi^* T^{\mathbb{C}}N$$

into the pulled-back bundle $\varphi^* T^{\mathbb{C}} N$. Then the $(0,1)$-*exterior derivative* ${}^h\nabla''\partial\varphi$ of $\partial\varphi$ is defined by

$$({}^h\nabla''_{\overline{Z}}\partial\varphi)W := \nabla^{\varphi}_{\overline{Z}}(\partial\varphi(W)) - \partial\varphi(\bar{\partial}_{\overline{Z}}W), \quad Z, W \in \Gamma(T^{(1,0)}M).$$

Definition 12. A smooth map $\varphi : (M, J) \to (N, h)$ is said to be a *pluriharmonic map* if ${}^h\nabla''\partial\varphi = 0$.

By using the complex structure J, the complexification of second fundamental form is decomposed as

$${}^h\nabla \mathrm{d}\varphi = ({}^h\nabla \mathrm{d}\varphi)^{(2,0)} + ({}^h\nabla \mathrm{d}\varphi)^{(1,1)} + ({}^h\nabla \mathrm{d}\varphi)^{(0,2)}.$$

It is easy to see that $({}^h\nabla \mathrm{d}\varphi)^{(1,1)}(W; \overline{Z}) = ({}^h\nabla''_{\overline{Z}}\partial\varphi)(W)$. Thus we have

Lemma 16. *Let (M, J) be a complex manifold and (N, h) a Riemannian manifold. Then a map $\varphi : M \to N$ is pluriharmonic if and only if $({}^h\nabla \mathrm{d}\varphi)^{(1,1)} = 0$.*

Remark 6. The partial differential equation $({}^h\nabla \mathrm{d}\varphi)^{(1,1)} = 0$ makes sense for maps from a complex manifold into a manifold with a linear connection.

Let Σ be a Riemann surface, (M, J) a complex manifold and $\varphi : M \to (N, h)$ a map into a Riemannian manifold. Assume that (M, J) admits a Kähler metric g compatible to J. Take a map $\psi : \Sigma \to M$. Then the second fundamental form of the composition map $\varphi \circ \psi$ is given by

$${}^h\nabla \mathrm{d}(\varphi \circ \psi) = \mathrm{d}\varphi({}^g\nabla \mathrm{d}\psi) + ({}^h\nabla \mathrm{d}\varphi)(\mathrm{d}\psi; \mathrm{d}\psi).$$

Here we take any Kähler metric in the conformal class of Σ to define the second fundamental form ${}^g\nabla \mathrm{d}\psi$. This composition formula implies the following well known result (We can take a Kähler metric on a small neighborhood of (M, J)).

Proposition 17. *Let (M, J) be complex manifold and (N, h) a Riemannian manifold. Then a map $\varphi : M \to N$ is pluriharmonic if and only if for any holomorphic curve $\iota : \Sigma \to M$, the composite $\varphi \circ \iota$ is harmonic.*

Let $\varphi : M \to N$ be a pluriharmonic map. If M admits Kähler metrics, then φ is harmonic with respect to any Kähler metrics on M.

Take $Z = X - \sqrt{-1}JX$, $W = Y - \sqrt{-1}JY \in \Gamma(T^{(1,0)}M)$. Then we have

$$\begin{aligned}({}^h\nabla''_{\bar{W}}\partial\varphi)Z &= \{({}^h\nabla \mathrm{d}\varphi)(X; Y) + ({}^h\nabla \mathrm{d}\varphi)(JX; JY)\} \\ &\quad + \sqrt{-1}\{({}^h\nabla \mathrm{d}\varphi)(X; JY) - ({}^h\nabla \mathrm{d}\varphi)(JX; Y)\}.\end{aligned}$$

The following result is easily verified.

Lemma 18. *For a smooth map $\varphi : (M, J, g) \to (N, h)$ from a Kähler manifold into a Riemannian manifold, the following properties are mutually equivalent.*

- φ *is pluriharmonic,*
- $({}^h\nabla \mathrm{d}\varphi)(X; Y) + ({}^h\nabla \mathrm{d}\varphi)(JX; JY) = 0$ *on* TM,
- $({}^h\nabla \mathrm{d}\varphi)(X; JY) - ({}^h\nabla \mathrm{d}\varphi)(JX; Y) = 0$ *on* TM.

Based on these observations, we arrive at the following definition.

Definition 13. Let (M, J) be an almost Hermitian manifold and $(N, {}^N\nabla)$ a manifold with a linear connection. Then a map $\varphi : M \to N$ is said to be an *affine* $(1, 1)$*-harmonic map* if $({}^N\nabla \mathrm{d}\varphi)^{(1,1)} = 0$.

