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Twistor holomorphic affine surfaces and projective invariants

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Abstract. We study affine immersions with twistor lifts. Using a decomposition of a connection, we obtain several projective invariants for such affine immersions. In particular, affine immersions with holomorphic twistor lifts are considered. We can show the property that an affine immersion has holomorphic twistor lifts is invariant under projective transformations and characterize immersions with holomorphic twistor lifts by vanishing of some of projective invariants. In the case of compact affine surfaces with holomorphic twistor lifts, we see a quantization phenomenon for one of the projective invariants which we obtain. Moreover, we prove that a real analytic twistor holomorphic affine surface with the symmetric Ricci tensor with respect to both complex structures is totally geodesic or totally umbilic.

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§1. Introduction.

The twistor space is important and useful to study conformal geometry since we can translate certain conformal objects into complex ones. It is also important for the study of surfaces in even dimensional Riemannian manifolds (see [1], [2] and [3], for example). Some notions related to the twistor spaces can be considered in affine differential geometry replacing by projective objects instead of conformal ones. For example, the independence of the almost complex structure on the twistor space under conformal transformations can be replaced by the projective invariance. Then it is interesting to study affine immersions with holomorphic twistor lifts, which are invariant under projective transformations of the ambient manifolds. Using the decomposition of connections (see [5]), we obtain several projective invariants for affine immersions with twistor lifts. Consequently, an affine immersion with holomorphic twistor lift can be characterized by the vanishing of some of these invariants. In the case of compact affine surfaces with holomorphic twistor lifts, we see a quantization phenomenon for one of the projective invariants which we obtain (Theorem 4.3).

In Riemannian geometry, twistor holomorphic immersions with vanishing normal connection are totally umbilic, which are one of the simplest twistor holomorphic surfaces, and hence, the rank of the first normal space is 0 or 1. However, corresponding conditions do not imply this property in affine differential geometry. In fact, we can find an example (Example 4.4) of a twistor holomorphic affine surface with vanishing transversal connection whose rank of first normal space equals to 2. On the other hand, we can show that a real analytic twistor holomorphic affine surface with the symmetric Ricci tensor with respect to $\pm$-both complex structures is totally geodesic or totally umbilic surface whose rank of the first normal space is 1 (Corollary 4.12).

In Section 2, we recall fundamental facts for the decomposition of connections on complex vector bundles. We study the twistor space and define the twistor lift for affine immersions in Section 3. In Section 4, we consider twistor holomorphic surfaces.

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§2. Complex vector bundles.

Throughout this paper, all manifolds and maps are assumed to be smooth unless otherwise mentioned. Let E be a vector bundle over a manifold M and E_x the fiber of E over $x \in M$. We write TM (resp. T^*M) for the tangent (resp. cotangent) bundle of M. For vector bundles E, E' over M, we denote the homomorphism bundle whose fiber is the space of linear mappings E_x to E'_x by $\mathrm{Hom}(E, E')$, and set $\mathrm{End}(E) := \mathrm{Hom}(E, E)$. Let $\varphi : N \to M$ be a smooth map and F a fiber bundle over M. The pull back bundle of F over N by φ is denoted by $\varphi^{\#}F$. The set of all connections of a vector bundle E is denoted by $\mathcal{C}(E)$. The space of all sections of a fiber bundle F is denoted by $\Gamma(F)$. Let $\Lambda^k(E)$ be the set of all E-valued k-forms on M.

In this section, we summarize the fundamental results for the decomposition of connection on a complex vector bundle (see [5]) and prove several lemmas which we use in the later sections. Let M be an almost complex manifold with an almost complex structure J and E a vector bundle over M with $I \in \Gamma(\mathrm{End}(E))$ satisfying $I^2 = -id$ and $D \in \mathcal{C}(E)$. We do not assume that I

is parallel with respect to D in general. We define

$$D'_X\zeta := \frac{1}{2}(D_X\zeta - ID_{JX}\zeta), \;\; D''_X\zeta := \frac{1}{2}(D_X\zeta + ID_{JX}\zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. It is easy to see $D = D' + D''$.

The connection D on E induces the connection $\bar{D}$ on $\mathrm{End}(E)$, which is given as follows. For $S \in \Gamma(\mathrm{End}(E))$, $\bar{D}_X S \in \Gamma(\mathrm{End}(E))$ is defined by

$$(\bar{D}_X S)(\zeta) := [D_X, S](\zeta) = D_X S(\zeta) - S(D_X\zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Let $\nabla \in \mathcal{C}(TM)$ be a torsion free connection on M. Also we can define the connection (we use the same letter $\bar{D}$) on $T^*M \otimes \mathrm{End}(E)$, that is,

$$\begin{aligned}(\bar{D}_X T)_Y\zeta &:= D_X(T_Y\zeta) - T_{\nabla_X Y}\zeta - T_Y(D_X\zeta) = [D_X, T_Y](\zeta) - T_{\nabla_X Y}\zeta \\ &= (\bar{D}_X T_Y)(\zeta) - T_{\nabla_X Y}\zeta\end{aligned}$$

for $T \in \Gamma(T^*M \otimes \mathrm{End}(E))$. We define $D^I \in \mathcal{C}(E)$ by

$$D^I_X\zeta := D_X\zeta - \frac{1}{2}I(\bar{D}_X I)(\zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Then we have

$$D^{I\prime}_X\zeta = \frac{1}{2}(D'_X\zeta - ID'_X I\zeta), \;\; D^{I\prime\prime}_X\zeta = \frac{1}{2}(D''_X\zeta - ID''_X I\zeta)$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. Set

$$A^{D\prime}_X\zeta := \frac{1}{2}(D'_X\zeta + ID'_X I\zeta), \;\; A^{D\prime\prime}_X\zeta := \frac{1}{2}(D''_X\zeta + ID''_X I\zeta),$$

for $X \in TM$ and $\zeta \in \Gamma(E)$. We see that $D' = D^{I\prime} + A^{D\prime}$ and $D'' = D^{I\prime\prime} + A^{D\prime\prime}$, and hence $D = D^{I\prime} + D^{I\prime\prime} + A^{D\prime} + A^{D\prime\prime}$. It is easy to see that $A^{D\prime}$, $A^{D\prime\prime} \in \Lambda^1(\mathrm{End(E)})$. Note that I is parallel with respect to D^I. The operators $A^{D\prime}$ and $A^{D\prime\prime}$ are explicitly given as follows :

$$A^{D\prime}_X = \frac{1}{4}(I(\bar{D}_X I) + (\bar{D}_{JX} I)), \;\; A^{D\prime\prime}_X = \frac{1}{4}(I(\bar{D}_X I) - (\bar{D}_{JX} I))$$

for $X \in TM$. Let R^D be the curvature form of the connection D.

Lemma 2.1. *We have*

$$\mathrm{Tr}R^D_{X,Y}I \;=\; \mathrm{Tr}R^{D^I}_{X,Y}I + \frac{1}{2}\mathrm{Tr}(\bar{D}_X I)(\bar{D}_Y I)I \tag{2.1}$$

for X, $Y \in TM$.

