

The complexifications of pseudo-Riemannian manifolds and anti-Kaehler geometry

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Abstract. In this paper, we first define the complexification of a real analytic map between real analytic Koszul manifolds and show that the complexified map is the holomorphic extension of the original map. Next we define an anti-Kaehler metric compatible with the adapted complex structure on the complexification of a real analytic pseudo-Riemannian manifold. In particular, for a pseudo-Riemannian homogeneous space, we define another complexification and a (complete) anti-Kaehler metric on the complexification. One of main purposes of this paper is to find the interesting relation between these two complexifications (equipped with the anti-Kaehler metrics) of a pseudo-Riemannian homogeneous space. Another of main purposes of this paper is to show that almost all principal orbits of some isometric action on the first complexification (equipped with the anti-Kaehler metric) of a semi-simple pseudo-Riemannian symmetric space are curvature-adapted isoparametric submanifolds with flat section in the sense of this paper.

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§1. Introduction

Any C^ω -manifold M admits its complexification, that is, a complex manifold equipped with an anti-holomorphic involution σ whose fixed point set is C^ω -diffeomorphic to M , where C^ω means the real analyticity. To get a canonical complexification of M one needs some extra structure on M . For example, if M equips with a C^ω -Riemannian metric g , then so-called adapted complex structure J^g is defined on a tubular neighborhood U^g (which we take as largely as possible) of the zero section of the tangent bundle TM of M and (U^g, J^g) gives a complexification of M under the identification of M with the zero section (see [15,18]). See [4,5,6,7,15,16,18,19,20,21] the basic facts for the

adapted complex structure. We denote (U^g, J^g) by $M_g^{\mathbb{C}}$. In more general, R. Szöke ([22]) extended the notion of the adapted complex structure to the case where M equips with a C^ω -Koszul connection ∇ , where a C^ω -Koszul connection means a C^ω -linear connection of TM . In this paper, we denote this complex structure by J^∇ , its domain by U^∇ and (U^∇, J^∇) by $M_{\nabla}^{\mathbb{C}}$, which is a complexification of M . We shall call a manifold equipped with a Koszul connection a *Koszul manifold*. Thus we get a canonical complexification of a C^ω -Koszul manifold (as a special case, a C^ω -pseudo-Riemannian manifold). On the complexification $M_g^{\mathbb{C}} := (U^g, J^g)$ of a C^ω -pseudo-Riemannian manifold (M, g) of index ν , a pseudo-Kaehler metric g_K of index ν compatible with J^g which satisfies $\iota^*g_K = \frac{1}{2}g$ (ι : the inclusion map of M into $M_g^{\mathbb{C}}$) is defined in terms of the energy function $E : TM \rightarrow \mathbb{R}$ (see [22] in detail), where E is defined by $E(v) := \frac{1}{2}g(v, v)$ ($v \in TM$).

In [10], we defined the (extrinsic) complexification of a complete C^ω -Riemannian submanifold (M, g) immersed by f in a Riemannian symmetric space $N = G/K$ of non-compact type as follows. First we defined the complexification $f^{\mathbb{C}}$ of f as a map of a tubular neighborhood $(M_g^{\mathbb{C}})_f$ of M in the complexification $M_g^{\mathbb{C}}$ of M into the anti-Kaehler symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$. Next we showed that $f^{\mathbb{C}}$ is an immersion over a tubular neighborhood $(M_g^{\mathbb{C}})_{f:i}$ of the zero section in $M_g^{\mathbb{C}}$. We called an anti-Kaehler submanifold $((M_g^{\mathbb{C}})_{f:i}, (f^{\mathbb{C}}|_{(M_g^{\mathbb{C}})_{f:i}})^*\langle \cdot, \cdot \rangle)$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$ the *extrinsic complexification* of the Riemannian submanifold (M, g) . Also, in [10], we showed that complex focal radii of M introduced in [9] are the quantities which indicate the position of focal points of $((M_g^{\mathbb{C}})_{f:i}, (f^{\mathbb{C}}|_{(M_g^{\mathbb{C}})_{f:i}})^*\langle \cdot, \cdot \rangle)$. Furthermore, by imposing a condition related to complex focal radii, we defined the notions of a complex equifocal submanifold and proper complex equifocal submanifold. It is conjectured that this notion coincides with that of an isoparametric submanifold with flat section introduced by Heintze-Liu-Olmos in [9]. In [10], [11] and [12], we obtained some results for a complex equifocal submanifold by investigating the lift of the complexification of the submanifold to some path space.

L. Geatti and C. Gorodski [6] showed that a polar representation of a real reductive algebraic group on a pseudo-Euclidean space has the same closed orbits as the isotropy representation (i.e., the linear isotropy action) of a pseudo-Riemannian symmetric space (see Theorem 1 of [6]). Also, they showed that the principal orbits of the polar representation through a semi-simple element (i.e., the orbit through a regular element (in the sense of [6])) is an isoparametric submanifold in the sense of [6] by investigating the complexified representation (see Theorem 11 (also Example 12) of [6]), where an isoparametric submanifold in the sense of [6] means the finite dimensional version of a proper complex isoparametric submanifold in a pseudo-Hilbert space defined in [9]. All isoparametric submanifold (in a pseudo-Euclidean space) in this sense are

isoparametric manifolds with flat section in the sense of [8]. On the other hand, we [14] showed that, for a Hermann type action $H \curvearrowright G/K$ (i.e., H is a symmetric subgroup of G) on a (semi-simple) pseudo-Riemannian symmetric space G/K , the principal H -orbits through $\exp_G(w)K$ (w : a semi-simple element, \exp_G : the exponential map of G) are curvature-adapted proper complex equifocal submanifolds (hence isoparametric submanifolds with flat section in the sense of [8]).

In this paper, we shall first define the complexification $f^{\mathbb{C}}$ of a C^ω -map of a C^ω -Koszul manifold (M, ∇) into another C^ω -Koszul manifold $(\widetilde{M}, \widetilde{\nabla})$ as a map of a tubular neighborhood $(M_{\nabla}^{\mathbb{C}})_f$ of M in $M_{\nabla}^{\mathbb{C}}$ into $\widetilde{M}_{\widetilde{\nabla}}^{\mathbb{C}}$ and show that $f^{\mathbb{C}}$ is holomorphic and that, if f is an immersion, then $f^{\mathbb{C}}$ also is an immersion on a tubular neighborhood $(M_{\nabla}^{\mathbb{C}})_{f:i}$ of M in $(M_{\nabla}^{\mathbb{C}})_f$ (see Section 4). Let (M, g) be a C^ω -pseudo-Riemannian manifold. Next, on a tubular neighborhood $(M_g^{\mathbb{C}})_A = (U_A^g, J^g)$ (which we take as largely as possible) of M in $M_g^{\mathbb{C}}$, we define an anti-Kaehler metric g_A compatible with J^g (i.e., $g_A(J^g X, J^g Y) = -g_A(X, Y)$ ($X, Y \in TU_A^g$), $\nabla J^g = 0$) satisfying $\iota^* g_A = g$, where ∇ is the Levi-Civita connection of g_A and ι is the inclusion map of M into $(M_g^{\mathbb{C}})_A$. Note that g_A is defined uniquely. We show that, for a C^ω -isometric immersion $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ between C^ω -pseudo-Riemannian manifolds, $f^{\mathbb{C}} : ((M_g^{\mathbb{C}})_A \cap (M_g^{\mathbb{C}})_{f:i}, g_A) \rightarrow ((\widetilde{M}_g^{\mathbb{C}})_A, \widetilde{g}_A)$ is a holomorphic and isometric (that is, an anti-Kaehler) immersion. Next, for a pseudo-Riemannian homogeneous space, we define its another complexification as the quotient of the complexification of its isometry group by the complexification of its isotropy group, where we assume that the isometry group and the isotropy group have faithful real representations. Note that this quotient has a natural anti-Kaehler structure. The first purpose of this paper is to find an interesting relation between two complexifications (see Theorem 6.1). The second purpose of this paper is to define the dual of a C^ω -pseudo-Riemannian manifold (M, g) at each point and the dual of a totally geodesic C^ω -submanifold of (M, g) in the anti-Kaehler manifold $((M_g^{\mathbb{C}})_A, g_A)$ (see Definitions 2 and 3 in Section 7). Next we define the notions of a complex Jacobi field in an anti-Kaehler manifold and a complex focal radius of an anti-Kaehler submanifold and show some facts related to them (see Section 8). Furthermore, we define the notions of a complex equifocal submanifold and an isoparametric one in a pseudo-Riemannian homogeneous space and investigate the equivalence between their notions for a C^ω -submanifold in a pseudo-Riemannian symmetric space (see Section 9). The third purpose of this paper is to show that, almost all orbits of the G -action on the complexification $((G/K)_g^{\mathbb{C}})_A, g_A$ of a pseudo-Riemannian symmetric space $(G/K, g)$ are curvature-adapted isoparametric submanifolds with flat section such that the shape operators are complex diagonalizable (see Theorem 9.3).

Future plan of research. We plan to solve both of various problems (for example, problems for harmonic analysis) in a C^ω -pseudo-Riemannian manifold (M, g) and the corresponding problems in the dual of (M, g) by solving the corresponding problems in $((M_g^{\mathbb{C}})_A, g_A)$.

§2. Basic notions and facts

In this section, we shall recall basic notions and facts. Let (M, ∇) be a C^∞ -Koszul manifold and $\pi : TM \rightarrow M$ be the tangent bundle of M . Denote by $\overset{\circ}{TM}$ the punctured tangent bundle $TM \setminus M$, where M is identified with the zero section of TM . Denote by \mathfrak{V} the vertical distribution on TM and by \mathfrak{H} the horizontal distribution on TM with respect to ∇ . Also, denote by $w_u^V (\in \mathfrak{V}_u)$ the vertical lift of $w \in T_{\pi(u)}M$ to u . Let Φ_t be the geodesic flow of ∇ and X^S be the vector field on TM associated with Φ_t . Define a distribution \mathfrak{L}^∇ on $\overset{\circ}{TM}$ by $\mathfrak{L}_u^\nabla := \text{Span}\{u_u^V, X_u^S\}$ ($u \in \overset{\circ}{TM}$). This distribution \mathfrak{L}^∇ is involutive and hence defines a foliation on $\overset{\circ}{TM}$. This foliation is called the *Koszul foliation* and we denote it by \mathfrak{F}^∇ . In particular, if ∇ is the Levi-Civita connection of a pseudo-Riemannian metric, then we call it a *Levi-Civita foliation*. These terminologies are used in [22]. Let $\gamma : I \rightarrow M$ be a maximal geodesic. The image $\gamma_*(TI)$ yields two leaves of \mathfrak{F}^∇ and all leaves of \mathfrak{F}^∇ are obtained in this way. Let ξ be a vector field along γ_* . If there exists a geodesic variation γ_t in M satisfying $\gamma_0 = \gamma$ and $\frac{d}{dt}|_{t=0}\gamma_{t*} = \xi$, then ξ is called a *parallel vector field*. Note that ξ is an extension of the Jacobi field $\frac{d}{dt}|_{t=0}\gamma_t$ along γ . If (M, ∇) is a C^ω -Koszul manifold, then there uniquely exists a complex structure J^∇ on a suitable domain U^∇ of TM containing M such that for each maximal geodesic γ in (M, ∇) , $\gamma_* : \gamma_*^{-1}(U^\nabla) \rightarrow (U^\nabla, J^\nabla)$ is holomorphic (see Theorem 0.3 of [22]), where $\gamma_*^{-1}(U^\nabla)$ is regarded as an open set of \mathbb{C} under the natural identification of $T\mathbb{R}$ with \mathbb{C} . We take U^∇ as largely as possible. This complex structure J^∇ is called the *adapted complex structure*. We denote this complex manifold (U^∇, J^∇) by $M_{\nabla}^{\mathbb{C}}$ and call it the complexification of (M, ∇) . In particular, if ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g , then U^∇, J^∇ and $M_{\nabla}^{\mathbb{C}}$ are denoted by U^g, J^g and $M_g^{\mathbb{C}}$, respectively. Denote by R the curvature tensor of ∇ . According to Remark 2.2 of [5] and the statement (b) of Page 8 of [5], we see that, if (M, ∇) is locally symmetric (i.e., ∇ : torsion-free and $\nabla R = 0$) and the spectrum of $R(\cdot, X)X$ contains no negative number for each $X \in TM$, then the adapted complex structure J^∇ is defined on TM (i.e., $U^\nabla = TM$).