When N is a Riemannian manifold and ${}^N\nabla$ is the Levi-Civita connection, $(1, 1)$-*harmonic map* have been called $(1, 1)$-*geodesics maps* (see [41]). In [13], a loop group method for constructing all $(1, 1)$-harmonic maps from non compact simply connected Riemannn surfaces into Lie groups equipped with bi-invariant affine connection is established.

In general, for maps between Kähler manifolds, pluriharmonicity is weaker than holomorphicity and stronger than harmonicity. Under some differential geometric conditions, harmonicity implies pluriharmonicity and also pluriharmonicity implies holomorphicity. For more informations on pluriharmonic maps we refer to Udagawa [52] and references therein.

§9. $(1, 1)$-harmonic maps

In this section we extend the notion of affine $(1, 1)$-harmonic maps to more general target spaces. Rawnsley showed the following result.

Proposition 19 ([41])**.** *Let (M, J, g) be a $(1, 2)$-symplectic almost Hermitian manifold and (N, F, h) a Riemannian f-manifold satisfying the condition* (A). *Then every f-holomorphic map $\varphi : M \to N$ is a $(1, 1)$-harmonic map.*

On the other hand Black obtained the following result.

Proposition 20 ([3])**.** *Let (M, J, g) be a cosymplectic almost Hermitian manifold and (N, F, h) a Riemannian f-manifold satisfying $(\mathrm{d}^\nabla F)^{(1,1)} = 0$. Then every f-holomorphic map $\varphi : M \to N$ is a $(1, 1)$-harmonic map.*

Iwamatsu generalized above results due to Rawnsley and Black.

Proposition 21 ([22])**.** *Let (M, J, g) be a $(1, 2)$-symplectic almost Hermitian manifold and (N, F, h) a Riemannian f-manifold satisfying $(\mathrm{d}^\nabla F)^{(1,1)} = 0$. Then every f-holomorphic map $\varphi : M \to N$ is a $(1, 1)$-harmonic map.*

There exist Riemannian f-manifolds satisfying $(\mathrm{d}^\nabla F)^{(1,1)} = 0$ but do not satisfy the condition (A). In fact, one easily check that quasi-Sasakian manifolds, especially Sasakian manifolds satisfy $(\mathrm{d}^\nabla F)^{(1,1)} = 0$ but do not satisfy the condition (A). Iwamatsu exhibited examples of partial flag manifolds satisfying these properties [22].

Corollary 7. *Let (M, J, g) be a $(1, 2)$-symplectic almost Hermitian manifold and N a quasi-Sasakian manifold, then every holomorphic map $\varphi : M \to N$ is a $(1, 1)$-harmonic map.*

Note that Kenmotsu manifolds never satisfy the condition $(\mathrm{d}^\nabla F)^{(1,1)} = 0$.
More generally, for maps from Riemannian f-manifolds, Bejan and Benyounes considered the following properties.

Definition 14 ([2])**.** Let (M, F, g) be a Riemannian f-manifold and (N, h) a Riemannian manifold. A map $\varphi : M \to N$ is said to be

1. f-$(1, 1)$-*harmonic* if

$$({}^h\nabla \mathrm{d}\varphi)(X; Y) + ({}^h\nabla \mathrm{d}\varphi)(FX; FY) = 0, \quad X, Y \in \Gamma(TM). \tag{9.1}$$

2. f-*pluriharmonic* if

$$(D^+_W \mathrm{d}^-\varphi)(\bar{Z}) := {}^h\nabla^\varphi_W(\mathrm{d}^-\varphi)(\overline{Z}) - (\mathrm{d}^-\varphi)(\Pi_-({}^g\nabla_W \overline{Z})) = 0 \tag{9.2}$$

and

$$({}^h\nabla \mathrm{d}\varphi)(\nu, X) = 0, \quad X \in \Gamma(TM), \quad \nu \in \Gamma(\mathcal{F}). \tag{9.3}$$

Note that f-$(1, 1)$-harmonic maps are called f-$(1, 1)$-*geodesic maps* in [2].

Proposition 22. *A map $\varphi : (M, F, g) \to (N, h)$ is f-$(1, 1)$-harmonic if and only if it satisfies* (9.3) *and its restriction to any holomorphic curve is harmonic.*

Here we introduce the following notion.

Definition 15. Let (M, F, g) be a Riemannian f-manifold with associated almost *CR*-structure $\mathcal{S}$. Denote by (P, J) the real expression of $\mathcal{S}$. Then a map $\varphi : (M, F, g) \to (N, h)$ into a Riemannian manifold is said to be $\mathcal{S}$-$(1, 1)$-*harmonic* if

$$({}^h\nabla \mathrm{d}\varphi)(X; Y) + ({}^h\nabla \mathrm{d}\varphi)(FX; FY) = 0, \quad X, Y \in \Gamma(P). \tag{9.4}$$

Obviously, $\mathcal{S}$-$(1, 1)$-harmonic property is weaker than f-$(1, 1)$-harmonic property.

