Doctoral Thesis
Doctor of Science

Studies on minimal mass blow-up solutions for nonlinear Schrödinger equations

（非線形 Schrödinger 方程式の最小質量爆発解についての研究）

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March, 2023

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Acknowledgement

First of all, I would like to express my sincere gratitude to my supervisor, Professor Masahito Ohta. He has taught me a great deal, from basics of Schrödinger equation to finer details involved in forms of research.

I would also like to thank Professor Hiromichi Itou, Professor Keiichi Kato, Professor Naoyuki Koike, and Professor Tomomi Yokota for examining this thesis.

I would also like to express my deepest gratitude to Doctor Noriyoshi Fukaya. Thanks to the many advices he gave me, I was able to write superior papers and this thesis.

Chapter 1

Introduction

This thesis is based on [11, 12].

As stated by [3, 19], nonlinear Schrödinger equations occur naturally in fields such as nonlinear optics and plasma physics. The phenomenon of blow-up of solutions, which we deal with in this thesis, used to be less emphasised in physics. This is because a finite-time blow-up is a theoretical phenomenon that loses smoothness and generates a singularity at a certain finite time, but in reality no singularity occurs. There is also a lack of model validity in the neighbourhood of this blow-up time, as the assumptions made when modelling the nonlinear Schrödinger equation are not fulfilled. In other words, this leads to the expectation that in the neighbourhood of the theoretical blow-up time there are perturbations that in reality suppress the blow-up. Once an blow-up is foreseen and its conditions determined, it may be possible to reconstruct a non-blow-up solution from that front. This is very useful when analysing problems with non-integrable systems that do not have an analytical solution. In addition, even if the solution does not ultimately blow up in reality, its effects do not disappear. In order to ensure that the blow-up is controlled and to study the behaviour that follows, it is essential to be able to analyse the behaviour in the neighbourhood of the blow-up time properly.

In mathematical analysis, the studies of the nonlinear Schrödinger equation have mainly focused on the locally well-posedness of initial value problems, the stability of solitary wave, the scattering for solutions, and the blow-up of solutions. However, there has been little investigation of the detailed behaviour of blow-up solutions, and other than the classical generalisation, the results are only for some several equations. These results will be presented in Section 1.1. In the following parts, the equations treated in this thesis and their properties will be presented.

In this thesis, we consider the following nonlinear Schrödinger equation with real-valued potentials:

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}}u - Vu = 0 \tag{NLS}$$

in $\mathbb{R}^N$, where V satisfies the following condition:

$$V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \left(p \geq 1 \text{ and } p > \frac{N}{2}\right). \tag{1.1}$$

It is well known that (NLS) is locally well-posed in $H^1(\mathbb{R}^N)$ from [6, Proposition 3.2.2, Proposition 3.2.5, Theorem 3.3.9, and Proposition 4.2.3]. Namely, the following properties hold:

- For any $u_0 \in H^1(\mathbb{R}^N)$, there exists a maximal solution $u \in C((T_*, T^*), H^1(\mathbb{R}^N)) \cap C^1((T_*, T^*), H^{-1}(\mathbb{R}^N))$ for (NLS) with $u(0) = u_0$. Moreover, the solution is unique.

- There is a blow-up alternative:

$$T^* < \infty \text{ implies } \lim_{t \nearrow T^*} \|u(t)\|_{H^1} = \infty.$$

- The solution depends continuously on the initial values. Namely, for a sequence $(u_{0,n})_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^N)$ such that $u_{0,n} \to u_0$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, let u_n and u be solutions for (NLS) with $u_n(0) = u_{0,n}$ and $u(0) = u_0$, respectively. Then for any bounded closed interval $I \subset (T_*, T^*)$, u_n are defined on I if n is sufficiently large and

$$u_n \to u \quad \text{in } C(I, H^1(\mathbb{R}^N))$$

holds, where T_* and T^* are infimum and supremum of the maximal interval of u, respectively.

Furthermore, the mass (i.e., L^2-norm) and energy E of the solution are conserved by the flow, where

$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u(x)|^2 dx.$$

We define Hilbert spaces Σ^k by

$$\Sigma^k := \left\{u \in H^k(\mathbb{R}^N) \middle| |x|^k u \in L^2(\mathbb{R}^N)\right\}, \quad \|u\|_{\Sigma^k}^2 := \|u\|_{H^k}^2 + \||x|^k u\|_2^2.$$

We call Σ^1 the virial space. If $u_0 \in \Sigma^1$, then the solution u for (NLS) with $u(0) = u_0$ belongs to $C((T_*, T^*), \Sigma^1)$ from [6, Lemma 6.5.2].

Moreover, we consider the following condition instead of (1.1):

$$V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \quad \left(p \geq 2 \text{ and } p > \frac{N}{2}\right). \tag{1.2}$$

Under this condition, if $u_0 \in H^2(\mathbb{R}^N)$, then the corresponding solution u belongs to $C((T_*, T^*), H^2(\mathbb{R}^N)) \cap C^1((T_*, T^*), L^2(\mathbb{R}^N))$. Furthermore, if $u_0 \in \Sigma^2$, then the solution u for (NLS) with $u(0) = u_0$ belongs to $C((T_*, T^*), \Sigma^2)$ and $|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$ from the same proof as in [6, Lemma 6.5.2].

1.1 Previous results

Firstly, results based on the general theory of blow-up for the nonlinear Schrödinger equation are presented. In [6, Theorem 6.5.4], the virial identity is used to give sufficient conditions for the solution to blow up. Applied to (NLS), if $u_0 \in \Sigma^1$ and

$$V + \frac{1}{2}x \cdot \nabla V \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \quad E(u_0) < 0,$$

then the corresponding solution blows up at a finite time. However, the assumptions in the main result Theorem 3.1 allow the choice of potentials and initial values that do not satisfy the conditions, and in Theorem 4.1 the potential in fact do not satisfy the conditions. In particular, Theorem 4.1 shows that for any given energy level it is possible to construct a blow-up solution with the energy. Therefore, although it is a relatively simple sufficient condition, it is not very wide in its application.

Secondly, [6, Theorem 6.5.13] gives an estimate from below of the blow-up rates of finite-time blow-up solutions. Applied to (NLS), if $V = 0$ (i.e., mass-critical problem) and a solution u for (NLS) blows up as $t \nearrow T$ $(< \infty)$, then

$$\|\nabla u(t)\|_2 \gtrsim \frac{1}{(T-t)^{\frac{1}{2}}} \quad (t \nearrow T).$$

This result derives concrete information namely the blow-up rate. However, as discussed (1.3), it has been shown that blow-up rates of finite-time blow-up solutions for the critical mass is $|T-t|^{-1}$ in the critical problem.

The generalisations above do not describe the detailed behaviour of blow-up solutions. On the other hand, specific information may be known about some blow-up solutions of several equations. Firstly, we describe the results regarding the mass-critical problem:

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{CNLS}$$

It is well known ([2, 9, 21]) that there exists a unique classical solution Q for

$$-\Delta Q + Q - |Q|^{\frac{4}{N}} Q = 0, \quad Q \in H^1(\mathbb{R}^N), \quad Q > 0, \quad Q \text{ is radial},$$

which is called the ground state. If $\|u\|_2 = \|Q\|_2$ ($\|u\|_2 < \|Q\|_2$, $\|u\|_2 > \|Q\|_2$), we say that u has the *critical mass* (*subcritical mass*, *supercritical mass*, respectively).

We note that $E_{\text{crit}}(Q) = 0$, where E_{crit} is the energy with respect to (CNLS). Moreover, the ground state Q attains the best constant in the Gagliardo-Nirenberg inequality

$$\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \leq \left(1 + \frac{2}{N}\right)\left(\frac{\|v\|_2}{\|Q\|_2}\right)^{\frac{4}{N}} \|\nabla v\|_2^2 \quad \text{for } v \in H^1(\mathbb{R}^N).$$

Therefore, for all $v \in H^1(\mathbb{R}^N)$,

$$E_{\text{crit}}(v) \geq \frac{1}{2}\|\nabla v\|_2^2 \left(1 - \left(\frac{\|v\|_2}{\|Q\|_2}\right)^{\frac{4}{N}}\right)$$

holds. This inequality and the mass and energy conservations imply that all subcritical-mass solutions for (CNLS) are global and bounded in $H^1(\mathbb{R}^N)$.

Regarding the critical-mass case, we apply the pseudo-conformal transformation

$$u(t,x) \ \mapsto \ \frac{1}{|t|^{\frac{N}{2}}} u\left(-\frac{1}{t}, \pm\frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}$$

to the solitary wave solution $u(t,x) := Q(x)e^{it}$. Then we obtain

$$S(t,x) := \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) e^{-\frac{i}{t}} e^{i\frac{|x|^2}{4t}},$$

which is also a solution for (CNLS) and satisfies

$$\|S(t)\|_2 = \|Q\|_2, \quad \|\nabla S(t)\|_2 \sim \frac{1}{|t|} \quad (t \nearrow 0). \tag{1.3}$$

Namely, S is a minimal-mass blow-up solution for (CNLS). Moreover, S is the only finite time blow-up solution for (CNLS) with critical mass, up to the symmetries of the flow (see [13]).

Regarding the supercritical-mass case, there exists a solution u for (CNLS) such that

$$\|\nabla u(t)\|_2 \sim \sqrt{\frac{\log\left|\log|T^* - t|\right|}{T^* - t}} \quad (t \nearrow T^*)$$

(see [15, 16]).

Secondly, we describe previous results regarding the following nonlinear Schrödinger equation with a real-valued potential:

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}} u - V(x)u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{PNLS}$$

Carles and Nakamura [5] deal with the case where V is a Stark potential, i.e., $V(x) = \xi\cdot x$ for some $\xi \in \mathbb{R}^N$. Carles [4] deals with the case where $V(x) = \pm\omega^2|x|^2$ for $\omega \in \mathbb{R}$. By using the Avron-Herbst formula for the former and the generalised lens transform for the latter, solutions for (CNLS) can be transformed into solutions for (PNLS). Therefore, in these cases, the minimal-mass blow-up solution for (PNLS) can be constructed from the minimal-mass blow-up solution S for (CNLS). The results are therefore similar to the result of the mass-critical problem. Moreover, Csobo and Genoud [7] and Mukherjee, Nam, and Nguyen [17] deal with $N \geq 3$ and $V(x) = -\frac{c}{|x|^2}$ for some $0 < c \leq \frac{(N-2)^2}{4}$. Although these results use a different ground state from Q, as in the mass-critical problem, they obtain the minimal-mass blow-up solution by applying a pseudo-conformal transformation.

Banica, Carles, and Duyckaerts [1] presents the following result for

$$i\frac{\partial u}{\partial t} + \Delta u + g(x)|u|^{\frac{4}{N}} u - V(x)u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{INLS}$$

Theorem 1.1 (Banica, Carles, and Duyckaerts [1])**.** Let $N = 1$ or 2, $V \in C^2(\mathbb{R}^N,\mathbb{R})$, and $g \in C^4(\mathbb{R}^N,\mathbb{R})$. Assume $\left(\frac{\partial}{\partial x}\right)^\beta V \in L^\infty(\mathbb{R}^N)$ $(|\beta| \leq 2)$, $\left(\frac{\partial}{\partial x}\right)^\beta g \in L^\infty(\mathbb{R}^N)$ $(|\beta| \leq 4)$, and

$$g(0) = 1, \quad \frac{\partial g}{\partial x_j}(0) = \frac{\partial^2 g}{\partial x_j \partial x_k}(0) = 0 \quad (1 \leq j,k \leq N).$$

Then there exist $T > 0$ and a solution $u \in C((0,T),\Sigma^1)$ for (INLS) such that

$$\left\| u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\frac{|x|^2}{4t} - i\theta\left(\frac{1}{t}\right) - itV(0)} \right\|_{\Sigma^1} \to 0 \quad (t \searrow 0),$$

where θ and λ are continuous real-valued functions and x is a continuous $\mathbb{R}^N$-valued function such that

$$\theta(\tau) = \tau + o(\tau) \quad \text{as } \tau \to +\infty,$$
$$\lambda(t) \sim t \text{ and } |x(t)| = o(t) \quad \text{as } t \searrow 0.$$

Results of [4, 5, 7, 17] construct blow-up solutions by applying the pseudo-conformal transformation to the ground states. In contrast to these, the seminal work Raphaël and Szeftel [18] constructs a minimal-mass blow-up solution for

$$i\frac{\partial u}{\partial t} + \Delta u + k(x)|u|^{\frac{4}{N}}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

without using the pseudo-conformal transformation. Le Coz, Martel, and Raphaël [10] based on the methodology of [18] obtains the following results for

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}}u \pm |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{DPNLS}$$

Theorem 1.2 (Le Coz, Martel, and Raphaël [10])**.** Let $N = 1, 2, 3$, $1 < p < 1 + \frac{4}{N}$, and $\pm = +$. Then for any energy level $E_0 \in \mathbb{R}$, there exist $t_0 < 0$ and a radially symmetric initial value $u_0 \in H^1(\mathbb{R}^N)$ with

$$\|u_0\|_2 = \|Q\|_2, \quad E(u_0) = E_0$$

such that the corresponding solution u for (DPNLS) with $u(t_0) = u_0$ blows up at $t = 0$ with a blow-up rate of

$$\|\nabla u(t)\|_2 = \frac{C(p) + o_{t\nearrow 0}(t)}{|t|^\sigma},$$

where $\sigma = \frac{4}{4+N(p-1)}$ and $C(p) > 0$.

Theorem 1.3 ([10])**.** Let $N = 1, 2, 3$, $1 < p < 1 + \frac{4}{N}$, and $\pm = -$. If an initial value has critical mass, then the corresponding solution for (DPNLS) with $u(0) = u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

This result means that minimal-mass blow-up solutions do not exist ([10, Lemma 1.2]).

These results show that the perturbation term, which is a small power-type nonlinearity, affects the existence and non-existence of the minimal-mass blow-up solution, and furthermore affects the blow-up rate of the minimal-mass blow-up solution if it exists.

1.2 Organisation of this thesis

This thesis is henceforth structured as follows.

Firstly, in Chapter 2, the definitions and properties of the symbols used in this thesis are described, as well as the lemmas necessary for the proofs of the main results. The proof of Lemma 2.2 is given in Chapter 5.

Next, in Chapter 3, the result is described for the case where the potential V is smooth.

Finally, in Chapter 4, the results are described for the case where the potential V is a inverse power potential.

Chapter 2

Preliminaries

2.1 Notations

Let

$$\mathbb{N} := \mathbb{Z}_{\geq 1}, \quad \mathbb{N}_0 := \mathbb{Z}_{\geq 0}.$$

We define

$$(u,v)_2 := \operatorname{Re} \int_{\mathbb{R}^N} u(x)\overline{v}(x)dx, \qquad \|u\|_p := \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}},$$

$$f(z) := |z|^{\frac{4}{N}} z, \qquad F(z) := \frac{1}{2+\frac{4}{N}} |z|^{2+\frac{4}{N}} \quad \text{for } z \in \mathbb{C}.$$

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, we denote the differentials of f and F by df and dF, respectively. We define

$$\Lambda := \frac{N}{2} + x \cdot \nabla, \quad L_+ := -\Delta + 1 - \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}}, \quad L_- := -\Delta + 1 - Q^{\frac{4}{N}}.$$

Namely, Λ is the generator of L^2-scaling, and L_+ and L_- come from the linearised Schrödinger operator around Q. Then

$$L_- Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_- |x|^2 Q = -4\Lambda Q, \quad L_+ \rho = |x|^2 Q, \quad L_- xQ = -\nabla Q$$

hold, where $\rho \in \mathcal{S}(\mathbb{R}^N)$ is the unique radial solution for $L_+\rho = |x|^2 Q$. Note that there exist $C_\alpha, \kappa_\alpha > 0$ such that

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha Q(x) \right| \leq C_\alpha Q(x), \quad \left| \left(\frac{\partial}{\partial x} \right)^\alpha \rho(x) \right| \leq C_\alpha (1+|x|)^{\kappa_\alpha} Q(x).$$

for any multi-index α. Furthermore, there exists $\mu > 0$ such that for all $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} &\langle L_+ \operatorname{Re} u, \operatorname{Re} u \rangle + \langle L_- \operatorname{Im} u, \operatorname{Im} u \rangle \\ &\geq \mu \|u\|_{H^1}^2 - \frac{1}{\mu} \left((\operatorname{Re} u, Q)_2{}^2 + |(\operatorname{Re} u, xQ)_2|^2 + (\operatorname{Re} u, |x|^2 Q)_2{}^2 + (\operatorname{Im} u, \rho)_2{}^2 \right) \end{aligned} \tag{2.1}$$

(e.g., see [14,15,18,20]). We denote by $\mathcal{Y}$ the set of functions $g \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N) \cap H^1_{\text{rad}}(\mathbb{R}^N)$ such that

$$\exists C_\alpha, \kappa_\alpha > 0, \ |x| \geq 1 \Rightarrow \left| \left(\frac{\partial}{\partial x} \right)^\alpha g(x) \right| \leq C_\alpha (1+|x|)^{\kappa_\alpha} Q(x)$$

for any multi-index α. Moreover, we defined by $\mathcal{Y}'$ the set of functions $g \in \mathcal{Y}$ such that

$$g \in H^2(\mathbb{R}^N) \quad \text{and} \quad \Lambda g \in C(\mathbb{R}^N).$$

Finally, we use the notation $\lesssim$ and $\gtrsim$ when the inequalities hold up to a positive constant. We also use the notation $\approx$ when $\lesssim$ and $\gtrsim$ hold. Moreover, positive constants C and ϵ are sufficiently large and small, respectively.

2.2 Lemmas

In this section, the key lemmas in the proofs of the main results are described.

Firstly, we consider a more general Schrödinger equation

$$i\frac{\partial u}{\partial t} + \Delta u + g(u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{GNLS}$$

For $g = g_1 + \cdots + g_k$, we consider the following assumptions:

(a) There exists $G_j \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ such that $G_j' = g_j$.

(b) There exist $r_j, \rho_j \in [2, 2^*)$ such that for any $M < \infty$, there exists $L(M) < \infty$ such that

$$\|g_j(u) - g_j(v)\|_{\rho_j'} \le L(M)\|u - v\|_r$$

for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1} + \|v\|_{H^1} \le M$.

(c) For any $u \in H^1(\mathbb{R}^N)$,

$$\operatorname{Im} g_j(u)\overline{u} = 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Here, p' is the Hölder conjugate and 2^* is the Sobolev conjugate, i.e., $2^* := \frac{2N}{N-2}$ $(N \ge 3)$, $2^* := \infty$ $(N = 1, 2)$. Then the following property analogous to continuous dependence holds:

Lemma 2.1. Let $g = g_1 + \cdots + g_k$ satisfy (a), (b), and (c). For $(\varphi_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ and $\varphi \in H^1(\mathbb{R}^N)$, let u_n and u be solutions for (GNLS) with $u_n(0) = \varphi_n$ and $u(0) = \varphi$, respectively. Moreover, we assume that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^N)$ and that for any bounded closed interval $J \subset (T_*(\varphi), T^*(\varphi))$, there exists $m \in \mathbb{N}$ such that $\sup_{n\ge m} \|u_n\|_{L^\infty(J,H^1)} < \infty$. Then

$$u_n \to u \quad \text{in } L^\infty_{\text{loc}}((-T_{\min}(\varphi), T_{\max}(\varphi)), L^2(\mathbb{R}^N)) \quad (n \to \infty).$$

In particular, $u_n(t) \rightharpoonup u(t)$ weakly in $H^1(\mathbb{R}^N)$ for any $t \in (T_*(\varphi), T^*(\varphi))$.

Proof. We may assume that $T_1, T_2 > 0$ and $J = [-T_1, T_2]$. Then we define

$$M := \|u\|_{L^\infty(J,H^1)} + \sup_{n\ge m} \|u_n\|_{L^\infty(J,H^1)}.$$

Furthermore, we define

$$\mathcal{G}_j(u)(t) := i\int_0^t \mathcal{T}(t-s)g_j(u(s))ds, \quad \mathcal{H}(u)(t) := \mathcal{T}(t)\varphi + \mathcal{G}_1(u)(t) + \cdots + \mathcal{G}_k(u)(t),$$

where $\mathcal{T}(t) := e^{it\Delta}$. Similarly, we define $\mathcal{G}_j(u_n)$ and $\mathcal{H}(u_n)$. According to Duhamel's principle, we have $u = \mathcal{H}(u)$ and $u_n = \mathcal{H}(u_n)$.

Let $n \ge m$ and $0 < T \le \min\{T_1, T_2\}$. Moreover, let (q, r), (q_j, r_j), and (γ_j, ρ_j) be admissible pairs. Then, according to the Strichartz estimate and (b), we have

$$\|\mathcal{T}(t)\varphi_n - \mathcal{T}(t)\varphi\|_{L^q(\mathbb{R},L^r)} \le C\|\varphi_n - \varphi\|_{L^2},$$
$$\|\mathcal{G}_j(u_n) - \mathcal{G}_j(u)\|_{L^q((-T,T),L^r)} \le C(M)T^{\frac{1}{\gamma_j'} - \frac{1}{q_j}}\|u_n - u\|_{L^{q_j}((-T,T),L^{r_j})}.$$

For $v, w \in C([-T,T], H^1(\mathbb{R}^N))$, we define

$$d(v,w) := \|v - w\|_{L^\infty((-T,T),L^2)} + \sum_{j=1}^k \|v - w\|_{L^{q_j}((-T,T),L^{r_j})}.$$

Then we have

$$d(u_n, u) = d(\mathcal{H}(u_n), \mathcal{H}(u)) \le C\|\varphi_n - \varphi\|_{L^2} + d(u_n, u)C(M)\sum_{j=1}^k T^{\frac{1}{\gamma_j'} - \frac{1}{q_j}}.$$

Since there exists $T(M) > 0$ such that $C(M)\sum_{j=1}^k T(M)^{\frac{1}{\gamma_j'} - \frac{1}{q_j}} \le \frac{1}{2}$, we obtain

$$\|u_n - u\|_{L^\infty((-T(M),T(M)),L^2)} \le d(u_n, u) \le C\|\varphi_n - \varphi\|_{L^2} \to 0 \quad (n \to \infty),$$

which yields the conclusion.

Finally, $(u_n(t))_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and converges to $u(t)$ in $L^2(\mathbb{R}^N)$ for any $t \in (T_*(\varphi), T^*(\varphi))$. Therefore, $(u_n(t))_{n\in\mathbb{N}}$ weakly converges to $u(t)$ in $H^1(\mathbb{R}^N)$. □

Let be $P^{\pm}_{j,k} \in \mathcal{Y}'$, $\lambda > 0$, $b \in \mathbb{R}$, and $K \in \mathbb{N}_0$ and define

$$P(y;\lambda,b) := Q(y) + \sum_{0\le j+k\le K} \left(b^{2j}\lambda^{(k+1)\alpha} P^{+}_{j,k}(y) + ib^{2j+1}\lambda^{(k+1)\alpha} P^{-}_{j,k}(y) \right).$$

Then the following lemma holds. The proof is described in Chapter 5.

Lemma 2.2 (Decomposition)**.** There exists $\bar{l}, \overline{C} > 0$ such that the following statement holds. Let I be an interval and $\delta > 0$ be sufficiently small. We assume that $u \in C(I, H^1(\mathbb{R}^N)) \cap C^1(I, H^{-1}(\mathbb{R}^N))$ satisfies

$$\forall\, t \in I,\ \left\| \lambda(t)^{\frac{N}{2}} u\left(t, \lambda(t)y - w(t)\right) e^{-i\gamma(t)} - Q \right\|_{H^1} < \delta$$

for some functions $\lambda : I \to (0, \bar{l})$, $\gamma : I \to \mathbb{R}$, and $w : I \to \mathbb{R}^N$. Then there exist unique functions $\tilde{\lambda} : I \to (0,\infty)$, $\tilde{b} : I \to \mathbb{R}$, $\tilde{\gamma} : I \to \mathbb{R}/2\pi\mathbb{Z}$, and $\tilde{w}(t) : I \to \mathbb{R}^N$ such that

$$u(t,x) = \frac{1}{\tilde{\lambda}(t)^{\frac{N}{2}}} \left(P(\cdot;\tilde{\lambda}(t),\tilde{b}(t)) + \tilde{\varepsilon} \right) \left(t, \frac{x+\tilde{w}(t)}{\tilde{\lambda}(t)} \right) e^{-i\frac{\tilde{b}(t)}{4}\frac{|x+\tilde{w}(t)|^2}{\tilde{\lambda}(t)^2} + i\tilde{\gamma}(t)},$$

$$\left| \frac{\tilde{\lambda}(t)}{\lambda(t)} - 1 \right| + \left| \tilde{b}(t) \right| + |\tilde{\gamma}(t) - \gamma(t)|_{\mathbb{R}/2\pi\mathbb{Z}} + \left| \frac{\tilde{w}(t) - w(t)}{\tilde{\lambda}(t)} \right| < \overline{C}$$

hold, where $|\cdot|_{\mathbb{R}/2\pi\mathbb{Z}}$ is defined by

$$|c|_{\mathbb{R}/2\pi\mathbb{Z}} := \inf_{m\in\mathbb{Z}} |c + 2\pi m|,$$

and that $\tilde{\varepsilon}$ satisfies the orthogonal conditions

$$(\tilde{\varepsilon}, i\Lambda P)_2 = \left(\tilde{\varepsilon}, |y|^2 P\right)_2 = (\tilde{\varepsilon}, i\rho)_2 = 0, \quad (\tilde{\varepsilon}, yP)_2 = 0$$

on I. In particular, $\tilde{\lambda}$, $\tilde{b}$, $\tilde{\gamma}$, and $\tilde{w}$ are C^1 functions and independent of λ, γ, and w.