Proof. By a straightforward calculation, we have

$$R^D_{X,Y} = R^{D^I}_{X,Y} + \frac{1}{4}(\bar{D}_X I)(\bar{D}_Y I) - \frac{1}{4}(\bar{D}_Y I)(\bar{D}_X I) + \frac{1}{2}I(R^{\bar{D}}_{X,Y} I)$$

for X, $Y \in TM$. Hence it holds

$$\mathrm{Tr} R^D_{X,Y} I = \mathrm{Tr} R^{D^I}_{X,Y} I + \frac{1}{2}\mathrm{Tr}(\bar{D}_X I)(\bar{D}_Y I)I.$$

□

Note that the first Chern form $c_1(E, D^I)$ is given by

$$4\pi c_1(E, D^I)(X, Y) = \mathrm{Tr} R^{D^I}_{X,Y} I \tag{2.2}$$

for all X, $Y \in TM$. Since

$$\begin{aligned} 16\mathrm{Tr} A^{D\prime}_X A^{D\prime}_X &= \mathrm{Tr}(\bar{D}_X I)(\bar{D}_X I) + 2\mathrm{Tr} I(\bar{D}_X I)(\bar{D}_{JX} I) \\ &\quad + \mathrm{Tr}(\bar{D}_{JX} I)(\bar{D}_{JX} I) \end{aligned} \tag{2.3}$$

$$\begin{aligned} 16\mathrm{Tr} A^{D\prime\prime}_X A^{D\prime\prime}_X &= \mathrm{Tr}(\bar{D}_X I)(\bar{D}_X I) - 2\mathrm{Tr} I(\bar{D}_X I)(\bar{D}_{JX} I) \\ &\quad + \mathrm{Tr}(\bar{D}_{JX} I)(\bar{D}_{JX} I), \end{aligned} \tag{2.4}$$

we have

$$\begin{aligned} 8(\mathrm{Tr} A^{D\prime}_X A^{D\prime}_X + \mathrm{Tr} A^{D\prime\prime}_X A^{D\prime\prime}_X) &= \mathrm{Tr}(\bar{D}_X I)(\bar{D}_X I) \\ &\quad + \mathrm{Tr}(\bar{D}_{JX} I)(\bar{D}_{JX} I) \end{aligned} \tag{2.5}$$

and

$$2(\mathrm{Tr} A^{D\prime}_X A^{D\prime}_X - \mathrm{Tr} A^{D\prime\prime}_X A^{D\prime\prime}_X) = \mathrm{Tr} R^D_{X,JX} I - \mathrm{Tr} R^{D^I}_{X,JX} I \tag{2.6}$$

for $X \in \Gamma(TM)$.

Next assume that a vector bundle E is decomposed into $E = E_1 \oplus E_2$ and a complex structure I preserves the decomposition. Then the complex structure I_i on E_i ($i \in \{1, 2\}$) can be naturally defined. We can also define the connection D^i on each E_i by $D^i_X \xi_i = \pi_i D_X \iota_i \xi$ for $X \in TM$ and $\xi \in \Gamma(E_i)$, where $\pi_i : E \to E_i$ is the projection and $\iota_i : E_i \to E$ is the inclusion ($i \in \{1, 2\}$). Moreover, define $B^j_X \xi_i := \pi_j(D_X \iota_i \xi_i)$ for $X \in TM$ and $\xi_i \in \Gamma(E_i)$ ($i, j \in \{1, 2\}$ and $i \neq j$). Then we have

$$D_X \xi = \iota_1 D^1_X \pi_1(\xi) + \iota_2 B^2_X \pi_1(\xi) + \iota_2 D^2_X \pi_2(\xi) + \iota_1 B^1_X \pi_2(\xi)$$

for $X \in TM$ and $\xi \in \Gamma(E)$. By a straightforward calculation, we have

Lemma 2.2. *We have*

$$(\bar{D}_X I)(\iota_i\xi_i) = \iota_i(\bar{D}^i_X I_i)(\xi_i) + \iota_j(B^j_X I_i\xi_i - I_j B^j_X\xi_i)$$

for $X \in TM$ *and* $\xi_i \in \Gamma(E_i)$ $(i, j \in \{1, 2\}$ *and* $i \neq j)$.

Using this lemma, we obtain

Lemma 2.3. *We have*

$$D^I_X\iota_i\xi_i = \iota_i(D^{i\,I_i}_X\xi_i) - \frac{1}{2}\iota_j(B^j_X\xi_i - I_j B^j_X I_i\xi_i)$$

for $X \in TM$ *and* $\xi_i \in \Gamma(E_i)$ $(i, j \in \{1, 2\}$ *and* $i \neq j)$.

Hence we see that $D^I \neq \iota_1 D^{1\,I_1}\pi_1 + \iota_2 D^{2\,I_2}\pi_2$ in general. But, by Lemmas 2.1 and 2.3, we have the following lemma.

Lemma 2.4. *For all* X, $Y \in TM$, *we have*

$$\mathrm{Tr}R^{D^I}_{X,Y}I = \mathrm{Tr}R^{D^{1I_1}}_{X,Y}I_1 + \mathrm{Tr}R^{D^{2I_2}}_{X,Y}I_2.$$

Proof. By a straightforward calculation, we have

$$\pi_i R^D_{X,Y}\iota_i\xi_i = R^{D^i}_{X,Y}\xi_i + B^i_X B^j_Y\xi_i - B^i_Y B^j_X\xi_i$$

for X, $Y \in TM$ and $\xi_i \in E_i$ $(i, j \in \{1, 2\}$ and $i \neq j)$. From Lemma 2.3, it holds that

$$\begin{aligned}
& \pi_i(\bar{D}_X I)(\bar{D}_Y I)(\iota_i\xi_i) \\
= \; & \pi_i(\bar{D}_X I)(\iota_i(\bar{D^i}_Y I_i)(\xi_i) + \iota_j B^j_Y I_i\xi_i - \iota_j I_j B^j_Y\xi_i) \\
= \; & (\bar{D^i}_X I_i)(\bar{D^i}_Y I_i)(\xi_i) + B^i_X I_j(B^j_Y I_i\xi_i - I_j B^j_Y\xi_i) - I_i B^i_X(B^j_Y I_i\xi_i - I_j B^j_Y\xi_i) \\
= \; & (\bar{D^i}_X I_i)(\bar{D^i}_Y I_i)(\xi_i) + B^i_X I_j B^j_Y I_i\xi_i + B^i_X B^j_Y\xi_i \\
& - I_i B^i_X B^j_Y I_i\xi_i + I_i B^i_X I_j B^i_Y\xi_i
\end{aligned}$$

for X, $Y \in TM$ and $\xi_i \in E_i$ $(i, j \in \{1, 2\}$ and $i \neq j)$. Therefore, we obtain

$$\begin{aligned}
\mathrm{Tr}R^D_{X,Y}I &= \mathrm{Tr}\pi_1 R^D_{X,Y}\iota_1 I_1\pi_1 + \mathrm{Tr}\pi_2 R^D_{X,Y}\iota_2 I_2\pi_2 \\
&= \mathrm{Tr}R^{D^1}_{X,Y}I_1 + \mathrm{Tr}B^1_X B^2_Y I_1 - \mathrm{Tr}B^1_Y B^2_X I_1 \\
&\quad + \mathrm{Tr}R^{D^2}_{X,Y}I_2 + \mathrm{Tr}B^2_X B^1_Y I_2 - \mathrm{Tr}B^2_Y B^1_X I_2
\end{aligned}$$

and

$$\begin{aligned}
& \mathrm{Tr}(\bar{D}_X I)(\bar{D}_Y I)I \\
= \; & \mathrm{Tr}(\bar{D^1}_X I_1)(\bar{D^1}_Y I_1)I_1 + 2\mathrm{Tr}B^1_X B^2_Y I_1 - 2\mathrm{Tr}B^1_Y B^2_X I_1 \\
& + \mathrm{Tr}(\bar{D^2}_X I_2)(\bar{D^2}_Y I_2)I_2 + 2\mathrm{Tr}B^2_X B^1_Y I_2 - 2\mathrm{Tr}B^2_Y B^1_X I_2,
\end{aligned}$$

which mean the conclusion from Lemma 2.1. □

§3. Twistor spaces, twistor lifts and projective invariants.