Proposition 23. *A map $\varphi : (M, F, g) \to (N, h)$ is f-$(1,1)$-harmonic if and only if it is $\mathcal{S}$-$(1,1)$-harmonic and satisfies* (9.3)*.*

Direct computation shows the formula

$$(D^{+}_{W}\mathrm{d}^{-}\varphi)(\bar{Z}) = ({}^{h}\nabla \mathrm{d}\varphi)(\bar{Z}; W) + \mathrm{d}\varphi(\hat{\Pi}({}^{g}\nabla_{W}\bar{Z})).$$

Hence we obtain

Proposition 24. ([2]) *Any two of the following conditions imply the third one*:

- *φ is f-$(1,1)$-harmonic,*
- *φ is f-pluriharmonic,*
- *φ satisfies $({}^{h}\nabla \mathrm{d}\varphi)(\nu, X) = 0$ and*

$$\mathrm{d}\varphi(\widehat{\Pi}({}^{g}\nabla_{W}\overline{Z})) = 0, \quad Z, W \in \Gamma(S).$$

Here $\widehat{\Pi} : T^{\mathbb{C}}M \to \hat{T}(M) = \mathcal{S} \oplus \mathcal{F}$ is the projection.

Remark 7. A Riemannian f-manifold (M, F, g) satisfies condition (A) if and only if $\hat{\Pi}({}^{g}\nabla_{W}\bar{Z}) = 0$ for any $Z, W \in \Gamma(\mathcal{S})$. Hence for maps from a Riemannian f-manifold satisfying condition (A), then the notion of f-(1,1)-harmonic map coincides with that of f-pluriharmonic map.

9.1. $(1,1)$-harmonic maps in contact geometry

Let $(M; F, \xi, \eta, g)$ be an almost contact Riemannian manifold and $\varphi : M \to (N, h)$ a map into a Riemannian manifold. Then the tension field of φ is computed as

$$\tau(\varphi) = \sum_{i=1}^{n}\{({}^{h}\nabla \mathrm{d}\varphi)(e_i; e_i) + ({}^{h}\nabla \mathrm{d}\varphi)(Fe_i; Fe_i)\} + ({}^{h}\nabla \mathrm{d}\varphi)(\xi; \xi).$$

Here we take a local orthonormal frame field on M of the form
$\{e_1, e_2, \cdots, e_n; Fe_1, Fe_2, \cdots, Fe_n, \xi\}$.

Corollary 8. *Every f-$(1,1)$-harmonic map from an almost contact Riemannian manifold into a Riemannian manifold is harmonic.*

Proposition 25 ([21])**.** *Let M and N be Sasakian manifolds and $\varphi : M \to N$ a $\pm$ holomorphic map. Then φ is $\mathcal{S}$-$(1,1)$-harmonic. Moreover, a holomorphic map is f-$(1,1)$-harmonic if and only if φ is an isometric immersion.*

One can check that this proposition is still valid for $\pm$ holomorphic maps between quasi-Sasakian manifolds (see (6.1)).

9.2. *CR*-pluriharmonic maps

Let M be a strongly pseudo convex *CR*-manifold, *i.e.*, integrable contact Riemannian manifold. The *Tanaka-Webster connection* $\hat{\nabla}$ on M is defined by

$$\hat{\nabla}_X Y = {}^g\nabla_X Y + \eta(X)FY - \eta(Y){}^g\nabla_X \xi + \{({}^g\nabla_X \eta)Y\}\xi.$$

One can see that $\hat{\nabla}F = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla}\eta = 0$ and $\hat{\nabla}g = 0$ (Tanaka [46], Webster [55, 56]).

The complexified tangent bundle $T^{\mathbb{C}}M$ is decomposed as

$$T^{\mathbb{C}}M = \mathcal{S} \oplus \overline{\mathcal{S}} \oplus \mathcal{F}, \quad \mathcal{F} = \bigcup_{x\in M} \mathbb{C}\,\xi_x.$$

Hereafter in this subsection we assume that M is a *normal* strongly pseudo convex *CR* manifold. According to Tanaka [46], a strongly pseudo convex *CR* manifold $M = (M; F, \xi, \eta, g)$ is said to be *normal* if $[\xi, \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S})$ and $[X, JY] = J[X, Y]$ for all X, $Y \in \Gamma(\mathcal{D})$. It is known that a strongly pseudo convex *CR*-manifold M is normal if and only if it is a Sasakian manifold.