§3. Anti-Kaehler manifolds

Let M be a C^∞ -manifold, J be a complex structure on M and g be a pseudo-Riemannian metric on M . Denote by ∇ the Levi-Civita connection of g . If $g(JX, JY) = -g(X, Y)$ for any tangent vectors X and Y of M , then (M, J, g) is called a *anti-Hermitian manifold*. Furthermore, if $\nabla J = 0$, then it is called an *anti-Kaehler manifold*. See [1,2] about the basic facts for the anti-Kaehler manifold. For an anti-Kaehler manifold, the following remarkable fact holds.

Proposition 3.1. *Let (M, J, g) be an anti-Kaehler manifold and \exp_p be the exponential map at $p \in M$. Then $\exp_p : (T_pM, J_p) \rightarrow (M, J)$ is holomorphic.*

Proof. Let $u \in T_pM$ and $X \in T_u(T_pM)$. Define a geodesic variation δ (resp. $\bar{\delta}$) by $\delta(t, s) := \exp_p(t(u + sX))$ (resp. $\bar{\delta}(t, s) := \exp_p(t(u + sJ_pX))$) for $(t, s) \in [0, 1]^2$. Let $Y := \delta_*(\frac{\partial}{\partial s}|_{s=0})$ and $\bar{Y} := \bar{\delta}_*(\frac{\partial}{\partial s}|_{s=0})$, which are Jacobi fields along the geodesic γ_u with $\dot{\gamma}_u(0) = u$. Since (M, J, g) is anti-Kaehler, we have $\nabla J = 0$ and $R(Jv, w) = JR(v, w)$ ($v, w \in TM$) (by Lemma 5.2 of [1]), where R is the curvature tensor of g . Hence we have

$$\nabla_{\dot{\gamma}_u} \nabla_{\dot{\gamma}_u} (JY) + R(JY, \dot{\gamma}_u)\dot{\gamma}_u = J(\nabla_{\dot{\gamma}_u} \nabla_{\dot{\gamma}_u} Y + R(Y, \dot{\gamma}_u)\dot{\gamma}_u) = 0,$$

that is, JY is also a Jacobi field along γ_u . Also, we have $JY(0) = \bar{Y}(0) = 0$ and $\nabla_{\dot{\gamma}_u(0)} JY = \nabla_{\dot{\gamma}_u(0)} \bar{Y} = J_pX$. Hence we have $JY = \bar{Y}$. On the other hand, we have $JY(1) = J_{\gamma_u(1)}(\exp_p)_{*u}(X)$ and $\bar{Y}(1) = (\exp_p)_{*u}(J_pX)$. Therefore $J_{\gamma_u(1)} \circ (\exp_p)_{*u} = (\exp_p)_{*u} \circ J_p$ follows from the arbitrariness of X . Since this relation holds for any $u \in T_pM$, $\exp_p : (T_pM, J_p) \rightarrow (M, J)$ is holomorphic. □

According to this fact, we can define so-called normal holomorphic coordinate around each point p of a real $2n$ -dimensional anti-Kaehler manifold (M, J, g) as follows. Let \tilde{U} be a neighborhood of the origin of T_pM such that $\exp_p|_{\tilde{U}}$ is a diffeomorphism and $(e_1, J_p e_1, \dots, e_n, J_p e_n)$ be a J_p -base of T_pM . Define $\tilde{\phi} : \mathbb{C}^n \rightarrow T_pM$ by $\tilde{\phi}(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) = \sum_{i=1}^n (x_i e_i + y_i J_p e_i)$. Set $U := \exp_p(\tilde{U})$ and $\phi := \tilde{\phi}^{-1} \circ (\exp_p|_{\tilde{U}})^{-1}$. According to Proposition 3.1, (U, ϕ) is a holomorphic local coordinate of (M, J, g) . We call such a coordinate a *normal holomorphic coordinate* of (M, J, g) . Let $v \in T_pM$ and define a map $\gamma_v^{\mathbb{C}} : D \rightarrow M$ by $\gamma_v^{\mathbb{C}}(z) = \exp_p((\operatorname{Re} z)v + (\operatorname{Im} z)J_p v)$ ($z \in D$), where D is an open neighborhood of 0 in \mathbb{C} . We may assume that $\gamma_v^{\mathbb{C}}$ is an immersion by shrinking D if necessary. According to Proposition 3.1, $\gamma_v^{\mathbb{C}}$ is the holomorphic extension of γ_v and hence it is totally geodesic. We call $\gamma_v^{\mathbb{C}}$ a *complex geodesic* in (M, J, g) .

Next we give examples of an anti-Kaehler manifold. Let (G, K) be a semi-simple symmetric pair and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g} := \text{Lie } G$ associated with (G, K) . Denote by g the G -invariant pseudo-Riemannian metric on a quotient manifold G/K arising from the restriction $B|_{\mathfrak{p} \times \mathfrak{p}}$ to \mathfrak{p} of the Killing form B of \mathfrak{g} . Then $(G/K, g)$ and $(G/K, -g)$ are (semi-simple) pseudo-Riemannian symmetric spaces. Note that $(G/K, -g)$ is a Riemannian symmetric space of compact type if (G, K) is a Riemannian symmetric pair of compact type and that $(G/K, g)$ is a Riemannian symmetric space of non-compact type if (G, K) is a Riemannian symmetric pair of non-compact type. Let $G^{\mathbb{C}}, K^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}}$ be the complexifications of $G, K, \mathfrak{g}, \mathfrak{k}$ and \mathfrak{p} , respectively. For the complexification $B^{\mathbb{C}} (: \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \rightarrow \mathbb{C})$ of B , $2\text{Re } B^{\mathbb{C}}$ is the Killing form of $\mathfrak{g}^{\mathbb{C}}$ regarded as a real Lie algebra, where $\text{Re } B^{\mathbb{C}}$ is the real part of $B^{\mathbb{C}}$. The pair $(G^{\mathbb{C}}, K^{\mathbb{C}})$ is a semi-simple symmetric pair, where $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ are regarded as real Lie groups. Denote by \tilde{g} the $G^{\mathbb{C}}$ -invariant pseudo-Riemannian metric on $G^{\mathbb{C}}/K^{\mathbb{C}}$ arising from $2\text{Re } B^{\mathbb{C}}|_{\mathfrak{p}^{\mathbb{C}} \times \mathfrak{p}^{\mathbb{C}}}$ and by J the $G^{\mathbb{C}}$ -invariant complex structure arising from $j : \mathfrak{p}^{\mathbb{C}} \rightarrow \mathfrak{p}^{\mathbb{C}} \underset{\text{def}}{(\Leftrightarrow} jX = \sqrt{-1}X)$. Then $(G^{\mathbb{C}}/K^{\mathbb{C}}, J, \tilde{g})$ and $(G^{\mathbb{C}}/K^{\mathbb{C}}, J, -\tilde{g})$ are anti-Kaehler manifolds. We call these anti-Kaehler manifolds the *anti-Kaehler symmetric spaces associated with $(G/K, g)$ and $(G/K, -g)$* , respectively. See [13] about general theory of anti-Kaehler symmetric spaces.

§4. A complexification of a C^ω -map between Koszul manifolds

In this section, we shall define the complexification of a C^ω -map between C^ω -Koszul manifolds and investigate it. Let $f : (M, \nabla) \rightarrow (\tilde{M}, \tilde{\nabla})$ be a C^ω -map between C^ω -Koszul manifolds. First we shall recall the definition of the (maximal) holomorphic extension α^h of a C^ω -curve $\alpha : (a, b) \rightarrow \tilde{M}$ in $\tilde{M}_{\tilde{\nabla}}^{\mathbb{C}}$. Fix $t_0 \in (a, b)$ and take a holomorphic local coordinate $(V, \phi = (z_1, \dots, z_m))$ of $\tilde{M}_{\tilde{\nabla}}^{\mathbb{C}}$ around $\alpha(t_0)$ satisfying $\tilde{M} \cap V = \phi^{-1}(\mathbb{R}^m)$, where $m = \dim \tilde{M}$. Let $(\phi \circ \alpha)(t) = (\alpha_1(t), \dots, \alpha_m(t))$. Since $\alpha_i(t)$ ($i = 1, \dots, m$) are of class C^ω , we get their holomorphic extensions $\alpha_i^h : D_i \rightarrow \mathbb{C}$ ($i = 1, \dots, m$), where D_i is a neighborhood of t_0 in \mathbb{C} . Define $\alpha_{t_0}^h : \left(\bigcap_{i=1}^m D_i\right) \cap (\alpha_1^h \times \dots \times \alpha_m^h)^{-1}(\phi(V)) \rightarrow M_{\nabla}^{\mathbb{C}}$ by $\alpha_{t_0}^h(z) := \phi^{-1}(\alpha_1^h(z), \dots, \alpha_m^h(z))$. This complex curve $\alpha_{t_0}^h$ is a holomorphic extension of $\alpha|_{(t_0-\varepsilon, t_0+\varepsilon)}$, where ε is a sufficiently small positive number. For each $t \in (a, b)$, we get a holomorphic extension α_t^h of $\alpha|_{(t-\varepsilon', t+\varepsilon')}$, where ε' is a sufficiently small positive number. By patching $\{\alpha_t^h\}_{t \in (a, b)}$, we get a holomorphic extension of α and furthermore, by extending the holomorphic extension to the maximal one, we get the maximal holomorphic extension α^h . Now we shall define the complexification $f^{\mathbb{C}}$ of f .

Definition. Let $(M_{\nabla}^{\mathbb{C}})_f := \{v \in M_{\nabla}^{\mathbb{C}} \mid \sqrt{-1} \in \text{Dom}((f \circ \gamma_v)^h)\}$, where γ_v is the geodesic in (M, ∇) with $\dot{\gamma}_v(0) = v$, $(f \circ \gamma_v)^h$ is the (maximal) holomorphic extension of $f \circ \gamma_v$ in $\widetilde{M}_{\nabla}^{\mathbb{C}}$ and $\text{Dom}((f \circ \gamma_v)^h)$ is the domain of $(f \circ \gamma_v)^h$. This set $(M_{\nabla}^{\mathbb{C}})_f$ is a tubular neighborhood of M in $M_{\nabla}^{\mathbb{C}}$. We define $f^{\mathbb{C}} : (M_{\nabla}^{\mathbb{C}})_f \rightarrow \widetilde{M}_{\nabla}^{\mathbb{C}}$ by $f^{\mathbb{C}}(v) := (f \circ \gamma_v)^h(\sqrt{-1})$ ($v \in (M_{\nabla}^{\mathbb{C}})_f$).

For this complexification $f^{\mathbb{C}}$, the following facts hold.