Let V be a real vector space of dimension $2n$. A complex structure on V is an endomorphism $J : V \to V$ such that $J^2 = -id$. We denote the set of all complex structures on V by $W(V)$. Let $\mathrm{GL}(V)$ be the general linear group acting on V. Choose $J_0 \in W(V)$ and set $\mathrm{GL}(V, J_0) := \{A \in \mathrm{GL}(V) \mid AJ_0 = J_0 A\}$. It is clear that $\mathrm{GL}(V, J_0)$ is the general linear group of V as a complex vector space with respect to J_0. We have

$$W(V) \cong \mathrm{GL}(V)/\mathrm{GL}(V, J_0).$$

The tangent space to $W(V)$ at J can be identified with the vector subspace $\{a \in \mathrm{gl}(V) \mid aJ = -Ja\}$, where $\mathrm{gl}(V)$ is the Lie algebra of $\mathrm{GL}(V)$. We define an almost complex structure $\mathcal{J}$ on $W(V)$ by

$$\mathcal{J}(a) = \frac{1}{2}[J, a]$$

for $a \in T_J W(V)$. Note that we have $\mathcal{J}(a) = (1/2)[J, a] = (1/2)(Ja - aJ) = Ja = -aJ$ for all $a \in T_J W(V)$.

Let $(\tilde{M}, \tilde{\nabla})$ be a $2n$-dimensional manifold $\tilde{M}$ with a torsion free affine connection $\tilde{\nabla}$. We define the twistor space $\mathcal{Z}(\tilde{M})$ of $\tilde{M}$ by

$$\mathcal{Z}(\tilde{M}) := \bigcup_{x \in \tilde{M}} W(T_x \tilde{M}).$$

The bundle projection $p : \mathcal{Z}(\tilde{M}) \to \tilde{M}$ and the connection $\tilde{\nabla}$ induce the horizontal subbundle of $T\mathcal{Z}(\tilde{M})$. The almost complex structure $J^{\mathcal{Z}}_{\tilde{\nabla}}$ on the twistor space is defined by $(J^{\mathcal{Z}}_{\tilde{\nabla}})_J(X) = (J(p_*(X)))^h_J$ for all horizontal vectors X at $J \in \mathcal{Z}(\tilde{M})$ and $(J^{\mathcal{Z}}_{\tilde{\nabla}})_J(Y) = \mathcal{J}(Y)$ for all vertical vectors Y at $J \in \mathcal{Z}(\tilde{M})$, where $(\,\cdot\,)^h$ stands for the horizontal lift and $\mathcal{J}$ is the almost complex structure defined above.

To prove the invariance of this almost complex structure under the projective changes of connections, we recall the fundamental facts for the tangent bundle TE of a vector bundle $p : E \to P$ with a connection D of E. Let $K^D : TE \to E$ be the connection map with respect to D. The horizontal lift X^h of $X \in TP$ can be characterized by the equations $K^D(X^h) = 0$ and $p_*(X^h) = X$. On the other hand, the vertical lift ξ^v of $\xi \in E$ can be characterized by the equations $K^D(\xi^v) = \xi$ and $p_*(\xi^v) = 0$. Let $(x^1, \ldots, x^m)$ be a local coordinate system on $U \subset P$, where $m = \dim P$. Take $(e_1, \ldots, e_r)$ a local frame of E and denote its dual frame $(\omega^1, \ldots, \omega^r)$, where $r = \mathrm{rank} E$. Setting $p_i = x^i \circ p$ $(i = 1, \ldots, m)$, we can consider a local coordinate system $(p_1, \ldots, p_m, \omega^1, \ldots, \omega^r)$. In terms of this local coordinate system, for

$X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i} \in TP$ and $\xi \in E$, the both lifts can be described by

$$(X^h)_u = \sum_i X^i(p(u)) \frac{\partial}{\partial p_i}\bigg|_u - \sum_{i,k,l} X^i(p(u))\omega^l(u)\Gamma^k_{il} \frac{\partial}{\partial \omega^k}\bigg|_u \tag{3.1}$$

and

$$(\xi^v)_u = \sum_k \omega^k(\xi) \frac{\partial}{\partial \omega^k}\bigg|_u \tag{3.2}$$

at $u \in E$, where

$$\Gamma^k_{il} = \omega^k(D_{\frac{\partial}{\partial x^i}} e_l).$$

Hence we can calculate

$$\begin{aligned} (X^h)_u &= \sum_i X^i(p(u)) \frac{\partial}{\partial p_i}\bigg|_u - \sum_{k,l} \omega^l(u)\omega^k(D_{X_{p(u)}} e_l) \frac{\partial}{\partial \omega^k}\bigg|_u \\ &= \sum_i X^i(p(u)) \frac{\partial}{\partial p_i}\bigg|_u - \sum_l \omega^l(u)(D_{X_{p(u)}} e_l)^v \end{aligned} \tag{3.3}$$

for $X \in TP$.

Here we give the following operators. Let E and E' be complex vector bundles with complex structures $I \in \Gamma(\mathrm{End}(E))$ and $I' \in \Gamma(\mathrm{End}(E'))$ respectively. For $T \in \Lambda^1(\mathrm{Hom}(E, E'))$, we set

$$\begin{aligned} T_X^{(2,0)}\xi &:= \frac{1}{4}(T_X\xi - I'T_{JX}\xi - I'T_X I\xi - T_{JX}I\xi), \\ T_X^{(1,1)+}\xi &:= \frac{1}{4}(T_X\xi + I'T_{JX}\xi - I'T_X I\xi + T_{JX}I\xi), \\ T_X^{(1,1)-}\xi &:= \frac{1}{4}(T_X\xi - I'T_{JX}\xi + I'T_X I\xi + T_{JX}I\xi), \\ T_X^{(0,2)}\xi &:= \frac{1}{4}(T_X\xi + I'T_{JX}\xi + I'T_X I\xi - T_{JX}I\xi). \end{aligned}$$

for all $X \in TM$ and $\xi \in E$.