Remark 2.3. In particular, if u and $P(\cdot;\lambda,b)$ are spherically symmetrical, then $\tilde{w} = 0$.

2.3 Outline of proofs

We prove main theorems by using a simplified version with modification of the method of Le Coz, Martel, and Raphaël [10], which is based on seminal work of Raphaël and Szeftel [18]. We proceed in the following steps:

Step 1. For a solution u for (NLS), we consider the following transformation:

$$u(t,x) = \frac{1}{\lambda(s)^{\frac{N}{2}}} v\left(s,y\right) e^{-i\frac{b(s)|y|^2}{4} + i\gamma(s)}, \quad y = \frac{x+w(s)}{\lambda(s)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(s)^2}. \tag{2.2}$$

Then v satisfies

$$0 = i\frac{\partial v}{\partial s} + \Delta v - v + f(v) - \lambda^2 V(\lambda y - w)v + \text{modulation terms}.$$

Step 2. We construct a blow-up profile P as an approximate solution for

$$i\frac{\partial P}{\partial s} + \Delta P - P + f(P) - \lambda^2 V(\lambda\cdot - w)P + \theta\frac{|y|^2}{4}P = 0,$$

where $\theta\frac{|y|^2}{4}P$ is a some correction term.

Step 3. Let $v = P + \varepsilon$ for some error function ε. Then we obtain the equation of ε:

$$0 = i\frac{\partial \varepsilon}{\partial s} + \Delta\varepsilon - \varepsilon + f(P+\varepsilon) - f(P) - \lambda^2 V(\lambda y - w)\varepsilon + \theta\frac{|y|^2}{4}\varepsilon + \text{modulation terms} + \text{error terms}.$$

Step 4. By using the modulation terms and ε, we estimate the parameters λ, b, γ, and w.

Step 5. We construct a sequence of suitable solutions for (NLS) and show that the limit of the sequence is the desired minimal-mass blow-up solution.

Chapter 3

Case of smooth potentials

3.1 Problem and Main result

In this chapter, for the potential V in the equation

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{\frac{4}{N}}u - Vu = 0, \tag{NLS}$$

we assume the following:

$$V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \qquad \left(p \geq 2 \text{ and } p > \frac{N}{2}\right), \tag{3.1}$$

$$V \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N), \tag{3.2}$$

$$\nabla V, \nabla^2 V \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \qquad (q \geq 2 \text{ and } q > N). \tag{3.3}$$

Then we obtain the following result:

Theorem 3.1 ([12]). Let the potential V satisfy (3.1), (3.2), and (3.3). Then there exist $t_0 < 0$ and a initial value $u_0 \in \Sigma^1$ with

$$\|u_0\|_2 = \|Q\|_2$$

such that the corresponding solution u for (NLS) with $u(t_0) = u_0$ blows up at $t = 0$. Moreover,

$$\left\| u(t,x) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x + w(t)}{\lambda(t)}\right) e^{-i\frac{b(t)}{4}\frac{|x+w(t)|^2}{\lambda(t)^2} + i\gamma(t)} \right\|_{\Sigma^1} \to 0 \quad (t \nearrow 0)$$

holds for some C^1 functions $\lambda : (t_0, 0) \to (0, \infty)$, $b, \gamma : (t_0, 0) \to \mathbb{R}$, and $w : (t_0, 0) \to \mathbb{R}^N$ such that

$$\lambda(t) = |t|\,(1 + o(1))\,, \quad b(t) = |t|\,(1 + o(1))\,, \quad \gamma(t) \sim |t|^{-1}, \quad |w(t)| = O(|t|^2)$$

as $t \nearrow 0$.

Firstly, the assumptions in Theorem 3.1 are weaker than those in Theorem 1.1 with $g = 1$. Theorem 3.1 has no restrictions on spatial dimensions. On the other hand, according to the lack of regularity of the nonlinearity $|u|^{\frac{4}{N}}u$, Theorem 1.1 requires the restriction $N = 1$ or 2. Although Theorem 3.1 is also affected by the lack of regularity, we overcome this difficulty by using the properties of the ground state. In Theorem 3.1, the assumption (3.2) plays an important role. We use to (3.2) to apply Taylor's theorem to V. When V does not satisfy (3.2), blow-up rates should change as the result of Le Coz, Martel, and Raphaël [10] or Theorem 4.1.

Secondly, we improve some parts of the arguments in Le Coz, Martel, and Raphaël [10] and Raphaël and Szeftel [18]. Although the authors of [10,18] introduce the Morawetz functional ([10, Section 5] and [18, Lemma 3.3]) and apply a *truncation* procedure to the functional, we avoid using the functional by modifying the definition of ε. As a result, without the truncation, we work directly in the virial space Σ^1. Moreover, the authors of [10] use the continuous dependence on the initial value for (DPNLS) in $H^s(\mathbb{R}^N)$ for some $s \in [0, 1)$. Although this continuous dependence is an important fact in the proof of the main result in [10], it is not obvious for (NLS). Therefore, instead of proving the continuous dependence for (NLS) in $H^s(\mathbb{R}^N)$ for some $s \in [0, 1)$, we use Lemma 2.1, which gives a kind of the continuous dependence. Consequently, we provide a simpler and more general proof.

3.2 Proof of Theorem 3.1

The error term Ψ is defined by

$$\Psi(y;\lambda,w) := \lambda^2 V(\lambda y - w)Q(y)$$

for $\lambda > 0$ and $w \in \mathbb{R}^N$. Moreover, we define κ by

$$\kappa := 1 - \frac{N}{q} > 0.$$

Without loss of generality, we may assume that $V(0) = 0$.

Proposition 3.2. There exists a sufficiently small constant $\epsilon' > 0$ such that

$$\left\| e^{\epsilon'|y|}\Psi \right\|_2 + \left\| e^{\epsilon'|y|}\nabla\Psi \right\|_2 \lesssim \lambda^3 + \lambda^\kappa |w|^2 \tag{3.4}$$

for $0 < \lambda \ll 1$ and $w \in \mathbb{R}^N$. Moreover, for any radial function $\varphi \in L^2(\mathbb{R}^N)$,

$$|(\Psi,\varphi)_2| \lesssim \lambda^2|w| + \lambda^{1+\kappa}\left(\lambda^2 + |w|^2\right).$$

Proof. For the sake of simplicity, we assume $\nabla^2 V \in L^q(\mathbb{R}^N)$.

By using Taylor's theorem and $V(0) = 0$, we write

$$\lambda^2 V(\lambda y - w) = \lambda^2(\lambda y - w)\cdot\nabla V(0) + \sum_{|\alpha|=2}\int_0^1 \lambda^2(\lambda y - w)^\alpha \frac{\partial^\alpha V}{\partial x^\alpha}(\tau(\lambda y - w))(1-\tau)d\tau,$$

$$\lambda^3\frac{\partial V}{\partial x_j}(\lambda y - w) = \lambda^3\frac{\partial V}{\partial x_j}(0) + \int_0^1 \lambda^3(\lambda y - w)\cdot\left(\nabla\frac{\partial V}{\partial x_j}\right)(\tau(\lambda y - w))d\tau.$$

Therefore, we have

$$|\Psi(y)| \lesssim \lambda^2(\lambda|y| + |w|)Q(y) + \lambda^2(\lambda|y| + |w|)^2\int_0^1 \left|\nabla^2 V(\tau(\lambda y - w))\right| d\tau Q(y),$$

$$|\nabla\Psi(y)| \lesssim \lambda^2\left(\lambda(1+|y|) + |w|\right)Q(y) + \lambda^2(\lambda(1+|y|) + |w|)^2\int_0^1 \left|\nabla^2 V(\tau(\lambda y - w))\right| d\tau Q(y).$$

According to (3.3) and the exponential decay of Q from [6, Theorem 8.1.1], there exists a sufficiently small constant $\epsilon' > 0$ such that

$$\left\| e^{\epsilon'|y|}\nabla^2 V(\tau(\lambda y - w))\left(1+|y|\right)Q(y) \right\|_2 \lesssim \tau^{-\frac{N}{q}}\lambda^{-\frac{N}{q}}\left\|\nabla^2 V\right\|_q \|Q^{\frac{1}{2}}\|_{\frac{2q}{q-2}}.$$

Therefore,

$$\left\| e^{\epsilon'|y|}\Psi \right\|_2 + \left\| e^{\epsilon'|y|}\nabla\Psi \right\|_2 \lesssim \lambda^2(\lambda + |w|) + \lambda^{2-\frac{N}{q}}(\lambda + |w|)^2\int_0^1 \tau^{-\frac{N}{q}}d\tau \lesssim \lambda^3 + \lambda^\kappa|w|^2.$$

Finally, since $(yQ,\varphi)_2 = 0$ for any radial function $\varphi \in L^2(\mathbb{R}^N)$, we obtain

$$(\Psi,\varphi)_2 = -\lambda^2 w\cdot\nabla V(0)(Q,\varphi)_2 + \sum_{|\alpha|=2}\int_0^1 \lambda^2\left((\lambda y - w)^\alpha\frac{\partial^\alpha V}{\partial x^\alpha}(\tau(\lambda y - w))Q,\varphi\right)_2(1-\tau)d\tau.$$

Therefore, we obtain conclusion. □

Remark 3.3. For the estimate of (3.4) in [12], there is a term $\lambda^{1+\kappa}|w|^2$, but this is correct for $\lambda^\kappa|w|^2$.

Next, we give a uniform estimate of the modulation terms.

Let s_0 be sufficiently large. Given $t_1 < 0$ which is sufficiently close to 0, we define $s_1 := -{t_1}^{-1}$ and $\lambda_1 = b_1 = {s_1}^{-1}$. Let $u(t)$ be the solution for (NLS) with an initial value

$$u(t_1,x) := \frac{1}{{\lambda_1}^{\frac{N}{2}}}Q\left(\frac{x}{\lambda_1}\right)e^{-i\frac{b_1}{4}\frac{|x|^2}{{\lambda_1}^2}}. \tag{3.5}$$

Note that $u \in C((T_*, T^*), \Sigma^2(\mathbb{R}^N))$ and $|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$. Moreover,

$$\operatorname{Im} \int_{\mathbb{R}^N} u(t_1, x)\nabla \overline{u}(t_1, x)dx = 0$$

holds. Moreover, u satisfies the assumption in Lemma 2.2 in a neighbourhood of t_1. Therefore, by applying Lemma 2.2 with $P = Q$ (i.e., $P^{\pm}_{j,k} = 0$ for all j, k) to u, there exist decomposition parameters $\tilde{\lambda}_{t_1}$, $\tilde{b}_{t_1}$, $\tilde{\gamma}_{t_1}$, $\tilde{w}_{t_1}$, and $\tilde{\varepsilon}_{t_1}$ such that

$$u(t, x) = \frac{1}{\tilde{\lambda}_{t_1}(t)^{\frac{N}{2}}} (Q + \tilde{\varepsilon}_{t_1}) \left(t, \frac{x + \tilde{w}_{t_1}(t)}{\tilde{\lambda}_{t_1}(t)}\right) e^{-i\frac{\tilde{b}_{t_1}(t)}{4}\frac{|x+\tilde{w}_{t_1}(t)|^2}{\tilde{\lambda}_{t_1}(t)^2} + i\tilde{\gamma}_{t_1}(t)}, \tag{3.6}$$

$$(\tilde{\varepsilon}_{t_1}, i\Lambda Q)_2 = \left(\tilde{\varepsilon}_{t_1}, |y|^2 Q\right)_2 = (\tilde{\varepsilon}_{t_1}, i\rho)_2 = 0, \quad (\tilde{\varepsilon}_{t_1}, yQ)_2 = 0 \tag{3.7}$$

hold in the neighbourhood of t_1. We define the rescaled time s_{t_1} by

$$s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\tilde{\lambda}_{t_1}(\tau)^2} d\tau.$$

Moreover, let I_{t_1} be the maximal interval of the existence of the decomposition such that (3.6) and (3.7) hold and we define

$$J_{s_1} := s_{t_1}(I_{t_1}).$$

Then, since $s_{t_1} : I_{t_1} \to J_{s_1}$ is strictly monotonically increasing, we can define inverse function ${s_{t_1}}^{-1} : J_{s_1} \to I_{t_1}$. Furthermore, we define

$$t_{t_1} := {s_{t_1}}^{-1}, \qquad \lambda_{t_1}(s) := \tilde{\lambda}_{t_1}(t_{t_1}(s)), \qquad b_{t_1}(s) := \tilde{b}_{t_1}(t_{t_1}(s)),$$
$$\gamma_{t_1}(s) := \tilde{\gamma}_{t_1}(t_{t_1}(s)), \qquad w_{t_1}(s) := \tilde{w}_{t_1}(t_{t_1}(s)), \qquad \varepsilon_{t_1}(s, y) := \tilde{\varepsilon}_{t_1}(t_{t_1}(s), y)$$

for $s \in J_{s_1}$. In addition, although it is an abuse of the symbol, we define

$$\Psi(s, y) := \Psi(y; \lambda(s), w(s)).$$

For the sake of clarity in notation, we often omit the subscript t_1. Additionally, for sufficiently large s_1 ($\geq s_0$), we define

$$s' := \max\{s_0, \inf J_{s_1}\}.$$

Let K be sufficiently large and L and M be defined by

$$L := \frac{3}{2} + \frac{1}{K}, \quad 1 < M < 2(L - 1).$$

Moreover, we define s_* by

$$s_* := \inf\{\sigma \in (s', s_1] \mid (3.8) \text{ holds on } [\sigma, s_1]\},$$

where

$$\begin{cases} \|\varepsilon(s)\|_{H^1}^2 + b(s)^2\||y|\varepsilon(s)\|_2^2 < s^{-2L}, \\ |s\lambda(s) - 1| < s^{-M}, \quad |sb(s) - 1| < s^{-M}, \quad |w(s)| < s^{-\frac{3}{2}}. \end{cases} \tag{3.8}$$

Note that for all $s \in (s_*, s_1]$, we have

$$s^{-1}(1 - s^{-M}) < \lambda(s), b(s) < s^{-1}(1 + s^{-M}).$$

In Lemma 3.12, we will show that $s_0 = s' = s_*$ holds for any $s_1 > s_0$ if s_0 is sufficiently large.

By direct calculations, we obtain

$$\begin{aligned}\Psi = {} & i\frac{\partial \varepsilon}{\partial s} + \Delta\varepsilon - \varepsilon + f(Q + \varepsilon) - f(Q) - \lambda^2 V(\lambda y - w)\varepsilon \\ & - i\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right)\Lambda(Q + \varepsilon) + \left(1 - \frac{\partial \gamma}{\partial s}\right)(Q + \varepsilon) + \left(\frac{\partial b}{\partial s} + b^2\right)\frac{|y|^2}{4}(Q + \varepsilon) \\ & - \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right) b\frac{|y|^2}{2}(Q + \varepsilon) + i\frac{1}{\lambda}\frac{\partial w}{\partial s} \cdot \nabla(Q + \varepsilon) + \frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s} \cdot y(Q + \varepsilon)\end{aligned} \tag{3.9}$$

on J_{s_1}. Finally, we define

$$\operatorname{Mod}(s) := \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2, 1 - \frac{\partial \gamma}{\partial s}, \frac{\partial w}{\partial s}\right).$$

In order to obtain a uniform estimate of the modulation terms, the following lemma is first presented.

Lemma 3.4. For all $s \in (s_*, s_1]$,

$$|(\operatorname{Im} \varepsilon(s), \nabla Q)_2| \lesssim s^{-2}. \tag{3.10}$$

Proof. According to a direct calculation, we have

$$\frac{d}{dt} \operatorname{Im} \int_{\mathbb{R}^N} u(t,x) \nabla \overline{u}(t,x) dx = -2 \left(V u(t), \nabla u(t) \right)_2 = \left\langle \nabla V u(t), u(t) \right\rangle.$$

Moreover, according to (3.3) and (3.6), we obtain

$$\begin{aligned} |\langle \nabla V u(t), u(t) \rangle| &= \left| \left\langle (\nabla V)(\tilde{\lambda}(t) y - \tilde{w}(t))(Q + \tilde{\varepsilon}(t)), Q + \tilde{\varepsilon}(t) \right\rangle \right| \\ &\lesssim \left(1 + \tilde{\lambda}(t)^{1-\frac{N}{q}} \right) \|Q\|_{H^1}^2 + \left(\tilde{\lambda}(t)^{-\frac{N}{q}} + 1 \right) \left(\|Q\|_{H^1} + \|\tilde{\varepsilon}(t)\|_{H^1} \right) \|\tilde{\varepsilon}(t)\|_{H^1} \\ &\lesssim 1. \end{aligned}$$

Accordingly, we obtain

$$\begin{aligned} \left| \operatorname{Im} \int_{\mathbb{R}^N} u(t(s), x) \nabla \overline{u}(t(s), x) dx \right| &\lesssim \int_s^{s_1} \lambda(\sigma)^2 \left| \langle \nabla V u(t(\sigma)), u(t(\sigma)) \rangle \right| d\sigma \\ &\lesssim \int_s^{s_1} \sigma^{-2} d\sigma \lesssim s^{-1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &2(\operatorname{Im} \varepsilon(s), \nabla Q)_2 + (\varepsilon(s), i \nabla \varepsilon(s))_2 + \frac{b}{2} \int_{\mathbb{R}^N} y \left| Q(y) + \varepsilon(s,y) \right|^2 dy \\ &= \lambda \operatorname{Im} \int_{\mathbb{R}^N} u(t(s), x) \nabla \overline{u}(t(s), x) dx \\ &= O\left(s^{-2} \right) \end{aligned}$$

Moreover, from (3.8) and the orthogonal conditions (3.7), we obtain

$$\begin{aligned} 2(\varepsilon(s), i \nabla \varepsilon(s))_2 + b \int_{\mathbb{R}^N} y \left| Q(y) + \varepsilon(s,y) \right|^2 dy &= 2(\varepsilon(s), i \nabla \varepsilon(s))_2 + b \int_{\mathbb{R}^N} y |\varepsilon(s,y)|^2 dy \\ &= O(s^{-2L}). \end{aligned}$$

Consequently, we obtain (3.10). □

Lemma 3.5 (Estimation of modulation terms)**.** For all $s \in (s_*, s_1]$,

$$2(\varepsilon(s), Q)_2 = - \|\varepsilon(s)\|_2^2, \tag{3.11}$$

$$|\operatorname{Mod}(s)| \lesssim s^{-3}, \tag{3.12}$$

$$\left| \frac{\partial b}{\partial s} + b^2 \right| + \left| \frac{1}{\lambda} \frac{\partial \lambda}{\partial s} + b \right| \lesssim s^{-2L}. \tag{3.13}$$

Proof. According to the mass conservation, we have

$$2 \left(\varepsilon, Q \right)_2 = \|u\|_2^2 - \|Q\|_2^2 - \|\varepsilon\|_2^2 = - \|\varepsilon\|_2^2,$$

meaning (3.11) holds.

For $v = \Lambda Q$, $i|y|^2 Q$, ρ, or $y_j Q$, the following estimates hold:

$$|f(Q + \varepsilon) - f(Q) - df(Q)(\varepsilon)||v| \lesssim |\varepsilon|^2, \quad |(\lambda^2 V(\lambda y - w)\varepsilon, v)_2| \lesssim \left(\lambda^3 + \lambda^\kappa |w|^2 \right) \|\varepsilon\|_2.$$

By differentiating the orthogonal conditions (3.7) with respect to the time variable s, we obtain

$$0 = \frac{d}{ds} \left(i\varepsilon, \Lambda Q \right)_2 = \left(i \frac{\partial \varepsilon}{\partial s}, \Lambda Q \right)_2 \tag{3.14}$$

$$= \frac{d}{ds} \left(i\varepsilon, i|y|^2 Q \right)_2 = \left(i \frac{\partial \varepsilon}{\partial s}, i|y|^2 Q \right)_2 \tag{3.15}$$

$$= \frac{d}{ds} \left(i\varepsilon, \rho \right)_2 = \left(i \frac{\partial \varepsilon}{\partial s}, \rho \right)_2 \tag{3.16}$$

$$= \frac{d}{ds} \left(i\varepsilon, i y_j Q \right)_2 = \left(i \frac{\partial \varepsilon}{\partial s}, i y_j Q \right)_2. \tag{3.17}$$

For the first line of (3.9),

$$\begin{aligned}&-\Delta\varepsilon+\varepsilon-\left(f\left(Q+\varepsilon\right)-f\left(Q\right)\right)+\lambda^2V(\lambda y-w)\varepsilon\\=&L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon-\left(f\left(Q+\varepsilon\right)-f\left(Q\right)-df(Q)(\varepsilon)\right)+\lambda^2V(\lambda y-w)\varepsilon\end{aligned}$$

holds. Therefore, we have

$$\begin{aligned}i\frac{\partial\varepsilon}{\partial s}=&L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon-\left(f\left(Q+\varepsilon\right)-f\left(Q\right)-df(Q)(\varepsilon)\right)+\lambda^2V(\lambda y-w)\varepsilon\\&+i\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)\Lambda(Q+\varepsilon)-\left(1-\frac{\partial\gamma}{\partial s}\right)(Q+\varepsilon)-\left(\frac{\partial b}{\partial s}+b^2\right)\frac{|y|^2}{4}(Q+\varepsilon)\\&+\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)b\frac{|y|^2}{2}(Q+\varepsilon)-i\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot\nabla(Q+\varepsilon)-\frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s}\cdot y(Q+\varepsilon)+\Psi.\end{aligned}$$

From (3.14), we have

$$\left(L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon,\Lambda Q\right)_2=\left(\operatorname{Re}\varepsilon,L_+\Lambda Q\right)_2=-2\left(\operatorname{Re}\varepsilon,Q\right)_2=-2\left(\varepsilon,Q\right)_2=\|\varepsilon\|_2^2.$$

Therefore,

$$\begin{aligned}&\frac{1}{4}\||y|Q\|_2^2\left(\frac{\partial b}{\partial s}+b^2\right)=-\left(\left(\frac{\partial b}{\partial s}+b^2\right)\frac{|y|^2}{4}Q,\Lambda Q\right)_2\\=&\Bigg(i\frac{\partial\varepsilon}{\partial s}-L_+\operatorname{Re}\varepsilon-iL_-\operatorname{Im}\varepsilon+f\left(Q+\varepsilon\right)-f\left(Q\right)-df(Q)(\varepsilon)-\lambda^2V(\lambda y-w)\varepsilon\\&\quad-i\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)\Lambda(Q+\varepsilon)+\left(1-\frac{\partial\gamma}{\partial s}\right)(Q+\varepsilon)+\left(\frac{\partial b}{\partial s}+b^2\right)\frac{|y|^2}{4}\varepsilon-\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)b\frac{|y|^2}{2}(Q+\varepsilon)\\&\quad+i\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot\nabla(Q+\varepsilon)+\frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s}\cdot y(Q+\varepsilon)-\Psi,\Lambda Q\Bigg)_2\end{aligned}$$

and according to orthogonal conditions (3.7), Equation (3.9), Proposition 3.2, and Lemma 3.4, we see that

$$\left|\frac{\partial b}{\partial s}+b^2\right|\lesssim s^{-2L}+s^{-\left(\frac{1}{2}+\frac{1}{K}\right)}|\mathrm{Mod}(s)|.$$

For (3.15), (3.16), and (3.17), we similarly obtain

$$\left|\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right|+\left|1-\frac{\partial\gamma}{\partial s}\right|\lesssim s^{-2L}+s^{-\left(\frac{1}{2}+\frac{1}{K}\right)}|\mathrm{Mod}(s)|,\quad\left|\frac{\partial w}{\partial s}\right|\lesssim s^{-3}+s^{-1}|\mathrm{Mod}(s)|.$$

Note that Lemma 3.4 is used to obtain $\frac{\partial w}{\partial s}$ from (3.17). Therefore,

$$|\mathrm{Mod}(s)|\lesssim s^{-3}+\epsilon\,|\mathrm{Mod}(s)|\,.$$

For detail of the proof of the inequality, see [10, Lemma 4.1]. Consequently, we obtain (3.12) and (3.13). □

Let m, ϵ_1, and ϵ_2 satisfy

$$1<1+\epsilon_1<\frac{m}{2}\leq L,\quad 0<\epsilon_2<\frac{m\mu\epsilon_1}{16},$$

where μ is from the coercivity (2.1) of L_+ and L_-. Moreover, we define

$$\begin{aligned}H(s,\varepsilon)&:=\frac{1}{2}\|\varepsilon\|_{H^1}^2+\epsilon_2b(s)^2\||y|\varepsilon\|_2^2-\int_{\mathbb{R}^N}\left(F(Q(y)+\varepsilon(y))-F(Q(y))-dF(Q(y))(\varepsilon(y))\right)dy\\&\qquad+\frac{1}{2}\lambda(s)^2\int_{\mathbb{R}^N}V(\lambda(s)y-w(s))|\varepsilon(y)|^2dy,\\S(s,\varepsilon)&:=\frac{1}{\lambda(s)^m}H(s,\varepsilon).\end{aligned}$$

Lemma 3.6 (Coercivity)**.** For all $s\in(s_*,s_1]$,

$$H(s,\varepsilon)\geq\frac{\mu}{4}\|\varepsilon\|_{H^1}^2+\epsilon_2b^2\||y|\varepsilon\|_2^2$$

holds. Moreover,

$$\frac{1}{\lambda^m}\left(\frac{\mu}{4}\|\varepsilon\|_{H^1}^2+\epsilon_2b^2\||y|\varepsilon\|_2^2\right)\leq S(s,\varepsilon)\lesssim\frac{1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right)$$

holds.