On a normal strongly pseudo convex *CR*-manifold, *i.e.*, Sasakian manifold, the Tanaka-Webster connection has the form

$$\hat{\nabla}_X Y = {}^g\nabla_X Y + \eta(X)FY + \eta(Y)FX + g(X, FY)\xi.$$

In particular the torsion tensor field $\hat{T}$ of $\hat{\nabla}$ satisfies $F(\hat{T}(X, Y)) = 0$ for all X, $Y \in \mathfrak{X}(M)$.

The Tanaka-Webster connection $\hat{\nabla}$ induces a holomorphic structure on the bundle $\mathcal{S}$. In fact, the Cauchy-Riemann operator $\bar{\partial}$ is defined by

$$\bar{\partial}_{\overline{W}} Z = \hat{\nabla}_{\overline{W}} Z, \quad Z, W \in \Gamma(\mathcal{S}).$$

Hence the pair $(\mathcal{S}, \bar{\partial})$ is a holomorphic vector bundle over M.

Now let $\varphi : M \to (N, h)$ be a map from a Sasakian manifold M into a Riemannian manifold. Then as in the case of complex manifolds, the $(0, 1)$-exterior derivative $D''\partial\varphi$ can be defined by

$$(D''_{\overline{W}}\partial\varphi)Z := {}^h\nabla^{\varphi}_{\overline{W}}(\partial\varphi(Z)) - \partial\varphi(\bar{\partial}_{\overline{W}} Z), \quad Z, W \in \Gamma(\mathcal{S}).$$

Here we used the notation $\partial\varphi := \mathrm{d}^+\varphi$. We call a map φ a *CR-pluriharmonic map* if it has vanishing $(0, 1)$-exterior derivative. We note that the equation $D''\partial\varphi = 0$ makes sense for maps into a manifold with a linear connection. This motivates us to give the following definition.

Definition 16. Let $(M, \mathcal{S})$ be a normal strongly pseudo convex *CR*-manifold and $\varphi : M \to (N, {}^N\nabla)$ a smooth map into a manifold with a linear connection. Then φ is said to be *CR-pluriharmonic* if $D''\partial\varphi = 0$.

Here we recall the notion of pseudo-second fundamental form introduced by Petit [40].

Let $(M; F, \xi, \eta, g)$ be a strongly pseudo convex *CR*-manifold with Tanaka-Webster connection $\hat{\nabla}$ and $(N, {}^N\nabla)$ a manifold with a linear connection. For a smooth map $\varphi : M \to N$, the *pseudo-second fundamental form* ${}^N\hat{\nabla}\mathrm{d}\varphi$ of φ with respect to ${}^N\nabla$ is defined by

$$({}^N\hat{\nabla}\mathrm{d}\varphi)(Y; X) = {}^N\nabla^{\varphi}_X \mathrm{d}\varphi(Y) - \mathrm{d}\varphi(\hat{\nabla}_X Y). \tag{9.5}$$

Here $\hat{\nabla}$ is the Tanaka-Webster connection of M. In case the target manifold N is a Riemannian manifold and choose ${}^N\nabla = {}^h\nabla$ as the Levi-Civita connection, then we denote the pseudo-second fundamental form by ${}^h\hat{\nabla}\mathrm{d}\varphi$. When we consider a map $\varphi : M \to N$ between strongly pseudo convex *CR*-manifolds $M = (M; F, \xi, \eta, g, \hat{\nabla})$ and $N = (N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h, \hat{\nabla})$ equipped with Tanaka-Webster connection, we denote the pseudo-second fundamental form of φ with respect to $\hat{\nabla}$ by $\hat{\nabla}\mathrm{d}\varphi$.

Here we rewrite the *CR*-pluriharmonicity equation in terms of the pseudo-second fundamental form as follows:

Proposition 26. *Let $(\mathcal{D}, J)$ denote the real expression of a normal strongly pseudo convex CR-manifold $(M, \mathcal{S})$. Then a smooth map $\varphi : (M, \mathcal{S}) \to (N, h)$ into a Riemannian manifold is CR-pluriharmonic if and only if*

$$({}^h\hat{\nabla}\mathrm{d}\varphi)(X; Y) + ({}^h\hat{\nabla}\mathrm{d}\varphi)(JX; JY) = 0, \quad X, Y \in \Gamma(\mathcal{D}). \tag{9.6}$$