Proposition 4.1. *Let $f : (M, \nabla) \rightarrow (\widetilde{M}, \widetilde{\nabla})$ be a C^ω -map between C^ω -Koszul manifolds. Then $f^{\mathbb{C}} : (M_{\nabla}^{\mathbb{C}})_f \rightarrow \widetilde{M}_{\nabla}^{\mathbb{C}}$ is the (maximal) holomorphic extension of f . Also, if f is an immersion, then $f^{\mathbb{C}}$ is an immersion on a tubular neighborhood (which is denoted by $(M_{\nabla}^{\mathbb{C}})_{f:i}$ in the sequel) of M in $(M_{\nabla}^{\mathbb{C}})_f$.*

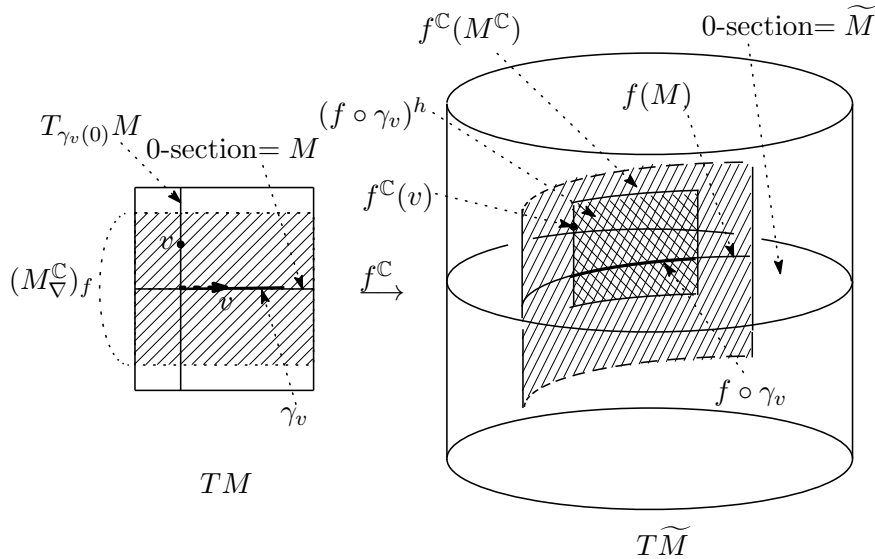


Figure 1.

Proof. First we shall show $f^{\mathbb{C}}|_M = f$. Take an arbitrary $p(=0_p) \in M$ (=the zero section of TM), where 0_p is the zero vector of T_pM . We have $f^{\mathbb{C}}(p) = f^{\mathbb{C}}(0_p) = (f \circ \gamma_{0_p})^h(\sqrt{-1}) = f(p)$. Thus $f^{\mathbb{C}}|_M = f$ holds. Next we shall show that $f^{\mathbb{C}}$ is holomorphic. According to Theorem 3.4 of [19], we suffice to show that, for each geodesic γ in (M, ∇) , $f^{\mathbb{C}} \circ \gamma_*$ is holomorphic. For each $z = x + \sqrt{-1}y \in \text{Dom}(f^{\mathbb{C}} \circ \gamma_*)$, we have

$$\begin{aligned} (f^{\mathbb{C}} \circ \gamma_*)(z) &= (f^{\mathbb{C}} \circ \gamma_*)(y(\frac{d}{dt})_x) = f^{\mathbb{C}}(y\dot{\gamma}(x)) \\ &= (f \circ \gamma_{y\dot{\gamma}(x)})^h(\sqrt{-1}) = (f \circ \gamma)^h(z), \end{aligned}$$

where we note that the tangent bundle $T\mathbb{R}$ is identified with \mathbb{C} under the correspondence $y(\frac{d}{dt})_x \leftrightarrow x + \sqrt{-1}y$. That is, we get $f^{\mathbb{C}} \circ \gamma_* = (f \circ \gamma)^h$. Hence

$f^{\mathbb{C}} \circ \gamma_*$ is holomorphic. Thus the first-half part of the statement is shown. The second-half part of the statement is trivial. \square

Let $(\widetilde{M}, \widetilde{\nabla})$ be an m -dimensional C^ω -Koszul manifold, F be a \mathbb{R}^k -valued C^ω -function over an open set V of \widetilde{M} ($k < m$) and \mathbf{a} be a regular value of F . Let $M := F^{-1}(\mathbf{a})$ and ι be the inclusion map of M into \widetilde{M} . Take an arbitrary C^ω -Koszul connection ∇ of M . Then we have the following fact.

Proposition 4.2. *The image $\iota^{\mathbb{C}}((M_{\nabla}^{\mathbb{C}})_\iota)$ is an open portion of $(F^h)^{-1}(\mathbf{a})$, where F^h is the (maximal) holomorphic extension of F to $\widetilde{M}_{\nabla}^{\mathbb{C}}$ (which is a \mathbb{C}^k -valued holomorphic function on a tubular neighborhood \widetilde{V} of V in $\widetilde{M}_{\nabla}^{\mathbb{C}}$).*

Here we shall explain the (maximal) holomorphic extension F^h of F to $\widetilde{M}_{\nabla}^{\mathbb{C}}$. Fix $p_0 \in V$ and take a holomorphic local coordinate $(W_{p_0}, \phi = (z_1, \dots, z_m))$ of $\widetilde{M}_{\nabla}^{\mathbb{C}}$ about p_0 satisfying $\widetilde{M} \cap W_{p_0} = \phi^{-1}(\mathbb{R}^m)$ and $\widetilde{M} \cap W_{p_0} \subset V$. Since $F \circ (\phi|_{\widetilde{M} \cap W_{p_0}})^{-1}$ is of class C^ω , we get its holomorphic extension

$$(F \circ (\phi|_{\widetilde{M} \cap W_{p_0}})^{-1})^h : D \rightarrow \mathbb{C}^k,$$

where D is a neighborhood of $\phi(p_0)$ in \mathbb{C}^m . Define $F_{p_0}^h : \phi^{-1}(D \cap \phi(W_{p_0})) \rightarrow \mathbb{C}^k$ by

$$F_{p_0}^h := (F \circ (\phi|_{\widetilde{M} \cap W_{p_0}})^{-1})^h \circ \phi|_{\phi^{-1}(D \cap \phi(W_{p_0}))}.$$

This \mathbb{C}^k -valued function $F_{p_0}^h$ is a holomorphic extension of $F|_{\widetilde{M} \cap W_{p_0}}$ to $\widetilde{M}_{\nabla}^{\mathbb{C}}$. For each $p \in V$, we get a holomorphic extension F_p^h of $F|_{V_p}$ (V_p : a sufficiently small neighborhood of p in V). By patching $\{F_p^h\}_{p \in V}$, we get a holomorphic extension of F and furthermore, by extending to the holomorphic extension to the maximal one, we get the desired (maximal) holomorphic extension F^h .

Proof of Proposition 4.2. Take $X \in M_{\nabla}^{\mathbb{C}} (\subset TM)$ and $\gamma_X : (-\varepsilon, \varepsilon) \rightarrow M$ be the geodesic in (M, ∇) with $\dot{\gamma}_X(0) = X$. Since $\gamma_X(t) \in M$, we have $F(\gamma_X(t)) = \mathbf{a}$, where $t \in (-\varepsilon, \varepsilon)$. Let $(\iota \circ \gamma_X)^h : D \rightarrow \widetilde{M}_{\nabla}^{\mathbb{C}}$ be the (maximal) holomorphic extension of $\iota \circ \gamma_X$ in $\widetilde{M}_{\nabla}^{\mathbb{C}}$. Since $F^h \circ (\iota \circ \gamma_X)^h : ((\iota \circ \gamma_X)^h)^{-1}(\widetilde{V}) \rightarrow \mathbb{C}^k$ is holomorphic and $(F^h \circ (\iota \circ \gamma_X)^h)(t) = \mathbf{a}$ ($t \in (-\varepsilon, \varepsilon)$), we see that $F^h \circ (\iota \circ \gamma_X)^h$ is identically equal to \mathbf{a} . Hence we get $F^h(\iota^{\mathbb{C}}(X)) = F^h((\iota \circ \gamma_X)^h(\sqrt{-1})) = \mathbf{a}$, that is, $\iota^{\mathbb{C}}(X) \in (F^h)^{-1}(\mathbf{a})$. From the arbitrariness of X , it follows that $\iota^{\mathbb{C}}((M_{\nabla}^{\mathbb{C}})_\iota) \subset (F^h)^{-1}(\mathbf{a})$. Furthermore, since $\dim \iota^{\mathbb{C}}((M_{\nabla}^{\mathbb{C}})_\iota) = \dim (F^h)^{-1}(\mathbf{a})$, $\iota^{\mathbb{C}}((M_{\nabla}^{\mathbb{C}})_\iota)$ is an open portion of $(F^h)^{-1}(\mathbf{a})$. \square

Remark 4.1. Take another C^ω -Koszul connection $\widehat{\nabla}$ of M . Let $\widehat{\iota}^{\mathbb{C}}$ be the

complexification of ι as a map of $M_{\nabla}^{\mathbb{C}}$ into $\widetilde{M}_{\nabla}^{\mathbb{C}}$. Take $X \in (M_{\nabla}^{\mathbb{C}})_{\iota} \cap (\widetilde{M}_{\nabla}^{\mathbb{C}})_{\iota} (\subset TM)$. Then $\iota^{\mathbb{C}}(X)$ and $\widehat{\iota}^{\mathbb{C}}(X)$ are mutually distinct in general but they belong to $(F^h)^{-1}(\mathbf{a})$.

Example. Let $S^n(r) := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = r^2\}$ and g be the standard Riemannian metric of $S^n(r)$. Denote by ι the inclusion map of $S^n(r)$ into \mathbb{R}^{n+1} . Then we have

$$\iota^{\mathbb{C}}(S^n(r)_g^{\mathbb{C}}) = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1^2 + \dots + z_{n+1}^2 = r^2\}.$$

§5. The anti-Kaehler metric on the complexification of a pseudo-Riemannian manifold

Let (M, g) be an m -dimensional C^ω -pseudo-Riemannian manifold and $M_g^{\mathbb{C}} = (U^g, J^g)$ be its complexification. We shall construct an anti-Hermitian metric associated with J^g on a tubular neighborhood of M in $M_g^{\mathbb{C}}$. Fix $p_0 \in M$. Take a holomorphic local coordinate $(V, \phi = (z_1, \dots, z_m))$ of $M_g^{\mathbb{C}}$ around p_0 satisfying $M \cap V = \phi^{-1}(\mathbb{R}^m)$. Let $\phi|_{M \cap V} = (x_1, \dots, x_m)$. As $g|_{M \cap V} = \sum_{i=1}^n \sum_{j=1}^n g_{ij} dx_i dx_j$, we define a holomorphic metric g^{h,p_0} on a neighborhood of

$M \cap V$ in V by $g^{h,p_0} := \sum_{i=1}^n \sum_{j=1}^n g_{ij}^h dz_i dz_j$, where g_{ij}^h is a holomorphic extension

of g_{ij} . Thus, for each $p \in M$, we can define a holomorphic metric $g^{h,p}$ on a neighborhood of p in $M_g^{\mathbb{C}}$. By patching $g^{h,p}$'s ($p \in M$), we get a holomorphic metric on a tubular neighborhood of M in $M_g^{\mathbb{C}}$. Furthermore, we extend this holomorphic metric to the maximal one. Denote by g^h this maximal holomorphic metric.

Notation 1. Denote by $(M_g^{\mathbb{C}})_A$ the domain of g^h .

Note that g^h is a holomorphic section of the holomorphic vector bundle $(T^*((M_g^{\mathbb{C}})_A) \otimes T^*((M_g^{\mathbb{C}})_A))^{(2,0)} (\subset (T^*((M_g^{\mathbb{C}})_A) \otimes T^*((M_g^{\mathbb{C}})_A))^{\mathbb{C}})$ consisting of all complex $(0, 2)$ -tensors of type $(2, 0)$ of $(M_g^{\mathbb{C}})_A$. From g^h , we define an anti-Kaehler metric associated with J^g as follows.

Definition 1. Define $\overline{g^h}$ by $\overline{g^h}(Z_1, Z_2) = \overline{g^h(\overline{Z_1}, \overline{Z_2})}$ ($Z_1, Z_2 \in (T(M_g^{\mathbb{C}})_A)^{\mathbb{C}}$), where $\overline{(\cdot)}$ is the conjugation of (\cdot) . Then $(g^h + \overline{g^h})|_{T((M_g^{\mathbb{C}})_A) \times T((M_g^{\mathbb{C}})_A)}$ is an anti-Kaehler metric on $(M_g^{\mathbb{C}})_A$ (by Theorem 2.2 of [1]). We denote this anti-Kaehler metric by g_A .