Proof. Firstly, we have

$$\int_{\mathbb{R}^N}\left(F(Q(y)+\varepsilon(y))-F(Q(y))-dF(Q(y))(\varepsilon(y))-\frac{1}{2}d^2F(Q(y))(\varepsilon(y),\varepsilon(y))\right)dy$$
$$=O\left(\|\varepsilon\|_{H^1}^3+\|\varepsilon\|_{H^1}^{2+\frac{4}{N}}\right).$$

Furthermore, according to (3.1), we have

$$\left|\lambda^2\int_{\mathbb{R}^N}V(\lambda y-w)|\varepsilon(y)|^2dy\right|\leq\epsilon\|\varepsilon\|_{H^1}^2.$$

Finally, since

$$\|\varepsilon\|_{H^1}^2-\int_{\mathbb{R}^N}d^2F(Q(y))(\varepsilon(y),\varepsilon(y))dy=\left(L_+\operatorname{Re}\varepsilon,\operatorname{Re}\varepsilon\right)_2+\left(L_-\operatorname{Im}\varepsilon,\operatorname{Im}\varepsilon\right)_2,$$

we obtain Lemma 3.6. □

Lemma 3.7. For all $s\in(s_*,s_1]$,

$$|(f(Q+\varepsilon)-f(Q),\Lambda\varepsilon)_2|\lesssim\|\varepsilon\|_{H^1}^2,\tag{3.18}$$
$$\left|\lambda^2\left(V(\lambda y-w)\varepsilon,\Lambda\varepsilon\right)_2\right|\lesssim s^{-(1+\kappa)}\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right).\tag{3.19}$$

Proof. For (3.18), see [10, Section 5.4] or Lemma 4.16. For (3.19), a direct calculation shows

$$\left(V(\lambda y-w)\varepsilon,\Lambda\varepsilon\right)_2=-\frac{1}{2}\left(\lambda y\cdot(\nabla V)(\lambda y-w)\varepsilon,\varepsilon\right)_2.$$

Therefore, from (3.3), we obtain (3.19). □

Lemma 3.8 (Derivative of H in time)**.** For all $s\in(s_*,s_1]$,

$$\frac{d}{ds}H(s,\varepsilon(s))\geq-b\left(\frac{4\epsilon_2}{\epsilon_1}\|\varepsilon\|_{H^1}^2+\left(\frac{m}{2}+1+\epsilon_1\right)\epsilon_2b^2\,\||y|\varepsilon\|_2^2+Cs^{-(2+L)}\right).$$

Remark 3.9. The term Cs^{-4} is present in [12], but has been corrected to $Cs^{-(2+L)}$.

Proof. Firstly,

$$\begin{aligned}\frac{d}{ds}H(s,\varepsilon(s))&=\frac{\partial H}{\partial s}(s,\varepsilon(s))+\left(i\frac{\partial H}{\partial\varepsilon}(s,\varepsilon(s)),i\frac{\partial\varepsilon}{\partial s}(s)\right)_2,\\ \frac{\partial H}{\partial\varepsilon}(s,\varepsilon)&=-\Delta\varepsilon+\varepsilon+2\epsilon_2b^2|y|^2\varepsilon-(f(Q+\varepsilon)-f(Q))+\lambda^2V(\lambda y-w)\varepsilon\\ &=L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon+2\epsilon_2b^2|y|^2\varepsilon-(f(Q+\varepsilon)-f(Q)-df(Q)(\varepsilon))+\lambda^2V(\lambda y-w)\varepsilon,\\ \frac{\partial H}{\partial s}(s,\varepsilon)&=2\epsilon_2b\frac{\partial b}{\partial s}\,\||y|\varepsilon\|_2^2+\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}\lambda^2\int_{\mathbb{R}^N}V(\lambda y-w)|\varepsilon|^2dy\\ &\quad+\frac{1}{2}\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}\lambda^3\int_{\mathbb{R}^N}y\cdot(\nabla V)(\lambda y-w)|\varepsilon|^2dy-\frac{1}{2}\lambda^3\int_{\mathbb{R}^N}\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot(\nabla V)(\lambda y-w)|\varepsilon|^2dy\end{aligned}$$

holds. Additionally, we define

$$\begin{aligned}\operatorname{Mod}_{\mathrm{op}}v:=&i\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)\Lambda v-\left(1-\frac{\partial\gamma}{\partial s}\right)v-\left(\frac{\partial b}{\partial s}+b^2\right)\frac{|y|^2}{4}v+\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)b\frac{|y|^2}{2}v\\ &-i\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot\nabla v-\frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s}\cdot yv.\end{aligned}$$

Then,

$$i\frac{\partial\varepsilon}{\partial s}=\frac{\partial H}{\partial\varepsilon}-2\epsilon_2b^2|y|^2\varepsilon+\operatorname{Mod}_{\mathrm{op}}(Q+\varepsilon)+\Psi$$

holds.

Then, we have

$$2\epsilon_2b\frac{\partial b}{\partial s}\,\||y|\varepsilon\|_2^2\geq-2\epsilon_2(1+\epsilon)b^3\,\||y|\varepsilon\|_2^2.$$

According to (3.1), (3.3), and Lemma 3.7, we have

$$\frac{\partial H}{\partial s}\geq-2\epsilon_2(1+\epsilon)b^3\,\||y|\varepsilon\|_2^2-\epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right).$$

Next, since $\frac{\partial H}{\partial \varepsilon} \in L^2(\mathbb{R}^N)$,

$$\left(i\frac{\partial H}{\partial \varepsilon}, \frac{\partial H}{\partial \varepsilon} \right) = 0$$

holds.

For $\left(i\frac{\partial H}{\partial \varepsilon}, -2\epsilon_2 b^2 |y|^2 \varepsilon\right)_2$,

$$\left(i\frac{\partial H}{\partial \varepsilon}, -2\epsilon_2 b^2 |y|^2 \varepsilon \right)_2 = 4\epsilon_2 b^2 (i\nabla\varepsilon, y\varepsilon)_2 + \left(i(|Q+\varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}})Q, -2\epsilon_2 b^2 |y|^2 \varepsilon \right)_2 .$$

Therefore, we have

$$\begin{aligned} \left| \left(i\frac{\partial H}{\partial \varepsilon}, -2\epsilon_2 b^2 |y|^2 \varepsilon \right)_2 \right| &\leq 4\epsilon_2 b^2 \, \| |y|\varepsilon \|_2 \, \|\nabla\varepsilon\|_2 + C\epsilon_2 b^2 \|\varepsilon\|_2^2 \\ &\leq 4\epsilon_2 b^2 \, \| |y|\varepsilon \|_2 \, \|\nabla\varepsilon\|_2 + \epsilon b \|\varepsilon\|_2^2 . \end{aligned}$$

For $\left(i\frac{\partial H}{\partial \varepsilon}, \mathrm{Mod}_{\mathrm{op}}\, Q\right)_2$, by using orthogonal properties (3.7) and Lemma 3.5, we have

$$\left| \left(i\frac{\partial H}{\partial \varepsilon}, i\Lambda Q \right)_2 \right| + \left| \left(i\frac{\partial H}{\partial \varepsilon}, Q \right)_2 \right| + \left| \left(i\frac{\partial H}{\partial \varepsilon}, |y|^2 Q \right)_2 \right| + \frac{1}{\lambda} \left| \left(i\frac{\partial H}{\partial \varepsilon}, i\frac{\partial Q}{\partial y_j} \right)_2 \right| + \left| \left(i\frac{\partial H}{\partial \varepsilon}, y_j Q \right)_2 \right| \lesssim \|\varepsilon\|_2^2 + s^{-2}.$$

Therefore, based on the definition of $\mathrm{Mod}_{\mathrm{op}}\, Q$ and (3.12), we have

$$\left| \left(i\frac{\partial H}{\partial \varepsilon}, \mathrm{Mod}_{\mathrm{op}}\, Q \right)_2 \right| \lesssim |\mathrm{Mod}(s)| \left(\|\varepsilon\|_2^2 + s^{-2} \right) \leq \epsilon b \|\varepsilon\|_2^2 + C s^{-5}.$$

For $\left(i\frac{\partial H}{\partial \varepsilon}, \mathrm{Mod}_{\mathrm{op}}\, \varepsilon\right)_2$, by Lemma 3.7, we have

$$\left| \left(i\frac{\partial H}{\partial \varepsilon}, i\Lambda\varepsilon \right)_2 \right| \lesssim \|\varepsilon\|_{H^1}^2 + b^2 \| | \cdot | \varepsilon \|_2^2 .$$

Next,

$$\left(i\frac{\partial H}{\partial \varepsilon}, \varepsilon \right)_2 = \left(\left(|Q+\varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}} \right) Q, i\varepsilon \right)_2 .$$

Therefore, we have

$$\left| \left(i\frac{\partial H}{\partial \varepsilon}, \varepsilon \right)_2 \right| \lesssim \|\varepsilon\|_2^2 .$$

Next, since

$$\left(i\frac{\partial H}{\partial \varepsilon}, i\frac{\partial \varepsilon}{\partial y_j} \right)_2 = -4\epsilon_2 b^2 (y_j\varepsilon, \varepsilon)_2 - \left(f(Q+\varepsilon) - f(Q), \frac{\partial \varepsilon}{\partial y_j} \right)_2 - \frac{\lambda^3}{2} \left(\frac{\partial V}{\partial y_j}(\lambda y - w)\varepsilon, \varepsilon \right)_2 ,$$

$$\left(f(Q+\varepsilon), \frac{\partial \varepsilon}{\partial y_j} \right)_2 = -\left(f(Q+\varepsilon), \frac{\partial Q}{\partial y_j} \right)_2 , \quad \left(f(Q), \frac{\partial Q}{\partial y_j} \right)_2 = 0,$$

and

$$\begin{aligned} \left(f(Q+\varepsilon) - f(Q), \frac{\partial \varepsilon}{\partial y_j} \right)_2 &= -\left(f(Q+\varepsilon) - f(Q), \frac{\partial Q}{\partial y_j} \right)_2 - \left(f(Q), \frac{\partial \varepsilon}{\partial y_j} \right)_2 \\ &= -\left(f(Q+\varepsilon) - f(Q), \frac{\partial Q}{\partial y_j} \right)_2 + \left(df(Q)(\varepsilon), \frac{\partial Q}{\partial y_j} \right)_2 , \end{aligned}$$

we have

$$\frac{1}{\lambda} \left| \left(i\frac{\partial H}{\partial \varepsilon}, i\frac{\partial \varepsilon}{\partial y_j} \right)_2 \right| \lesssim s^{-(2L-1)} .$$

Finally,

$$\left(i\frac{\partial H}{\partial \varepsilon}, y_j\varepsilon \right)_2 = -2 \left(\frac{\partial \varepsilon}{\partial y_j}, i\varepsilon \right)_2 + \left(\left(|Q+\varepsilon|^{\frac{4}{N}} - Q^{\frac{4}{N}} \right) Q, iy_j\varepsilon \right)_2 .$$

Therefore, we have

$$\left| \left(i\frac{\partial H}{\partial \varepsilon}, y_j\varepsilon \right)_2 \right| \lesssim \|\varepsilon\|_{H^1}^2 .$$

According to the definition of $\mathrm{Mod}_{\mathrm{op}}\,\varepsilon$ and (3.12), we have

$$\begin{aligned}\left|\left(i\frac{\partial H}{\partial\varepsilon},\mathrm{Mod}_{\mathrm{op}}\,\varepsilon\right)_2\right| &\leq C\,|\mathrm{Mod}(s)|\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2+Cs^{-(2L-1)}\right)\\ &\leq \epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right)+Cs^{-(2L+2)}.\end{aligned}$$

Finally, we have

$$\left|\left(i\frac{\partial H}{\partial\varepsilon},\Psi\right)_2\right|\lesssim s^{-(3+L)}.$$

Consequently, we have

$$\begin{aligned}\left|\left(i\frac{\partial H}{\partial\varepsilon},i\frac{\partial\varepsilon}{\partial s}\right)_2\right| &\leq 4\epsilon_2 b^2\,\||y|\varepsilon\|_2\,\|\nabla\varepsilon\|_2+\epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right)+Cs^{-(3+L)}\\ &\leq 4\epsilon_2 b^2\,\||y|\varepsilon\|_2\,\|\nabla\varepsilon\|_2+\epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right)+Cbs^{-(2+L)}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{ds}H(s,\varepsilon(s)) &\geq -2\epsilon_2(1+\epsilon)b^3\,\||y|\varepsilon\|_2^2-\epsilon b\|\varepsilon\|_{H^1}^2-4\epsilon_2 b^2\,\||\cdot|\varepsilon\|_2\,\|\nabla\varepsilon\|_2-\epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\||y|\varepsilon\|_2^2\right)-Cbs^{-(2+L)}\\ &\geq -2(1+\epsilon+\epsilon_1)\epsilon_2 b^3\,\||y|\varepsilon\|_2^2-\frac{2\epsilon_2}{\epsilon_1}b\|\nabla\varepsilon\|_2^2-\epsilon b\left(\|\varepsilon\|_{H^1}^2+b^2\,\||y|\varepsilon\|_2^2\right)-Cbs^{-(2+L)}.\end{aligned}$$

□

We define κ' by

$$\kappa':=\frac{1}{4}-\frac{2}{K}.$$

Lemma 3.10 (Derivative of S in time)**.** For all $s\in(s_*,s_1]$,

$$\frac{d}{ds}S(s,\varepsilon(s))\gtrsim\frac{b}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2+b^2\,\||y|\varepsilon\|_2^2-s^{-(2L+\kappa')}\right).$$

Proof. According to (3.12) and Lemmas 3.7 and 3.8, we have

$$\begin{aligned}\frac{d}{ds}S(s,\varepsilon(s)) &= m\frac{b}{\lambda^m}H(s,\varepsilon(s))-m\frac{1}{\lambda^m}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)H(s,\varepsilon(s))+\frac{1}{\lambda^m}\frac{d}{ds}H(s,\varepsilon(s))\\ &\geq\frac{b}{\lambda^m}\left(\left(\frac{m\mu}{4}-\frac{4\epsilon_2}{\epsilon_1}\right)\|\varepsilon\|_{H^1}^2+\left(\frac{m}{2}-(1+\epsilon_1)\right)\epsilon_2 b^2\,\||y|\varepsilon\|_2^2-\epsilon s^{-(2L+\kappa')}\right).\end{aligned}$$

Therefore, we obtain Lemma 3.10. □

Lemma 3.11. There exists a sufficiently small $\epsilon_3>0$ such that for all $s\in(s_*,s_1]$,

$$\|\varepsilon(s)\|_{H^1}^2+b(s)^2\,\||y|\varepsilon(s)\|_2^2\lesssim s^{-(2L+\kappa')},\tag{3.20}$$
$$|s\lambda(s)-1|<(1-\epsilon_3)s^{-M},\tag{3.21}$$
$$|sb(s)-1|<(1-\epsilon_3)s^{-M},\tag{3.22}$$
$$|w(s)|\lesssim s^{-2}.\tag{3.23}$$

Proof. By using Lemmas 3.6 and 3.10 as in the proof of [10, Lemma 6.1], we see that (3.20). Indeed, we prove (3.20) by contradiction. Let $C_\dagger>0$ be sufficiently large and define

$$s_\dagger:=\inf\left\{\sigma\in(s_*,s_1]\;\middle|\;\|\varepsilon(\tau)\|_{H^1}^2+b(\tau)^2\,\||y|\varepsilon(\tau)\|_2^2\leq C_\dagger\tau^{-(2L+\kappa')}\quad(\tau\in[\sigma,s_1])\right\}.$$

Then $s_\dagger<s_1$ holds. Here, we assume that $s_\dagger>s_*$. Then we have

$$\|\varepsilon(s_\dagger)\|_{H^1}^2+b(s_\dagger)^2\,\||y|\varepsilon(s_\dagger)\|_2^2=C_\dagger {s_\dagger}^{-(2L+\kappa')}.$$

Let define

$$s_\ddagger:=\sup\left\{\sigma\in(s_*,s_1]\;\middle|\;\|\varepsilon(\tau)\|_{H^1}^2+b(\tau)^2\,\||y|\varepsilon(\tau)\|_2^2\geq\tau^{-(2L+\kappa')}\quad(\tau\in[s_\dagger,\sigma])\right\}.$$

Then $s_\ddagger > s_\dagger$ holds. Furthermore,

$$\|\varepsilon(s_\ddagger)\|_{H^1}^2 + b(s_\ddagger)^2 \||y|\varepsilon(s_\ddagger)\|_2^2 = {s_\ddagger}^{-(2L+\kappa')}.$$

Then, according to Corollary 3.6 and Lemma 3.10, we have

$$\frac{C_1'}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2\right) \le S(s,\varepsilon) \le \frac{C_2'}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2\right),$$
$$\frac{C_1 b}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2 - s^{-(2L+\kappa'')}\right) \le \frac{d}{ds}S(s,\varepsilon(s))$$

for $s \in (s_*, s_1]$. Here, $S(\cdot,\varepsilon(\cdot))$ is monotonically increasing on $[s_\dagger, s_\ddagger]$. Therefore, we have

$$\begin{aligned}
C_1' C_\dagger {s_\dagger}^{-(2L+\kappa')} &= C_1'\left(\|\varepsilon(s_\dagger)\|_{H^1}^2 + b(s_\dagger)^2 \||y|\varepsilon(s_\dagger)\|_2^2\right) \le \lambda(s_\dagger)^m S\left(s_\dagger, \varepsilon(s_\dagger)\right) \le \lambda(s_\dagger)^m S\left(s_\ddagger, \varepsilon(s_\ddagger)\right)\\
&\le \frac{\lambda(s_\dagger)^m}{\lambda(s_\ddagger)^m} C_2'\left(\|\varepsilon(s_\ddagger)\|_{H^1}^2 + b(s_\ddagger)^2 \||y|\varepsilon(s_\ddagger)\|_2^2\right) = \frac{\lambda(s_\dagger)^m}{\lambda(s_\ddagger)^m} C_2' {s_\ddagger}^{-(2L+\kappa')}\\
&\le (1+\epsilon) C_2' \left(\frac{s_\ddagger}{s_\dagger}\right)^{-(2L-m+\kappa')} {s_\dagger}^{-(2L+\kappa')}.
\end{aligned}$$

Accordingly, we have $C_1' C_\dagger \le (1+\epsilon) C_2'$, which yields a contradiction if $C_\dagger$ is sufficiently large. Consequently, $s_\dagger \le s_*$. Moreover, since $s_* \le s_\dagger$ clearly holds by definition, we have $s_* = s_\dagger$. Therefore, (3.20) holds.

We prove (3.21). Since

$$\left|\frac{d}{ds}(s\lambda)\right| \le s^{-1}(1+\epsilon)\left(s^{-M} + Cs^{-(2L-1)}\right) \le (1+\epsilon) s^{-(M+1)}$$

and $\lambda(s_1) = {s_1}^{-1}$, we have

$$|s\lambda - 1| \le \int_s^{s_1} (1+\epsilon)\sigma^{-(M+1)} d\sigma \le \frac{1+\epsilon}{M} s^{-M}.$$

Therefore, (3.21) holds since $M > 1$. Next, we prove (3.22). Since

$$\left|\frac{b}{\lambda} - 1\right| \lesssim \int_s^{s_1} \sigma^{-(2L-1)} d\sigma \lesssim s^{-2(L-1)}$$

from (3.13), we have

$$|sb - s\lambda| \lesssim s^{-2(L-1)}.$$

Consequently, we have

$$|sb - 1| \le |sb - s\lambda| + |s\lambda - 1| \le \frac{1+\epsilon}{M} s^{-M} + Cs^{-2(L-1)}.$$

Therefore, (3.22) holds. Finally, since

$$|w(s)| \le \int_s^{s_1} |\mathrm{Mod}(\sigma)| d\sigma \lesssim \int_s^{s_1} \sigma^{-3} d\sigma \lesssim s^{-2},$$

we obtain (3.23). □

Consequently, (3.8) holds on $[s_0, s_1]$:

Lemma 3.12. If s_0 is sufficiently large, then $s_* = s' = s_0$ for any $s_1 > s_0$.

Proof. From Lemma 3.11, $s_* = s'$ is obvious.

We prove $s' \le s_0$ by contradiction. Assume that for any $s_0 \gg 1$, there exists $s_1 > s_0$ such that $s' > s_0$. In the following, we consider the initial value (3.5) in response to such s_1 and the corresponding solution u for (NLS).

Let $t' := \inf I_{t_1}$. Then $s' = \inf J_{s_1} > s_0$ holds. Furthermore, we have

$$\left\|\lambda(s)^{\frac{N}{2}} u(s, \lambda(s)y - w(s)) e^{-i\gamma(s)} - Q(y)\right\|_{H^1} \le \frac{\delta}{4}$$

for all $s \in (s', s_1]$. Since $t_{t_1}((s', s_1]) = (t', t_1]$, we have

$$\left\|\tilde{\lambda}(t)^{\frac{N}{2}} u(t, \tilde{\lambda}(t)y - \tilde{w}(t)) e^{-i\tilde{\gamma}(t)} - Q(y)\right\|_{H^1} \le \frac{\delta}{4}$$

for all $t \in (t', t_1]$. We consider three cases $t' > T_*$, $t' = T_* > -\infty$, and $t' = -\infty$.

Firstly, assume $t' > T_*$. Then λ and $\tilde{\lambda}$ are bounded on $(s', s_1]$ and $(t', t_1]$, respectively, according to (3.8) and $s_* = s'$. Then, by setting t sufficiently close to t', we have

$$\left\| \tilde{\lambda}(t)^{\frac{N}{2}} u(t', \tilde{\lambda}(t)y - \tilde{w}(t))e^{-i\tilde{\gamma}(t)} - Q(y) \right\|_{H^1} < \delta.$$

Therefore, there exists the decomposition of u in a neighbourhood of t' according to Lemma 2.2. Its existence contradicts the maximality of I_{t_1}.

Next, assume $t' = T_* > -\infty$. Then $\|\nabla u(t)\|_2 \to \infty$ $(t \searrow t')$ holds according to the blow-up alternative. Also, $\|\nabla u(s)\|_2 \to \infty$ $(s \searrow s')$ holds. Then since

$$\|u(s)\|_2 + \lambda(s)\|\nabla u(s)\|_2 \lesssim 1,$$

we have $\lambda(s) \to 0$ $(s \searrow s')$. Therefore, we obtain

$$|s\lambda(s) - 1| \to 1, \quad s^{-M} \to s'^{-M} < \frac{1}{2} \quad (s \searrow s'),$$

which contradicts (3.21).

Finally, assume $t' = -\infty$. Then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ that converges to s' such that $\lim_{n\to\infty} \lambda(s_n) = \infty$ holds. Therefore, we obtain

$$|s_n\lambda(s_n) - 1| \to \infty, \quad {s_n}^{-M} \to s'^{-M} < 1 \quad (n \to \infty),$$

which contradicts (3.21).

Consequently, we obtain $s' \leq s_0$. □

The estimates obtained with Lemmas 3.11 and 3.12 are in the time variable s. Therefore, they need to be rewritten in the time variable t. To do so, for any t_1 sufficiently close to 0, each decomposition parameters must be defined on a sufficient interval.

Lemma 3.13. Let s_0 be sufficiently large. Then there exists $t_0 < 0$ such that

$$[t_0, t_1] \subset {s_{t_1}}^{-1}([s_0, s_1]), \quad \left|s_{t_1}(t)^{-1} - |t|\right| \lesssim |t|^{M+1} \quad (t \in [t_0, t_1])$$

hold for all $t_1 \in (t_0, 0)$.