Lemma 3.1. *For two connections $\tilde{\nabla}^1$, $\tilde{\nabla}^2 \in \mathcal{C}(T\tilde{M})$, set $T := \tilde{\nabla}^1 - \tilde{\nabla}^2$. If $T^{(0,2)} = 0$ for all $J \in \mathcal{Z}(\tilde{M})$, then we have $J^{\mathcal{Z}}_{\tilde{\nabla}^1} = J^{\mathcal{Z}}_{\tilde{\nabla}^2}$.*

Proof. Denote the horizontal subbundle and vertical subbundle of $T\mathcal{Z}(\tilde{M})$ by $\mathcal{H}^1$ (resp. $\mathcal{H}^2$) and $\mathcal{V}$ with respect to $\tilde{\nabla}^1$ (resp. $\tilde{\nabla}^2$). So we have $T_J\mathcal{Z}(\tilde{M}) = \mathcal{H}^1_J \oplus \mathcal{V}_J = \mathcal{H}^2_J \oplus \mathcal{V}_J$ at each point $J \in \mathcal{Z}(\tilde{M})$. Let h_i and v_i be the projections onto $\mathcal{H}^i$ and $\mathcal{V}$ with respect to the decomposition $\mathcal{H}^i \oplus \mathcal{V}$ $(i = 1, 2)$. It is easy to see $h_i + v_i = \mathrm{id}$ and $v_i v_j = v_j$ for any $i, j \in \{1, 2\}$ and $i \neq j$. Using these equations, we have $v_i + v_j h_i = v_j$ for any $i, j \in \{1, 2\}$ and $i \neq j$. From the definition of $J^{\mathcal{Z}}_{\tilde{\nabla}^i}$, we have $J^{\mathcal{Z}}_{\tilde{\nabla}^i}(X) = J(v_iX) + (Jp_*(X))^{h_i}$ for $X \in T_J\mathcal{Z}$. It holds that the $\mathcal{H}^1$-horizontal components of $J^{\mathcal{Z}}_{\tilde{\nabla}^i}(X)$ $(i = 1, 2)$ coincide. In fact, we have $h_1(J^{\mathcal{Z}}_{\tilde{\nabla}^1}(X)) = (Jp_*(X))^{h_1}$ and $h_1(J^{\mathcal{Z}}_{\tilde{\nabla}^2}(X)) = h_1((Jp_*(X))^{h_2}) = (Jp_*(X))^{h_1}$ for all tangent vectors X on $\mathcal{Z}(\tilde{M})$. So it is need to show $v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^1}(X)) = v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^2}(X))$. To prove this equation, it is sufficient to see $J(v_1Y^{h_2}) = v_1(JY)^{h_2}$ at $J \in \mathcal{Z}(\tilde{M})$ for all $Y \in TM$. In fact, we can obtain

$$v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^1}(X)) = J(v_1X)$$

and

$$\begin{aligned} v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^2}(X)) &= v_1(Jv_2X) + v_1(J(p_*X))^{h_2} \\ &= J(v_1 + v_2h_1)X + v_1(J(p_*X))^{h_2} \\ &= J(v_1X) - J(v_1h_2X) + v_1(J(p_*X))^{h_2}. \end{aligned}$$

Then $v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^1}(X)) = v_1(J^{\mathcal{Z}}_{\tilde{\nabla}^2}(X))$ if and only if $J(v_1h_2X) = v_1(J(p_*X))^{h_2}$. It is easy to see $(\tilde{\nabla}^1)\bar{}_X\phi - (\tilde{\nabla}^2)\bar{}_X\phi = [T_X, \phi]$ for $\phi \in \Gamma(\mathrm{End}(T\tilde{M}))$ and $X \in TM$. By the definitions of the horizontal and the vertical lifts and (3.3), it holds that

$$J(v_1Y^{h_2}) = -J(Y^{h_1} - Y^{h_2}) = (J[T_Y, J])^v,$$

and similarly $v_1(JY)^{h_2} = ([T_{JY}, J])^v$ at $J \in \mathcal{Z}(\tilde{M})$. Since $T^{(0,2)} = 0$ for all $J \in \mathcal{Z}(\tilde{M})$, we have $J^{\mathcal{Z}}_{\tilde{\nabla}^1} = J^{\mathcal{Z}}_{\tilde{\nabla}^2}$. □

Corollary 3.2. *If connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ are projectively equivalent, then $J^{\mathcal{Z}}_{\tilde{\nabla}^1} = J^{\mathcal{Z}}_{\tilde{\nabla}^2}$.*

Proof. If $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ are projectively equivalent, there exist a 1-form σ such that the difference tensor T satisfies $T_XY = \sigma(X)Y + \sigma(Y)X$ for $X, Y \in TM$. It is easy to see $T^{(0,2)} = 0$ for all $J \in \mathcal{Z}(\tilde{M})$. □

Then we are allowed to write $J^{\mathcal{Z}}$ for $J^{\mathcal{Z}}_{\tilde{\nabla}}$ with no confusions in projective geometry.

We define the twistor lift for an almost complex submanifold in an even-dimensional manifold in affine differential geometry, which is similar to Riemannian one. We need to recall the definition of affine immersions with transversal bundle. Let (M, ∇) and $(\tilde{M}, \tilde{\nabla})$ be smooth manifolds with torsion free affine connections and $f : M \to \tilde{M}$ an immersion. In this paper, we omit the symbol of the differential map f_* for an immersion f. An immersion $f : M \to \tilde{M}$ is called an immersion with a transversal bundle N if $f^{\#}T\tilde{M} = TM \oplus N$ holds. Let π_{TM} and π_N be the projections from $f^{\#}T\tilde{M}$

onto TM and N, respectively. We say that $f : M \to \tilde{M}$ is an affine immersion with a transversal bundle N if f is an immersion with a transversal bundle N and $\pi_{TM}((f^{\#}\tilde{\nabla})_X Y) = \nabla_X Y$ for all X, $Y \in \Gamma(TM)$, where $f^{\#}\tilde{\nabla}$ is the pull back connection of $\tilde{\nabla}$ by f. Set $\alpha(X,Y) := \pi_N((f^{\#}\tilde{\nabla})_X Y)$, $S_\xi X := -\pi_{TM}((f^{\#}\tilde{\nabla})_X \xi)$ and $\nabla^N_X \xi := \pi_N((f^{\#}\tilde{\nabla})_X \xi)$ for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$. Then α, S and ∇^N are called the affine fundamental form, the affine shape operator and the transversal connection, respectively. We see that

$$(f^{\#}\tilde{\nabla})_X Y = \nabla_X Y + \alpha(X,Y) \quad \text{and} \quad (f^{\#}\tilde{\nabla})_X \xi = -S_\xi X + \nabla^N_X \xi$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$. We refer to [6] and [7] for affine immersions. Let I be an almost complex structure on M. Assume that $\tilde{M}$ is a $2n$-dimensional manifold. Consider an affine immersion $f : (M,\nabla) \to (\tilde{M},\tilde{\nabla})$ with a transversal bundle N. We assume that there exists a complex structure I^N on N. Note that we do not assume $\bar{\nabla} I = 0$, $\bar{\nabla}^N I^N = 0$ in general. We define $\tilde{I}$ by $\tilde{I}(X) = I(X)$ and $\tilde{I}(\xi) = I^N(\xi)$ for $X \in TM$ and $\xi \in N$. The section $\tilde{I} \in \Gamma(f^{\#}(\mathcal{Z}(\tilde{M})))$ is called a *twistor lift* of f (or M). Hereafter we often omit the symbol $f^{\#}$ for the induced objects for f if there is no confusion.