Proof. Take $Z = X - \sqrt{-1}JX$, $W = Y - \sqrt{-1}Y \in \Gamma(\mathcal{D})$. Then the $(0,1)$-exterior derivative of $\partial\varphi$ is computed as

$$\begin{aligned}(D''_{\overline{W}}\partial\varphi)Z = &\left\{({}^h\hat{\nabla}\mathrm{d}\varphi)(X; Y) + ({}^h\hat{\nabla}\mathrm{d}\varphi)(JX; JY)\right\} \\ &- \sqrt{-1}\left\{({}^h\hat{\nabla}\mathrm{d}\varphi)(X; JY) - ({}^h\hat{\nabla}\mathrm{d}\varphi)(JX; Y)\right\}.\end{aligned}$$

This shows the required result. □

Note that the condition (9.6) is different from $\mathcal{S}$-$(1,1)$-harmonic condition. Because, we used the Tanaka-Webster connection $\hat{\nabla}$ instead of Levi-Civita connection ${}^g\nabla$.

Proposition 27. *A smooth map $\varphi : (M, \mathcal{S}) \to (N, h)$ of a normal strongly pseudo convex CR-manifold is CR-pluriharmonic if and only if its restriction to any holomorphic curve is harmonic.*

Proof. Let $\Sigma = (\Sigma, J_\Sigma)$ be a Riemann surface with complex structure J_Σ and $\iota : \Sigma \to M$ be a holomorphic immersion. Then we have

$${}^h\nabla\mathrm{d}\psi = ({}^h\nabla\mathrm{d}\varphi)(\mathrm{d}\iota; \mathrm{d}\iota) + \mathrm{d}\varphi(\hat{\nabla}\mathrm{d}\iota)$$

for the composite $\psi = \varphi \circ \iota$. Take a local complex coordinate z on Σ, the above composition law implies

$$({}^h\nabla''\partial\psi)(\partial_z;\partial_{\bar{z}}) = (D''\partial\varphi)(\iota_*\partial_z;\iota_*\partial_{\bar{z}}) + \mathrm{d}\varphi((\hat{\nabla}\mathrm{d}\iota)(\partial_z;\partial_{\bar{z}})).$$

Next, from the holomorphicity $\mathrm{d}\iota \circ J_\Sigma = F \circ \mathrm{d}\iota$ and the normality of M, one can check that $(\hat{\nabla}\mathrm{d}\iota)(\partial_z;\partial_{\bar{z}}) = 0$. Thus if φ is *CR*-pluriharmonic, then ψ satisfies ${}^h\nabla''\partial\psi = 0$, that is, ψ is harmonic. □

Unfortunately, strongly pseudo convex *CR*-manifolds do not admit nonconstant holomorphic curves. On the other hand, Bryant studied the space of holomorphic curves in *Lorentzian CR-manifolds* (nondegenerate pseudo-Hermitian manifolds whose Levi-form have Lorentzian signature) [8]. Note that $(\mathrm{U}\mathbb{H}^3, F_1)$ equipped with the Killing metric is a Lorentzian *CR*-manifold.

§10. Pseudo-harmonic maps

The notion of pseudo-harmonicity was introduced by Barletta, Dragomir and Urakawa.

Definition 17 ([1])**.** Let $\varphi : (M; F, \xi, \eta, g) \to (N, {}^N\nabla)$ be a map from a strongly pseudo convex *CR*-manifold into a manifold with a linear connection. The *pseudo-tension field* $\hat{\tau}(\varphi)$ of φ is defined by

$$\hat{\tau}(\varphi) = \mathrm{tr}_g\{\Pi_{\mathcal{D}}({}^N\hat{\nabla}\mathrm{d}\varphi)\}.$$

Here $\Pi_{\mathcal{D}}({}^N\hat{\nabla}\mathrm{d}\varphi)$ is the restriction of the pseudo-second fundamental form ${}^N\hat{\nabla}\mathrm{d}\varphi$ to $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D})$. A smooth map φ is said to be *pseudo-harmonic* if its pseudo-tension field vanishes.

Barletta, Dragomir and Urakawa [1] have given a variational characterization of pseudo-harmonicity.

Now we obtain the following result.