Proof. Firstly, $[t_{t_1}(s_0), t_1] = {s_{t_1}}^{-1}([s_0, s_1])$ holds. For all $s \in [s_0, s_1]$, we have

$$t_1 - t_{t_1}(s) = s^{-1} - {s_1}^{-1} + \int_s^{s_1} \sigma^{-2} \left(\sigma\lambda_{t_1}(\sigma) + 1\right) \left(\sigma\lambda_{t_1}(\sigma) - 1\right) d\sigma$$

since $-{s_1}^{-1} = t_1 = t_{t_1}(s_1)$. Therefore, we have

$$\frac{1}{2}s^{-1} \leq s^{-1}\left(1 - 3s^{-M}\right) \leq |t_{t_1}(s)| \leq s^{-1}\left(1 + 3s^{-M}\right) \leq 2s^{-1}.$$

Accordingly, we obtain $\frac{1}{2}|t_{t_1}(s)| \leq s^{-1} \leq 2|t_{t_1}(s)|$. According to ${s_{t_1}}^{-1} = t_{t_1}$, we obtain

$$\frac{1}{2}|t| \leq s_{t_1}(t)^{-1} \leq 2|t|. \tag{3.24}$$

Consequently, according to (3.24), we obtain

$$\left||t| - s_{t_1}(t)^{-1}\right| \leq 3s_{t_1}(t)^{-(M+1)} \leq 3 \cdot 2^{M+1}|t|^{M+1}.$$

Furthermore, since

$$t_{t_1}(s_0) = -|t_{t_1}(s_0)| \leq -\frac{1}{2}s_{t_1}(t_{t_1}(s_0))^{-1} = -\frac{1}{2}{s_0}^{-1}$$

and s_0 is independent of t_1 according to Lemma 3.12, we obtain Lemma 3.13. □

Lemma 3.14 (Conversion of estimates)**.** For any $t_1 \in (t_0, 0)$ and $t \in [t_0, t_1]$,

$$\tilde{\lambda}_{t_1}(t) = |t|\left(1 + \epsilon_{\tilde{\lambda},t_1}(t)\right), \qquad \tilde{b}_{t_1}(t) = |t|\left(1 + \epsilon_{\tilde{b},t_1}(t)\right), \qquad |\tilde{w}_{t_1}(t)| \lesssim |t|^2,$$

$$\|\tilde{\varepsilon}_{t_1}(t)\|_{H^1} \lesssim |t|^{L+\frac{\kappa'}{2}}, \qquad \||y|\tilde{\varepsilon}_{t_1}(t)\|_2 \lesssim |t|^{L+\frac{\kappa'}{2}-1}$$

hold for some functions $\epsilon_{\tilde{\lambda},t_1}$ and $\epsilon_{\tilde{b},t_1}$. Furthermore,

$$\sup_{t_1\in[t,0)} \left|\epsilon_{\tilde{\lambda},t_1}(t)\right| \lesssim |t|^M, \quad \sup_{t_1\in[t,0)} \left|\epsilon_{\tilde{b},t_1}(t)\right| \lesssim |t|^M.$$

Proof. Firstly, we define $\epsilon_{\tilde{\lambda},t_1}(t) := \frac{\tilde{\lambda}_{t_1}(t)}{|t|} - 1$. According to (3.24) and Lemma 3.13, we have

$$\left|\epsilon_{\tilde{\lambda},t_1}(t)\right| = \left|\left(s_{t_1}(t)\tilde{\lambda}_{t_1}(t) - 1\right)\frac{1}{s_{t_1}(t)|t|} + \frac{1}{s_{t_1}(t)|t|} - 1\right| \lesssim |t|^M.$$

Similarly, we define $\epsilon_{\tilde{b},t_1}(t) := \frac{\tilde{b}_{t_1}(t)}{|t|} - 1$ and obtain estimates of $\tilde{b}_{t_1}(t)$ and $\tilde{w}_{t_1}(t)$. □

Finally, this chapter ends with the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $(t_n)_{n\in\mathbb{N}} \subset (t_0, 0)$ be a increasing sequence such that $\lim_{n\nearrow\infty} t_n = 0$. For each $n \in \mathbb{N}$, let u_n be the solution for (NLS) with the initial value

$$u_n(t_n, x) := \frac{1}{{\lambda_{1,n}}^{\frac{N}{2}}} Q\left(\frac{x}{\lambda_{1,n}}\right) e^{-i\frac{b_{1,n}}{4}\frac{|x|^2}{{\lambda_{1,n}}^2}}$$

at t_n, where $b_{1,n} = \lambda_{1,n} = {s_n}^{-1} = -t_n$.

According to Lemma 2.2, there exists the decomposition

$$u_n(t,x) = \frac{1}{\tilde{\lambda}_n(t)^{\frac{N}{2}}} (Q + \tilde{\varepsilon}_n)\left(t, \frac{x + \tilde{w}_n(t)}{\tilde{\lambda}_n(t)}\right) e^{-i\frac{\tilde{b}_n(t)}{4}\frac{|x+\tilde{w}_n(t)|^2}{\tilde{\lambda}_n(t)^2} + i\tilde{\gamma}_n(t)}$$

on $[t_0, t_n]$. Then $(u_n(t_0))_{n\in\mathbb{N}}$ is bounded in Σ^1. Therefore, up to a subsequence, there exists $u_\infty(t_0) \in \Sigma^1$ such that

$$u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{weakly in } \Sigma^1.$$

Moreover, from Fréchet-Kolmogorov theorem, we see that

$$u_n(t_0) \to u_\infty(t_0) \quad \text{in } L^2(\mathbb{R}^N) \quad (n \to \infty).$$

Let u_∞ be the solution for (NLS) with the initial value $u_\infty(t_0)$ and T^* be the supremum of the maximal existence interval of u_∞. Moreover, we define $T := \min\{0, T^*\}$. For any $T' \in [t_0, T)$, we have $[t_0, T'] \subset [t_0, t_n]$ if n is sufficiently large. Then there exists n_0 such that

$$\sup_{n\geq n_0} \|u_n\|_{L^\infty([t_0,T'],\Sigma^1)} \lesssim \left(1 + |T'|^{-1}\right)\left(1 + |t_0|^L\right)$$

holds. According to Lemma 2.1,

$$u_n \to u_\infty \quad \text{in } C([t_0, T'], L^2(\mathbb{R}^N)) \quad (n \to \infty)$$

holds. In particular, $u_n(t) \rightharpoonup u_\infty(t)$ in Σ^1 for any $t \in [t_0, T)$. Furthermore, we have

$$\|u_\infty(t)\|_2 = \|u_\infty(t_0)\|_2 = \lim_{n\to\infty} \|u_n(t_0)\|_2 = \lim_{n\to\infty} \|u_n(t_n)\|_2 = \|Q\|_2.$$

According to weak convergence in Σ^1 and Lemma 2.2, we decompose u_∞ to

$$u_\infty(t,x) = \frac{1}{\tilde{\lambda}_\infty(t)^{\frac{N}{2}}} (Q + \tilde{\varepsilon}_\infty)\left(t, \frac{x + \tilde{w}_\infty(t)}{\tilde{\lambda}_\infty(t)}\right) e^{-i\frac{\tilde{b}_\infty(t)}{4}\frac{|x+\tilde{w}_\infty(t)|^2}{\tilde{\lambda}_\infty(t)^2} + i\tilde{\gamma}_\infty(t)}$$

on $[t_0, T)$. Furthermore, as $n \to \infty$,

$$\begin{aligned}
&\tilde{\lambda}_n(t) \to \tilde{\lambda}_\infty(t), \quad \tilde{b}_n(t) \to \tilde{b}_\infty(t), \quad \tilde{w}_n(t) \to \tilde{w}_\infty(t), \quad e^{i\tilde{\gamma}_n(t)} \to e^{i\tilde{\gamma}_\infty(t)}, \\
&\tilde{\varepsilon}_n(t) \rightharpoonup \tilde{\varepsilon}_\infty(t) \quad \text{weakly in } \Sigma^1
\end{aligned}$$

hold for any $t \in [t_0, T)$. Therefore, we obtain

$$\begin{aligned}
&\tilde{\lambda}_\infty(t) = |t|\,(1 + \epsilon_{\tilde{\lambda},0}(t)), && \tilde{b}_\infty(t) = |t|\,(1 + \epsilon_{\tilde{b},0}(t)), && |\tilde{w}_\infty(t)| \lesssim |t|^2, \\
&\|\tilde{\varepsilon}_\infty(t)\|_{H^1} \lesssim |t|^{L+\frac{\kappa'}{2}}, && \||y|\tilde{\varepsilon}_\infty(t)\|_2 \lesssim |t|^{L+\frac{\kappa'}{2}-1}, && \left|\epsilon_{\tilde{\lambda},0}(t)\right| \lesssim |t|^M, \quad \left|\epsilon_{\tilde{b},0}(t)\right| \lesssim |t|^M
\end{aligned}$$

from the uniform estimates in Lemma 3.14. Consequently, we obtain Theorem 3.1. □

Chapter 4

Case of inverse power potential

4.1 Problem and Main results

In this chapter, we consider the following equation

$$i\frac{\partial u}{\partial t}+\Delta u+|u|^{\frac{4}{N}}u\pm\frac{1}{|x|^{2\sigma}}u=0, \tag{NLS}$$

where

$$0<\sigma<\min\left\{\frac{N}{4},1\right\}. \tag{4.1}$$

Then we obtain the following results.

Theorem 4.1 ([11]). Assume (4.1). Then for any energy level $E_0\in\mathbb{R}$, there exist $t_0<0$ and a radially symmetric initial value $u_0\in\Sigma^1$ with

$$\|u_0\|_2=\|Q\|_2,\quad E(u_0)=E_0$$

such that the corresponding solution u for (NLS) with $\pm=+$ and $u(t_0)=u_0$ blows up at $t=0$. Moreover,

$$\left\|u(t)-\frac{1}{\lambda(t)^{\frac{N}{2}}}P\left(t,\frac{x}{\lambda(t)}\right)e^{-i\frac{b(t)}{4}\frac{|x|^2}{\lambda(t)^2}+i\gamma(t)}\right\|_{\Sigma^1}\to 0\quad (t\nearrow 0)$$

holds for some blow-up profile P and C^1 functions $\lambda:(t_0,0)\to(0,\infty)$ and $b,\gamma:(t_0,0)\to\mathbb{R}$ such that

$$P(t)\to Q\quad\text{in }\Sigma^1,\qquad \lambda(t)=C_1(\sigma)|t|^{\frac{1}{1+\sigma}}\left(1+o(1)\right),$$
$$b(t)=C_2(\sigma)|t|^{\frac{1-\sigma}{1+\sigma}}\left(1+o(1)\right),\qquad \gamma(t)^{-1}=O\left(|t|^{\frac{1-\sigma}{1+\sigma}}\right)$$

as $t\nearrow 0$.

On the other hand, the following holds when $\pm=-$.

Theorem 4.2 ([11]). Assume $N\geq 2$ and $0<\sigma<1$. If $u_0\in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\|u_0\|_2=\|Q\|_2$, the corresponding solution u for (NLS) with $\pm=-$ and $u(0)=u_0$ is global and bounded in $H^1(\mathbb{R}^N)$.

As in Theorem 1.3, this result means that there is no spherically symmetric minimal-mass blow-up solution.

In terms of blow-up rate expectations, we compare (NLS) and (DPNLS). We consider the transformation (2.2). Then v for solution u for (DPNLS) satisfies

$$0=i\frac{\partial v}{\partial s}+\Delta v-v+|v|^{\frac{4}{N}}v\pm\lambda^{\alpha}|v|^{p-1}v+\text{modulation terms}+\text{error terms}$$

with $\alpha=2-\frac{N}{2}(p-1)$. Theorem 1.2 states that there exists a minimal-mass blow-up solution with a blow-up rate $|t|^{-\frac{2}{4-\alpha}}$ with $\pm=+$, and Theorem 1.3 states that there exists no minimal-mass blow-up solution with $\pm=-$. On the other hand, v for solution u for (NLS) satisfies

$$0=i\frac{\partial v}{\partial s}+\Delta v-v+|v|^{\frac{4}{N}}v\pm\lambda^{\alpha}\frac{1}{|y|^{2\sigma}}v+\text{modulation terms}+\text{error terms}$$

with $\alpha = 2 - 2\sigma$. Therefore, we expect (NLS) to behave similarly to (DPNLS) regarding blow-up solutions. Namely, there may exist a minimal-mass blow-up solution with a blow-up rate of $|t|^{-\frac{2}{4-\alpha}}$ with $\pm = +$, and there may exist no minimal-mass blow-up solution with $\pm = -$. Moreover, the method in this paper could also be applied to nonlinear terms of the form $|x|^{-2\sigma}|u|^{p-1}u$.

In results [4, 5, 7, 17], the blow-up solutions are explicitly constructed by applying a special transformation to the solitary wave. In contrast, (NLS) has no such transformation. Therefore, we need a non-classical method that does not use the pseudo-conformal transformation, such as [10] or [18]. As a result, we have constructed a blow-up solution with a non-trivial rate $|t|^{-\frac{1}{1+\sigma}}$. In particular, Theorem 4.1 is the first result for an unbounded potential without algebraic properties.

The potential in Theorem 3.1 is smooth and the blow-up rate is $|t|^{-1}$. In contrast, the potential in Theorem 4.1 is singular at the origin and the blow-up rate is $|t|^{-\frac{1}{1+\sigma}}$. The smoothness at the origin (or more precisely at the blow-up point) is what makes the difference between the two blow-up rates.

In terms of blow-up rates, we construct a blow-up solution with a rate of $|t|^{-\frac{1}{1+\sigma}}$. Here, we have $|t|^{-\frac{1}{1+\sigma}} \to |t|^{-\frac{1}{2}}$ as $\sigma \to 1$. This blow-up rate is different from the rate $|t|^{-1}$ in results in [7, 17]. If $\sigma = 1$, then the inverse power potential term cannot be treated as a perturbation because the scaling is balanced by the Laplacian unlike $\sigma < 1$ and (DPNLS). Consequently, when $\sigma = 1$, by using a different ground state from the one used in this paper and the pseudo-conformal transformation, we obtain a blow-up solution with a blow-up rate $|t|^{-1}$. Moreover, since $C_1(\sigma) \to \infty$ as $\sigma \to 1$, the limit dose not make sense.

In the proof of Lemmas 4.17 and 4.18, we use Σ^2 regularity of the error function ε. Therefore, we assume (4.1) is for the error function ε to belong to Σ^2. However, the behaviour of blow-up in Theorem 4.1 is described in Σ^1. Accordingly, it may be not essential.

4.2 Construction of blow-up profile

For $K \in \mathbb{N}$, we define

$$\Sigma_K := \left\{ (j,k) \in {\mathbb{N}_0}^2 \ \middle| \ j+k \leq K \right\}.$$

Proposition 4.3. Let $K, K' \in \mathbb{N}$ be sufficiently large. Let $\lambda(s) > 0$ and $b(s) \in \mathbb{R}$ be C^1 functions of s such that $\lambda(s) + |b(s)| \ll 1$.

(i). *Existence of blow-up profile.* For any $(j,k) \in \Sigma_{K+K'}$, there exist $P_{j,k}^+, P_{j,k}^- \in \mathcal{Y}'$, $\beta_{j,k} \in \mathbb{R}$, and $\Psi \in H^1(\mathbb{R}^N)$ such that P satisfies

$$i\frac{\partial P}{\partial s} + \Delta P - P + f(P) + \lambda^\alpha \frac{1}{|y|^{2\sigma}} P + \theta \frac{|y|^2}{4} P = \Psi,$$

where $\alpha = 2 - 2\sigma$, and P and θ are defined by

$$P(s,y) := Q(y) + \sum_{(j,k)\in\Sigma_{K+K'}} \left(b(s)^{2j}\lambda(s)^{(k+1)\alpha} P_{j,k}^+(y) + ib(s)^{2j+1}\lambda(s)^{(k+1)\alpha} P_{j,k}^-(y) \right),$$

$$\theta(s) := \sum_{(j,k)\in\Sigma_{K+K'}} b(s)^{2j}\lambda(s)^{(k+1)\alpha}\beta_{j,k}.$$

Moreover, for some sufficiently small $\epsilon' > 0$,

$$\left\| e^{\epsilon'|y|}\Psi \right\|_{H^1} \lesssim \lambda^\alpha \left(\left| \frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| \right) + (b^2 + \lambda^\alpha)^{K+2}$$

holds.

(ii). *Mass and energy properties of blow-up profile.* Let define

$$P_{\lambda,b,\gamma}(s,x) := \frac{1}{\lambda(s)^{\frac{N}{2}}} P\left(s, \frac{x}{\lambda(s)}\right) e^{-i\frac{b(s)}{4}\frac{|x|^2}{\lambda(s)^2} + i\gamma(s)}.$$

Then

$$\left| \frac{d}{ds} \|P_{\lambda,b,\gamma}\|_2^2 \right| \lesssim \lambda^\alpha \left(\left| \frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| \right) + (b^2 + \lambda^\alpha)^{K+2},$$

$$\left| \frac{d}{ds} E(P_{\lambda,b,\gamma}) \right| \lesssim \frac{1}{\lambda^2} \left(\left| \frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b \right| + \left| \frac{\partial b}{\partial s} + b^2 - \theta \right| + (b^2 + \lambda^\alpha)^{K+2} \right)$$

hold. Moreover,

$$\left|8E(P_{\lambda,b,\gamma}) - \||y|Q\|_2^2\left(\frac{b^2}{\lambda^2} - \frac{2\beta}{2-\alpha}\lambda^{\alpha-2}\right)\right| \lesssim \frac{\lambda^\alpha(b^2+\lambda^\alpha)}{\lambda^2} \tag{4.2}$$

holds, where

$$\beta := \beta_{0,0} = \frac{4\sigma\||y|^{-\sigma}Q\|_2^2}{\||y|Q\|_2^2}.$$

Remark 4.4. Incorrect construction of profile in [11] has been corrected. In the definition of Φ in [11], $b^{2j}\lambda^{(k+2)\alpha}\frac{1}{|y|^{2\sigma}}P_{j,k}^+$ and $ib^{2j+1}\lambda^{(k+2)\alpha}\frac{1}{|y|^{2\sigma}}P_{j,k}^-$ are added only when $j=0$ and $k=K+K'$, but in reality they must be added when $j+k=K+K'$. As a result, Proposition 4.10 is required.

Proof. See [10, Proposition 2.1] for details of proofs.

We prove (i). We set

$$Z := \sum_{(j,k)\in\Sigma_{K+K'}} b^{2j}\lambda^{k\alpha}P_{j,k}^+ + i\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j+1}\lambda^{k\alpha}P_{j,k}^-.$$

Then $P = Q + \lambda^\alpha Z$ holds. Moreover, let set

$$\Theta(s) := \sum_{(j,k)\in\Sigma_{K+K'}} b(s)^{2j}\lambda(s)^{(k+1)\alpha}c_{j,k}^+ \tag{4.3}$$

and we consider

$$i\frac{\partial P}{\partial s} + \Delta P - P + f(P) + \lambda^\alpha\frac{1}{|y|^{2\sigma}}P + \theta\frac{|y|^2}{4}P + \Theta Q = 0,$$

where $P_{j,k}^+, P_{j,k}^- \in \mathcal{Y}'$ and $\beta_{j,k}, c_{j,k}^+ \in \mathbb{R}$ are to be determined.

Firstly, we have

$$\begin{aligned} i\frac{\partial P}{\partial s} &= -i\sum_{(j,k)\in\Sigma_{K+K'}}((k+1)\alpha+2j)b^{2j+1}\lambda^{(k+1)\alpha}P_{j,k}^+ \\ &\quad + i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha}F_{j,k}^{\frac{\partial P}{\partial s},-} + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha}F_{j,k}^{\frac{\partial P}{\partial s},+} + \Phi^{\frac{\partial P}{\partial s}}, \end{aligned}$$

where

$$\begin{aligned} \Phi^{\frac{\partial P}{\partial s}} &= \left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s} + b\right)\sum_{(j,k)\in\Sigma_{K+K'}}(k+1)\alpha b^{2j}\lambda^{(k+1)\alpha}(iP_{j,k}^+ - bP_{j,k}^-) \\ &\quad + \left(\frac{\partial b}{\partial s} + b^2 - \theta\right)\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j-1}\lambda^{(k+1)\alpha}(2jiP_{j,k}^+ - (2j+1)bP_{j,k}^-) \end{aligned}$$

and for $j,k\geq 0$, $F_{j,k}^{\frac{\partial P}{\partial s},\pm}$ consists of $P_{j',k'}^\pm$ and $\beta_{j',k'}$ for $(j',k')\in\Sigma_{K+K'}$ such that $k'\leq k-1$ and $j'\leq j+1$ or $k'\leq k$ and $j'\leq j-1$. Only a finite number of these functions are non-zero. In particular, $F_{j,k}^{\frac{\partial P}{\partial s},\pm}$ belongs to $\mathcal{Y}'$ and

$$F_{j,0}^{\frac{\partial P}{\partial s},+} = (2j+\alpha+1)P_{j-1,0}^-, \quad F_{j,0}^{\frac{\partial P}{\partial s},-} = 0$$

for any $j\geq 0$.

Next, we have

$$\begin{aligned} \Delta P - P + |P|^{\frac{4}{N}}P &= -\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j}\lambda^{(k+1)\alpha}L_+P_{j,k}^+ - i\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j+1}\lambda^{(k+1)\alpha}L_-P_{j,k}^- \\ &\quad + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha}F_{j,k}^{f,+} + i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha}F_{j,k}^{f,-} + \Phi^f, \end{aligned}$$

where

$$\Phi^f = f(Q+\lambda^\alpha Z) - \sum_{k=0}^{K+K'+1}\frac{1}{k!}d^kf(Q)(\lambda^\alpha Z,\cdots,\lambda^\alpha Z)$$

and for $j,k\geq 0$, $F_{j,k}^{f,\pm}$ consists of Q, $P_{j',k'}^\pm$, and $\beta_{j',k'}$ for $(j',k')\in\Sigma_{K+K'}$ such that $k'\leq k-1$ and $j'\leq j$. Only a finite number of these functions are non-zero. In particular, $F_{j,k}^{f,\pm}$ belongs to $\mathcal{Y}'$ and $F_{j,0}^{f,\pm} = 0$ for any $j\geq 0$.

Next, we have

$$\lambda^\alpha \frac{1}{|y|^{2\sigma}} P = \sum_{j+k\geq 0} \left(b^{2j}\lambda^{(k+1)\alpha} \frac{1}{|y|^{2\sigma}} F_{j,k}^{\sigma,+} + ib^{2j+1}\lambda^{(k+1)\alpha}\frac{1}{|y|^{2\sigma}} F_{j,k}^{\sigma,-} \right),$$

where

$$F_{j,k}^{\sigma,+} = \begin{cases} Q & (j=k=0) \\ 0 & (j\geq 1,\ k=0) \\ P_{j,k-1}^{+} & (k\geq 1) \end{cases}, \quad F_{j,k}^{\sigma,-} = \begin{cases} 0 & (k=0) \\ P_{j,k-1}^{-} & (k\geq 1) \end{cases}.$$

Finally, we have

$$\theta \frac{|y|^2}{4} P = \sum_{(j,k)\in\Sigma_{K+K'}} b^{2j}\lambda^{(k+1)\alpha}\beta_{j,k}\frac{|y|^2}{4}Q + \sum_{j,k\geq 0} b^{2j}\lambda^{(k+1)\alpha} F_{j,k}^{\theta,+} + i\sum_{j,k\geq 0} b^{2j+1}\lambda^{(k+1)\alpha}F_{j,k}^{\theta,-}$$

and for $j,k\geq 0$, $F_{j,k}^{\theta,\pm}$ consists of Q, $P_{j',k'}^{\pm}$, and $\beta_{j',k'}$ for $(j',k')\in\Sigma_{K+K'}$ such that $k'\leq k-1$ and $j'\leq j$. Only a finite number of these functions are non-zero. In particular, $F_{j,k}^{\theta,\pm}$ belongs to $\mathcal{Y}'$ and $F_{j,0}^{\theta,\pm}=0$ for any $j\geq 0$.