Remark 5.1. (i) For $X, Y \in T(M_g^{\mathbb{C}})$, we have $g_A(X, Y) = 2\text{Re}(g^h(X, Y))$.

(ii) If (M, g) is Einstein, then $((M_g^{\mathbb{C}})_A, g_A)$ also is Einstein (see Section 5 of [1]). Hence $((((M_g^{\mathbb{C}})_A)_{g_A}^{\mathbb{C}})_A, (g_A)_A)$ also is Einstein. Thus we get an inductive construction of an Einstein (anti-Kaehler) manifold.

Notation 2. For a C^ω -map $f : (M, g) \rightarrow (\widetilde{M}, \widetilde{g})$ between C^ω -pseudo-Riemannian manifolds, we set $(M_g^{\mathbb{C}})_{A,f:i} := (M_g^{\mathbb{C}})_A \cap (M_g^{\mathbb{C}})_{f:i}$.

For the complexification of a C^ω -isometric immersion between C^ω -pseudo-Riemannian manifolds, we have the following fact.

Theorem 5.1. *Let $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ be a C^ω -isometric immersion between C^ω -pseudo-Riemannian manifolds. Then the complexified map $f^{\mathbb{C}} : ((M_g^{\mathbb{C}})_{A,f:i} \cap (f^{\mathbb{C}})^{-1}((\widetilde{M}_g^{\mathbb{C}})_A), g_A) \rightarrow (\widetilde{M}_g^{\mathbb{C}})_A, \widetilde{g}_A$ is a holomorphic and isometric immersion.*

Proof. For simplicity, we set $(M_g^{\mathbb{C}})'_{A,f:i} := (M_g^{\mathbb{C}})_{A,f:i} \cap (f^{\mathbb{C}})^{-1}((\widetilde{M}_g^{\mathbb{C}})_A)$. We suffice to show that $(f^{\mathbb{C}})^* \widetilde{g}_A = g_A$. Let g^h (resp. \widetilde{g}^h) be a holomorphic metric arising from g (resp. \widetilde{g}). Since $f^{\mathbb{C}}$ is holomorphic by Proposition 4.1, $((f^{\mathbb{C}})^{\mathbb{C}})^* \widetilde{g}^h$ is the holomorphic $(0, 2)$ -tensor field on $(M_g^{\mathbb{C}})'_{A,f:i}$. Also, it is clear that $((f^{\mathbb{C}})^{\mathbb{C}})^* \widetilde{g}^h|_{TM \times TM} = f^* \widetilde{g} (= g)$. Hence we get $((f^{\mathbb{C}})^{\mathbb{C}})^* \widetilde{g}^h = g^h$ on $(M_g^{\mathbb{C}})'_{A,f:i}$ and furthermore

$$\begin{aligned} (f^{\mathbb{C}})^* \widetilde{g}_A &= (f^{\mathbb{C}})^* \left((\widetilde{g}^h + \overline{\widetilde{g}^h})|_{T((\widetilde{M}_g^{\mathbb{C}})_A) \times T((\widetilde{M}_g^{\mathbb{C}})_A)} \right) \\ &= \left(((f^{\mathbb{C}})^{\mathbb{C}})^* \widetilde{g}^h + \overline{((f^{\mathbb{C}})^{\mathbb{C}})^* \widetilde{g}^h} \right) |_{T((M_g^{\mathbb{C}})'_{A,f:i}) \times T((M_g^{\mathbb{C}})'_{A,f:i})} \\ &= (g^h + \overline{g^h})|_{T((M_g^{\mathbb{C}})'_{A,f:i}) \times T((M_g^{\mathbb{C}})'_{A,f:i})} = g_A \end{aligned}$$

on $(M_g^{\mathbb{C}})'_{A,f:i}$. □

Definition 2. We call the anti-Kaehler submanifold $f^{\mathbb{C}} : ((M_g^{\mathbb{C}})'_{A,f:i}, g_A) \hookrightarrow ((\widetilde{M}_g^{\mathbb{C}})_A, \widetilde{g}_A)$ the *complexification of the Riemannian submanifold* $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$.

§6. Complete complexifications of pseudo-Riemannian homogeneous spaces

Let $(G/K, g)$ be a pseudo-Riemannian homogeneous space. Here we assume that G and K admit faithful real representations. Hence the complexifications $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ of G and K are defined. Since g_{eK} is invariant with respect to the K -action on $T_{eK}(G/K)$, its complexification $g_{eK}^{\mathbb{C}}$ is invariant

with respect to the $K^{\mathbb{C}}$ -action on $T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}})(= (T_{eK}(G/K))^{\mathbb{C}})$. Hence we obtain a $G^{\mathbb{C}}$ -invariant holomorphic metric \tilde{g}^h on $G^{\mathbb{C}}/K^{\mathbb{C}}$ from the \mathbb{C} -bilinear extension of $g_{eK}^{\mathbb{C}}$ to $(T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}))^{\mathbb{C}} \times (T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}))^{\mathbb{C}}$. Set $\tilde{g}_A := 2\text{Re}\tilde{g}^h|_{T(G^{\mathbb{C}}/K^{\mathbb{C}}) \times T(G^{\mathbb{C}}/K^{\mathbb{C}})}$, which is also $G^{\mathbb{C}}$ -invariant. Define a linear map $j : T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}) \rightarrow T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}})$ by $j(X) := \sqrt{-1}X$ ($X \in T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}})$). Since j is invariant with respect to the $K^{\mathbb{C}}$ -action on $T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}})$, we obtain a $G^{\mathbb{C}}$ -invariant almost complex structure \tilde{J} of $G^{\mathbb{C}}/K^{\mathbb{C}}$ from j . Then it is shown that (\tilde{J}, \tilde{g}_A) is an anti-Kaehler structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Also, it is clear that $(G^{\mathbb{C}}/K^{\mathbb{C}}, \tilde{J}, \tilde{g}_A)$ is geodesically complete. By identifying G/K with $G(eK^{\mathbb{C}})$, $G^{\mathbb{C}}/K^{\mathbb{C}}$ is regarded as the complete complexification of G/K . Define $\Phi : T(G/K) \rightarrow G^{\mathbb{C}}/K^{\mathbb{C}}$ by $\Phi(v) := \exp_p(\tilde{J}_p v)$ for $v \in T(G/K)$, where p is the base point of v and \exp_p is the exponential map of the anti-Kaehler manifold $(G^{\mathbb{C}}/K^{\mathbb{C}}, \tilde{J}, \tilde{g}_A)$ at p ($\in G/K = G(eK^{\mathbb{C}}) \subset G^{\mathbb{C}}/K^{\mathbb{C}}$). Note that this map Φ is called the polar map in [5].

Remark 6.1. For a C^ω -isometric immersion f of a C^ω -Riemannian manifold (M, g) into a Riemannian symmetric space $(G/K, g)$ of non-compact type, we [10] defined its complexification as an immersion of a tubular neighborhood of M in $(M_g^{\mathbb{C}})_{f:i}$ into $G^{\mathbb{C}}/K^{\mathbb{C}}$. It is shown that the complexification defined in [10] is equal to the composition of the complexification $f^{\mathbb{C}} : (M_g^{\mathbb{C}})_{f:i} \rightarrow (G/K)_g^{\mathbb{C}}$ defined in Section 4 and the polar map Φ .

Set $\tilde{\Omega} := \bigcup_{v \in T^\perp G(eK^{\mathbb{C}})} \{\exp(sv) \mid 0 \leq s < r_v\}$, where \exp is the exponential map of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and r_v is the first focal radius of $G(eK^{\mathbb{C}}) \subset G^{\mathbb{C}}/K^{\mathbb{C}}$ along γ_v . We have the following fact for Φ .

Theorem 6.1. *The restriction $\Phi|_{((G/K)_g^{\mathbb{C}})_A}$ of Φ to $((G/K)_g^{\mathbb{C}})_A$ is a diffeomorphism onto $\tilde{\Omega}$ and, each point of the boundary $\partial((G/K)_g^{\mathbb{C}})_A$ of $((G/K)_g^{\mathbb{C}})_A$ in $T(G/K)$ is a critical point of Φ . Furthermore, $\Phi|_{((G/K)_g^{\mathbb{C}})_A}$ is a holomorphic isometry (that is, $(\Phi|_{((G/K)_g^{\mathbb{C}})_A})^* \tilde{J} = J^g$ and $(\Phi|_{((G/K)_g^{\mathbb{C}})_A})^* \tilde{g}_A = g_A$).*

Proof. Let Ω be the connected component of $T(G/K)$ containing the 0-section ($= G/K$) of the set of all regular points of Φ . From the definition of Φ , it is easy to show that $v \in T_p(G/K) \subset T(G/K)$ is a critical point of Φ if and only if $\Phi(v)$ is a focal point of the orbit $G(eK^{\mathbb{C}})$ along γ_v or a conjugate point of p along γ_v . Hence we see that $\Phi(\Omega) = \tilde{\Omega}$ and that $\Phi|_\Omega$ is a diffeomorphism onto $\tilde{\Omega}$. Now we shall show that $\Phi|_\Omega$ is a holomorphic isometry. Let γ be a geodesic in G/K . We have

$$\begin{aligned} (\Phi \circ \gamma_*)(s + t\sqrt{-1}) &= \Phi(t\gamma'(s)) = \exp_{\gamma(s)}(\tilde{J}_{\gamma(s)}(t\gamma'(s))) \\ &= (\gamma_{t\gamma'(s)})^{\mathbb{C}}(\sqrt{-1}) = \gamma^{\mathbb{C}}(s + t\sqrt{-1}), \end{aligned}$$

where $(\gamma_{t\gamma'(s)})^{\mathbb{C}}$ (resp. $\gamma^{\mathbb{C}}$) is the complexification of $\gamma_{t\gamma'(s)}$ (resp. γ) in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Thus $\Phi \circ \gamma_*(: T\mathbb{R} = \mathbb{C} \rightarrow (G^{\mathbb{C}}/K^{\mathbb{C}}, \tilde{\mathcal{J}}))$ is holomorphic. Therefore, according to Theorem 3.4 of [19], $\Phi|_{(G/K)_g^{\mathbb{C}}}$ is holomorphic, that is, $(\Phi|_{(G/K)_g^{\mathbb{C}}})^*\tilde{\mathcal{J}} = J_A$. On the other hand, it is clear that $(\Phi|_{\Omega})^*\tilde{\mathcal{J}}$ is equal to J_A on Ω . Hence we have $\Omega \subset (G/K)_g^{\mathbb{C}}$. Since $(\Phi|_{\Omega})^*\tilde{g}^h$ is the non-extendable holomorphic metric arising from g . Hence we have $\Omega = ((G/K)_g^{\mathbb{C}})_A$. Hence the statement of this theorem follows. \square

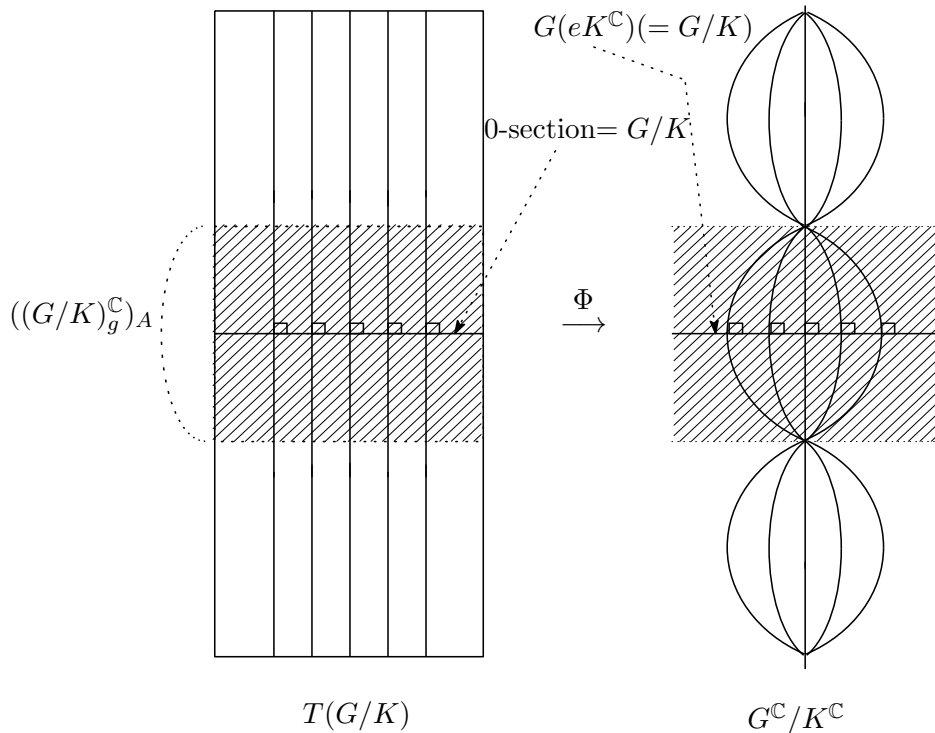


Figure 2.