Proposition 28. *Let $(M; F, \xi, \eta, g)$ be a normal strongly pseudo convex CR-manifold and $\varphi : M \to (N, {}^N\nabla)$ a smooth map into a manifold with a linear connection. If φ is CR-pluriharmonic, then φ is pseudo-harmonic.*

Proof. Let us compute the pseudo-tension field of a map $\varphi : (M, \mathcal{S}) \to (N, {}^N\nabla)$. Take a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; Je_1, Je_2, \ldots, Je_n\}$ of $\mathcal{D}$. Then we have

$$\hat{\tau}(\varphi) = \sum_{i=1}^{m}\{({}^N\hat{\nabla}\mathrm{d}\varphi)(e_i;e_i) + ({}^N\hat{\nabla}\mathrm{d}\varphi)(Je_i;Je_i)\}.$$

This formula implies that if φ is *CR*-pluriharmonic then $\hat{\tau}(\varphi) = 0$. □

Now let us investigate *CR*-pluriharmonicity of holomorphic maps between normal strongly pseudo convex *CR*-manifolds equipped with Tanaka-Webster connections. For a holomorphic map $\varphi : (M; F, \xi, \eta, g, \hat{\nabla}) \to (N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h, \hat{\nabla})$ between normal strongly pseudo-convex *CR*-manifolds equipped with Tanaka-Webster connection and X, $Y \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned}
\hat{\nabla}_{\varphi_*(FX)}\varphi_*(FY) &= \hat{\nabla}_{\varphi_*(FX)}\tilde{F}(\varphi_*Y) = \tilde{F}\hat{\nabla}_{\varphi_*(FX)}(\varphi_*Y) \\
&= \tilde{F}\left(\hat{\nabla}_{\varphi_*Y}\varphi_*(FX) + [\varphi_*FX, \varphi_*Y] + \hat{T}(\varphi_*FX, \varphi_*Y)\right) \\
&= \tilde{F}\left(\hat{\nabla}_{\varphi_*Y}\tilde{F}(\varphi_*X) + [\varphi_*FX, \varphi_*Y]\right) \\
&= \tilde{F}^2\hat{\nabla}_{\varphi_*Y}(\varphi_*X) + \tilde{F}[\varphi_*FX, \varphi_*Y] \\
&= -\hat{\nabla}_{\varphi_*Y}(\varphi_*X) + \tilde{\eta}(\hat{\nabla}_{\varphi_*Y}(\varphi_*X))\tilde{\xi} + \tilde{F}[\varphi_*FX, \varphi_*Y] \\
&= -\hat{\nabla}_{\varphi_*Y}(\varphi_*X) + \tilde{F}[\varphi_*FX, \varphi_*Y],
\end{aligned}$$

since $\mathrm{d}\varphi(\Gamma(\mathcal{D})) \subset \Gamma(\tilde{\mathcal{D}})$. From this we get

$$\begin{aligned}
\hat{\nabla}_{\varphi_*(FX)}\varphi_*(FY) &= -\hat{\nabla}_{\varphi_*Y}(\varphi_*X) + \tilde{F}[\varphi_*FX, \varphi_*Y] \\
&= -\left(\hat{\nabla}_{\varphi_*X}(\varphi_*Y) + [\varphi_*Y, \varphi_*X] + \hat{T}(\varphi_*Y, \varphi_*X)\right) \\
&\quad + \tilde{F}[\varphi_*FX, \varphi_*Y].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hat{\nabla}_{FX}(FY) &= F\hat{\nabla}_{FX}Y \\
&= F\left(\hat{\nabla}_Y(FX) + [FX, Y] + \hat{T}(FX, Y)\right) \\
&= F^2(\hat{\nabla}_Y X) + F[FX, Y] \\
&= -\hat{\nabla}_Y X + \eta(\hat{\nabla}_Y X)\xi + F[FX, Y] \\
&= -\hat{\nabla}_Y X + F[FX, Y] \\
&= -\left(\hat{\nabla}_X Y + [Y, X] + \hat{T}(Y, X)\right) + F[FX, Y].
\end{aligned}$$

From these computations, one obtains

$$(\hat{\nabla}\mathrm{d}\varphi)(FY; FX) = -(\hat{\nabla}\mathrm{d}\varphi)(Y; X).$$

Namely φ is *CR*-pluriharmonic with respect to Tanaka-Webster connection.

Proposition 29. *Let $M = (M; F, \xi, \eta, g, \hat{\nabla})$ and $(N; \tilde{F}, \tilde{\xi}, \tilde{\eta}, h, \hat{\nabla})$ be normal strongly pseudo convex CR-manifolds equipped with Tanaka-Webster connection. Then every holomorphic map $\varphi : M \to N$ is CR-pluriharmonic with respect to Tanaka-Webster connection.*

Analogues to Kähler geometry, for smooth maps between Sasakian manifolds, *CR*-pluriharmonicity is stronger than pseudo-harmonicity and weaker than holomorphicity.

Remark 8. S. D. Jung and M. J. Jung [25] showed that any transversally holomorphic map between Kähler foliations is transversally harmonic with the minimum transversal f_k-energy in its foliated homotopy class. It would be interesting to introduce the notion of *transversally pluiharmonic maps* from Riemannian manifolds equipped with Kähler-foliation to foliated Riemannian manifolds.