Here, we define

$$\begin{aligned} F_{j,k}^{\pm} &:= F_{j,k}^{\frac{\partial P}{\partial s},\pm} + F_{j,k}^{f,\pm} + F_{j,k}^{\theta,\pm}, \\ \Phi^{>K+K'} &:= \sum_{(j,k)\notin\Sigma_{K+K'}} b^{2j}\lambda^{(k+1)\alpha}F_{j,k}^{+} + i\sum_{(j,k)\notin\Sigma_{K+K'}} b^{2j+1}\lambda^{(k+1)\alpha}F_{j,k}^{-}, \\ \Phi &:= \Phi^{\frac{\partial P}{\partial s}} + \Phi^f + \Phi^{>K+K'} + \sum_{j+k=K+K'}\left(b^{2j}\lambda^{(k+2)\alpha}\frac{1}{|y|^{2\sigma}}P_{j,k}^{+} + ib^{2j+1}\lambda^{(k+2)\alpha}\frac{1}{|y|^{2\sigma}}P_{j,k}^{-}\right). \end{aligned}$$

Then $\Phi^{>K+K'}$ is a finite sum and we obtain

$$\begin{aligned} & i\frac{\partial P}{\partial s} + \Delta P - P + f(P) + \lambda^\alpha\frac{1}{|y|^{2\sigma}}P + \theta\frac{|y|^2}{4}P + \Theta Q \\ =& \sum_{(j,k)\in\Sigma_{K+K'}} b^{2j}\lambda^{(k+1)\alpha}\left(-L_+P_{j,k}^{+} + \beta_{j,k}\frac{|y|^2}{4}Q + \frac{1}{|y|^{2\sigma}}F_{j,k}^{\sigma,+} + F_{j,k}^{+} + c_{j,k}^{+}Q \right) \\ & + i\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j+1}\lambda^{(k+1)\alpha}\left(-L_-P_{j,k}^{-} - ((k+1)\alpha+2j)P_{j,k}^{+} + \frac{1}{|y|^{2\sigma}}F_{j,k}^{\sigma,-} + F_{j,k}^{-}\right) \\ & + \Phi. \end{aligned}$$

For each $(j,k)\in\Sigma_{K+K'}$, we choose recursively $P_{j,k}^{\pm}\in\mathcal{Y}'$ and $\beta_{j,k}, c_{j,k}^{+}\in\mathbb{R}$ that are solutions for the systems

$$(S_{j,k}) \quad \begin{cases} L_+P_{j,k}^{+} - F_{j,k}^{+} - \beta_{j,k}\frac{|y|^2}{4}Q - \frac{1}{|y|^{2\sigma}}F_{j,k}^{\sigma,+} - c_{j,k}^{+}Q = 0 \\ L_-P_{j,k}^{-} - F_{j,k}^{-} + ((k+1)\alpha+2j)P_{j,k}^{+} - \frac{1}{|y|^{2\sigma}}F_{j,k}^{\sigma,-} = 0 \end{cases}$$

and satisfy

$$c_{j,k}^{+} = 0 \quad (j+k\leq K), \qquad \frac{1}{|y|^2}P_{j,k}^{\pm}, \frac{1}{|y|}|\nabla P_{j,k}^{\pm}| \in L^\infty(\mathbb{R}^N) \quad (j+k=K+K').$$

Such solutions $(P_{j,k}^{+}, P_{j,k}^{-}, \beta_{j,k}, c_{j,k}^{+})$ are obtained from the later Propositions 4.7 and 4.8 and Corollary 4.11.

In the same way as [10, Proposition 2.1], for some sufficiently small $\epsilon'>0$, we have

$$\begin{aligned} \left\| e^{\epsilon'|y|}\Phi^{\frac{\partial P}{\partial s}} \right\|_{H^1} &\lesssim \lambda^\alpha\left(\left|\frac{1}{\lambda}\frac{\partial\lambda}{\partial s} + b\right| + \left|\frac{\partial b}{\partial s} + b^2 - \theta\right|\right), \\ \left\| e^{\epsilon'|y|}\Phi^{f} \right\|_{H^1} &\lesssim \lambda^{(K+K'+2)\alpha}, \\ \left\| e^{\epsilon'|y|}\Phi^{>K+K'} \right\|_{H^1} &\lesssim \left(b^2+\lambda^\alpha\right)^{K+K'+2}. \end{aligned}$$

Moreover,

$$\left\| e^{\epsilon'|y|}\Theta Q \right\|_{H^1} \lesssim \left(b^2+\lambda^\alpha\right)^{K+2}$$

holds. Therefore, we have

$$\left\|e^{\epsilon'|y|}\Psi\right\|_{H^1} \lesssim \lambda^\alpha \left(\left|\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right| + \left|\frac{\partial b}{\partial s} + b^2 - \theta\right|\right) + \left(b^2 + \lambda^\alpha\right)^{K+2},$$

where $\Psi := \Phi - \Theta Q$.

Next, we prove only (4.2) of (ii). The rest is the same as in [10]. We have

$$\begin{aligned}\lambda^2 E(P_{\lambda,b,\gamma}) =& \frac{1}{2}\left\|\nabla Q + \lambda^\alpha \nabla Z\right\|_2^2 - \int_{\mathbb{R}^N} F(Q+\lambda^\alpha Z)dy - \frac{\lambda^\alpha}{2}\left\||y|^{-\sigma}Q + \lambda^\alpha |y|^{-\sigma} Z\right\|_2^2 \\ &- \frac{b}{2}(iQ + i\lambda^\alpha Z, \Lambda Q + \lambda^\alpha \Lambda Z)_2 + \frac{b^2}{8}\left\||y|Q + \lambda^\alpha |y| Z\right\|_2^2.\end{aligned}$$

Here,

$$\frac{1}{2}\|\nabla Q\|_2^2 = \int_{\mathbb{R}^N} F(Q)dy, \qquad (\nabla Q, \lambda^\alpha \nabla Z)_2 = -(Q, \lambda^\alpha Z)_2 + \int_{\mathbb{R}^N} dF(Q)(\lambda^\alpha Z)dy,$$

$$\frac{1}{2}\left\||y|^{-\sigma}Q\right\|_2^2 = \frac{1}{8}\left\||y|Q\right\|_2^2 \frac{2\beta}{2-\alpha}, \qquad (iQ, \Lambda Q)_2 = 0$$

hold and we have

$$\begin{aligned}(Q, \lambda^\alpha Z)_2 &= \sum_{(j,k)\in\Sigma_{K+K'},\ j+k\geq 1} b^{2j}\lambda^{(k+1)\alpha}\left(Q, P_{j,k}^+\right)_2 = O(\lambda^\alpha(b^2+\lambda^\alpha)), \\ b(i\lambda Z, \Lambda Q)_2 &= -b\sum_{(j,k)\in\Sigma_{K+K'}} b^{2j+1}\lambda^{(k+1)\alpha}\left(P_{j,k}^-, \Lambda Q\right)_2 = O(b^2\lambda^\alpha).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\lambda^2 E(P_{\lambda,b,\gamma}) =& -\int_{\mathbb{R}^N}\left(F(Q+\lambda^\alpha Z) - F(Q) - dF(Q)(\lambda^\alpha Z)\right)dy \\ &- \frac{\lambda^\alpha}{8}\left\||y|Q\right\|_2^2\frac{2\beta}{2-\alpha} + \frac{b^2}{8}\left\||y|Q\right\|_2^2 + O(\lambda^\alpha(b^2+\lambda^\alpha))\end{aligned}$$

and

$$\int_{\mathbb{R}^N}\left(F(Q+\lambda^\alpha Z) - F(Q) - dF(Q)(\lambda^\alpha Z)\right)dy = O(\lambda^{2\alpha}).$$

Consequently, we have the conclusion. □

In the rest of this section, we construct solutions $(P_{j,k}^+, P_{j,k}^-, \beta_{j,k}, c_{j,k}^+) \in \mathcal{Y}'^2 \times \mathbb{R}^2$ for systems $(S_{j,k})$ in the proof of Proposition 4.3.

Proposition 4.5. For any $\psi \in H^{-1}(\mathbb{R}^N)$ such that $\left\langle \psi, \frac{\partial Q}{\partial x_j}\right\rangle = 0$ $(j = 1,\ldots,N)$, there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $L_+\varphi = \psi$ in $H^{-1}(\mathbb{R}^N)$. Similarly, for any $\psi \in H^{-1}(\mathbb{R}^N)$ such that $\langle \psi, Q\rangle = 0$, there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $L_-\varphi = \psi$ in $H^{-1}(\mathbb{R}^N)$.

Proof. Let ϕ_+ be the ground state of L_+ and μ_+ be the eigenvalue of ϕ_+. Then $\mu_+ < 0$ and we may assume $\|\phi_+\|_2 = 1$. Let define $H_\pm$ that are subspaces of $H^1(\mathbb{R}^N)$ by

$$H_+ := \mathrm{Span}\left\{\phi_+, \frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial Q}{\partial x_N}\right\}^\perp, \quad H_- := \mathrm{Span}\left\{Q\right\}^\perp,$$

then $H_\pm$ are Hilbert spaces and

$$\exists C_\pm > 0\ \forall \varphi \in H_\pm,\ \langle L_\pm \varphi, \varphi\rangle \geq C_\pm \|\varphi\|_{H^1}^2$$

hold, where double sign correspond. Therefore, from the Lax-Milgram theorem,

$$\forall \psi \in H_\pm^*\ \exists! \tilde{\varphi}_\pm \in H_\pm,\ L_\pm\tilde{\varphi}_\pm = \psi \quad \text{in } H_\pm^*$$

hold, where double sign correspond.

Here, let $\left\langle \psi, \frac{\partial Q}{\partial x_j} \right\rangle = 0$, $\varphi := \tilde{\varphi} + \frac{\langle \psi,\phi_+ \rangle}{\mu_+}\phi_+$, and $\tilde{\chi} := \chi - (\chi,\phi_+)\phi_+ - (\chi,\nabla Q)\cdot\nabla Q$ for each $\chi \in H^1(\mathbb{R}^N)$. Then $\tilde{\chi} \in H_+$ and we have

$$\begin{aligned}\langle L_+\varphi, \chi\rangle &= \langle \varphi, L_+\chi\rangle = \langle \varphi, L_+\tilde{\chi} + \mu_+(\chi,\phi_+)\phi_+\rangle \\ &= \langle L_+\tilde{\varphi}, \tilde{\chi}\rangle + \left\langle \frac{\langle \psi,\phi_+\rangle}{\mu_+}\phi_+, \mu_+(\chi,\phi_+)\phi_+ \right\rangle \\ &= \langle \psi, \tilde{\chi}\rangle + (\chi,\phi_+)\langle \psi,\phi_+\rangle + (\chi,\nabla Q)\cdot\langle \psi, \nabla Q\rangle \\ &= \langle \psi, \chi\rangle.\end{aligned}$$

This means that $L_+\varphi = \psi$ in $H^{-1}(\mathbb{R}^N)$.

The same is proved in the case of $\langle \psi, Q\rangle = 0$. □

Proposition 4.6. For any $\psi,\chi \in \mathcal{Y}$, there exists $\varphi \in \mathcal{Y}$ such that $L_+\varphi = \psi + |y|^{-2\sigma}\chi$. Similarly, for any $\psi,\chi \in \mathcal{Y}$ such that $\langle \psi + |y|^{-2\sigma}\chi, Q\rangle = 0$, there exists $\varphi \in \mathcal{Y}$ such that $L_-\varphi = \psi + |y|^{-2\sigma}\chi$.

Proof. We prove only for L_+. Since $\mathcal{Y} \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$, the existence of H^1-solution is clearly from Proposition 4.5.

Firstly, based on a classical argument of elliptic partial differential equations, we have $\varphi \in C^\infty(\mathbb{R}^N \setminus \{0\})$. From the maximum principal,

$$\exists C_\alpha, \kappa_\alpha > 0,\ |x| \geq 1 \Rightarrow \left| \left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x) \right| \leq C_\alpha(1+|x|^{\kappa_\alpha})Q(x)$$

holds for any multi-index α. Since $\psi + |y|^{-2\sigma}\chi \in L^p(\mathbb{R}^N)$ for some $p > \max\{\frac{N}{2}, 1\}$, we have $\varphi \in L^\infty(\mathbb{R}^N)$ (see [8, Theorem 8.15]). Furthermore, since

$$-\Delta\varphi + \varphi = \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}}\varphi + \psi + \frac{1}{|y|^{2\sigma}}\chi \in L^p(\mathbb{R}^N),$$

we have $\varphi \in W^{2,p}(\mathbb{R}^N) \hookrightarrow C^{0,\gamma}(\mathbb{R}^N)$ for some $\gamma \in (0,1)$. Namely, $\varphi \in \mathcal{Y}$. □

Proposition 4.7. The system $(S_{j,k})$ has a solution $(P^+_{j,k}, P^-_{j,k}, \beta_{j,k}, c^+_{j,k}) \in \mathcal{Y}^2 \times \mathbb{R}^2$.

Proof. We solve

$$(S_{j,k}) \begin{cases} L_+P^+_{j,k} - F^+_{j,k} - \beta_{j,k}\dfrac{|y|^2}{4}Q - \dfrac{1}{|y|^{2\sigma}}F^{\sigma,+}_{j,k} - c^+_{j,k}Q = 0, \\ L_-P^-_{j,k} - F^-_{j,k} + ((k+1)\alpha + 2j)P^+_{j,k} - \dfrac{1}{|y|^{2\sigma}}F^{\sigma,-}_{j,k} = 0. \end{cases}$$

For $(S_{j,k})$, we consider the following two systems:

$$(\tilde{S}_{j,k}) \begin{cases} L_+\tilde{P}^+_{j,k} - F^+_{j,k} - \beta_{j,k}\dfrac{|y|^2}{4}Q - \dfrac{1}{|y|^{2\sigma}}F^{\sigma,+}_{j,k} = 0, \\ L_-\tilde{P}^-_{j,k} - F^-_{j,k} + ((k+1)\alpha + 2j)\tilde{P}^+_{j,k} - \dfrac{1}{|y|^{2\sigma}}F^{\sigma,-}_{j,k} = 0. \end{cases}$$

and

$$(S'_{j,k}) \begin{cases} P^+_{j,k} = \tilde{P}^+_{j,k} - \dfrac{c^+_{j,k}}{2}\Lambda Q, \\ P^-_{j,k} = \tilde{P}^-_{j,k} - c^-_{j,k}Q - \dfrac{((k+1)\alpha + 2j)c^+_{j,k}}{8}|y|^2 Q. \end{cases}$$

Then by applying $(S'_{j,k})$ to a solution for $(\tilde{S}_{j,k})$, we obtain a solution for $(S_{j,k})$.

Firstly, we solve

$$(\tilde{S}_{0,0}) \begin{cases} L_+\tilde{P}^+_{0,0} - \beta_{0,0}\dfrac{|y|^2}{4}Q - \dfrac{1}{|y|^{2\sigma}}Q = 0, \\ L_-\tilde{P}^-_{0,0} + \alpha\tilde{P}^+_{0,0} = 0. \end{cases}$$

For any $\beta_{0,0} \in \mathbb{R}$, there exists a solution $\tilde{P}^+_{0,0} \in \mathcal{Y}$ from Proposition 4.6. Let

$$\beta_{0,0} := \frac{4\sigma\||y|^{-\sigma}Q\|_2^2}{\||y|Q\|_2^2}.$$

Then since

$$\left(\tilde{P}^+_{0,0}, Q\right)_2 = -\frac{1}{2}\left\langle L_+\tilde{P}^+_{0,0}, \Lambda Q\right\rangle = \frac{1}{2}\left(\frac{\beta_{0,0}}{4}\||y|Q\|_2^2 - \sigma\||y|^{-\sigma}Q\|_2^2\right) = 0,$$

there exists a solution $\tilde{P}^-_{0,0} \in \mathcal{Y}$. By taking $c^+_{0,0} = 0$, we obtain a solution $(P^+_{0,0}, P^-_{0,0}, \beta_{0,0}, c^+_{0,0}) \in \mathcal{Y}^2 \times \mathbb{R}^2$ for $(S_{0,0})$. Here, let $H(j_0, k_0)$ denote by that

$$\forall (j,k) \in \Sigma_{K+K'},\ k < k_0 \text{ or } (k = k_0 \text{ and } j < j_0)$$
$$\Rightarrow (S_{j,k}) \text{ has a solution } (P^+_{j,k}, P^-_{j,k}, \beta_{j,k}, c^+_{j,k}) \in \mathcal{Y}^2 \times \mathbb{R}^2.$$

From the above discuss, $H(1,0)$ is true. If $H(j_0,k_0)$ is true, then $F^{\pm}_{j_0,k_0}$ is defined and belongs to $\mathcal{Y}$. Moreover, for any β_{j_0,k_0}, there exists a solution $\tilde{P}^+_{j_0,k_0}$. Let be β_{j_0,k_0} such that

$$\left\langle -F^-_{j_0,k_0} + ((k_0+1)\alpha + 2j_0)\tilde{P}^+_{j_0,k_0} - \frac{1}{|y|^{2\sigma}}F^{\sigma,-}_{j_0,k_0}, Q\right\rangle = 0.$$

Then we obtain a solution $\tilde{P}^-_{j_0,k_0}$. Here, we define

$$c^-_{j_0,k_0} := \begin{cases} \frac{\tilde{P}^-_{j_0,k_0}(0)}{Q(0)} & (j_0+k_0 \neq K+1), \\ 0 & (j_0+k_0 = K+1, \text{ and } \tilde{P}^-_{j_0,k_0}(0) \neq 0), \\ 1 & (j_0+k_0 = K+1, \text{ and } \tilde{P}^-_{j_0,k_0}(0) = 0), \end{cases}$$
$$c^+_{j_0,k_0} := \begin{cases} 0 & (j_0+k_0 \leq K), \\ 0 & (j_0+k_0 = K+1, \text{ and } \tilde{P}^+_{j_0,k_0}(0) \neq 0), \\ 1 & (j_0+k_0 = K+1, \text{ and } \tilde{P}^+_{j_0,k_0}(0) = 0), \\ \frac{2\tilde{P}^+_{j_0,k_0}(0)}{Q(0)} & (j_0+k_0 \geq K+2). \end{cases}$$

Then we obtain a solution for (S_{j_0,k_0}). This means that $H(j_0+1,k_0)$ is true if $j_0+k_0 \leq K+K'-1$ and $H(0,k_0+1)$ is true if $j_0+k_0 = K+K'$. In particular, $H(0,K+K'+1)$ means that for any $(j,k) \in \Sigma_{K+K'}$, there exists a solution $(P^+_{j,k}, P^-_{j,k}, \beta_{j,k}, c^+_{j,k}) \in \mathcal{Y}^2 \times \mathbb{R}^2$.

Furthermore, $P^{\pm}_{j,k}(0) \neq 0$ for $j+k = K+1$ and $P^{\pm}_{j,k}(0) = 0$ for $j+k \geq K+2$ hold. □

Proposition 4.8. For $P^{\pm}_{j,k}$,

$$P^{\pm}_{j,k} \in H^2(\mathbb{R}^N) \quad \text{and} \quad \Lambda P^{\pm}_{j,k} \in C(\mathbb{R}^N).$$

Namely, $P^{\pm}_{j,k} \in \mathcal{Y}'$.

Proof. Firstly, since $P^{\pm}_{j,k} \in \mathcal{Y}$ and $P^{\pm}_{j,k}$ is solution for $(S_{j,k})$, $\Delta P^{\pm}_{j,k} \in L^2(\mathbb{R}^N)$. Therefore, $P^{\pm}_{j,k} \in H^2(\mathbb{R}^N)$.

Regarding $\Lambda P^{\pm}_{j,k} \in C(\mathbb{R}^N)$, proving $y \cdot \nabla P^{\pm}_{j,k} \in C(\mathbb{R}^N)$ is sufficient. Firstly,

$$\begin{aligned} L_+(y\cdot\nabla P^+_{j,k}) &= y\cdot\nabla(F^+_{j,k} + \beta_{j,k}\frac{|y|^2}{4}Q + \frac{1}{|y|^{2\sigma}}F^{\sigma,+}_{j,k} + c^+_{j,k}Q) \\ &\quad + 2(F^+_{j,k} + \beta_{j,k}\frac{|y|^2}{4}Q + \frac{1}{|y|^{2\sigma}}F^{\sigma,+}_{j,k} + c^+_{j,k}Q) \\ &\quad - 2P^+_{j,k} + 2\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}}P^+_{j,k} - \frac{4}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q P^+_{j,k} \end{aligned}$$

holds. Since $|y|^{-2\sigma}y\cdot\nabla F^{\sigma,+}_{j,k} \in L^p(\mathbb{R}^N)$ for some $p > \max\{\frac{N}{2}, 1\}$, we have $L_+(y\cdot\nabla P^+_{j,k}) \in L^p(\mathbb{R}^N)$. Therefore, we have $y\cdot\nabla P^+_{j,k} \in C(\mathbb{R}^N)$. Similarly, we have $y\cdot\nabla P^-_{j,k} \in C(\mathbb{R}^N)$. □

Proposition 4.9. For any $0 \leq j \leq K+1$, there exists $k_j \geq 1$ such that

$$\frac{1}{r^2}P^{\pm}_{j,k}, \frac{1}{r}\frac{\partial P^{\pm}_{j,k}}{\partial r} \in L^\infty(\mathbb{R}^N)$$

for any $k \geq k_j$, where $r = |y|$.

Proof. We prove only for $P^+_{j,k}$.

Let $f_k := P^+_{j,K-j+k}$ for $k \in \mathbb{N}$. Then $f_1(0) \neq 0$ and $f_k(0) = 0$ for $k \geq 2$ hold. Moreover, Let

$$F_k := f_k - \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}} f_k - F^+_{j,K-j+k} - \beta_{j,K-j+k} \frac{r^2}{4} Q - c^+_{j,K-j+k} Q.$$

If $r^{-q} f_k$ converges to non-zero as $r \to +0$ for some $q \in [0, 2\sigma)$ or $r^{-q} f_k$ converges as $r \to +0$ for some $q \geq 2\sigma$, then $r^{N-1} \frac{\partial f_{k+1}}{\partial r}$ converges to 0 as $r \to +0$. Indeed, if $N = 1$, then $f_k \in W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1(\mathbb{R}^N)$ for some $p > 1$. Therefore, since f_k is an even function, $\frac{\partial f_k}{\partial r}(0) = 0$ holds. On the other hand, for $N \geq 2$,

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial f_{k+1}}{\partial r} \right) = F_{k+1} - \frac{1}{r^{2\sigma}} f_k \tag{4.4}$$

holds. If $r^{-q} f_k$ converges as $r \to +0$ for some $q \geq 2\sigma$, then $r^{-2\sigma} f_k$ is bounded. Therefore, for some sufficiently large p, we have $f_{k+1} \in W^{2,p}(\mathbb{R}^N) \hookrightarrow C^1(\mathbb{R}^N)$. Accordingly, $r^{N-1} \frac{\partial f_{k+1}}{\partial r}$ converges to 0 as $r \to +0$. On the other hand, if $r^{-q} f_k$ converges to non-zero as $r \to +0$ for some $q \in [0, 2\sigma)$, the right hand of (4.4) diverge $+\infty$ or $-\infty$ as $r \to +0$. Therefore, $r^{N-1} \frac{\partial f_{k+1}}{\partial r}$ is increasing or decreasing as $r \to +0$, meaning $r^{N-1} \frac{\partial f_{k+1}}{\partial r}$ converges in $[-\infty, \infty]$. Let

$$C := \lim_{r \to +0} r^{N-1} \left| \frac{\partial f_{k+1}}{\partial r} \right|.$$

Then for any $\epsilon > 0$, there exists $r_0 > 0$ such that $\left| \frac{\partial f_{k+1}}{\partial r} \right| \geq (C - \epsilon) r^{-(N-1)}$ for any $r \in (0, r_0)$. On the other hand, $f_{k+1} \in W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,N}(\mathbb{R}^N)$ for some $p > \frac{N}{2}$ and $\left| \frac{\partial f_{k+1}}{\partial r} \right| = |\nabla f_{k+1}|$. Therefore, we have

$$\infty > \int_{B(0,r_0)} |\nabla f_{k+1}(x)|^N dx \geq C_N \int_0^{r_0} \frac{C - \epsilon}{r^{N(N-1)}} dr.$$

Since $\int_0^{r_0} r^{-N(N-1)} dr = \infty$, we obtain $C - \epsilon \leq 0$. Consequently, we have $C \leq 0$, meaning $C = 0$.