§7. Duals of a pseudo-Riemannian manifolds

In this section, we shall define the dual of a C^ω -pseudo-Riemannian manifold and the dual of a totally geodesic C^ω -pseudo-Riemannian submanifold. Let (M, g) be a C^ω -pseudo-Riemannian manifold. For each $p \in M$, we set $M_p^* := (M_g^{\mathbb{C}})_A \cap T_pM$ and denote the inclusion map of M_p^* into $(M_g^{\mathbb{C}})_A$ by ι_p . For M_p^* , the following fact holds.

Proposition 7.1. *Let \exp_p be the exponential map of $((M_g^{\mathbb{C}})_A, g_A)$ at p and $D_p (\subset T_p((M_g^{\mathbb{C}})_A))$ be its domain. The above set M_p^* coincides with the*

geodesic umbrella $\exp_p(T_p(M_p^*) \cap D)$.

Proof. For each $X \in M_p^*$, we get $\text{id}_M^{\mathbb{C}}(X) = \gamma_X^{\mathbb{C}}(\sqrt{-1}) = \exp_p(J_p^g X)$. On the other hand, it is clear that $\text{id}_M^{\mathbb{C}} = \text{id}_{M_g^{\mathbb{C}}}$. Hence we get

$$X = \exp_p(J_p^g X) \in \exp_p(T_p(M_p^*) \cap D).$$

From the arbitrariness of X , we get $M_p^* \subset \exp_p(T_p(M_p^*) \cap D)$. It is clear that this relation implies $M_p^* = \exp_p(T_p(M_p^*) \cap D)$. □

Definition 3. We call the pseudo-Riemannian manifold $(M_p^*, \iota_p^* g_A)$ the *dual of (M, g) at p* .

The following question is proposed naturally:

Are (M, g) and $(M_p^*, \iota_p^* g_A)$ totally geodesic in $((M_g^{\mathbb{C}})_A, g_A)$?

For this question, we can show the following fact.

Proposition 7.2. *The submanifold (M, g) is totally geodesic in $((M_g^{\mathbb{C}})_A, g_A)$.*

Proof. Define $\sigma : M_g^{\mathbb{C}} \rightarrow M_g^{\mathbb{C}}$ by $\sigma(X) = -X$ ($X \in (M_g^{\mathbb{C}})_A$). It is clear that σ is an isometry of $((M_g^{\mathbb{C}})_A, g_A)$. Hence, since M is a component of the fixed point set of σ , (M, g) is totally geodesic in $((M_g^{\mathbb{C}})_A, g_A)$. □

Also, we can show the following fact in the case where (M, g) is a pseudo-Riemannian symmetric space.

Theorem 7.3. *Let $(G/K, g)$ be a pseudo-Riemannian symmetric space associated with a semi-simple symmetric pair (G, K) . Then $((G/K)_p^*, \iota_p^* g_A)$ is totally geodesic in $((G/K)_g^{\mathbb{C}})_A, g_A$.*

Proof. We suffice to show the statement in case of $p = eK (= eK^{\mathbb{C}})$ (e : the identity element of G). Let \mathfrak{g} be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition associated with (G, K) . Then $T_{eK}(G^{\mathbb{C}}/K^{\mathbb{C}})$ is identified with $\mathfrak{p}^{\mathbb{C}}$. Let Φ be as in Section 6. It follows from the definition of Φ that $\exp_{eK^{\mathbb{C}}}(\sqrt{-1}\mathfrak{p}) \supset \Phi((G/K)_{eK}^*)$. Since $\sqrt{-1}\mathfrak{p}$ is a Lie triple system of $\mathfrak{p}^{\mathbb{C}}$, $\exp_{eK^{\mathbb{C}}}(\sqrt{-1}\mathfrak{p})$ is totally geodesic in $G^{\mathbb{C}}/K^{\mathbb{C}}$. Hence, since $\Phi|_{((G/K)_g^{\mathbb{C}})_A}$ is an isometry into $G^{\mathbb{C}}/K^{\mathbb{C}}$ by Theorem 6.1, $(G/K)_{eK}^*$ is totally geodesic in $((G/K)_g^{\mathbb{C}})_A, g_A$. □

Let $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ be a C^ω -isometric immersion between C^ω -pseudo-Riemannian manifolds and set $(M_p^*)_f := M_p^* \cap (M_g^{\mathbb{C}})_f$. Then the following question is proposed naturally:

Is $f^{\mathbb{C}}((M_p^)_f)$ contained in $\widetilde{M}_{f(p)}^*$ for each $p \in M$?*

For this problem, we have the following fact.

Theorem 7.4. *If f is totally geodesic, then $f^{\mathbb{C}}((M_p^*)_f)$ is contained in $\widetilde{M}_{f(p)}^*$ for each $p \in M$.*

Proof. Let $X \in (M_p^*)_f$. Denote by $\exp_{f(p)}$ the exponential map of $((\widetilde{M}_g^{\mathbb{C}})_A, \widetilde{g}_A)$ at $f(p)$. Since f is totally geodesic and $\exp_{f(p)}$ is holomorphic, we have

$$\begin{aligned} f^{\mathbb{C}}(X) &= (f \circ \gamma_X)^h(\sqrt{-1}) = (\gamma_{f_*(X)})^{\mathbb{C}}(\sqrt{-1}) = \exp_{f(p)}(J_{f(p)}^{\widetilde{g}}(f_*(X))) \\ &\in \exp_{f(p)}(T_{f(p)}\widetilde{M}_{f(p)}^* \cap D), \end{aligned}$$

where γ_X (resp. $\gamma_{f_*(X)}$) is the geodesic in (M, g) (resp. $(\widetilde{M}, \widetilde{g})$) with $\dot{\gamma}_X(0) = X$ (resp. $\dot{\gamma}_{f_*(X)}(0) = f_*(X)$) and D is the domain of $\exp_{f(p)}$. According to Proposition 7.1, $\exp_{f(p)}(T_{f(p)}\widetilde{M}_{f(p)}^* \cap D)$ is equal to $\widetilde{M}_{f(p)}^*$. Therefore, we get $f^{\mathbb{C}}((M_p^*)_f) \subset \widetilde{M}_{f(p)}^*$. □

Definition 4. For a totally geodesic C^ω -pseudo-Riemannian submanifold $f(M)$ in $(\widetilde{M}, \widetilde{g})$, we call a submanifold $f^{\mathbb{C}}((M_p^*)_f)$ in $(\widetilde{M}_{f(p)}^*, \iota_{f(p)}^*\widetilde{g}_A)$ the *dual of $f(M)$* .

Example. Let G/K be a pseudo-Riemannian symmetric space, H be a symmetric subgroup of G , θ be the involution of G with $(\text{Fix } \theta)_0 \subset K \subset \text{Fix } \theta$ and σ be the involution of G with $(\text{Fix } \sigma)_0 \subset H \subset \text{Fix } \sigma$, where $(\text{Fix } \theta)_0$ (resp. $(\text{Fix } \sigma)_0$) is the identity component of $\text{Fix } \theta$ (resp. $\text{Fix } \sigma$). Assume that $\theta \circ \sigma = \sigma \circ \theta$. Also, let G^* be the dual of G with respect to K and H^* be the dual of H with respect to $H \cap K$. Then the orbit $H(eK) (\subset G/K)$ is totally geodesic and hence $\iota^{\mathbb{C}}((H(eK))_{eK}^*)$ is contained in $(G/K)_{eK}^* (= G^*/K)$, where $\iota^{\mathbb{C}}$ is the complexification of the inclusion map of $H(eK)$ into G/K . Furthermore, $\iota^{\mathbb{C}}((H(eK))_{eK}^*)$ coincides with the orbit $H^*(eK) (\subset G^*/K = (G/K)_{eK}^*)$.

§8. Complex focal radii

In this section, we shall introduce the notions of a complex Jacobi field along a complex geodesic in an anti-Kaehler manifold. Also, we give a new definition of a complex focal radius of anti-Kaehler submanifold by using the notion of

a complex Jacobi field and show that the notion of a complex focal radius by this new definition coincides with one defined in [10] (see Proposition 8.4). Next we show a fact which is very useful to calculate the complex focal radii of an anti-Kaehler submanifold with section in an anti-Kaehler symmetric space (see Proposition 8.5). Also, we show that a complex focal radius of a C^ω -Riemannian submanifold in a Riemannian symmetric space G/K of non-compact type (see Definition 6 about the definition of this notion) coincides with one defined in [9] (see Proposition 8.6). Let (M, J, g) be an anti-Kaehler manifold, ∇ (resp. R) be the Levi-Civita connection (resp. the curvature tensor) of g and $\nabla^{\mathbb{C}}$ (resp. $R^{\mathbb{C}}$) be the complexification of ∇ (resp. R). Let $(TM)^{(1,0)}$ be the holomorphic vector bundle consisting of complex vectors of M of type $(1, 0)$. Note that the restriction of $\nabla^{\mathbb{C}}$ to $(TM)^{(1,0)}$ is a holomorphic connection of $(TM)^{(1,0)}$ (see Theorem 2.2 of [1]). For simplicity, assume that (M, J, g) is complete even if the discussion of this section is valid without the assumption of the completeness of (M, J, g) . Let $\gamma : \mathbb{C} \rightarrow M$ be a complex geodesic, that is, $\gamma(z) = \exp_{\gamma(0)}((\operatorname{Re} z)\gamma_*((\frac{\partial}{\partial s})_0) + (\operatorname{Im} z)J_{\gamma(0)}\gamma_*((\frac{\partial}{\partial s})_0))$, where (z) is the complex coordinate of \mathbb{C} and $s := \operatorname{Re} z$. Let $Y : \mathbb{C} \rightarrow (TM)^{(1,0)}$ be a holomorphic vector field along γ . That is, Y assigns $Y_z \in (T_{\gamma(z)}M)^{(1,0)}$ to each $z \in \mathbb{C}$ and, for each holomorphic local coordinate $(U, (z_1, \dots, z_n))$ of M with $U \cap \gamma(\mathbb{C}) \neq \emptyset$, $Y_i : \gamma^{-1}(U) \rightarrow \mathbb{C}$ ($i = 1, \dots, n$) defined by $Y_z = \sum_{i=1}^n Y_i(z)(\frac{\partial}{\partial z_i})_{\gamma(z)}$ are holomorphic.