We study influences of projective changes for geometric objects of affine immersions. On $\tilde{M}$, we take two torsion free affine connections $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ which are projectively equivalent, that is, there exist a 1-form σ on $\tilde{M}$ satisfying

$$\tilde{\nabla}^2_X Y = \tilde{\nabla}^1_X Y + \sigma(X)Y + \sigma(Y)X$$

for X, $Y \in \Gamma(T\tilde{M})$. By a straightforward calculation, we have

$$\begin{aligned}
\tilde{\nabla}^{2\tilde{I}\prime}_X Y &= \tilde{\nabla}^{1\tilde{I}\prime}_X Y + \frac{1}{2}(\sigma(X)Y + \sigma(Y)X \\
&\quad -\sigma(\tilde{I}X)\tilde{I}Y - \sigma(\tilde{I}Y)\tilde{I}X), && (3.4)\\
\tilde{\nabla}^{2\tilde{I}\prime\prime}_X Y &= \tilde{\nabla}^{1\tilde{I}\prime\prime}_X Y + \frac{1}{2}(\sigma(X)Y + \sigma(\tilde{I}X)\tilde{I}Y), && (3.5)\\
A^{\tilde{\nabla}^2\prime}_X Y &= A^{\tilde{\nabla}^1\prime}_X Y + \frac{1}{2}(\sigma(Y)X + \sigma(\tilde{I}Y)\tilde{I}X), && (3.6)\\
A^{\tilde{\nabla}^2\prime\prime}_X Y &= A^{\tilde{\nabla}^1\prime\prime}_X Y && (3.7)
\end{aligned}$$

for X, $Y \in \Gamma(T\tilde{M})$. Since the affine fundamental form α (resp. the affine shape operator S) can be viewed as an element of $\Lambda^1(\mathrm{Hom}(TM,N))$ (resp. $\Lambda^1(\mathrm{Hom}(N,TM))$), we have the following lemma.

Lemma 3.3. *Let $f : M \to \tilde{M}$ be an affine immersion with a transversal* *bundle N and a twistor lift $\tilde{I}$. We have*

$$\begin{aligned}
\tilde{\nabla}^{\tilde{I}\prime}_X Y &= \nabla^{I\prime}_X Y + \alpha^{(2,0)}(X,Y), \\
\tilde{\nabla}^{\tilde{I}\prime\prime}_X Y &= \nabla^{I\prime\prime}_X Y + \alpha^{(1,1)+}(X,Y), \\
A^{\tilde{\nabla}\prime}_X Y &= A^{\nabla\prime}_X Y + \alpha^{(1,1)-}(X,Y), \\
A^{\tilde{\nabla}\prime\prime}_X Y &= A^{\nabla\prime\prime}_X Y + \alpha^{(0,2)}(X,Y)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\nabla}^{\tilde{I}\prime}_X \xi &= -S^{(2,0)}_\xi X + (\nabla^N)^{I^N\prime}_X \xi, \\
\tilde{\nabla}^{\tilde{I}\prime\prime}_X \xi &= -S^{(1,1)+}_\xi X + (\nabla^N)^{I^N\prime\prime}_X \xi, \\
A^{\tilde{\nabla}\prime}_X \xi &= -S^{(1,1)-}_\xi X + A^{\nabla^N\prime}_X \xi, \\
A^{\tilde{\nabla}\prime\prime}_X \xi &= -S^{(0,2)}_\xi X + A^{\nabla^N\prime\prime}_X \xi
\end{aligned}$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(N)$.

By the equations (3.4)-(3.7) and Lemma 3.3, we have the following proposition.

Proposition 3.4. *Let $f : M \to \tilde{M}$ be an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. The following objects are invariant under projective change of $\tilde{\nabla}$: α, $A^{\nabla\prime\prime}$, $\bar{\nabla}^N I^N$, $S^{(1,1)+}$ and $S^{(0,2)}$.*

From Lemma 2.1 and Proposition 3.4, we have the following corollary.

Corollary 3.5. *For X, $Y \in TM$, $\mathrm{Tr} R^{\nabla^N}_{X,Y} I^N$ is invariant under projective change of $\tilde{\nabla}$. In particular, $\mathrm{Tr} R^{\nabla^{N I^N}} I^N (= 4\pi c_1(N, \nabla^{N I^N}))$ is given by projective invariants.*

Proof. If $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ are projective equivalent connections described by σ, then we have ${}^2\nabla^N_X \xi = {}^1\nabla^N_X \xi + \sigma(X)\xi$ for all $\xi \in \Gamma(N)$, where ${}^i\nabla^N$ is the connection on N induced by $\tilde{\nabla}^i$ $(i = 1, 2)$. Then it holds that $\mathrm{Tr} R^{{}^1\nabla^N}_{X,Y} I^N = \mathrm{Tr} R^{{}^2\nabla^N}_{X,Y} I^N$. By Lemma 2.1 and Proposition 3.4, $\mathrm{Tr} R^{\nabla^{N I^N}} I^N$ is given by projective invariants. □

The Ricci tensor of a connection $\tilde{\nabla}$ is denoted by $Ric^{\tilde{\nabla}}$, which is not necessary symmetric. For a $(0,2)$-tensor ρ on M, set $\rho^s(X,Y) := (1/2)(\rho(X,Y) + \rho(Y,X))$ for X, $Y \in TM$. We define a symmetric $(0,2)$-tensor Φ on M by

$$\begin{aligned}
\Phi(X,Y) := &-\mathrm{Tr}((f^\#\tilde{\nabla})^{-}_X \tilde{I})(f^\#\tilde{\nabla})^{-}_Y \tilde{I}) - (f^\#\tilde{\nabla})^{-}_{IX} \tilde{I})(f^\#\tilde{\nabla})^{-}_{IY} \tilde{I})) \\
&+ \frac{4}{2n-1}((f^* Ric^{\tilde{\nabla}})^s(X,Y) + (f^* Ric^{\tilde{\nabla}}))^s(IX,IY)) \\
&-2\mathrm{Tr} R^{\nabla^I}_{X,IY} I + 2\mathrm{Tr} R^{\nabla^I}_{IX,Y} I
\end{aligned}$$

for X, $Y \in TM$. It is easy to see that $\Phi(X,Y) = \Phi(IX,IY)$ for all X and $Y \in TM$.

Lemma 3.6. *Let $f : M \to \tilde{M}$ be an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. Then we have*

$$16\mathrm{Tr}A_X^{f^\#\tilde{\nabla}''}A_X^{f^\#\tilde{\nabla}''} - 4\mathrm{Tr}R_{X,IX}^{\nabla^{NI^N}} I^N + 4\mathrm{Tr}W_{X,IX}^{\tilde{\nabla}}\tilde{I} = -\Phi(X,X)$$

for all $X \in TM$, where $W^{\tilde{\nabla}}$ is the projective curvature tensor of $\tilde{\nabla}$.