Let $\sigma_1 := 0$ and $C_1 := f_1(0)$. Moreover, let

$$\sigma_{k+1} := \begin{cases} 1 - \sigma + \sigma_k & (\sigma_k < \sigma) \\ 1 & (\sigma_k \geq \sigma) \end{cases}, \quad C_{k+1} := \begin{cases} \frac{-C_k}{2\sigma_{k+1}(N - 2(\sigma - \sigma_k))} & (\sigma_k < \sigma) \\ \frac{F_{k+1}(0) - 0^{2(\sigma_k - \sigma)} C_k}{2N} & (\sigma_k \geq \sigma) \end{cases}.$$

In particular, $\sigma_k = \min\{1, (1 - \sigma)(k - 1)\}$ and $C_k \neq 0$ if $\sigma_k < \sigma$. Then

$$\lim_{r \to +0} \frac{1}{r^{2\sigma_k}} f_k(r) = C_k \tag{4.5}$$

holds. For $k = 1$, it clearly holds. Moreover, for $k \geq 2$,

$$\lim_{r \to +0} \frac{1}{r^{2\sigma_k - 1}} \frac{\partial f_k}{\partial r}(r) = 2\sigma_k C_k$$

holds. Indeed, if (4.5) holds for some k, then $r^{N-1} \frac{\partial f_{k+1}}{\partial r}$ converges to 0 as $r \to +0$ in both cases $\sigma_k < \sigma$ and $\sigma_k \geq \sigma$ from the above discuss. We assume $\sigma_k < \sigma$. Since

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial f_{k+1}}{\partial r} \right) = F_{k+1} - \frac{1}{r^{2(\sigma - \sigma_k)}} \frac{1}{r^{2\sigma_k}} f_k,$$

for any $\epsilon > 0$, there exists $r_0 > 0$ such that

$$(-C_k - \epsilon) r^{N-1-2(\sigma - \sigma_k)} \leq \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial f_{k+1}}{\partial r} \right) \leq (-C_k + \epsilon) r^{N-1-2(\sigma - \sigma_k)}$$

for any $r \in (0, r_0)$. Integrating in $[0, r]$, we have

$$\frac{(-C_k - \epsilon) r^{1-2(\sigma - \sigma_k)}}{N - 2(\sigma - \sigma_k)} \leq \frac{\partial f_{k+1}}{\partial r} \leq \frac{(-C_k + \epsilon) r^{1-2(\sigma - \sigma_k)}}{N - 2(\sigma - \sigma_k)}.$$

Integrating in $[0, r]$ again, we have

$$\frac{(-C_k - \epsilon) r^{2-2(\sigma - \sigma_k)}}{(2 - 2(\sigma - \sigma_k))(N - 2(\sigma - \sigma_k))} \leq f_{k+1} \leq \frac{(-C_k + \epsilon) r^{2-2(\sigma - \sigma_k)}}{(2 - 2(\sigma - \sigma_k))(N - 2(\sigma - \sigma_k))}.$$

Therefore, we have

$$\lim_{r\to+0}\frac{1}{r^{2\sigma_{k+1}-1}}\frac{\partial f_{k+1}}{\partial r}(r)=2\sigma_{k+1}C_{k+1},\quad \lim_{r\to+0}\frac{1}{r^{2\sigma_{k+1}}}f_{k+1}(r)=C_{k+1}.$$

On the other hand, we assume $\sigma_k\geq\sigma$. Then for any $\epsilon>0$, there exists $r_0>0$ such that

$$(F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k-\epsilon)r^{N-1}\leq\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial f_{k+1}}{\partial r}\right)\leq(F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k+\epsilon)r^{N-1}$$

for any $r\in(0,r_0)$. Integrating in the same way as for $\sigma_k<\sigma$, we have

$$\frac{F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k-\epsilon}{N}r\leq\frac{\partial f_{k+1}}{\partial r}\leq\frac{F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k+\epsilon}{N}r.$$

Moreover, since

$$\frac{F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k-\epsilon}{2N}r^2\leq f_{k+1}\leq\frac{F_{k+1}(0)-0^{2(\sigma_k-\sigma)}C_k+\epsilon}{2N}r^2,$$

we have

$$\lim_{r\to+0}\frac{1}{r^{2\sigma_{k+1}-1}}\frac{\partial f_{k+1}}{\partial r}(r)=2\sigma_{k+1}C_{k+1},\quad \lim_{r\to+0}\frac{1}{r^{2\sigma_{k+1}}}f_{k+1}(r)=C_{k+1}.$$

Consequently, we obtain Proposition 4.9 if $k_j\geq\frac{1}{1-\sigma}+K+1-j>0$. □

Proposition 4.10. For any $j\geq K+2$ and $k\geq0$,

$$\frac{1}{r^2}P^{\pm}_{j,k},\ \frac{1}{r}\frac{\partial P^{\pm}_{j,k}}{\partial r}\in L^\infty(\mathbb{R}^N)$$

hold.

Proof. Firstly, we consider

$$(S_{j,0})\quad\begin{cases}L_+P^+_{j,0}-(2(j-1)+\alpha+1)P^-_{j-1,0}-\beta_{j,0}\frac{|y|^2}{4}Q-\frac{1}{|y|^{2\sigma}}F^{\sigma,+}_{j,0}-c^+_{j,0}Q=0,\\ L_-P^-_{j,0}+(\alpha+2j)P^+_{j,0}-\frac{1}{|y|^{2\sigma}}F^{\sigma,-}_{j,0}=0.\end{cases}$$

From Proposition 4.8, the solutions $P^{\pm}_{j,0}$ for $(S_{j,0})$ are continuous functions. In particular, since $F^{\sigma,\pm}_{j,0}=0$ when $j\geq1$, we see that $P^{\pm}_{j,0}\in C^2(\mathbb{R}^N)$ when $j\geq2$. Moreover, since $P^{\pm}_{j,k}(0)=0$ when $j+k\geq K+2$ and $P^{\pm}_{j,k}$ are spherically symmetrical, we obtain

$$P^{\pm}_{j,0}(y)\sim|y|^2$$

as $|y|\to0$ when $j\geq K+2$.

Next, for some $k\geq0$, we assume that

$$P^{\pm}_{j,k}(y)\sim|y|^2$$

holds as $|y|\to0$ when $j\geq K+2$. Since $P^{\pm}_{j,k+1}$ is the solutions for

$$(S_{j,k+1})\quad\begin{cases}L_+P^+_{j,k+1}-F^+_{j,k+1}-\beta_{j,k+1}\frac{|y|^2}{4}Q-\frac{1}{|y|^{2\sigma}}P^+_{j,k}-c^+_{j,k+1}Q=0,\\ L_-P^-_{j,k+1}-F^-_{j,k+1}+((k+2)\alpha+2j)P^+_{j,k+1}-\frac{1}{|y|^{2\sigma}}P^-_{j,k}=0.\end{cases}$$

and $P^{\pm}_{j,k+1}(0)=0$, we obtain

$$\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}P^{\pm}_{j,k}\right)=G^{\pm}_{j,k},$$

where $r:=|y|$ and

$$G^+_{j,k}:=P^+_{j,k+1}-\left(1+\frac{4}{N}\right)Q^{\frac{4}{N}}P^+_{j,k+1}-F^+_{j,k+1}-\beta_{j,k+1}\frac{r^2}{4}Q-\frac{1}{r^{2\sigma}}P^+_{j,k}-c^+_{j,k+1}Q,$$

$$G^-_{j,k}:=P^-_{j,k+1}-Q^{\frac{4}{N}}P^-_{j,k+1}-F^-_{j,k+1}+((k+2)\alpha+2j)P^+_{j,k+1}-\frac{1}{r^{2\sigma}}P^-_{j,k}.$$

Consequently, the rest can be proved in the similar way as Proposition 4.9. □

Corollary 4.11. For some sufficiently large K',

$$\frac{1}{r^2}P^{\pm}_{j,k},\ \frac{1}{r}\frac{\partial P^{\pm}_{j,k}}{\partial r}\in L^\infty(\mathbb{R}^N)$$

for any $j,k\geq 0$ such that $j+k=K+K'$.

Proof. Let $K':=\max_{0\leq j\leq K+1}\{k_j+1\}\geq 2$. If $j\geq K+2$, then it is obvious from Proposition 4.10. On the other hand, if $j\leq K+1$, it holds from Proposition 4.9 since $k\geq K'-1\geq k_j$. □

4.3 Proof of Theorem 4.1

We expect the modulation terms to be sufficiently small. Namely, we expect the parameters λ and b in the decomposition to approximately satisfy

$$\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b=\frac{\partial b}{\partial s}+b^2-\theta=0.$$

Therefore, the approximation functions λ_{app} and b_{app} of the parameters λ and b will be determined by the following lemma:

Lemma 4.12. Let

$$\lambda_{\mathrm{app}}(s):=\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}}s^{-\frac{2}{\alpha}},\quad b_{\mathrm{app}}(s):=\frac{2}{\alpha s}.$$

Then $(\lambda_{\mathrm{app}},b_{\mathrm{app}})$ is a solution for

$$\frac{\partial b}{\partial s}+b^2-\beta\lambda^\alpha=0,\quad \frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b=0$$

in $s>0$.

Furthermore, the following lemma determines $\lambda(s_1)$ and $b(s_1)$ for a given energy level E_0 and a sufficiently large s_1.

Lemma 4.13. Let define $C_0:=\frac{8E_0}{\||y|Q\|_2^2}$ and $0<\lambda_0\ll 1$ such that $\frac{2\beta}{2-\alpha}+C_0{\lambda_0}^{2-\alpha}>0$. For $\lambda\in(0,\lambda_0]$, we set

$$\mathcal{F}(\lambda):=\int_\lambda^{\lambda_0}\frac{1}{\mu^{\frac{\alpha}{2}+1}\sqrt{\frac{2\beta}{2-\alpha}+C_0\mu^{2-\alpha}}}d\mu.$$

Then for any $s_1\gg 1$, there exist $b_1,\lambda_1>0$ such that

$$\left|\frac{{\lambda_1}^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}(s_1)^{\frac{\alpha}{2}}}-1\right|+\left|\frac{b_1}{b_{\mathrm{app}}(s_1)}-1\right|\lesssim {s_1}^{-\frac{1}{2}}+{s_1}^{2-\frac{4}{\alpha}},\quad \mathcal{F}(\lambda_1)=s_1,\quad E(P_{\lambda_1,b_1,\gamma})=E_0.$$

Moreover,

$$\left|\mathcal{F}(\lambda)-\frac{2}{\alpha\lambda^{\frac{\alpha}{2}}\sqrt{\frac{2\beta}{2-\alpha}}}\right|\lesssim\lambda^{-\frac{\alpha}{4}}+\lambda^{2-\frac{3}{2}\alpha}$$

holds.

Proof. The method of choosing λ_1 and the estimate of $\mathcal{F}$ are the same as in [10]. In brief, since $\mathcal{F}(\lambda)\to\infty$ as $\lambda\to 0$, there exists such a λ_1 from the intermediate value theorem.

Setting $h(b):={\lambda_1}^2E(P_{\lambda_1,b,\gamma})$, from (4.2), we have

$$\begin{aligned}h(b)=&\frac{1}{8}\||y|Q\|_2^2\left(b^2-\frac{2\beta}{2-\alpha}{\lambda_1}^\alpha\right)+O({\lambda_1}^\alpha(b^2+{\lambda_1}^\alpha))\\=&\frac{1}{8}\||y|Q\|_2^2\left(b^2-b_{\mathrm{app}}(s_1)^2-\frac{2\beta}{2-\alpha}\left({\lambda_1}^\alpha-\lambda_{\mathrm{app}}(s_1)^\alpha\right)\right)+O({\lambda_1}^\alpha(b^2+{\lambda_1}^\alpha)).\end{aligned}$$

Then since λ_1 is sufficiently small if s_1 is sufficiently large, we have

$$\begin{aligned}h(0)-{\lambda_1}^2E_0&=-\frac{1}{8}\||y|Q\|_2^2\frac{2\beta}{2-\alpha}{\lambda_1}^\alpha-{\lambda_1}^2E_0+O({\lambda_1}^{2\alpha})<0,\\h(1)-{\lambda_1}^2E_0&=\frac{1}{8}\||y|Q\|_2^2\left(1-\frac{2\beta}{2-\alpha}{\lambda_1}^\alpha-{\lambda_1}^2C_0\right)+O({\lambda_1}^\alpha(1+{\lambda_1}^\alpha))>0.\end{aligned}$$

Therefore, there exists $b_1 \in (0,1)$ such that $h(b_1) = {\lambda_1}^2 E_0$ and we have

$$\begin{aligned}\left|{b_1}^2 - b_{\mathrm{app}}(s_1)^2\right| &\lesssim {\lambda_1}^2 + |{\lambda_1}^\alpha - \lambda_{\mathrm{app}}(s_1)^\alpha| + {\lambda_1}^\alpha \left(\left|{b_1}^2 - b_{\mathrm{app}}(s_1)^2\right| + \lambda_{\mathrm{app}}(s_1)^\alpha + {\lambda_1}^\alpha\right)\\ &\lesssim {s_1}^{-\frac{4}{\alpha}} + {s_1}^{-\frac{5}{2}}.\end{aligned}$$

Consequently, we have the conclusion. □

Let s_1 be sufficiently large and define

$$\mathcal{C} := \frac{\alpha}{4-\alpha}\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{4}{\alpha}}.$$

For $t_1 < 0$ that is sufficiently close to 0, we define

$$s_1 := |\mathcal{C}^{-1} t_1|^{-\frac{\alpha}{4-\alpha}}.$$

Additionally, let λ_1 and b_1 be given in Lemma 4.13 for s_1 and $\gamma_1 = 0$. Let u be the solution for (NLS) with $\pm = +$ with an initial value

$$u(t_1, x) := P_{\lambda_1, b_1, 0}(x) = \frac{1}{\lambda_1^{\frac{N}{2}}} P\left(s, \frac{x}{\lambda_1}\right) e^{-i\frac{b_1}{4}\frac{|x|^2}{\lambda_1^2}}.$$

Then u satisfies the assumption of Lemma 2.2 in a neighbourhood of t_1. By applying Lemma 2.2 to u, there exists a decomposition $(\tilde{\lambda}_{t_1}, \tilde{b}_{t_1}, \tilde{\gamma}_{t_1}, \tilde{\varepsilon}_{t_1})$ such that

$$u(t,x) = \frac{1}{\tilde{\lambda}_{t_1}(t)^{\frac{N}{2}}}\left(P + \tilde{\varepsilon}_{t_1}\right)\left(t, \frac{x}{\tilde{\lambda}_{t_1}(t)}\right) e^{-i\frac{\tilde{b}_{t_1}(t)}{4}\frac{|x|^2}{\tilde{\lambda}_{t_1}(t)^2} + i\tilde{\gamma}_{t_1}(t)}, \tag{4.6}$$

$$\left(\tilde{\varepsilon}_{t_1}, i\Lambda P\right)_2 = \left(\tilde{\varepsilon}_{t_1}, |y|^2 P\right)_2 = \left(\tilde{\varepsilon}_{t_1}, i\rho\right)_2 = 0 \tag{4.7}$$

in the neighbourhood of t_1. The rescaled time s_{t_1} is defined by

$$s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\tilde{\lambda}_{t_1}(\tau)^2} d\tau.$$

Then we define an inverse function ${s_{t_1}}^{-1} : s_{t_1}(I) \to I$. Moreover, we define

$$t_{t_1} := {s_{t_1}}^{-1}, \qquad \lambda_{t_1}(s) := \tilde{\lambda}(t_{t_1}(s)), \qquad b_{t_1}(s) := \tilde{b}(t_{t_1}(s)),$$
$$\gamma_{t_1}(s) := \tilde{\gamma}(t_{t_1}(s)), \qquad \varepsilon_{t_1}(s,y) := \tilde{\varepsilon}(t_{t_1}(s), y).$$

For the sake of clarity in notation, we often omit the subscript t_1. In particular, it should be noted that $u \in C((T_*, T^*), \Sigma^2(\mathbb{R}^N))$ and $|x|\nabla u \in C((T_*, T^*), L^2(\mathbb{R}^N))$. Furthermore, let I_{t_1} be the maximal interval such that a decomposition as (4.6) is obtained and we define

$$J_{s_1} := s\left(I_{t_1}\right).$$

Additionally, for sufficiently large $s_1 (\geq s_0)$, let

$$s' := \max\left\{s_0, \inf J_{s_1}\right\}.$$

Let

$$0 < M < \min\left\{\frac{1}{2}, \frac{4}{\alpha} - 2\right\}$$

and s_* be defined by

$$s_* := \inf\left\{\sigma \in (s', s_1] \mid (4.8) \text{ holds on } [\sigma, s_1]\right\},$$

where

$$\|\varepsilon(s)\|_{H^1}^2 + b(s)^2 \||y|\varepsilon(s)\|_2^2 < s^{-2K}, \quad \left|\frac{\lambda(s)^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| < s^{-M}. \tag{4.8}$$

Finally, we define

$$\mathrm{Mod}(s) := \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2 - \theta, 1 - \frac{\partial \gamma}{\partial s}\right).$$

By direct calculation, we obtain

$$\begin{aligned}-\Psi = i\frac{\partial \varepsilon}{\partial s} + \Delta\varepsilon - \varepsilon + f(P+\varepsilon) - f(P) + \lambda^\alpha \frac{1}{|y|^{2\sigma}}\varepsilon + \theta\frac{|y|^2}{4}\varepsilon \\ - i\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right)\Lambda(P+\varepsilon) + \left(1 - \frac{\partial \gamma}{\partial s}\right)(P+\varepsilon) \\ + \left(\frac{\partial b}{\partial s} + b^2 - \theta\right)\frac{|y|^2}{4}(P+\varepsilon) - \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right) b\frac{|y|^2}{2}(P+\varepsilon)\end{aligned} \tag{4.9}$$

holds on J_{s_1}.

Lemma 4.14. For $s \in (s_*, s_1]$,

$$|(\varepsilon(s), Q)| \lesssim s^{-(K+2)}, \quad |\operatorname{Mod}(s)| \lesssim s^{-(K+2)}, \quad \|e^{\varepsilon'|y|}\Psi\|_{H^1} \lesssim s^{-(K+4)}$$

hold.

Proof. Let

$$s_{**} := \inf\left\{ s \in [s_*, s_1] \;\middle|\; |(\varepsilon(\tau), P)_2| < \tau^{-(K+2)} \text{ holds on } [s, s_1]. \right\}.$$

We work below on the interval $[s_{**}, s_1]$.

According to the orthogonality properties (4.7), we have

$$0 = \frac{d}{ds}(i\varepsilon, \Lambda P)_2 = \left(i\frac{\partial \varepsilon}{\partial s}, \Lambda P\right)_2 + \left(i\varepsilon, \frac{\partial(\Lambda P)}{\partial s}\right)_2 \tag{4.10}$$

$$= \frac{d}{ds}\left(i\varepsilon, i|y|^2 P\right)_2 = \left(i\frac{\partial \varepsilon}{\partial s}, i|y|^2 P\right)_2 + \left(i\varepsilon, i|y|^2\frac{\partial P}{\partial s}\right)_2 \tag{4.11}$$

$$= \frac{d}{ds}(i\varepsilon, \rho)_2 = \left(i\frac{\partial \varepsilon}{\partial s}, \rho\right)_2. \tag{4.12}$$

For (4.10), we have

$$\left(i\varepsilon, \frac{\partial(\Lambda P)}{\partial s}\right)_2 = \left(i\varepsilon, \frac{\partial}{\partial s}(\lambda^\alpha \Lambda Z)\right)_2 = O(s^{-(K+3)}) + O(s^{-1}|\operatorname{Mod}(s)|) \tag{4.13}$$

and from Lemma (4.9),

$$\begin{aligned}&\left(i\frac{\partial \varepsilon}{\partial s}, \Lambda P\right)_2 \\ &= \Bigg(L_+ \operatorname{Re}\varepsilon + iL_- \operatorname{Im}\varepsilon - (f(P+\varepsilon) - f(P) - df(Q)(\varepsilon)) - \lambda^\alpha\frac{1}{|y|^{2\sigma}}\varepsilon - \theta\frac{|y|^2}{4}\varepsilon \\ &\qquad + i\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right)\Lambda(P+\varepsilon) - \left(1 - \frac{\partial \gamma}{\partial s}\right)(P+\varepsilon) - \left(\frac{\partial b}{\partial s} + b^2 - \theta\right)\frac{|y|^2}{4}(P+\varepsilon) \\ &\qquad + \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right) b\frac{|y|^2}{2}(P+\varepsilon) + \Psi, \Lambda P\Bigg)_2.\end{aligned}$$

According to $\Lambda P^{\pm}_{j,k} \in H^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and Proposition 4.3,

$$\begin{aligned}|(L_+ \operatorname{Re}\varepsilon, \Lambda P)_2| + |(iL_- \operatorname{Im}\varepsilon, \Lambda P)_2| + \left|\left(\lambda^\alpha\frac{1}{|y|^{2\sigma}}\varepsilon, \Lambda P\right)_2\right| + \left|\left(\theta\frac{|y|^2}{4}\varepsilon, \Lambda P\right)_2\right| &= O(s^{-(K+2)}), \\ (i\Lambda P, \Lambda P)_2 = (P, \Lambda P)_2 &= 0, \\ (\Psi, \Lambda P)_2 &= O(s^{-2(K+2)}) + O(s^{-1}|\operatorname{Mod}(s)|), \\ \left(|y|^2 P, \Lambda P\right)_2 &= -\||y|Q\|_2^2 + O(s^{-2})\end{aligned}$$

hold. Here, we have

$$f(P+\varepsilon) - f(P) - df(Q)(\varepsilon) = f(P+\varepsilon) - f(P) - df(P)(\varepsilon) + df(P)(\varepsilon) - df(Q)(\varepsilon).$$

Firstly, we consider $(f(P+\varepsilon) - f(P) - df(P)(\varepsilon))\,\Lambda\overline{P}$. For $N \leq 3$, according to Taylor's theorem, we have

$$\begin{aligned}\left|(f(P+\varepsilon) - f(P) - df(P)(\varepsilon))\,\Lambda\overline{P}\right| &\lesssim (1+|y|^\kappa)(P+|\varepsilon|)^{\frac{4}{N}-1}|\varepsilon|^2 Q \\ &\lesssim (1+|y|^\kappa)(Q+|\varepsilon|)^{\frac{4}{N}-1}|\varepsilon|^2 Q.\end{aligned}$$

On the other hand, we assume $N \geq 4$. If $Q < 3|\lambda^\alpha Z|$, then $1 \lesssim \lambda^\alpha(1+|y|^\kappa)$. Therefore, we have

$$\left|(f(P+\varepsilon) - f(P) - df(P)(\varepsilon))\,\Lambda\overline{P}\right| \lesssim \lambda^\alpha(1+|y|^\kappa)(Q^{\frac{4}{N}} + |\varepsilon|^{\frac{4}{N}})|\varepsilon|Q.$$

If $3|\lambda^\alpha Z| \leq Q$ and $Q < 3|\varepsilon|$, then we have

$$\left|(f(P+\varepsilon) - f(P) - df(P)(\varepsilon))\,\Lambda\overline{P}\right| \lesssim (1+|y|^\kappa)Q^{\frac{4}{N}}|\varepsilon|^2.$$

If $3|\varepsilon| \leq Q$, then $P - |\varepsilon| > \frac{1}{3}Q > 0$. According to Taylor's theorem, we have

$$\begin{aligned}\left|(f(P+\varepsilon) - f(P) - df(P)(\varepsilon))\,\Lambda\overline{P}\right| &\lesssim (1+|y|^\kappa)(P-|\varepsilon|)^{\frac{4}{N}-1}|\varepsilon|^2 Q\\ &\lesssim (1+|y|^\kappa)Q^{\frac{4}{N}}|\varepsilon|^2.\end{aligned}$$

Therefore, we have

$$(f(P+\varepsilon) - f(P) - df(P)(\varepsilon), \Lambda P)_2 = O(s^{-(K+2)}).$$

The same calculation for $(df(P)(\varepsilon) - df(Q)(\varepsilon))\,\Lambda\overline{P}$ yields

$$(df(P)(\varepsilon) - df(Q)(\varepsilon), \Lambda P)_2 = O(s^{-(K+2)}).$$

Accordingly, we have

$$\left(i\frac{\partial\varepsilon}{\partial s}, \Lambda P\right)_2 = -\frac{1}{4}\||y|Q\|\left(\frac{\partial b}{\partial s} + b^2 - \theta\right) + O(s^{-(K+2)}) + O(s^{-1}|\operatorname{Mod}(s)|)$$

and by (4.10) and (4.13),

$$\frac{\partial b}{\partial s} + b^2 - \theta = O(s^{-(K+2)}) + O(s^{-1}|\operatorname{Mod}(s)|).$$

The same calculations for (4.11) and (4.12) yield

$$\frac{1}{\lambda}\frac{\partial\lambda}{\partial s} + b = O(s^{-(K+2)}) + O(s^{-1}|\operatorname{Mod}(s)|), \quad 1 - \frac{\partial\gamma}{\partial s} = O(s^{-(K+2)}) + O(s^{-1}|\operatorname{Mod}(s)|).$$

Consequently, we have

$$|\operatorname{Mod}(s)| \lesssim s^{-(K+2)}, \quad \|e^{\epsilon'|y|}\Psi\|_{H^1} \lesssim s^{-(K+4)}.$$

Finally, since

$$\|P(s_1)\|_2^2 = \|P(s)\|_2^2 + 2(\varepsilon(s), P(s))_2 + \|\varepsilon(s)\|_2^2,$$

we have

$$\begin{aligned}|(\varepsilon(s), P(s))_2| &\lesssim \|\varepsilon(s)\|_2^2 + \int_s^{s_1}\left|\left.\frac{d}{ds}\right|_{s=\tau}\|P(s)\|_2^2\right|d\tau\\ &\lesssim s^{-2K} + \int_s^{s_1}\left(\tau^{-2}|\operatorname{Mod}(\tau)| + \tau^{-2(K+2)}\right)d\tau\\ &\lesssim s^{-(K+3)}.\end{aligned}$$

Therefore, if s_0 is sufficiently large, then we have $s_{**} = s_*$. Moreover, we have

$$|(\varepsilon(s), Q)_2| \lesssim |(\varepsilon(s), P(s))_2| + \lambda^\alpha\,|(\varepsilon(s), Z)_2| \lesssim s^{-(K+2)}.$$

□

Let $m > 0$ be sufficiently large and define

$$\begin{aligned}H(s,\varepsilon) &:= \frac{1}{2}\|\varepsilon\|_{H^1}^2 + \frac{b(s)^2}{2}\||y|\varepsilon\|_2^2 - \int_{\mathbb{R}^N}(F(P(s,y)+\varepsilon(y)) - F(P(s,y)) - dF(P(s,y))(\varepsilon(y)))\,dy\\ &\quad - \frac{1}{2}\lambda(s)^\alpha\left\||y|^{-\sigma}\varepsilon\right\|_2^2,\\ S(s,\varepsilon) &:= \frac{1}{\lambda(s)^m}H(s,\varepsilon).\end{aligned}$$

Lemma 4.15 (Estimates of S)**.** For $s \in (s_*, s_1]$,

$$\|\varepsilon\|_{H^1}^2 + b^2 \, \||y|\varepsilon\|_2^2 + O(s^{-2(K+2)}) \lesssim H(s,\varepsilon) \lesssim \|\varepsilon\|_{H^1}^2 + b^2 \, \||y|\varepsilon\|_2^2$$

hold. Moreover,

$$\frac{1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \, \||y|\varepsilon\|_2^2 + O(s^{-2(K+2)})\right) \lesssim S(s,\varepsilon) \lesssim \frac{1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \, \||y|\varepsilon\|_2^2\right)$$

hold.