References

[1] E. Barletta, S. Dragomir and H. Urakawa, Pseudoharmonic maps from nondegenerate CR manifolds to Riemannian manifolds, Indiana Univ. Math. J. **50** (2001), no. 2, 719–746.

[2] C. L. Bejan and M. Benyounes, f-pluriharmonic maps on manifolds with f-structures, SUT J. Math. **39** (2003), 161–170.

[3] M. Black, *Harmonic Maps into Homogeneous Spaces*, Pitman Research Notes in Mathematics Series, **255**, Longman Scientific & Technical, Harlow, 1991.

[4] D. E. Blair, The theory of quasi-Sasakian structures, J. Differ. Geom. **1** (1967), 331–345.

[5] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509** (1976), Springer-Verlag, Berlin.

[6] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. **203**, Birkhäuser, Boston, Basel, Berlin, 2002.

[7] D. E. Blair and S. Dragomir, Pseudohermitian geometry on contact Riemannian manifolds, Rend. Mat. (7) **22** (2002), 275–341.

[8] R. Bryant, Holomorphic curves in Lorentzian CR-manifolds, Trans. Amer. Math. Soc. **272** (1982), no. 1, 203–221.

[9] F. E. Burstall, *Non-linear Functional Analysis and Harmonic Maps*, Ph. D. Thesis, University of Warwick, 1987.

[10] J. T. Cho and J. Inoguchi, Contact metric hypersurfaces in complex space forms, in: *Proceedings of the Workshop on Differential Geometry of Submanifolds and its Related Topics Saga* (August 4-6, 2012), World Scientific, 2014, pp. 87–97.

[11] J. Dorfmeister and J.-H. Eschenburg, Pluriharmonic maps, loop groups and twistor theory, Ann. Global Anal. Geom. **24** (2003), 301–321.

[12] J. F. Dorfmeister, J. Inoguchi and S.-P. Kobayashi, Constant mean curvature surfaces in hyperbolic 3-space via loop groups, J. reine angew. Math. **686** (2014), no. 1, 1–36.

[13] J. F. Dorfmeister, J. Inoguchi and S.-P. Kobayashi, A loop group method for affine harmonic maps into Lie groups, preprint, 2014 (arXiv:1405.0333v1 [math.DG]).

[14] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, Regional Conference Series in Math. **50** (1983), Amer. Math. Soc.

[15] J. Eells and L. Lemaire, Another report on harmonic maps, Bull. London Math. Soc. **20** (1988), no. 5, 385–524.

[16] H. Endo, Invariant submanifolds in a contact Riemannian manifold, Tensor (N.S.) **42** (1985), no. 1, 86–89.

[17] J.-H. Eschenburg and R. Tribuzy, Associated families of pluriharmonic maps and isotropy, Manuscripta Math. **95** (1998), 295–310.

[18] C. Gherghe, S. Ianus and A. M. Pastore, CR-manifolds, harmonic maps and stability, J. Geom. **71** (2001), no. 1-2, 42–53.

[19] M. Harada, On Sasakian submanifolds, Tôhoku Math. J. **25** (1973), 103–109.

[20] S. Ianus, Sulle varietà di Cauchy-Riemann, Rend. Mat. Napoli **39** (1972), 191–195.

[21] S. Ianus and A. M. Pastore, Harmonic maps on contact metric manifolds, Ann. Math. Blaise Pascal **2** (1995), no. 2, 43–53.

[22] A. Iwamatsu, f-holomorphic maps and holomorphic maps into homogeneous spaces (in Japanese), Master Thesis, Ochanomizu University, 2000.

[23] D. Janssens and L. Vanheche, Almost contact structures and curvature tensors, Kōdai Math. J. **4** (1981), no. 1, 1–27.

[24] S. D. Jung, Variation formulas for transversally harmonic and biharmonic maps, J. Geom. Phys. **70** (2013) 9–20.

[25] S. D. Jung and M. J. Jung, Transversally holomorphic maps between Kähler foliations, J. Math. Anal. Appl. **416** (2014) 683–697.

[26] K. Kenmotsu, Invariant submanifolds in a Sasakian manifold, Tôhoku Math. J. (2) **21** (1969) 495–500.

[27] K. Kenmotsu, On Sasakian immersions, *Seminars on Contact Manifolds*, Publ. Study Group of Geometry, **4** (1970), 42–59.

[28] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tôhoku Math. J. **24** (1972), 93–103.

[29] C. R. LeBrun, Twistor CR manifolds and three-dimensional conformal geometry, Trans. Amer. Math. Soc. **284** (1984), 601–616.