Definition 5. If Y satisfies $\nabla_{\gamma_* (\frac{d}{dz})}^{\mathbb{C}} \nabla_{\gamma_* (\frac{d}{dz})}^{\mathbb{C}} Y + R^{\mathbb{C}}(Y, \gamma_* (\frac{d}{dz}))\gamma_* (\frac{d}{dz}) = 0$, then we call Y a *complex Jacobi field along γ* . Let $z_0 \in \mathbb{C}$. If there exists a (non-zero) complex Jacobi field Y along γ with $Y_0 = 0$ and $Y_{z_0} = 0$, then we call z_0 a *complex conjugate radius* of $\gamma(0)$ along γ . Let $\delta : \mathbb{C} \times D(\varepsilon) \rightarrow M$ be a holomorphic two-parameter map, where $D(\varepsilon)$ is the ε -disk centered at 0 in \mathbb{C} . Denote by z (resp. w) the first (resp. second) parameter of δ . If $\delta(\cdot, w_0) : \mathbb{C} \rightarrow M$ is a complex geodesic for each $w_0 \in D(\varepsilon)$, then we call δ a *complex geodesic variation*.

Easily we can show the following fact.

Proposition 8.1. *Let $\delta : \mathbb{C} \times D(\varepsilon) \rightarrow M$ be a complex geodesic variation. The complex variational vector field $Y := \delta_* (\frac{\partial}{\partial w} |_{w=0})$ is a complex Jacobi field along $\gamma := \delta(\cdot, 0)$.*

A vector field X on M is said to be *real holomorphic* if the Lie derivation $L_X J$ of J with respect to X vanishes. It is known that X is a real holomorphic vector field if and only if the complex vector field $X - \sqrt{-1}JX$ is holomorphic.

We have the following fact for a complex Jacobi field.

Proposition 8.2. *Let $\gamma : \mathbb{C} \rightarrow M$ be a complex geodesic.*

(i) *Let Y be a holomorphic vector field along γ and $Y_{\mathbb{R}}$ be the real part of Y . Then Y is a complex Jacobi field along γ if and only if, for any $z_0 \in \mathbb{C}$, $u \mapsto (Y_{\mathbb{R}})_{uz_0}$ is a Jacobi field along the geodesic γ_{z_0} (which is defined by $\gamma_{z_0}(u) := \gamma(uz_0)$).*

(ii) *A complex number z_0 is a complex conjugate radius of $\gamma(0)$ along γ if and only if $\gamma(z_0)$ is a conjugate point of $\gamma(0)$ along the geodesic γ_{z_0} .*

Proof. Let (z) ($z = s + t\sqrt{-1}$) be the natural coordinate of \mathbb{C} . Let $Y (= Y_{\mathbb{R}} - \sqrt{-1}JY_{\mathbb{R}})$ be a holomorphic vector field along γ . From $L_{Y_{\mathbb{R}}}J = 0$ and $\nabla J = 0$, we have

$$(8.1) \quad \begin{aligned} & \nabla_{\gamma_*\left(\frac{d}{dz}\right)}^{\mathbb{C}} \nabla_{\gamma_*\left(\frac{d}{dz}\right)}^{\mathbb{C}} Y + R^{\mathbb{C}}(Y, \gamma_*\left(\frac{d}{dz}\right))\gamma_*\left(\frac{d}{dz}\right) \\ &= \nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} \nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} Y_{\mathbb{R}} + R(Y_{\mathbb{R}}, \gamma_*\left(\frac{\partial}{\partial s}\right))\gamma_*\left(\frac{\partial}{\partial s}\right) \\ & \quad - \sqrt{-1}J \left(\nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} \nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} Y_{\mathbb{R}} + R(Y_{\mathbb{R}}, \gamma_*\left(\frac{\partial}{\partial s}\right))\gamma_*\left(\frac{\partial}{\partial s}\right) \right). \end{aligned}$$

Assume that Y is a complex Jacobi field. Then it follows from (8.1) that

$$\nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} \nabla_{\gamma_*\left(\frac{\partial}{\partial s}\right)} Y_{\mathbb{R}} + R(Y_{\mathbb{R}}, \gamma_*\left(\frac{\partial}{\partial s}\right))\gamma_*\left(\frac{\partial}{\partial s}\right) = 0.$$

Let $X := a\gamma_*\left(\frac{\partial}{\partial s}\right) + b\gamma_*\left(\frac{\partial}{\partial t}\right)$ ($a, b \in \mathbb{R}$). Furthermore, from $L_{Y_{\mathbb{R}}}J = 0$ and $\nabla J = 0$, we have

$$\nabla_X \nabla_X Y_{\mathbb{R}} + R(Y_{\mathbb{R}}, X)X = 0.$$

Hence we see that $u \mapsto (Y_{\mathbb{R}})_{uz_0}$ is a Jacobi field along γ_{z_0} for each $z_0 \in \mathbb{C}$. The converse also is shown in terms of (8.1), $L_{Y_{\mathbb{R}}}J = 0$ and $\nabla J = 0$ directly. Thus the statement (i) is shown. Assume that z_0 is a complex conjugate radius of $\gamma(0)$ along γ . Then there exists a non-trivial complex Jacobi field Y along γ with $Y_0 = 0$ and $Y_{z_0} = 0$. According to (i), $u \mapsto (Y_{\mathbb{R}})_{uz_0}$ is a Jacobi field along γ_{z_0} which vanishes at $u = 0, 1$. Furthermore, it is shown that $u \mapsto (Y_{\mathbb{R}})_{uz_0}$ is non-trivial because so is Y . Hence $\gamma(z_0)$ is a conjugate point of $\gamma(0)$ along γ_{z_0} . Conversely, assume that $\gamma(z_0)$ is a conjugate point of $\gamma(0)$ along γ_{z_0} . Then there exists a non-trivial Jacobi field \bar{Y} along γ_{z_0} with $\bar{Y}_0 = 0$ and $\bar{Y}_1 = 0$. There exists the complex Jacobi field Y along γ with $Y_0 = 0$ and $\nabla_{\gamma'_{z_0(0)}}^{\mathbb{C}} Y = \bar{Y}'_0 - \sqrt{-1}J\bar{Y}'_0$ by the existenceness of solutions of a complex ordinary differential equation. It is easy to show that $(Y_{\mathbb{R}})_{uz_0} = \bar{Y}_u$ for all $u \in \mathbb{R}$. Hence we have $(Y_{\mathbb{R}})_{z_0} = \bar{Y}_1 = 0$, that is, $Y_{z_0} = 0$. Therefore z_0 is a complex conjugate radius of $\gamma(0)$ along γ . Thus the statement (ii) is shown. □

Next we shall define the notion of the parallel translation along a holomorphic curve. Let $\alpha : D \rightarrow (M, J, g)$ be a holomorphic curve, where D is an open set of \mathbb{C} . Let Y be a holomorphic vector field along α . If $\nabla_{\alpha_* (\frac{d}{dz})}^{\mathbb{C}} Y = 0$, then we say that Y is *parallel*. For a parallel holomorphic vector field, we can show the following fact.

Proposition 8.3. *Let $\alpha : D \rightarrow (M, J, g)$ be a holomorphic curve. Take $z_0 \in D$ and $v \in (T_{\alpha(z_0)}M)^{(1,0)}$. Then the following statements (i) and (ii) hold.*

(i) *There uniquely exists a parallel holomorphic vector field Y along α with $Y_{z_0} = v$.*

(ii) *Let Y be a holomorphic vector field along α and $Y_{\mathbb{R}}$ be its real part. Then Y is parallel if and only if, for any (real) curve σ in D , $u \mapsto (Y_{\mathbb{R}})_{\sigma(u)}$ is parallel along $\alpha \circ \sigma$ with respect to ∇ .*

Proof. The statement (i) follows from the existenceness and the uniqueness of solutions of a complex ordinary differential equation. The statement (ii) is shown as follows. From $\nabla J = 0$ and $L_{Y_{\mathbb{R}}} J = 0$, we have $\nabla_{\alpha_* (\frac{d}{dz})}^{\mathbb{C}} Y = \frac{1}{2}(\nabla_{\alpha_* (\frac{\partial}{\partial s})} Y_{\mathbb{R}} - \sqrt{-1}J\nabla_{\alpha_* (\frac{\partial}{\partial s})} Y_{\mathbb{R}})$. Hence Y is parallel if and only if $\nabla_{\alpha_* (\frac{\partial}{\partial s})} Y_{\mathbb{R}} = 0$. Let $X := a\gamma_* (\frac{\partial}{\partial s}) + b\gamma_* (\frac{\partial}{\partial t})$ ($a, b \in \mathbb{R}$). From $\nabla J = 0$ and $L_{Y_{\mathbb{R}}} J = 0$, it follows that $\nabla_{\alpha_* (\frac{\partial}{\partial s})} Y_{\mathbb{R}} = 0$ is equivalent to $\nabla_X Y_{\mathbb{R}} = 0$. Therefore, the statement (ii) follows. □

Let α , z_0 and v be as in the statement of Proposition 8.3. There uniquely exists a parallel holomorphic vector field Y along α with $Y_{z_0} = v$. We denote Y_{z_1} by $(P_{\alpha})_{z_0, z_1}(v)$. It is clear that $(P_{\alpha})_{z_0, z_1}$ is a \mathbb{C} -linear isomorphism of $(T_{\alpha(z_0)}M)^{(1,0)}$ onto $(T_{\alpha(z_1)}M)^{(1,0)}$. We call $(P_{\alpha})_{z_0, z_1}$ the *parallel translation along α from z_0 to z_1* .

Let f be an immersion of an anti-Kaehler manifold (M, J, g) into another anti-Kaehler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. If $f_* \circ J = \widetilde{J} \circ f_*$ and $f^* \widetilde{g} = g$, then we call f an *anti-Kaehler immersion* and (M, J, g) an *anti-Kaehler submanifold immersed by f* . In the sequel, we omit the notation f_* . In [10], we introduced the notion of a complex focal radius of an anti-Kaehler submanifold. Now we shall define this notion in terms of a complex Jacobi field. Let $v \in T_{p_0}^{\perp} M$ and $\gamma_v^{\mathbb{C}} (: D \rightarrow \widetilde{M})$ be the (maximal) complex geodesic in $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with $(\gamma_v^{\mathbb{C}})_* ((\frac{d}{dz})_0) = \frac{1}{2}(v - \sqrt{-1}\widetilde{J}v)$, where $T_{p_0}^{\perp} M$ is the normal space of M at p_0 and D is a neighborhood of 0 in \mathbb{C} .

Definition 6. If there exists a complex Jacobi field Y along $\gamma_v^{\mathbb{C}}$ with $Y_0 (\neq 0) \in (T_{p_0}M)^{(1,0)}$ and $Y_{z_0} = 0$, then we call the complex number z_0 a *complex*

focal radius of M along $\gamma_v^{\mathbb{C}}$.

By imitating the proof of (ii) of Proposition 8.2, we can show the following fact.