Proof. If $N \leq 3$, then we have

$$\left|F(P+\varepsilon) - F(P) - dF(P)(\varepsilon) - \frac{1}{2}d^2F(P)(\varepsilon,\varepsilon)\right| \lesssim \left(|P|^{\frac{4}{N}-1} + |\varepsilon|^{\frac{4}{N}-1}\right)|\varepsilon|^3.$$

For $N \geq 4$, if $2|\varepsilon| \geq |P|$, then we have

$$\left|F(P+\varepsilon) - F(P) - dF(P)(\varepsilon) - \frac{1}{2}d^2F(P)(\varepsilon,\varepsilon)\right| \lesssim |\varepsilon|^{\frac{4}{N}+2}.$$

If $2|\varepsilon| < |P|$, then $|P| > 0$ and $|P| - |\varepsilon| > \frac{1}{2}|P|$. Therefore, we have

$$\left|F(P+\varepsilon) - F(P) - dF(P)(\varepsilon) - \frac{1}{2}d^2F(P)(\varepsilon,\varepsilon)\right| \lesssim (|P| - |\varepsilon|)^{\frac{4}{N}-1}|\varepsilon|^3 \lesssim |\varepsilon|^{\frac{4}{N}+2}.$$

Therefore, we obtain

$$\int_{\mathbb{R}^N}\left(F(P+\varepsilon) - F(P) - dF(P)(\varepsilon) - \frac{1}{2}d^2F(P)(\varepsilon,\varepsilon)\right)dy = o(\|\varepsilon\|_{H^1}^2).$$

Similarly, if $N \leq 3$, then we have

$$\left|\frac{1}{2}d^2F(P)(\varepsilon,\varepsilon) - \frac{1}{2}d^2F(Q)(\varepsilon,\varepsilon)\right| \lesssim \lambda^\alpha\left(Q^{\frac{4}{N}-1} + |\lambda^\alpha Z|^{\frac{4}{N}-1}\right)|\varepsilon|^2|Z|.$$

For $N \geq 4$, if $2|\lambda^\alpha Z| \geq Q$, then we have

$$\left|\frac{1}{2}d^2F(P)(\varepsilon,\varepsilon) - \frac{1}{2}d^2F(Q)(\varepsilon,\varepsilon)\right| \lesssim |\lambda^\alpha Z|^{\frac{4}{N}}|\varepsilon|^2.$$

If $2|\lambda^\alpha Z| < Q$, then $Q - |\lambda^\alpha Z| > \frac{1}{2}Q$. Therefore, we have

$$\left|\frac{1}{2}d^2F(P)(\varepsilon,\varepsilon) - \frac{1}{2}d^2F(Q)(\varepsilon,\varepsilon)\right| \lesssim \lambda^\alpha\left(Q - |\lambda^\alpha Z|\right)^{\frac{4}{N}-1}|\varepsilon|^2|Z| \lesssim (1+|y|^\kappa)\lambda^\alpha|\varepsilon|^2Q^{\frac{4}{N}}$$

and

$$\int_{\mathbb{R}^N}\left(\frac{1}{2}d^2F(P)(\varepsilon,\varepsilon) - \frac{1}{2}d^2F(Q)(\varepsilon,\varepsilon)\right)dy = o(\|\varepsilon\|_{H^1}^2).$$

Accordingly, we have

$$\begin{aligned}\|\varepsilon\|_{H^1}^2 - \int_{\mathbb{R}^N} d^2F(Q)(\varepsilon,\varepsilon)dy &= \langle L_+ \operatorname{Re}\varepsilon, \operatorname{Re}\varepsilon\rangle + \langle L_- \operatorname{Im}\varepsilon, \operatorname{Im}\varepsilon\rangle \\ &\geq \mu\|\varepsilon\|_{H^1}^2 - \frac{1}{\mu}\left((\operatorname{Re}\varepsilon, Q)_2^2 + (\operatorname{Re}\varepsilon, |y|^2Q)_2^2 + (\operatorname{Im}\varepsilon, \rho)_2^2\right) \\ &= \mu\|\varepsilon\|_{H^1}^2 - \frac{1}{\mu}\left((\varepsilon, Q)_2^2 + \left((\varepsilon, |y|^2P)_2 - \lambda^\alpha(\varepsilon, |y|^2Z)_2\right)^2 + (\varepsilon, i\rho)_2^2\right) \\ &= \mu\|\varepsilon\|_{H^1}^2 + O(s^{-2(K+2)}).\end{aligned}$$

Consequently, we have the lower estimate of H. The rest is obvious. □

Lemma 4.16. For $s \in (s_*, s_1]$,

$$|(f(P+\varepsilon) - f(P), \Lambda\varepsilon)_2| \lesssim \|\varepsilon\|_{H^1}^2 + s^{-3K}$$

holds.

Proof. Calculated in the same way as in [10, Section 5.4], we have

$$\begin{aligned}&\nabla\left(F(P+\varepsilon)-F(P)-dF(P)(\varepsilon)\right)\\=&\operatorname{Re}\left(f(P+\varepsilon)\nabla\left(\overline{P}+\overline{\varepsilon}\right)-f(P)\nabla\overline{P}-df(P)(\varepsilon)\nabla\overline{P}-f(P)\nabla\overline{\varepsilon}\right)\\=&\operatorname{Re}\left(\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)\nabla\overline{P}+\left(f(P+\varepsilon)-f(P)\right)\nabla\overline{\varepsilon}\right)\end{aligned}$$

and

$$\begin{aligned}\left(f(P+\varepsilon)-f(P),\Lambda\varepsilon\right)=\operatorname{Re}\int_{\mathbb{R}^N}\left(f(P+\varepsilon)-f(P)\right)\Lambda\overline{\varepsilon}dy&\\=\operatorname{Re}\int_{\mathbb{R}^N}\Big(\frac{N}{2}\left(f(P+\varepsilon)-f(P)\right)\overline{\varepsilon}-\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}&\\-N\left(F(P+\varepsilon)-F(P)-dF(P)(\varepsilon)\right)\Big)dy.&\end{aligned}$$

Firstly,

$$|\left(f(P+\varepsilon)-f(P)\right)\overline{\varepsilon}|+|F(P+\varepsilon)-F(P)-dF(P)(\varepsilon)|\lesssim((1+|y|^{\kappa})Q^{\frac{4}{N}}+|\varepsilon|^{\frac{4}{N}})|\varepsilon|^2$$

holds.

Next, we consider $\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}$. For $N\leq 3$, we have

$$\left|\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}\right|\lesssim(1+|y|^{\kappa})(Q+|\varepsilon|)^{\frac{4}{N}-1}|\varepsilon|^2Q.$$

For $N\geq 4$, if $Q<3|\lambda^{\alpha}Z|$, then $1\lesssim\lambda^{\alpha}(1+|y|^{\kappa})$. Therefore, we have

$$\left|\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}\right|\lesssim\lambda^{K\alpha}(1+|y|^{\kappa})(Q^{\frac{4}{N}}+|\varepsilon|^{\frac{4}{N}})|\varepsilon|Q.$$

If $3|\lambda^{\alpha}Z|\leq Q$ and $Q<3|\varepsilon|$, we have

$$\left|\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}\right|\lesssim(1+|y|^{\kappa})Q^{\frac{4}{N}}|\varepsilon|^2.$$

If $3|\varepsilon|\leq Q$, then $P-|\varepsilon|\geq\frac{1}{3}Q>0$. Therefore, we have

$$\left|\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)y\cdot\nabla\overline{P}\right|\lesssim(1+|y|^{\kappa})Q^{\frac{4}{N}}|\varepsilon|^2.$$

Consequently, we have the conclusion. □

Lemma 4.17 (Derivative of H in time)**.** For $s\in(s_*,s_1]$,

$$\frac{d}{ds}H(s,\varepsilon(s))\gtrsim-b\left(\|\varepsilon\|_{H^1}^2+b^2\left\||y|\varepsilon\right\|_2^2\right)+O(s^{-2(K+2)})$$

holds.

Proof. Firstly, we have

$$\frac{d}{ds}H(s,\varepsilon(s))=\frac{\partial H}{\partial s}(s,\varepsilon(s))+\left(i\frac{\partial H}{\partial\varepsilon}(s,\varepsilon(s)),i\frac{\partial\varepsilon}{\partial s}(s)\right)_2.$$

Here,

$$\begin{aligned}\frac{\partial H}{\partial\varepsilon}&=-\Delta\varepsilon+\varepsilon+b^2|y|^2\varepsilon-\left(f(P+\varepsilon)-f(P)\right)-\frac{\lambda^{\alpha}}{|y|^{2\sigma}}\varepsilon\\&=L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon+b^2|y|^2\varepsilon-\left(f(P+\varepsilon)-f(P)-df(Q)(\varepsilon)\right)-\frac{\lambda^{\alpha}}{|y|^{2\sigma}}\varepsilon,\\\frac{\partial H}{\partial s}&=b\frac{\partial b}{\partial s}\left\||y|\varepsilon\right\|_2^2-\operatorname{Re}\int_{\mathbb{R}^N}\left(f(P+\varepsilon)-f(P)-df(P)(\varepsilon)\right)\frac{\partial\overline{P}}{\partial s}dy-\frac{\alpha\lambda^{\alpha}}{2}\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}\left\||y|^{-\sigma}\varepsilon\right\|_2^2\end{aligned}$$

hold. Therefore, we have

$$\frac{\partial H}{\partial s}\gtrsim-b^3\left\||y|\varepsilon\right\|_2^2-s^{-2}b\|\varepsilon\|_{H^1}^2+O(s^{-3K}).$$

Let define

$$\operatorname{Mod}_{\mathrm{op}}v:=i\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)\Lambda v-\left(1-\frac{\partial\gamma}{\partial s}\right)v-\left(\frac{\partial b}{\partial s}+b^2-\theta\right)\frac{|y|^2}{4}v+\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)b\frac{|y|^2}{2}v.$$

Then

$$i\frac{\partial \varepsilon}{\partial s} = \frac{\partial H}{\partial \varepsilon} - b^2|y|^2\varepsilon - \theta\frac{|y|^2}{4}\varepsilon + \mathrm{Mod}_{\mathrm{op}}(P+\varepsilon) + \Psi$$

holds. Moreover, we have

$$\left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon(s)), i\frac{\partial \varepsilon}{\partial s}(s)\right)_2 = \left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon(s)), -b^2|y|^2\varepsilon - \theta\frac{|y|^2}{4}\varepsilon + \mathrm{Mod}_{\mathrm{op}}(P+\varepsilon) + \Psi\right)_2.$$

Secondly, we have

$$\begin{aligned}\left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon), -b^2|y|^2\varepsilon\right)_2 &= -2b^2\left(i\nabla\varepsilon, y\varepsilon\right)_2 + \left(i\left(|P+\varepsilon|^{\frac{4}{N}} - |P|^{\frac{4}{N}}\right)P, -2b^2|y|^2\varepsilon\right)_2\\ &= -2b^2\left(i\nabla\varepsilon, y\varepsilon\right)_2 + O(b^2\|\varepsilon\|_{H^1}^2 + s^{-3K})\\ &\gtrsim -b\left(\|\nabla\varepsilon\|_2^2 + b^2\||y|\varepsilon\|_2^2\right) + O(b^2\|\varepsilon\|_{H^1}^2 + s^{-3K}).\end{aligned}$$

Since $\theta \approx b^2$, we also have

$$\left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon), -\theta\frac{|y|^2}{4}\varepsilon\right)_2 \gtrsim -b\left(\|\nabla\varepsilon\|_2^2 + b^2\||y|\varepsilon\|_2^2\right) + O(b^2\|\varepsilon\|_{H^1}^2 + s^{-3K}).$$

Thirdly, from Lemma 4.14,

$$\left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon(s)), \mathrm{Mod}_{\mathrm{op}} P\right)_2 = O(s^{-2(K+2)}), \quad \left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon(s)), \Psi\right)_2 = O(s^{-2(K+2)})$$

hold.

Finally, since

$$|(f(P+\varepsilon) - f(P), \Lambda\varepsilon)_2| + \left|\left(f(P+\varepsilon) - f(P), i|y|^2\varepsilon\right)_2\right| = O(\|\varepsilon\|_{H^1}^2) + O(s^{-3K})$$

from Lemma 4.16, we have

$$\left(i\frac{\partial H}{\partial \varepsilon}(s,\varepsilon(s)), \mathrm{Mod}_{\mathrm{op}}\varepsilon\right)_2 = o\left(b\left(\|\varepsilon\|_{H^1}^2 + b^2\||y|\varepsilon\|_2^2\right)\right) + O(s^{-(5K+2)}).$$

Consequently, we have the conclusion. □

Lemma 4.18 (Derivative of S in time)**.** Let $m > 0$ be sufficiently large. Then

$$\frac{d}{ds}S(s,\varepsilon(s)) \gtrsim \frac{b}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2\||y|\varepsilon\|_2^2 + O(s^{-(2K+3)})\right)$$

holds for $s \in (s_*, s_1]$.

Proof. From Lemma 4.17, we have

$$\begin{aligned}\frac{d}{ds}S(s,\varepsilon(s)) &= -m\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}\frac{1}{\lambda^m}H(s,\varepsilon) + \frac{1}{\lambda^m}\frac{d}{ds}H(s,\varepsilon(s))\\ &= -m\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s} + b\right)\frac{1}{\lambda^m}H(s,\varepsilon) + m\frac{b}{\lambda^m}H(s,\varepsilon) + \frac{1}{\lambda^m}\frac{d}{ds}H(s,\varepsilon(s))\\ &\geq \frac{b}{\lambda^m}\left(\frac{mC}{2}\left(\|\varepsilon\|_{H^1}^2 + b^2\||y|\varepsilon\|_2^2\right) - C'\left(\|\varepsilon\|_{H^1}^2 + b^2\||y|\varepsilon\|_2^2\right) + O(s^{-(2K+3)})\right).\end{aligned}$$

Therefore, we have the conclusion if m is sufficiently large. □

We confirm (4.8) on $[s_0, s_1]$.

Lemma 4.19 (Re-estimation)**.** For $s \in (s_*, s_1]$,

$$\|\varepsilon(s)\|_{H^1}^2 + b(s)^2\||y|\varepsilon(s)\|_2^2 \lesssim s^{-(2K+2)}, \tag{4.14}$$

$$\left|\frac{\lambda(s)^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}} \tag{4.15}$$

holds.

Proof. We prove (4.14) by contradiction. Let $C_\dagger > 0$ be sufficiently large and define

$$s_\dagger := \inf\left\{\sigma \in (s_*, s_1] \;\middle|\; \|\varepsilon(\tau)\|_{H^1}^2 + b(\tau)^2 \||y|\varepsilon(\tau)\|_2^2 \le C_\dagger \tau^{-2(K+1)} \ (\tau \in [\sigma, s_1])\right\}.$$

Then $s_\dagger < s_1$ holds. Here, we assume that $s_\dagger > s_*$. Then we have

$$\|\varepsilon(s_\dagger)\|_{H^1}^2 + b(s_\dagger)^2 \||y|\varepsilon(s_\dagger)\|_2^2 = C_\dagger {s_\dagger}^{-2(K+1)}.$$

Let define

$$s_\ddagger := \sup\left\{\sigma \in (s_*, s_1] \;\middle|\; \|\varepsilon(\tau)\|_{H^1}^2 + b(\tau)^2 \||y|\varepsilon(\tau)\|_2^2 \ge \tau^{-2(K+1)} \ (\tau \in [s_\dagger, \sigma])\right\}.$$

Then we have $s_\ddagger > s_\dagger$. Furthermore,

$$\|\varepsilon(s_\ddagger)\|_{H^1}^2 + b(s_\ddagger)^2 \||y|\varepsilon(s_\ddagger)\|_2^2 = {s_\ddagger}^{-2(K+1)}.$$

Then according to Lemma 4.15 and Lemma 4.18, we have

$$\begin{aligned}\frac{C_1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2 - C' s^{-2(K+1)}\right) \le S(s,\varepsilon) \le \frac{C_2}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2\right),\\ \frac{b}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2 \||y|\varepsilon\|_2^2 - s^{-2(K+1)}\right) \lesssim \frac{d}{ds}S(s,\varepsilon).\end{aligned}$$

in $(s_*, s_1]$. Therefore, we have

$$\begin{aligned}C_1(C_\dagger - C'){s_\dagger}^{-2(K+1)} &= C_1\left(\|\varepsilon(s_\dagger)\|_{H^1}^2 + b(s_\dagger)^2 \||y|\varepsilon(s_\dagger)\|_2^2 - C'{s_\dagger}^{-2(K+1)}\right)\\ &\le \lambda(s_\dagger)^m S(s_\dagger, \varepsilon(s_\dagger))\\ &\le \lambda(s_\dagger)^m S(s_\ddagger, \varepsilon(s_\ddagger))\\ &\le C_2 \frac{\lambda(s_\dagger)^m}{\lambda(s_\ddagger)^m}\left(\|\varepsilon(s_\ddagger)\|_{H^1}^2 + b(s_\ddagger)^2 \||y|\varepsilon(s_\ddagger)\|_2^2\right)\\ &\le C_2 \frac{\lambda(s_\dagger)^m}{\lambda(s_\ddagger)^m} {s_\ddagger}^{-2(K+1)}\\ &\le 2C_2 \frac{{s_\dagger}^{-\frac{2m}{\alpha}}}{{s_\ddagger}^{-\frac{2m}{\alpha}}} \frac{{s_\ddagger}^{-2(K+1)}}{{s_\dagger}^{-2(K+1)}} {s_\dagger}^{-2(K+1)}\end{aligned}$$

and since $K - \frac{m}{\alpha} > 0$, we have

$$C_1(C_\dagger - C') \le 2C_2.$$

Since $C_\dagger$ is sufficiently large, it is a contradiction. Therefore, $s_\dagger \le s_*$. On the other hand, $s_\dagger \ge s_*$ is clearly. Accordingly, $s_* = s_\dagger$.

Next, since

$$|E(P_{\lambda,b,\gamma}(s)) - E_0| \le \left|\int_{s_1}^{s} \frac{d}{ds}\Big|_{s=\tau} E(P_{\lambda,b,\gamma}(s))d\tau\right| \le \int_{s}^{s_1} \tau^{-(K+2)+\frac{4}{\alpha}}d\tau \lesssim s^{-(K+1)+\frac{4}{\alpha}},$$

we have

$$\begin{aligned}\left|b^2 - \frac{2\beta}{2-\alpha}\lambda^\alpha - C_0\lambda^2\right| &\le \lambda^2\left(\left|\frac{b^2}{\lambda^2} - \frac{2\beta}{2-\alpha}\lambda^{\alpha-2} - \frac{8}{\||y|Q\|_2^2}E(P_{\lambda,b,\gamma})\right| + \frac{8}{\||y|Q\|_2^2}|E(P_{\lambda,b,\gamma}) - E_0|\right)\\ &\lesssim s^{-4}.\end{aligned} \tag{4.16}$$

From (4.16) and the definition of $\mathcal{F}$, we have

$$\left|\frac{\partial}{\partial s}\mathcal{F}(\lambda(s)) - 1\right| \lesssim s^{-2}.$$

Therefore, we have

$$|s - \mathcal{F}(\lambda(s))| \lesssim s^{-1}$$

since $\mathcal{F}(\lambda(s_1)) = s_1$. From definition λ_{app}, we have

$$\left|\frac{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}}{\lambda(s)^{\frac{\alpha}{2}}} - 1\right| \lesssim s^{-\frac{1}{2}} + s^{2-\frac{4}{\alpha}}$$

and

$$\left|\frac{\lambda(s)^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}}-1\right| \le \left|\frac{\lambda(s)^{\frac{\alpha}{2}}}{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}}\right|\left|\frac{\lambda_{\mathrm{app}}(s)^{\frac{\alpha}{2}}}{\lambda(s)^{\frac{\alpha}{2}}}-1\right| \lesssim s^{-\frac{1}{2}}+s^{2-\frac{4}{\alpha}}.$$

Finally, from (4.16) and the definitions of λ_{app} and b_{app}, we have

$$\left|b(s)^2-b_{\mathrm{app}}(s)^2\right| \lesssim s^{-4}+s^{-2-\frac{1}{2}}+s^{-\frac{4}{\alpha}}$$

and

$$\left|\frac{b(s)}{b_{\mathrm{app}}(s)}-1\right| \lesssim s^{-\frac{1}{2}}+s^{2-\frac{4}{\alpha}}.$$

Consequently, we obtain (4.15). □

Similar to the proof of Lemma Lemma 3.12, the following lemma are obtained.

Lemma 4.20. If s_0 is sufficiently large, then $s_* = s' = s_0$.

We rewrite the uniform estimates obtained for the time variable s in Lemma 4.19 into uniform estimates for the time variable t.

Lemma 4.21. If s_0 is sufficiently large, then there is $t_0 < 0$ that is sufficiently close to 0 such that for $t_1 \in (t_0, 0)$,

$$[t_0,t_1] \subset {s_{t_1}}^{-1}([s_0,s_1]), \quad \left|\mathcal{C}s_{t_1}(t)^{-\frac{4-\alpha}{\alpha}}-|t|\right| \lesssim |t|^{1+\frac{\alpha M}{4-\alpha}} \quad (t\in[t_0,t_1])$$

holds.

Proof. Since $t_{t_1}(s_1) = t_1$ and $s_1 = |\mathcal{C}^{-1}t_1|^{-\frac{\alpha}{4-\alpha}}$, we have

$$\begin{aligned}\int_s^{s_1}\lambda_{\mathrm{app}}(\tau)^2\left(\frac{\lambda_{t_1}(\tau)}{\lambda_{\mathrm{app}}(\tau)}-1\right)\left(\frac{\lambda_{t_1}(\tau)}{\lambda_{\mathrm{app}}(\tau)}+1\right)d\tau &= \int_s^{s_1}\left(\lambda_{t_1}(\tau)^2-\lambda_{\mathrm{app}}(\tau)^2\right)d\tau\\ &= t_{t_1}(s_1)-t_{t_1}(s)+\mathcal{C}({s_1}^{1-\frac{4}{\alpha}}-s^{1-\frac{4}{\alpha}})\\ &= |t_{t_1}(s)|-\mathcal{C}s^{-\frac{4-\alpha}{\alpha}}.\end{aligned}$$

Therefore, we have

$$\left||t_{t_1}(s)|-\mathcal{C}s^{-\frac{4-\alpha}{\alpha}}\right| \lesssim \int_s^{s_1}\lambda_{\mathrm{app}}(\tau)^2\tau^{-M}d\tau \lesssim \int_s^{s_1}\tau^{-\frac{4}{\alpha}-M}d\tau \le \frac{\alpha}{M+4-\alpha}s^{-\left(\frac{4-\alpha}{\alpha}+M\right)}.$$

Accordingly,

$$|t_{t_1}(s)| \approx s^{-\frac{4-\alpha}{\alpha}}, \quad \text{i.e., } |t| \approx s_{t_1}(t)^{-\frac{4-\alpha}{\alpha}}.$$

Moreover, there exists t_0 from Lemma 4.20. □

Lemma 4.22 (Conversion of estimates)**.** Let

$$\mathcal{C}_\lambda := \mathcal{C}^{-\frac{2}{4-\alpha}}\left(\frac{\alpha}{2}\sqrt{\frac{2\beta}{2-\alpha}}\right)^{-\frac{2}{\alpha}}, \quad \mathcal{C}_b := \frac{2}{\alpha}\mathcal{C}^{-\frac{\alpha}{4-\alpha}}.$$

For $t \in [t_0, t_1]$,

$$\tilde{\lambda}_{t_1}(t) = \mathcal{C}_\lambda|t|^{\frac{2}{4-\alpha}}\left(1+\epsilon_{\tilde{\lambda},t_1}(t)\right), \qquad \tilde{b}_{t_1}(t) = \mathcal{C}_b|t|^{\frac{\alpha}{4-\alpha}}\left(1+\epsilon_{\tilde{b},t_1}(t)\right),$$

$$\|\tilde{\varepsilon}_{t_1}(t)\|_{H^1} \lesssim |t|^{\frac{\alpha K}{4-\alpha}}, \qquad \||y|\tilde{\varepsilon}_{t_1}(t)\|_2 \lesssim |t|^{\frac{\alpha(K-1)}{4-\alpha}}$$

hold. Furthermore,

$$\sup_{t_1\in[t,0)}\left|\epsilon_{\tilde{\lambda},t_1}(t)\right| \lesssim |t|^{\frac{\alpha M}{4-\alpha}}, \quad \sup_{t_1\in[t,0)}\left|\epsilon_{\tilde{b},t_1}(t)\right| \lesssim |t|^{\frac{\alpha M}{4-\alpha}}.$$

Proof. Let

$$\epsilon_{\tilde{\lambda},t_1}(t) := \frac{\tilde{\lambda}_{t_1}(t)}{\mathcal{C}_\lambda|t|^{\frac{2}{4-\alpha}}}-1.$$

Then we have

$$\begin{aligned}\left|\epsilon_{\tilde{\lambda},t_1}(t)\right| &\le \left|\frac{\tilde{\lambda}_{t_1}(t)}{\lambda_{\mathrm{app}}(s_{t_1}(t))}-1\right|\left|\frac{\lambda_{\mathrm{app}}(s_{t_1}(t))}{\mathcal{C}_\lambda|t|^{\frac{2}{4-\alpha}}}\right| + \frac{1}{\mathcal{C}_\lambda|t|^{\frac{2}{4-\alpha}}}\left|\lambda_{\mathrm{app}}(s_{t_1}(t))-\mathcal{C}_\lambda|t|^{\frac{2}{4-\alpha}}\right|\\ &\lesssim |t|^{\frac{\alpha M}{4-\alpha}}.\end{aligned}$$

The same is done for $\epsilon_{\tilde{b},t_1}(t) := \tilde{b}_{t_1}(t)\mathcal{C}_b^{-1}|t|^{-\frac{\alpha}{4-\alpha}}-1$. □

proof of Theorem 4.1. Let $(t_n)_{n\in\mathbb{N}} \subset (t_0, 0)$ be a monotonically increasing sequence such that $\lim_{n\nearrow\infty} t_n = 0$. For each $n \in \mathbb{N}$, u_n is the solution for (NLS) with $\pm = +$ with an initial value

$$u_n(t_n, x) := P_{\lambda_{1,n}, b_{1,n}, 0}(x)$$

at t_n, where $b_{1,n}$ and $\lambda_{1,n}$ are given by Lemma 4.13 for t_n.