[30] A. Lichnerowicz, Applications harmoniques et variétés kähleriennes, in: *Symposia Mathematica*, Vol. III (INDAM, Rome, 1968/69) pp. 341–402, Academic Press, London.

[31] K. Ogiue, On almost contact structures, Kōdai Math. Sem. Rep. **19** (1967), 489–506.

[32] K. Ogiue, G-structures defined by tensor fields, Kōdai Math. Sem. Rep. **20** (1968), 54–75.

[33] K. Ogiue and M. Okumura, On cocomplex structures, Kōdai Math. Sem. Rep. **19** (1967), 507–512.

[34] M. Okumura, On contact metric immersion. Kōdai Math. Sem. Rep. **20** (1968) 389–409.

[35] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[36] Z. Olszak, On contact metric manifolds, Tôhoku Math. J. **31** (1979), 247–253.

[37] Z. Olszak, Curvature properties of quasi-Sasakian manifolds, Tensor N. S. **38** (1982), 19–28.

[38] H. Omori, *Infinite dimensional Lie Transformation Groups*, Lecture Notes in Math. **427** (1974), Springer-Verlag, Berlin.

[39] H. Omori, *Infinite-dimensional Lie Groups*, Translations of Mathematical Monographs, **158**, American Mathematical Society, Providence, RI, 1997.

[40] R. Petit, Harmonic maps and strictly pseudoconvex CR manifolds, Comm. Anal. Geom. **10** (2002), 575–610.

[41] J. H. Rawnsley, f-structures, f-twistor spaces and harmonic maps, in: *Geometry Seminar "Luigi Bianchi"* II 1984, Lecture Notes in Math. **1164** (1985), pp. 85–159, Springer-Verlag, Berlin.

[42] S. M. Salamon, Harmonic and holomorphic maps, in: *Geometry Seminar "Luigi Bianchi"* II 1984, Lecture Notes in Math. **1164** (1985), pp. 161–224, Springer-Verlag, Berlin.

[43] T. Saotome, A removability sigularity theorem of J-holomorphic mappings for strongly pseudo-convex manifolds, Hokkaido Math. J. **40** (2011), 33–49.

[44] R. E. Stong, The rank of an f-structure, Kōdai Math. Sem. Rep. **29** (1977), 207–209.

[45] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. **10**(1973), 495–506.

[46] N. Tanaka, *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. **9**. Kinokuniya Book-Store Co., Ltd., Tokyo, 1975.

[47] S. Tanno, Some transformations on manifolds with almost contact and contact metric structures I, II, Tôhoku Math. J. **15** (1963), 140–147, 322–331.

[48] S. Tanno, The automorphism group of almost contact Riemannian manifolds, Tôhoku Math. J. **21** (1969), 21–38.

[49] S. Tanno, Isometric immersions of Sasakian manifolds in spheres, Kōdai Math. Sem. Rep. **21** (1969) 448–458.

[50] S. Tanno, Quasi-Sasakian structures of rank $2p+1$, J. Differential Geom. **5** (1971), 317–324.

[51] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc. **314** (1989), 349–379.

[52] S. Udagawa, Pluriharmonic maps and minimal immersions of Kähler manifolds, J. London Math. Soc. (2) **37** (1988), 375–384.

[53] H. Urakawa, *Calculus of Variation and Harmonic Maps*, Translations in Math., Amer. Math. Soc., 1993.

[54] H. Urakawa, Variational problems over strongly pseudoconvex *CR* manifolds, in: *Differential Geometry, Proc. Symp. in honour of Prof. Su Buchin on his 90th birthday*, Shanghai 1991, (Gu, Hu, Xin eds), World Scientific, 1993, pp. 233–242.

[55] S. M. Webster, On the pseudo-conformal geometry of Kähler manifold, Math. Z. **157** (1977), 265–270.

[56] S. M. Webster, Pseudo-Hermitian structure on a real hypersurface, J. Differ. Geom. **13** (1978), 25–41.

[57] C. M. Wood, The Gauss section of a Riemannian immersion, J. London Math. Soc. (2) **33** (1986), no. 1, 157–168.

[58] Y. L. Xin, Some results on stable harmonic maps, Duke Math. J. **47** (1980), 609–613.

[59] K. Yano, On a structure defined by a tensor field of type $(1,1)$ satisfying $f^3+f=0$, Tensor **14** (1963), 99–109.

[60] K. Yano and S. Ishihara, Invariant submanifolds of an almost contact manifold, Kōdai Math. Sem. Rep. **21** (1969), 350–364.

Jun-ichi Inoguchi
Department of Mathematical Sciences, Yamagata University,
Yamagata, 990-8560, Japan
E-mail: inoguchi@sci.kj.yamagata-u.ac.jp