Proposition 8.4. *A complex number z_0 is a complex focal radius of M along the normal complex geodesic $\gamma_v^{\mathbb{C}}$ if and only if $\gamma_v^{\mathbb{C}}(z_0)$ is a focal point of M along the normal geodesic $(\gamma_v^{\mathbb{C}})_{z_0}$ (which is defined by $(\gamma_v^{\mathbb{C}})_{z_0}(u) := \gamma_v^{\mathbb{C}}(uz_0)$), that is, z_0 is a complex focal radius in the sense of [10].*

We consider the case where $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ is an anti-Kaehler symmetric space $G^{\mathbb{C}}/K^{\mathbb{C}}$ and where the anti-Kaehler submanifold M is a subset of $G^{\mathbb{C}}/K^{\mathbb{C}}$ (hence f is the inclusion map). For $v \in (T_{b_0K^{\mathbb{C}}}^{\perp}M)^{\mathbb{C}}$, we define \mathbb{C} -linear transformations \widehat{D}_v^{co} and \widehat{D}_v^{si} of $(T_{b_0K^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}))^{\mathbb{C}}$ by

$$\widehat{D}_v^{co} := b_{0*}^{\mathbb{C}} \circ \cos(\sqrt{-1}\text{ad}_{\mathfrak{g}^{\mathbb{C}}}^{\mathbb{C}}((b_{0*}^{\mathbb{C}})^{-1}v)) \circ (b_{0*}^{\mathbb{C}})^{-1}$$

and

$$\widehat{D}_v^{si} := b_{0*}^{\mathbb{C}} \circ \frac{\sin(\sqrt{-1}\text{ad}_{\mathfrak{g}^{\mathbb{C}}}^{\mathbb{C}}((b_{0*}^{\mathbb{C}})^{-1}v))}{\sqrt{-1}\text{ad}_{\mathfrak{g}^{\mathbb{C}}}^{\mathbb{C}}((b_{0*}^{\mathbb{C}})^{-1}v)} \circ (b_{0*}^{\mathbb{C}})^{-1},$$

respectively, where $\text{ad}_{\mathfrak{g}^{\mathbb{C}}}^{\mathbb{C}}$ is the complexification of the adjoint representation $\text{ad}_{\mathfrak{g}^{\mathbb{C}}}$ of $\mathfrak{g}^{\mathbb{C}}$. If, for each $bK^{\mathbb{C}} \in M$, $b_*^{-1}(T_{bK^{\mathbb{C}}}^{\perp}M) (\subset T_{eK^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}) \subset \mathfrak{g}^{\mathbb{C}})$ is a Lie triple system (resp. abelian subspace), that is, $\exp^{\perp}(T_{bK^{\mathbb{C}}}^{\perp}M)$ is totally geodesic (resp. flat and totally geodesic), then M is said to *have section* (resp. *have flat section*), where \exp^{\perp} is the normal exponential map of M .

Proposition 8.5. *Let M be an anti-Kaehler submanifold in $G^{\mathbb{C}}/K^{\mathbb{C}}$ with section and $v \in T_{b_0K^{\mathbb{C}}}^{\perp}M$. Set $v_{(1,0)} := \frac{1}{2}(v - \sqrt{-1}\widetilde{J}v)$. A complex number z_0 is a complex focal radius along $\gamma_v^{\mathbb{C}}$ if and only if*

$$\text{Ker} \left(\widehat{D}_{z_0v_{(1,0)}}^{co} - \widehat{D}_{z_0v_{(1,0)}}^{si} \circ (A^{\mathbb{C}})_{z_0v_{(1,0)}} \right) \Big|_{(T_{b_0K^{\mathbb{C}}}M)^{(1,0)}} \neq \{0\},$$

where $A^{\mathbb{C}}$ is the complexification of the shape tensor A of M .

Proof. Denote by $\widetilde{\nabla}$ (resp. \widetilde{R}) the Levi-Civita connection (resp. the curvature tensor) of $G^{\mathbb{C}}/K^{\mathbb{C}}$ and by $\widetilde{\nabla}^{\mathbb{C}}$ (resp. $\widetilde{R}^{\mathbb{C}}$) their complexification. Let Y be a holomorphic vector field along $\gamma_v^{\mathbb{C}}$. Define $\widehat{Y} : D \rightarrow (T_{b_0K^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}}))^{(1,0)}$ by $\widehat{Y}_z := (P_{\gamma_v^{\mathbb{C}}})_{z,0}(Y_z)$ ($z \in D$), where D is the domain of $\gamma_v^{\mathbb{C}}$. Easily we can show $\widetilde{\nabla}_{(\gamma_v^{\mathbb{C}})_*(\frac{d}{dz})}^{\mathbb{C}} \widehat{Y} = (P_{\gamma_v^{\mathbb{C}}})_{0,z}(\frac{d^2\widehat{Y}}{dz^2})$. From $\widetilde{\nabla}\widetilde{R} = 0$ (hence $\widetilde{\nabla}^{\mathbb{C}}\widetilde{R}^{\mathbb{C}} = 0$), we

have $\widetilde{R}^{\mathbb{C}}(Y, (\gamma_v^{\mathbb{C}})_*(\frac{d}{dz}))(\gamma_v^{\mathbb{C}})_*(\frac{d}{dz}) = (P_{\gamma_v^{\mathbb{C}}})_{0,z}(R_{b_0K^{\mathbb{C}}}^{\mathbb{C}}(\widehat{Y}_z, v_{(1,0)})v_{(1,0)})$. Hence Y is a complex Jacobi field if and only if $\frac{d^2\widehat{Y}}{dz^2} + R_{b_0K^{\mathbb{C}}}^{\mathbb{C}}(\widehat{Y}_z, v_{(1,0)})v_{(1,0)} = 0$ holds. By noticing

$$R_{b_0K^{\mathbb{C}}}^{\mathbb{C}}(\widehat{Y}_z, v_{(1,0)})v_{(1,0)} = -(b_{0*}^{\mathbb{C}} \circ \text{ad}_{\mathfrak{g}^{\mathbb{C}}}^{\mathbb{C}}((b_{0*}^{\mathbb{C}})^{-1}v_{(1,0)})^2 \circ (b_{0*}^{\mathbb{C}})^{-1})(\widehat{Y}_z)$$

and solving this complex ordinary differential equation, we have

$$\widehat{Y}_z = \widehat{D}_{zv(1,0)}^{co}(Y_0) + z\widehat{D}_{zv(1,0)}^{si} \left(\left. \frac{d\widehat{Y}}{dz} \right|_{z=0} \right).$$

Since M has section, both $\widehat{D}_{zv(1,0)}^{co}$ and $\widehat{D}_{zv(1,0)}^{si}$ preserve $(T_{b_0K^{\mathbb{C}}}M)^{\mathbb{C}}$ (and hence also $(T_{b_0K^{\mathbb{C}}}^{\perp}M)^{\mathbb{C}}$) invariantly. Hence, if $Y_0(\neq 0) \in (T_{b_0K^{\mathbb{C}}}M)^{\mathbb{C}}$ and $Y_{z_0} = 0$ for some z_0 , then we have $\frac{d\widehat{Y}}{dz}|_{z=0} \in (T_{b_0K^{\mathbb{C}}}M)^{\mathbb{C}}$, that is, $\frac{d\widehat{Y}}{dz}|_{z=0} = -(A^{\mathbb{C}})_{v(1,0)}(Y_0)$. Hence we have

$$(8.2) \quad Y_z = (P_{\gamma_v^{\mathbb{C}}})_{0,z}((\widehat{D}_{zv(1,0)}^{co} - \widehat{D}_{zv(1,0)}^{si} \circ (A^{\mathbb{C}})_{zv(1,0)})(Y_0)).$$

From this fact, the statement of this theorem follows. □

Let $f : (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$ be a C^ω -isometric immersion between C^ω -pseudo-Riemannian manifolds and $f^{\mathbb{C}} : ((M_g^{\mathbb{C}})'_{A,f;i}, g_A) \hookrightarrow ((\widetilde{M}_{\widetilde{g}}^{\mathbb{C}})_A, \widetilde{g}_A)$ be its complexification (see Definition 2).

Definition 7. For each normal vector $v(\neq 0)$ of M (in \widetilde{M}), we call a complex focal radius of $(M_g^{\mathbb{C}})'_{A,f;i}$ along $\gamma_v^{\mathbb{C}}$ a *complex focal radius of M along the normal geodesic γ_v* (in \widetilde{M}).

We consider the case where $(\widetilde{M}, \widetilde{g})$ is a Riemannian symmetric space G/K of non-compact type and where M has section. Let $v \in T_{b_0K^{\mathbb{C}}}^{\perp}M$ and $z(= s + t\sqrt{-1}) \in \mathbb{C}$. In [15], we defined the linear map D_{zv}^{co} (resp. D_{zv}^{si}) of $T_{b_0K^{\mathbb{C}}}(M_g^{\mathbb{C}})(= (T_{b_0K}M)^{\mathbb{C}})$ into $T_{b_0K^{\mathbb{C}}}(G^{\mathbb{C}}/K^{\mathbb{C}})(= (T_{b_0K}(G/K))^{\mathbb{C}})$ by

$$\left(\begin{array}{l} D_{zv}^{co} := b_{0*} \circ \cos \left(\sqrt{-1} \text{ad}_{\mathfrak{g}^{\mathbb{C}}}(b_{0*}^{-1}(sv + t\widetilde{J}v)) \right) \circ b_{0*}^{-1} \\ \text{resp. } D_{zv}^{si} := b_{0*} \circ \frac{\sin \left(\sqrt{-1} \text{ad}_{\mathfrak{g}^{\mathbb{C}}}(b_{0*}^{-1}(sv + t\widetilde{J}v)) \right)}{\sqrt{-1} \text{ad}_{\mathfrak{g}^{\mathbb{C}}}(b_{0*}^{-1}(sv + t\widetilde{J}v))} \circ b_{0*}^{-1} \end{array} \right).$$

The relations between these operators and the above operators \widehat{D}_{zv}^{co} and \widehat{D}_{zv}^{si} are as follows:

$$(8.3) \quad \widehat{D}_{zv(1,0)}^{co}(X - \sqrt{-1}JX) = D_{zv}^{co}(X) - \sqrt{-1}J(D_{zv}^{co}(X))$$

and

$$(8.4) \quad \widehat{D}_{zv(1,0)}^{si}(X - \sqrt{-1}JX) = D_{zv}^{si}(X) - \sqrt{-1}J(D_{zv}^{si}(X)),$$

where $X \in T_{b_0K^c}(M_g^C)$. From (8.2), (8.3) and (8.4), we have

$$(8.5) \quad (Y_{\mathbb{R}})_z = (P_{(\gamma_v^C)_z})_{0,1}((D_{zv}^{co} - D_{zv}^{si} \circ A_{zv}^C)((Y_{\mathbb{R}})_0))$$

for a complex Jacobi field Y along γ_v^C such that Y_0 and $\nabla_{(\gamma_v^C)_*((\frac{d}{dz})_0)}Y$ belong to $(T_{b_0K^c}(M_g^C))^C$, where $(P_{(\gamma_v^C)_z})_{0,1}$ is the parallel translation along $(\gamma_v^C)_z$ ($: u \mapsto \gamma_v^C(uz)$) from 0 to 1 and A is the shape tensor of (M, g) . Hence we have the following fact.

Proposition 8.6. *Let M be a C^ω -Riemannian submanifold in a Riemannian symmetric space G/K of non-compact type. Then $z(\in \mathbb{C})$ is a complex focal radius along γ_v (in the sense of Definition 7) if and only if $\text{Ker}(D_{zv}^{co} - D_{zv}^{si} \circ A_{zv}^C) \neq \{0\}$, where A is the shape tensor of M , that is, z is a complex focal radius along γ_v in the sense of [9].*

§9. Complex equifocal submanifolds and isoparametric ones

In [10], we defined the notion of a complex equifocal submanifold in a Riemannian symmetric space of non-compact type by imposing the condition related to complex focal radii. See [23] about the notion of equifocal submanifolds in Riemannian symmetric spaces. In the previous section, we defined the notion of a complex focal radius for C^ω -pseudo-Riemannian submanifold in a general C^ω -pseudo-Riemannian manifold. By imposing the same condition related to complex focal radii, we shall define the notion of a complex equifocal submanifold in a pseudo-Riemannian homogeneous space. Let M be a C^ω -pseudo-Riemannian submanifold in a C^ω -pseudo-Riemannian homogeneous space \widetilde{M} . If M has flat section, if the normal holonomy group of M is trivial and if, for any parallel normal vector field v of M , the complex focal radii along γ_{v_x} are independent of the choice of $x \in M$ (considering their multiplicities), then we call M a *complex equifocal submanifold*. If M has flat section, if the normal holonomy group of M is trivial and if, any sufficiently close parallel submanifolds of M have constant mean curvature with respect to the radial direction, then M is called an *isoparametric submanifold with flat section*. If, for each normal vector v of M , the Jacobi operator $R(\cdot, v)v$ preserves T_xM (x : the base point of v) invariantly and $[A_v, R(\cdot, v)v|_{T_xM}] = 0$, then M is called a *curvature-adapted submanifold*, where R is the curvature tensor of \widetilde{M} and A is the shape tensor of M . By imitating the proof of Theorem 15 in [10], we

can show the following facts for pseudo-Riemannian submanifolds in a semi-simple pseudo-Riemannian symmetric space. See [3,17] about the basic facts for pseudo-Riemannian symmetric spaces.