Proof. By (2.4) and Lemma 2.4, we have

$$\begin{aligned} & 16\mathrm{Tr}A_X^{f^\#\tilde{\nabla}''}A_X^{f^\#\tilde{\nabla}''} \\ = \; & \mathrm{Tr}((f^\#\tilde{\nabla})^-_X\tilde{I})((f^\#\tilde{\nabla})^-_X\tilde{I}) + \mathrm{Tr}((f^\#\tilde{\nabla})^-_{IX}\tilde{I})((f^\#\tilde{\nabla})^-_{IX}\tilde{I}) \\ & -4\mathrm{Tr}R_{X,IX}^{\tilde{\nabla}}\tilde{I} + 4\mathrm{Tr}R_{X,IX}^{\nabla^{NI^N}} I^N + 4\mathrm{Tr}R_{X,IX}^{\nabla^I} I \end{aligned}$$

for all $X \in TM$. The projective curvature tensor $W^{\tilde{\nabla}}$ of $\tilde{\nabla}$ is given by

$$W_{X,Y}^{\tilde{\nabla}}Z = R_{X,Y}^{\tilde{\nabla}}Z - (P(X,Y) - P(Y,X))Z - (P(Y,Z)X - P(X,Z)Y)$$

for X, Y, $Z \in T\tilde{M}$, where

$$P(X,Y) = \frac{1}{(2n)^2-1}\left(2nRic^{\tilde{\nabla}}(X,Y) + Ric^{\tilde{\nabla}}(Y,X)\right).$$

It holds that

$$\mathrm{Tr}R_{X,IX}^{\tilde{\nabla}}\tilde{I} \;=\; \mathrm{Tr}W_{X,IX}^{\tilde{\nabla}}\tilde{I} + \frac{1}{2n-1}\left((f^*Ric^{\tilde{\nabla}})(X,X) + (f^*Ric^{\tilde{\nabla}})(IX,IX)\right)$$

for all $X \in TM$. Then the conclusion can be obtained. □

From Corollary 3.5, Lemma 3.6 and the polarization for Φ, the following proposition can be obtained.

Proposition 3.7. *Let $f : M \to \tilde{M}$ be an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. Then Φ is invariant under projective change of $\tilde{\nabla}$.*

On a complex manifold M of $\dim_{\mathbf{R}} M = 2$, we can choose a volume form Ω on M, which satisfies $\Omega(X,IX) \neq 0$ for all nonzero $X \in TM$. For a symmetric $(0,2)$-tensor s on M, we define

$$a_\Omega(s) = \frac{s(X,X) + s(IX,IX)}{\Omega(X,IX)}$$

for a nonzero $X \in TM$. It is easy to see that $a_\Omega(s)$ is independent of the choice of X. Note that $a_\Omega(s)\Omega = a_{\Omega'}(s)\Omega'$ if $\Omega' = c\Omega$ for $c \neq 0$.

Theorem 3.8. *Let $f : M \to \tilde{M}$ an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. If M is a compact surface, then*

$$\int_M a_\Omega \left(-\mathrm{Tr}((f^\# \tilde{\nabla})^- \tilde{I})((f^\# \tilde{\nabla})^- \tilde{I}) + \frac{4}{2n-1} (f^* Ric^{\tilde{\nabla}})^s \right) \Omega \tag{3.8}$$

is a projective invariant.

Proof. We have

$$\begin{aligned} a_\Omega(\frac{1}{2}\Phi)\Omega &= a_\Omega \left(-\mathrm{Tr}((f^\# \tilde{\nabla})^- \tilde{I})((f^\# \tilde{\nabla})^- \tilde{I}) + \frac{4}{2n-1} (f^* Ric^{\tilde{\nabla}})^s \right) \Omega \\ &\quad -16\pi c_1(TM, \nabla^I) \end{aligned}$$

Since the left hand side of the above equation is a projective invariant and the Chern class is a topological invariant of M, we obtain the conclusion. □

Note that $-a_\Omega(\mathrm{Tr}((f^\# \tilde{\nabla})^- \tilde{I})((f^\# \tilde{\nabla})^- \tilde{I}))$ is a corresponding object to the (vertical) energy density of $\tilde{I}$ in the Riemannian case up to a constant.

§4. Twistor holomorphic affine surfaces.

In this section, we consider twistor holomorphic surfaces. Let $f : (M, \nabla, I) \to (\tilde{M}, \tilde{\nabla})$ be an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. If the twistor lift satisfies $\tilde{I}_* \circ I = J^{\mathcal{Z}} \circ \tilde{I}_*$, then f (or M) is called a *twistor holomorphic* immersion (or submanifold). By the definition of $J^{\mathcal{Z}}$, f is twistor holomorphic if and only if it holds that

$$(f^\# \tilde{\nabla})^-_{IX} \tilde{I} = \tilde{I}(f^\# \tilde{\nabla})^-_X \tilde{I} \tag{4.1}$$

for all $X \in TM$, which is equivalent to $A^{f^\# \tilde{\nabla}''} = 0$. In fact, taking the vertical components of $(\tilde{I}_* \circ I)(X) = (J^{\mathcal{Z}} \circ \tilde{I}_*)(X)$ for $X \in TM$, we have (4.1). It is easy to obtain the following equations

$$((f^\# \tilde{\nabla})^-_X \tilde{I})(Y) = (\bar{\nabla}_X I)(Y) + \alpha(X, IY) - I^N \alpha(X, Y) \tag{4.2}$$

and

$$((f^\# \tilde{\nabla})^-_X \tilde{I})(\xi) = (\bar{\nabla}^N_X I)(\xi) - S_{I^N \xi} X + I S_\xi X \tag{4.3}$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$. Using (4.1)-(4.3), we have the following proposition.

Proposition 4.1. *Let $f : M \to \tilde{M}$ be an affine immersion with a transversal bundle N and a twistor lift $\tilde{I}$. Then f is twistor holomorphic if and only if the following conditions hold :* (1) $A^{\nabla''} = 0$, (2) $A^{\nabla^N ''} = 0$, (3) $\alpha^{(0,2)} = 0$, (4) $S^{(0,2)} = 0$.

Although the following corollary is a direct consequence of Corollary 3.2, we can also obtain it by Propositions 3.4 and 4.1.

Corollary 4.2. *The property that f is twistor holomorphic is invariant under projective change of $\tilde{\nabla}$.*

By Lemma 3.6, Theorem 3.8 and (2.2), we have the following theorem.

Theorem 4.3. *Let $\tilde{\nabla}$ be a projectively flat connection on $\tilde{M}$ and (M, ∇, I) a compact surface with a complex structure I. Let $f : M \to \tilde{M}$ be a twistor holomorphic affine immersion with a transversal bundle N. Then the projective invariant (3.8) in Theorem 3.8 is an integer multiple of 16π.*

Proof. From the assumptions, it follows that $A^{f^{\#}\tilde{\nabla}''} = 0$ and $W^{\tilde{\nabla}} = 0$. By Lemma 3.6, we have

$$\begin{aligned} & a_{\Omega}\left(-\mathrm{Tr}(((f^{\#}\tilde{\nabla})^{-}\tilde{I})((f^{\#}\tilde{\nabla})^{-}\tilde{I}) + \frac{4}{2n-1}(f^{*}Ric^{\tilde{\nabla}})^{s}\right)\Omega \\ = \; & 16\pi c_1(TM, \nabla^{I}) + 16\pi c_1(N, \nabla^{NI^{N}}). \end{aligned}$$

□

An isometric immersion from an oriented surface with horizontal twistor lift is called superminimal. The volume of superminimal surface in an even dimensional unit sphere is an integer multiple of 2π, which is essentially proved in [2]. See also [5]. Using Theorem 4.3, we can obtain this result.

In the case of the isometric immersions, there are many twistor holomorphic immersions (see [3] and [4]). We give examples of twistor holomorphic affine immersions.