Proposition 9.1. *Let (M, g) be a C^ω -pseudo-Riemannian submanifold in a semi-simple pseudo-Riemannian symmetric space G/K equipped with the metric \tilde{g} induced from the Killing form of $\mathfrak{g} := \text{Lie } G$. Then the following statements (i) and (ii) hold:*

- (i) *If M is an isoparametric submanifold with flat section, then it is complex equifocal.*
- (ii) *Let M be a curvature-adapted complex equifocal submanifold. If, for any normal vector w of M , $R^{\mathbb{C}}(\cdot, w)|_{(T_x M)^{\mathbb{C}}}$ (x : the base point of w) and the complexified shape operator $A_w^{\mathbb{C}}$ are diagonalizable, then it is an isoparametric submanifold with flat section.*

Proof. Let M be a C^ω -pseudo-Riemannian submanifold with flat section in G/K whose normal holonomy group is trivial. Let v be a parallel normal vector field on M . Since M has flat section, $R(\cdot, v_x)v_x$ preserves $T_x M$ invariantly for each $x \in M$. Hence the \mathbb{C} -linear transformations $D_{zv_x}^{co}$ and $D_{zv_x}^{si}$ preserve $(T_x M)^{\mathbb{C}} (= T_x(M_g^{\mathbb{C}}))$ invariantly. Let $\eta_{sv} := \exp^\perp \circ sv (M \rightarrow G/K)$ and $M_{sv} := \eta_{sv}(M)$, where s is sufficiently close to zero. Define a function F_{sv} on M by $\eta_{sv}^* \omega_{sv} = F_{sv} \omega$, where ω (resp. ω_{sv}) is the volume element of M (resp. M_{sv}). Set $\widehat{F}_{v_x}(s) := F_{sv}(x)$ ($x \in M$). From (8.5), it follows that \widehat{F}_{v_x} ($x \in M$) has holomorphic extension (which is denoted by $\widehat{F}_{v_x}^h$) and that

$$(9.1) \quad \widehat{F}_{v_x}^h(z) = \det(D_{zv_x}^{co} - D_{zv_x}^{si} \circ A_{zv_x}^{\mathbb{C}}) \quad (z \in \mathbb{C}),$$

where $A^{\mathbb{C}}$ is the complexification of the shape tensor A of M , that is, the shape tensor of $M_g^{\mathbb{C}}$ and $D_{zv_x}^{co} - D_{zv_x}^{si} \circ A_{zv_x}^{\mathbb{C}}$ is regarded as a \mathbb{C} -linear transformation of $(T_x M)^{\mathbb{C}}$. By imitating the proof of Corollary 2.6 of [8], M is an isoparametric submanifold with flat section if and only if the projection from M to any (sufficiently close) parallel submanifold along the sections is volume preserving up to a constant factor (i.e., $\widehat{F}_{v_x}^h$ is independent of the choice of $x \in M$ for every parallel normal vector field v of M). On the other hand, the complex focal radii along the geodesic γ_{v_x} are caught as zero points of $\widehat{F}_{v_x}^h$. Hence we see that M is complex equifocal if and only if $(F_{v_x}^h)^{-1}(0)$ is independent of the choice of $x \in M$ for every parallel normal vector field v of M . From these facts, the statement (i) follows. Next we shall show the statement (ii). Let M be a curvature-adapted complex equifocal submanifold satisfying the conditions of the statement (ii), v be any parallel normal vector field of M and x be any point of M . Since M is curvature-adapted, $R^{\mathbb{C}}(\cdot, v_x)v_x$ preserves $(T_x M)^{\mathbb{C}}$ invariantly, $R^{\mathbb{C}}(\cdot, v_x)v_x|_{(T_x M)^{\mathbb{C}}}$ commutes with $A_{v_x}^{\mathbb{C}}$. Also, $R^{\mathbb{C}}(\cdot, v_x)v_x|_{(T_x M)^{\mathbb{C}}}$ and $A_{v_x}^{\mathbb{C}}$ are diagonalizable by the assumption. Hence they

are simultaneously diagonalizable. Hence, for each $x_0 \in M$, there exists a continuous orthonormal tangent frame field (e_1, \dots, e_n) of $(TM)^\mathbb{C}$ defined on a connected open neighborhood U of x_0 in M such that $R^\mathbb{C}(e_i, v)v = -\beta_i^2 e_i$ and $A_v^\mathbb{C}e_i = \lambda_i e_i$ ($i = 1, \dots, n$), where $n := \dim M$, β_i and λ_i ($i = 1, \dots, n$) are continuous complex-valued functions on U . From (9.1), we have

$$(9.2) \quad \widehat{F}_{v_x}^h(z) = \prod_{i=1}^n \left(\cos(\sqrt{-1}z\beta_i(x)) - \frac{\lambda_i(x) \sin(\sqrt{-1}z\beta_i(x))}{\sqrt{-1}\beta_i(x)} \right) \quad (x \in U).$$

Hence we have

$$(9.3) \quad (\widehat{F}_{v_x}^h)^{-1}(0) = \bigcup_{i=1}^n \left\{ z \mid \cos(\sqrt{-1}z\beta_i(x)) = \frac{\lambda_i(x) \sin(\sqrt{-1}z\beta_i(x))}{\sqrt{-1}\beta_i(x)} \right\} \quad (x \in U).$$

Since M is complex equifocal, we have $(\widehat{F}_{v_x}^h)^{-1}(0)$ is independent of the choice of $x \in U$. Hence, it follows from (9.3) that β_i and λ_i ($i = 1, \dots, n$) are constant on U . Furthermore, it follows from (9.2) that $\widehat{F}_{v_x}^h$ is independent of the choice of $x \in U$. From the arbitrariness of x_0 , $\widehat{F}_{v_x}^h$ is independent of the choice of $x \in M$. Thus M is an isoparametric submanifold with flat section. \square

According to Theorem A of [14], we have the following fact.

Proposition 9.2([14]). *Let G/K be a (semi-simple) pseudo-Riemannian symmetric space and H be a symmetric subgroup of G , τ (resp. σ) be an involution of G with $(\text{Fix } \tau)_0 \subset K \subset \text{Fix } \tau$ (resp. $(\text{Fix } \sigma)_0 \subset H \subset \text{Fix } \sigma$), $L := (\text{Fix}(\sigma \circ \tau))_0$ and $\mathfrak{l} := \text{Lie } L$, where $\text{Fix}(\cdot)$ is the fixed point group of (\cdot) and $\text{Fix}(\cdot)_0$ is the identity component of $\text{Fix}(\cdot)$. Assume that $\sigma \circ \tau = \tau \circ \sigma$. Let M be a principal orbit of the H -action on G/K through a point $\exp_G(v)K$ ($v \in \mathfrak{q}_K \cap \mathfrak{q}_H$ s.t. $\text{ad}(v)|_{\mathfrak{l}}$: semi-simple), where $\mathfrak{q}_K := \text{Ker}(\tau + \text{id})$ and $\mathfrak{q}_H := \text{Ker}(\sigma + \text{id})$. Then M is a curvature-adapted complex equifocal submanifold and, for each normal vector w of M , $R^\mathbb{C}(\cdot, w)w|_{(T_x M)^\mathbb{C}}$ (x : the base point of w) and $A_w^\mathbb{C}$ are diagonalizable. Also the orbit $H(eK)$ is a reflective focal submanifold of M .*

By using Theorem 6.1, Propositions 9.1 and 9.2, we prove the following fact.

Theorem 9.3. *Let $(G/K, g)$ be a (semi-simple) pseudo-Riemannian symmetric space. Then $((G/K)_g^\mathbb{C})_A$ is invariant with respect to the G -action on $T(G/K)$ and almost all principal orbits of this action are curvature-adapted isoparametric submanifolds with flat section in the anti-Kaehler manifold $((G/K)_g^\mathbb{C})_{A, g_A}$ such that the shape operators are complex diagonalizable. Also, the 0-section(= G/K) is a reflective focal submanifold of such principal orbits.*

Proof. Since G is a symmetric subgroup of $G^{\mathbb{C}}$ and the involutions associated with G and $K^{\mathbb{C}}$ commute, it follows from Proposition 9.2 that almost all principal orbits of the G -action on $G^{\mathbb{C}}/K^{\mathbb{C}}$ are curvature-adapted complex equifocal submanifold such that, for each normal vector w of M , $R^{\mathbb{C}}(\cdot, w)w|_{(T_x M)^{\mathbb{C}}}$ (x : the base point of w) and $A_w^{\mathbb{C}}$ are diagonalizable. Also $G(eK^{\mathbb{C}})(= G/K \subset G^{\mathbb{C}}/K^{\mathbb{C}})$ is a reflective focal submanifold of such principal orbits. By Proposition 9.1, such principal orbits are isoparametric submanifolds with flat section. For $g \in G$ and $v \in ((G/K)_g^{\mathbb{C}})_A \cap T_p(G/K)$, we have

$$\Phi(g_*v) = \exp_{g(p)}(\tilde{J}_{g(p)}(g_*v)) = g(\exp_p(\tilde{J}_p v)) = g(\Phi(v)),$$

where Φ is as in Section 6 and \tilde{J} is the complex structure of $G^{\mathbb{C}}/K^{\mathbb{C}}$. Thus Φ maps the G -orbits on $((G/K)_g^{\mathbb{C}})_A$ onto the G -orbits on $G^{\mathbb{C}}/K^{\mathbb{C}}$. Hence, since $\Phi|_{((G/K)_g^{\mathbb{C}})_A}$ is an isometry by Theorem 6.1, almost all principal orbits of the G -action on $((G/K)_g^{\mathbb{C}})_A$ are curvature-adapted isoparametric submanifolds with flat section and their shape operators are complex diagonalizable and the 0-section ($= G/K$) is a reflective focal submanifold of such principal orbits. \square

Concluding remark

We shall list up notations used in this paper.

| | |
|-------------------------------|---|
| J^{∇} | the adapted complex structure of ∇ |
| J^g | the adapted complex structure of g |
| g_A | the anti-Kaehler metric ass. with J^g |
| $M_{\nabla}^{\mathbb{C}}$ | the domain of J^{∇} |
| $M_g^{\mathbb{C}}$ | the domain of J^g |
| $(M_g^{\mathbb{C}})_f$ | the domain of $f^{\mathbb{C}}$ |
| $(M_g^{\mathbb{C}})_{f:i}$ | the domain such that $f^{\mathbb{C}}$ is an immersion |
| $(M_g^{\mathbb{C}})_A$ | the domain of (J^g, g_A) |
| $(M_g^{\mathbb{C}})_{A,f:i}$ | $(M_g^{\mathbb{C}})_A \cap (M_g^{\mathbb{C}})_{f:i}$ |
| $(M_g^{\mathbb{C}})'_{A,f:i}$ | $(M_g^{\mathbb{C}})_{A,f:i} \cap (f^{\mathbb{C}})^{-1}((\tilde{M}_g^{\mathbb{C}})_A)$ |

$$\left(\begin{array}{l} \nabla : C^{\omega}\text{-Koszul connection of } M \\ g : C^{\omega}\text{-pseudo-Riemannian metric of } M \\ f : C^{\omega}\text{-isometric immersion of } (M, g) \text{ into } (\tilde{M}, \tilde{g}) \end{array} \right)$$

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