Example 4.4. Let U be an open set of $\mathbf{R}^2$ and consider a graph immersion $f : U \to \mathbf{R}^4$ given by

$$f(x, y) = (x, y, F(x, y), G(x, y))$$

with a transversal bundle $N = \mathrm{Span}\{\xi_1, \xi_2\}$, where $F, G : U \to \mathbf{R}$ are smooth functions and $\xi_1 = (0, 0, 1, 0)$, $\xi_2 = (0, 0, 0, 1)$. A complex structure I (resp. I^N) of U (resp. N) is defined by $I\partial_x = \partial_y$ and $I\partial_y = -\partial_x$ (resp. $I^N\xi_1 = \xi_2$ and $I^N\xi_2 = -\xi_1$). We can see $S = 0$, and hence, $R^{\nabla} = 0$ and $R^{\nabla^N} = 0$. Moreover we have $\bar{\nabla}^N I^N = 0$ and $\alpha(\partial_u, \partial_v) = F_{uv}\xi_1 + G_{uv}\xi_2$, where $u, v \in \{x, y\}$. Then f is twistor holomorphic if and only if $F_{xx} - 2G_{xy} - F_{yy} = 0$ and $G_{xx} + 2F_{xy} - G_{yy} = 0$. For example, when $F(x, y) = x^2$ and $G(x, y) = xy$, we have $\alpha(\partial_x, \partial_x) = 2\xi_1$, $\alpha(\partial_x, \partial_y) = \xi_2$ and $\alpha(\partial_y, \partial_y) = 0$. Then it holds that $\alpha^{(0,2)} = 0$ and $\alpha^{(2,0)}(\partial_x, \partial_x) = 4\xi_1 \neq 0$. Moreover the first normal space satisfies $\dim N^1(x) = 2$ for all $x \in U$.

An affine immersion is said to be totally umbilic if there exists $\rho \in \Gamma(N^*)$ satisfying $S = \rho \otimes id_{TM}$. Note that any totally umbilic immersions satisfy $S^{(0,2)} = 0$ and $S^{(1,1)+} = 0$.

Example 4.5. Let (M, ∇, I) be a surface with a complex structure I and a connection ∇, and $f : M \to \mathbf{R}^3$ an affine immersion with the transversal bundle Span$\{f\}$. Consider the canonical totally geodesic embedding $\iota : \mathbf{R}^3 \to \mathbf{R}^{2n}$ ($n \geq 2$). We take parallel fields $\xi_1, \ldots, \xi_{2n-3}$ with respect to the standard connection on $\mathbf{R}^{2n}$ such that $(N :=)$Span$\{f, \xi_1, \ldots, \xi_{2n-3}\}$ is a transversal bundle for $\iota \circ f$. Define a complex structure I^N on N by $I^N(f) = \xi_1$, $I^N(\xi_1) = -f$, $I^N(\xi_2) = \xi_3$, $I^N(\xi_3) = -\xi_2$, $\ldots$, $I^N(\xi_{2n-4}) = \xi_{2n-3}$, $I^N(\xi_{2n-3}) = -\xi_{2n-4}$. Then we have $\alpha(X, Y) = Ric^{\nabla}(X, Y)f$, $S_f = id_{TM}$ and $S_{\xi_i} = 0$ ($i = 1, \ldots, 2n-3$) for the affine immersion $\iota \circ f$. Therefore $\iota \circ f : M \to \mathbf{R}^{2n}$ is a twistor holomorphic immersion if and only if Ric^{∇} is I-invariant, that is, $Ric^{\nabla}(IX, IY) = Ric^{\nabla}(X, Y)$ for all X, $Y \in TM$. Note that $\iota \circ f$ satisfies $\alpha^{(2,0)} = 0$ and $\alpha^{(0,2)} = 0$ if Ric^{∇} is I-invariant.

In the case of the isometric immersions, a twistor holomorphic surface in $\mathbf{R}^4$ with a flat normal connection is a open part of a plane or the standard 2-sphere, that is, it is totally umbilic. In this case, it holds the first normal space satisfies rank$N^1 = 0$ or 1. Example 4.4 implies that a twistor holomorphic affine surface with a flat transversal connection is not necessary rank$N^1 \leq 1$. We give a characterization of totally geodesic or totally umbilic surface with rank$N_1 = 1$ for twistor affine holomorphic surfaces. To prove this, we show some lemmas.

Lemma 4.6. *Let M be a surface with a complex structure I and a connection ∇. For non zero $X \in TM$, take $\omega \in T^*M$ satisfying $\omega(X) = 1$ and $\omega(IX) = 0$. Then we have*

$$\begin{aligned} Ric^{\nabla}(X, X) &= \omega(IR^{\nabla}_{X,IX}X), \\ Ric^{\nabla}(IX, IX) &= \omega(R^{\nabla}_{X,IX}IX), \\ Ric^{\nabla}(X, IX) &= \omega(IR^{\nabla}_{X,IX}IX), \\ Ric^{\nabla}(IX, X) &= \omega(R^{\nabla}_{X,IX}X). \end{aligned}$$

Lemma 4.7. *Let $f : M \to \mathbf{R}^m$ be an affine immersion. If the affine fundamental form α of f satisfies $\alpha(X, IX) = 0$ for all $X \in TM$, then α satisfies*

$$\begin{aligned} 2(\bar{\nabla}^N_X \alpha)(Y, Z) &= \alpha((\bar{\nabla}_Z I)(X), IY) + \alpha(IX, (\bar{\nabla}_Z I)(Y)) \\ &\quad + \alpha((\bar{\nabla}_{IY} I)(IX), IZ) - \alpha(X, (\bar{\nabla}_{IY} I)(Z)) \\ &\quad + \alpha((\bar{\nabla}_X I)(Y), Z) - \alpha(IY, (\bar{\nabla}_X I)(Z)) \end{aligned}$$

for all X, Y and $Z \in TM$.

Proof. From a straightforward calculation,

$$\begin{aligned}(\bar{\nabla}^N_X\alpha)(IY, IZ) &= (\bar{\nabla}^N_X\alpha)(Y, Z) - \alpha((\bar{\nabla}_X I)(Y), IZ) \\ &\quad -\alpha(IY, (\bar{\nabla}_X I)(Z))\end{aligned}$$

for all X, Y, $Z \in TM$. Using the Codazzi equation for α, we have

$$\begin{aligned}(\bar{\nabla}^N_{IY}\alpha)(X, IZ) &= -(\bar{\nabla}^N_{IY}\alpha)(IX, Z) + \alpha((\bar{\nabla}_{IY} I)(IX), IZ) \\ &\quad -\alpha(X, (\bar{\nabla}_{IY} I)(Z)) \\ &= -(\bar{\nabla}^N_{Z}\alpha)(IX, IY) + \alpha((\bar{\nabla}_{IY} I)(IX), IZ) \\ &\quad -\alpha(X, (\bar{\nabla}_{IY} I)(Z)) \\ &= -(\bar{\nabla}^N_{Z}\alpha)(X, Y) + \alpha((\bar{\nabla}_{Z} I)(X), IY) \\ &\quad +\alpha(IX, (\bar{\nabla}_{Z} I)(Y)) + \alpha((\bar{\nabla}_{IY} I)(IX), IZ) \\ &\quad -\alpha(X, (\bar{\nabla}_{IY} I)(Z))\end{aligned}$$

for all X, Y, $Z \in TM$. Using the Codazzi equation for α again, we have the desired conclusion. $\square$

By Lemma 4.7, we have

Lemma 4.8. *Let $f : M \to \mathbf{R}^m$ be an affine immersion. Assume that the affine fundamental form α of f satisfies $\alpha(X, IX) = 0$ for all $X \in TM$. Then for any non negative integer l, the all covariant derivatives of α of order l are in the first normal space $N_1(x)$ at each point x of M.*

By Lemma 4.7, we have the following.

Lemma 4.9. *Let $f : M \to \mathbf{R}^m$ be an affine immersion. Assume that the affine fundamental form α of f satisfies $\alpha(X, IX) = 0$ for all $X \in TM$. If the dimensions of first normal spaces at any points of M are constant, then N_1 is a parallel subbundle of N.*

Using above lemmas, we can obtain the following theorem.

Theorem 4.10. *Let (M, ∇, I) be a connected surface with a complex structure I such that the Ricci tensor Ric^{∇} of ∇ is symmetric. Let $f : M \to \mathbf{R}^m$ is a real analytic affine immersion satisfying $\alpha(X, IX) = 0$ for all $X \in TM$ and $[S_\xi, I] = 0$ for all $\xi \in N$. Then we see that*
(1) f is totally geodesic, that is, $\alpha = 0$ or
(2) There exist a 3-dimensional affine subspace V in $\mathbf{R}^m$ satisfying $f(M) \subset V$, and $f : M \to V$ is a non degenerate totally umbilic immersion such that Ric^{∇} is I-invariant .

Proof. Since $\alpha(X, IX) = 0$ for all $X \in TM$, we have $\dim N_1(x) \leq 1$ at any $x \in M$. At first, we show that the dimensions of the first normal spaces are constant on M. Assume that there exists a point $x \in M$ satisfying $\alpha_x = 0$, that is, $\dim N_1(x) = 0$. We may assume that $f(x) = 0$. Take a 1-form ω on $\mathbf{R}^{2n}$ satisfying $\mathrm{Ker}\omega = f_*(T_xM)$ and set $\varphi = \omega \circ f$. In fact, we see $X_1\varphi = \omega(f_*X_1) = 0$, $X_2X_1\varphi = \omega(\alpha(X_2, X_1))$, $X_3X_2X_1\varphi = \omega((\bar{\nabla}^N_{X_3}\alpha)(X_2, X_1) + \alpha(\nabla_{X_3}X_2, X_1) + \alpha(X_2, \nabla_{X_3}X_1))$ and so on. Note that $(\bar{\nabla}^N_{X_3}\alpha)(X_2, X_1)$ belongs to N_1. Hence, for any for any non negative integer l, we have $(X_l \cdots X_1\varphi)_x = 0$ for any $X_1, \dots, X_l \in \Gamma(TM)$ by Lemma 4.8. Since f is real analytic, we have $\varphi(M) = \{0\}$, that is, f is totally geodesic. Then we can conclude that the dimensions of the first normal spaces are constant on M. Since $\dim N_1$ is constant on M, N_1 is a smooth subbundle of N and parallel with respect to ∇^N by Lemma 4.9. Form the assumption of α, it holds that $\mathrm{rank}N_1 \leq 1$. If $\mathrm{rank}N_1 = 0$, then we have $\alpha = 0$, and hence, f is totally geodesic.

We assume $\mathrm{rank}N_1 = 1$. By the reduction theorem for affine immersions (see [6]), there exist a 3-dimensional affine subspace V in $\mathbf{R}^{2n}$ satisfying $f(M) \subset V$. Then $f : M \to V$ is non degenerate because of $\mathrm{rank}N_1 = 1$ and $\dim M = 2$. Next we show that $f : M \to V$ is totally umbilic. Take an arbitrary non zero tangent vector X at any point of M. By the Gauss equation, we have $R^{\nabla}_{X,IX}X = -S_{\alpha(X,X)}IX$. Let ω be the one form satisfying $\omega(X) = 1$ and $\omega(IX) = 0$. From the assumption $[S_\xi, I] = 0$ for all $\xi \in N$, we have $Ric^{\nabla}(X, IX) = \omega(S_{\alpha(X,X)}IX)$ and $Ric^{\nabla}(IX, X) = -\omega(S_{\alpha(X,X)}IX)$ by Lemma 4.6. Since the Ricci tensor of ∇ is symmetric, we have $\omega(S_{\alpha(X,X)}IX) = 0$. Therefore it holds that there exists $\mu(X) \in \mathbf{R}$ satisfying $S_{\alpha(X,X)}X = \mu(X)X$. Defining $\mu(X) = 0$ when $X = 0$, we can obtain a function $\mu_x : T_xM \to \mathbf{R}$ for each $x \in M$. We define

$$\tilde{\mu}(Y, Z) = \frac{1}{2}(\mu(Y + Z) - \mu(Y) - \mu(Z))$$

for all tangent vectors Y and Z. By a straightforward calculation, we have $\mu(aX + bIX) = (a^2 + b^2)\mu(X)$. So we see that $\tilde{\mu}$ is a symmetric tensor. Hence $f : M \to V$ is an isotropic affine immersion in the sense of Vrancken [8]. In particular, since $\dim V = 3$, $f : M \to V$ is totally umbilic, which means that Ric^{∇} is I-invariant. □

The assumptions $\alpha(X, IX) = 0$ for all $X \in TM$ and $[S_\xi, I] = 0$ for all $\xi \in N$ in Theorem 4.10 are invariant under projective transformations. In the case of isometric immersions, these conditions are equivalent each other. Note that if M is an isometrically immersed surface in $\tilde{M}$ satisfying $\alpha(X, IX) = 0$ for all $X \in TM$, then M is totally umbilic.

Corollary 4.11. *Let (M, ∇, I) be a connected surface with a complex structure I such that the Ricci tensor of ∇ is symmetric and $f : M \to \mathbf{R}^{2n}$ a real analytic* *affine immersion with a twistor lift. If* $\alpha^{(2,0)} = 0$*,* $\alpha^{(0,2)} = 0$*,* $S^{(0,2)} = 0$ *and* $S^{(1,1)+} = 0$*, then we have the same conclusion as in Theorem 4.10.*

Proof. Using $\alpha^{(2,0)} = 0$ and $\alpha^{(0,2)} = 0$, we have $\alpha(X, IX) = 0$ for all $X \in TM$. Moreover $[S_\xi, I] = 0$ for all $\xi \in N$ by $S^{(0,2)} = 0$ and $S^{(1,1)+} = 0$. From Theorem 4.10, we have the conclusion. □

If $f : M \to \mathbf{R}^{2n}$ a twistor holomorphic affine immersion with respect to I and $-I$, then we see that (1) I^N is parallel, (2) α satisfies $\alpha(X, IX) = 0$ for all $X \in TM$, (3) S satisfies $[S_\xi, I] = 0$ for all $\xi \in N$. Then we obtain

Corollary 4.12. *If* $f : M \to \mathbf{R}^{2n}$ *is a real analytic affine immersion from a connected surface with a complex structure* I *and symmetric Ricci tensor which is twistor holomorphic with respect to* I *and* $-I$*, then we have the same conclusion as in Theorem 4.10.*

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