According to Lemma 2.2 with an initial value $\tilde{\gamma}_n(t_n) = 0$, there exists a decomposition

$$u_n(t,x) = \frac{1}{\tilde{\lambda}_n(t)^{\frac{N}{2}}} (P + \tilde{\varepsilon}_n) \left(t, \frac{x}{\tilde{\lambda}_n(t)}\right) e^{-i\frac{\tilde{b}_n(t)}{4}\frac{|x|^2}{\tilde{\lambda}_n(t)^2} + i\tilde{\gamma}_n(t)}.$$

Then $(u_n(t_0))_{n\in\mathbb{N}}$ is bounded in Σ^1. Therefore, up to a subsequence, there exists $u_\infty(t_0) \in \Sigma^1$ such that

$$u_n(t_0) \rightharpoonup u_\infty(t_0) \quad \text{in } \Sigma^1, \quad u_n(t_0) \to u_\infty(t_0) \quad \text{in } L^2(\mathbb{R}^N) \quad (n \to \infty).$$

Let u_∞ be the solution for (NLS) with $\pm = +$ and an initial value $u_\infty(t_0)$, and let T^* be the supremum of the maximal existence interval of u_∞. Moreover, we define $T := \min\{0, T^*\}$. Then for any $T' \in [t_0, T)$, $[t_0, T'] \subset [t_0, t_n]$ if n is sufficiently large. Then there exist n_0 and $C(T', t_0) > 0$ such that

$$\sup_{n\geq n_0} \|u_n\|_{L^\infty([t_0,T'],\Sigma^1)} \leq C(T', t_0)$$

holds. Therefore, from Lemma 2.1,

$$u_n \to u_\infty \quad \text{in } C\left([t_0, T'], L^2(\mathbb{R}^N)\right) \quad (n \to \infty)$$

holds. In particular, $u_n(t) \rightharpoonup u_\infty(t)$ in Σ^1 for any $t \in [t_0, T)$. Furthermore, from the mass conservation, we have

$$\|u_\infty(t)\|_2 = \|u_\infty(t_0)\|_2 = \lim_{n\to\infty} \|u_n(t_0)\|_2 = \lim_{n\to\infty} \|u_n(t_n)\|_2 = \lim_{n\to\infty} \|P(t_n)\|_2 = \|Q\|_2.$$

Based on weak convergence in Σ^1 and Lemma 2.2, we decompose u_∞ to

$$u_\infty(t,x) = \frac{1}{\tilde{\lambda}_\infty(t)^{\frac{N}{2}}} (P + \tilde{\varepsilon}_\infty) \left(t, \frac{x}{\tilde{\lambda}_\infty(t)}\right) e^{-i\frac{\tilde{b}_\infty(t)}{4}\frac{|x|^2}{\tilde{\lambda}_\infty(t)^2} + i\tilde{\gamma}_\infty(t)},$$

on $[t_0, T)$. Furthermore, for any $t \in [t_0, T)$, as $n \to \infty$,

$$\tilde{\lambda}_n(t) \to \tilde{\lambda}_\infty(t), \quad \tilde{b}_n(t) \to \tilde{b}_\infty(t), \quad e^{i\tilde{\gamma}_n(t)} \to e^{i\tilde{\gamma}_\infty(t)}, \quad \tilde{\varepsilon}_n(t) \rightharpoonup \tilde{\varepsilon}_\infty(t) \quad \text{in } \Sigma^1$$

hold. Consequently, from the uniform estimate in Lemma 4.22, as $n \to \infty$, we have

$$\tilde{\lambda}_\infty(t) = \mathcal{C}_\lambda |t|^{\frac{2}{4-\alpha}} (1 + \epsilon_{\tilde{\lambda},0}(t)), \qquad \tilde{b}_\infty(t) = \mathcal{C}_b |t|^{\frac{\alpha}{4-\alpha}} (1 + \epsilon_{\tilde{b},0}(t)),$$

$$\|\tilde{\varepsilon}_\infty(t)\|_{H^1} \lesssim |t|^{\frac{\alpha K}{4-\alpha}}, \quad \||y|\tilde{\varepsilon}_\infty(t)\|_2 \lesssim |t|^{\frac{\alpha(K-1)}{4-\alpha}}, \qquad \left|\epsilon_{\tilde{\lambda},0}(t)\right| \lesssim |t|^{\frac{\alpha M}{4-\alpha}}, \quad \left|\epsilon_{\tilde{b},0}(t)\right| \lesssim |t|^{\frac{\alpha M}{4-\alpha}}.$$

Consequently, we obtain that u converges to the blow-up profile in Σ^1.

Finally, we check energy of u_∞. Since

$$E(u_n) - E\left(P_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n}\right) = \int_0^1 \left\langle E'(P_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n} + \tau\tilde{\varepsilon}_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n}), \tilde{\varepsilon}_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n}\right\rangle d\tau$$

and $E'(w) = -\Delta w - |w|^{\frac{4}{N}} w - |x|^{-2\sigma} w$, we have

$$E(u_n) - E\left(P_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n}\right) = O\left(|t|^{\frac{\alpha K - 4}{4-\alpha}}\right).$$

Similarly, we have

$$E(u_\infty) - E\left(P_{\tilde{\lambda}_\infty,\tilde{b}_\infty,\tilde{\gamma}_\infty}\right) = O\left(|t|^{\frac{\alpha K - 4}{4-\alpha}}\right).$$

From the continuity of E, we have

$$\lim_{n\to\infty} E\left(P_{\tilde{\lambda}_n,\tilde{b}_n,\tilde{\gamma}_n}\right) = E\left(P_{\tilde{\lambda}_\infty,\tilde{b}_\infty,\tilde{\gamma}_\infty}\right)$$

and from the conservation of energy,

$$E(u_n) = E(u_n(t_n)) = E\left(P_{\tilde{\lambda}_{1,n},\tilde{b}_{1,n},0}\right) = E_0.$$

Therefore, we have

$$E(u_\infty) = E_0 + o_{t\nearrow 0}(1)$$

and since $E(u_\infty)$ is constant for t, $E(u_\infty) = E_0$. □

4.4 Proof of Theorem 4.2

proof of Theorem 4.2. We assume that u is a critical-mass radial solution for (NLS) with $\pm = -$ and blows up at T^*. Let a sequence $(t_n)_{n\in\mathbb{N}}$ be such that $t_n \to T^*$ as $n \to T^*$ and define

$$\lambda_n := \frac{\|\nabla Q\|_2}{\|\nabla u(t_n)\|}, \quad v_n(x) := \lambda_n{}^{\frac{N}{2}} u(t_n, \lambda_n x).$$

Then

$$\|v_n\|_2 = \|Q\|_2, \quad \|\nabla v_n\|_2 = \|\nabla Q\|_2$$

hold. Moreover,

$$E_0 := E(u(t_n)) \geq E_{\mathrm{crit}}(u(t_n)) = \frac{E_{\mathrm{crit}}(v_n)}{\lambda_n{}^2}.$$

Therefore, we obtain

$$\limsup_{n\to\infty} E_{\mathrm{crit}}(v_n) \leq 0.$$

From the standard concentration argument (see [10,15]), there exist sequences $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$v_n(\cdot - x_n)e^{i\gamma_n} \to Q \quad \text{in } H^1(\mathbb{R}^N) \quad (n \to \infty).$$

Moreover, up to a subsequence, we have

$$v_n e^{i\gamma_n} \to Q \quad \text{in } H^1(\mathbb{R}^N) \quad (n \to \infty).$$

Indeed, if $(x_n)_{n\in\mathbb{N}}$ is unbounded, we may assume $x_n \to \infty$ as $n \to \infty$. Then since v_n decay uniformly by the radial lemma, we have

$$0 = \lim_{n\to\infty} \left\|v_n(\cdot - x_n)e^{i\gamma_n} - Q\right\|_{H^1}^2 = 2\|Q\|_{H^1}^2 - \lim_{n\to\infty} 2\left(v_n(\cdot - x_n)e^{i\gamma_n}, Q\right)_{H^1} = 2\|Q\|_{H^1}^2.$$

It is a contradiction. Therefore, $(x_n)_{n\in\mathbb{N}}$ is bounded. We may assume that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence. Let define $x_0 := \lim_{n\to\infty} x_n$. Then we have

$$v_n e^{i\gamma_n} \to Q(\cdot + x_0) \quad \text{in } H^1(\mathbb{R}^N) \quad (n \to \infty).$$

Since v_n and Q are radial, we obtain $x_0 = 0$.

Here, we have

$$\left\||x|^{-\sigma} u(t_n)\right\|_2^2 = \frac{\||x|^{-\sigma} v_n\|_2^2}{\lambda_n{}^{2\sigma}}.$$

Therefore, since $E_{\mathrm{crit}}(u) \geq 0$,

$$E_0 = E(u(t_n)) \geq \frac{\||x|^{-\sigma} v_n\|_2^2}{\lambda_n{}^{2\sigma}} \to \infty \quad (n \to \infty).$$

It is a contradiction. □

Chapter 5

Proof of decomposition lemma

In this chapter, we prove Lemma 2.2.

5.1 Decomposition by implicit function theorem

Definition 5.1. Let $\tilde{\varepsilon} : \mathbb{R}_{>0} \times \mathbb{R}^{2+N} \times H^1(\mathbb{R}^N) \times \mathbb{R}_{>0} \to H^1(\mathbb{R}^N)$ and $S : \mathbb{R}_{>0} \times \mathbb{R}^{2+N} \times H^1(\mathbb{R}^N) \times \mathbb{R}_{>0} \to \mathbb{R}^{3+N}$ define by

$$\begin{aligned}
\tilde{P}(\tilde{\lambda}, \tilde{b}, l)(y) &:= Q(y) + \sum_{(j,k)\in\Sigma_{K+K'}} \left(\tilde{b}^{2j} l^{k+1} \tilde{\lambda}^{(k+1)\alpha} P^+_{j,k}(y) + i\tilde{b}^{2j+1} l^{k+1} \tilde{\lambda}^{(k+1)\alpha} P^-_{j,k}(y)\right),\\
\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l)(y) &:= \tilde{\lambda}^{\frac{N}{2}} u(\tilde{\lambda} y - \tilde{w}) e^{i\tilde{b}\frac{|y|^2}{4} - i\tilde{\gamma}} - \tilde{P}(\tilde{\lambda}, \tilde{b}, l)(y),\\
S(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l) &:= \begin{pmatrix}
\left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), i\Lambda\tilde{P}(\tilde{\lambda}, \tilde{b}, l)\right)_2\\
\left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), |y|^2\tilde{P}(\tilde{\lambda}, \tilde{b}, l)\right)_2\\
\left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), i\rho\right)_2\\
\left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), y_1\tilde{P}(\tilde{\lambda}, \tilde{b}, l)\right)_2\\
\vdots\\
\left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), y_N\tilde{P}(\tilde{\lambda}, \tilde{b}, l)\right)_2
\end{pmatrix}.
\end{aligned}$$

The function $\tilde{S}$ obtained by the following proposition, i.e., $(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w})$ are the parameters of the decomposition lemma. The proof of the decomposition lemma is proved by the procedure of first showing that it can be obtained in a neighbourhood of the ground state Q and then extending it.

In the following, let $B_X(x, r)$ denote an open ball in X, with centre x and radius r.

Proposition 5.2. There exist $\overline{C}, \delta, l_0 > 0$ and a unique function $\tilde{S} : B_{H^1}(Q, \delta) \times (-l_0, l_0) \to (1 - \overline{C}, 1 + \overline{C}) \times (-\overline{C}, \overline{C})^{2+N}$ such that

$$\tilde{S}(Q, 0) = (1, 0, 0, 0) \quad \text{and} \quad S(\tilde{S}(u, l), u, l) = 0 \text{ for any } (u, l) \in B_{H^1}(Q, \delta) \times (-l_0, l_0).$$

Moreover, $\tilde{S}$ is C^1 function.

Proof. Firstly, since $\tilde{\varepsilon}(1, 0, 0, 0, Q, 0) = 0$, we obtain $S(1, 0, 0, 0, Q, 0) = 0$. Moreover, since

$$\frac{\partial\tilde{\varepsilon}}{\partial\tilde{\lambda}}(1, 0, 0, 0, Q, 0) = \Lambda Q, \qquad \frac{\partial\tilde{\varepsilon}}{\partial\tilde{b}}(1, 0, 0, 0, Q, 0) = i\frac{|y|^2}{4}Q,$$
$$\frac{\partial\tilde{\varepsilon}}{\partial\tilde{\gamma}}(1, 0, 0, 0, Q, 0) = -iQ, \qquad \frac{\partial\tilde{\varepsilon}}{\partial\tilde{w}}(1, 0, 0, 0, Q, 0) = -\nabla Q,$$

we obtain

$$
\begin{aligned}
D_{(\tilde{\lambda},\tilde{b},\tilde{\gamma},\tilde{w})}S(1,0,0,0,Q,0) &= \begin{pmatrix} (\Lambda Q, i\Lambda Q)_2 & (i\frac{|y|^2}{4}Q, i\Lambda Q)_2 & (-iQ, i\Lambda Q)_2 & (-\nabla Q, i\Lambda Q)_2 \\ (\Lambda Q, |y|^2 Q)_2 & (i\frac{|y|^2}{4}Q, |y|^2 Q)_2 & (-iQ, |y|^2 Q)_2 & (-\nabla Q, |y|^2 Q)_2 \\ (\Lambda Q, i\rho)_2 & (i\frac{|y|^2}{4}Q, i\rho)_2 & (-iQ, i\rho)_2 & (-\nabla Q, i\rho)_2 \\ (\Lambda Q, y_1 Q)_2 & (i\frac{|y|^2}{4}Q, y_1 Q)_2 & (-iQ, y_1 Q)_2 & (-\nabla Q, y_1 Q)_2 \\ \vdots & \vdots & \vdots & \vdots \\ (\Lambda Q, y_N Q)_2 & (i\frac{|y|^2}{4}Q, y_N Q)_2 & (-iQ, y_N Q)_2 & (-\nabla Q, y_N Q)_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{1}{4}\|yQ\|_2^2 & 0 & 0 \\ -\|yQ\|_2^2 & 0 & 0 & 0 \\ 0 & \frac{1}{4}(|y|^2 Q, \rho)_2 & -\frac{1}{2}\|yQ\|_2^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\|Q\|_2^2 I_N \end{pmatrix},
\end{aligned}
$$

where I_N is $N \times N$ identity matrix. In particular, $D_{(\tilde{\lambda},\tilde{b},\tilde{\gamma},\tilde{w})}S(1,0,0,0,Q,0)$ is regular.

By using implicit function theorem, we obtain a unique function $\tilde{S} : W \to V$ such thet

$$\tilde{S}(Q,0) = (1,0,0,0) \text{ and } S(\tilde{S}(u,l),u,l) = 0 \text{ for any } (u,l) \in W$$

for some open neighbourhoods $V \subset \mathbb{R}^{3+N}$ of $(1,0,0,0)$ and $W \subset H^1(\mathbb{R}^N) \times \mathbb{R}$ of $(Q,0)$. Moreover, since V is a neighbourhood of $(1,0,0,0)$,

$$(1,0,0,0) \in (1-\overline{C}, 1+\overline{C}) \times (-\overline{C}, \overline{C})^{2+N} \subset V$$

for some sufficiently small $\overline{C} > 0$. Since $\tilde{S}^{-1}((1-\overline{C}, 1+\overline{C}) \times (-\overline{C}, \overline{C})^{2+N})$ is open,

$$(Q,0) \in B_{H^1}(Q,\delta) \times (-l_0, l_0) \subset \tilde{S}^{-1}((1-\overline{C}, 1+\overline{C}) \times (-\overline{C}, \overline{C})^{2+N}) \subset W$$

for some $\delta, l_0 > 0$. Consequently, we obtain the conclusion. □

5.2 Preparation for extension

This section is dedicated to preparing for extending Proposition 5.2.

Definition 5.3. Let $T_{\lambda,\gamma,w} : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ define by

$$T_{\lambda,\gamma,w} u := \lambda^{\frac{N}{2}} u(\lambda \cdot - w) e^{-i\gamma}$$

for $\lambda > 0$, $\gamma \in \mathbb{R}$, and $w \in \mathbb{R}^N$.

Then, by direct calculation, we obtain the following properties.

Proposition 5.4. (i) $\|T_{\lambda,\gamma,w} u\|_2 = \|u\|_2$, $\|\nabla T_{\lambda,\gamma,w} u\|_2 = \lambda \|\nabla u\|_2$.

(ii) $T_{\lambda,\gamma,w} \in \mathcal{L}\left(H^1(\mathbb{R}^N)\right)$.

(iii) $T_{\lambda_1,\gamma_1,w_1} T_{\lambda_2,\gamma_2,w_2} = T_{\lambda_1\lambda_2,\gamma_1+\gamma_2,\lambda_2 w_1 + w_2}$.

(iv) For any $u \in H^1(\mathbb{R}^N)$, $\mathbb{R} \times \mathbb{R}^N \ni (\gamma, y) \mapsto T_{1,\gamma,w} u \in L^2(\mathbb{R}^N)$ is Lipschitz continuous.

(v) For any $u \in L^2(\mathbb{R}^N)$ such that $\Lambda u \in L^2(\mathbb{R}^N)$, $\mathbb{R} \ni \lambda \mapsto T_{\lambda,0,0} u \in L^2(\mathbb{R}^N)$ is locally Lipschitz continuous.

Lemma 5.5. Let $u \in H^1(\mathbb{R}^N)$ and $\delta \in (0, \|\nabla u\|_2)$. For $\lambda > 0$, if $B_{H^1}(u,\delta) \cap T_{\lambda,\gamma,w} B_{H^1}(u,\delta) \neq \emptyset$ for some $\gamma \in \mathbb{R}$ and $w \in \mathbb{R}^N$, then

$$\lambda \leq \frac{\|\nabla u\|_2 + \delta}{\|\nabla u\|_2 - \delta}$$

holds.

Proof. When $u = 0$, it is obvious. We may assume $u \neq 0$. Since

$$\|u - v\|_{H^1} < \delta, \quad \|u - T_{\lambda,b,w} v\|_{H^1} < \delta$$

for some $v \in B_{H^1}(u,\delta)$, we obtain

$$\begin{aligned}\delta \geq &\|T_{\lambda,\gamma,w}v - u\|_{H^1} \geq \|\nabla T_{\lambda,\gamma,w}v - \nabla u\|_2\\ \geq &\|\nabla T_{\lambda,\gamma,w}u - \nabla u\|_2 - \|\nabla T_{\lambda,\gamma,w}v - \nabla T_{\lambda,\gamma,w}u\|_2\\ \geq &\,|\lambda\|\nabla u\|_2 - \|\nabla u\|_2| - \lambda\|\nabla u - \nabla v\|_2\\ \geq &\lambda\|\nabla u\|_2 - \|\nabla u\| - \lambda\delta.\end{aligned}$$

Therefore, we obtain

$$\lambda \leq \frac{\|\nabla u\|_2 + \delta}{\|\nabla u\|_2 - \delta}.$$

□

Corollary 5.6. For any $\varepsilon > 0$ and $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists $\delta > 0$ such that for $\lambda > 0$, if $B_{H^1}(u,\delta) \cap T_{\lambda,\gamma,w}B_{H^1}(u,\delta) \neq \emptyset$ for some $\gamma \in \mathbb{R}$ and $w \in \mathbb{R}^N$, then $|1-\lambda| < \varepsilon$ holds.

Proof. Let $\delta := \frac{1}{2}\min\{\frac{1}{2}\|\nabla u\|_2, \frac{\varepsilon}{4}\|\nabla u\|_2\}$. From Lemma 5.5,

$$\lambda \leq \frac{\|\nabla u\|_2 + \delta}{\|\nabla u\|_2 - \delta} \leq \frac{\frac{3}{2}\|\nabla u\|_2}{\frac{1}{2}\|\nabla u\|_2} = 3$$

and from the proof of Lemma 5.5,

$$\delta \geq |\lambda - 1|\,\|\nabla u\|_2 - \lambda\delta \geq |\lambda - 1|\,\|\nabla u\|_2 - 3\delta.$$

Therefore, we obtain

$$|1-\lambda| \leq \frac{4\delta}{\|\nabla u\|_2} \leq \frac{4}{\|\nabla u\|_2}\frac{\varepsilon}{8}\|\nabla u\|_2 = \frac{\varepsilon}{2} < \varepsilon.$$

□

Lemma 5.7. For any $u \in L^2(\mathbb{R}^N)$,

$$\lim_{r\to\infty}\sup_{|w|\geq r}\sup_{\gamma\in\mathbb{R}}|(T_{1,\gamma,w}u,u)_2| = 0$$

holds.

Proof. If u has the compact support, then it is obvious. The rest can be shown by approximation. □

Lemma 5.8. For any $u \in L^2(\mathbb{R}^N)$, there exist $R > 0$ such that for $w \in \mathbb{R}^N$, if $B_{L^2}\left(u, \frac{1}{2}\|u\|_2\right) \cap T_{1,\gamma,w}B_{L^2}\left(u, \frac{1}{2}\|u\|_2\right) \neq \emptyset$ for some $\gamma \in \mathbb{R}$, then $|w| < R$ holds.

Proof. From Lemma 5.7, there exist $R > 0$ such that $\sup_{|w|\geq R}\sup_{\gamma\in\mathbb{R}}(T_{1,\gamma,w}u,u)_2 \leq \frac{1}{2}\|u\|_2^2$. Namely,

$$(T_{1,\gamma,w}u,u)_2 > \frac{1}{2}\|u\|_2^2 \;\Rightarrow\; |w| < R.$$

Next, from the assumption,

$$\|u-v\|_2 < \frac{1}{2}\|u\|_2, \quad \|u - T_{1,\gamma,w}v\|_2 < \frac{1}{2}\|u\|_2$$

holds for some $v \in B_{L^2}\left(u, \frac{1}{2}\|u\|_2\right)$. Therefore, since

$$\begin{aligned}\|T_{1,\gamma,w}u - u\|_2 =&\|T_{1,\gamma,w}u - T_{1,\gamma,w}v + T_{1,\gamma,w}v - u\|_2\\ \leq&\|u-v\|_2 + \|T_{1,\gamma,w}v - u\|_2\\ <&\|u\|_2,\end{aligned}$$

we obtain

$$\begin{aligned}\|u\|_2^2 >&\|T_{1,\gamma,w}u - u\|_2^2 = \|T_{1,\gamma,w}u\|_2^2 - 2(T_{1,\gamma,w}u,u)_2 + \|u\|_2^2\\ =&2\|u\|_2^2 - 2(T_{1,\gamma,w}u,u)_2.\end{aligned}$$

Consequently, we obtain

$$(T_{1,\gamma,w}u,u)_2 > \frac{1}{2}\|u\|_2^2$$

and it implies the conclusion. □

Lemma 5.9. Let $p \in [1, \infty)$. For $u \in L^p(\mathbb{R}^N)$, if $(\gamma, w) \neq (0,0)$ and $T_{1,\gamma,w}u = u$ for some $\gamma \in \mathbb{R}$ and $w \in \mathbb{R}^N$, then $u = 0$.

Proof. If $w = 0$, then $\gamma \neq 0$ and

$$T_{1,\gamma,0}u = u \ \Leftrightarrow\ (e^{-i\gamma} - 1)u = 0.$$

Namely, $u = 0$.

Let $w \neq 0$. Then we may assume $w = e_1$, where e_1 is the standard basis. Since

$$|u(\cdot - e_1)| = |u(\cdot - e_1)e^{i\gamma}| = |T_{1,\gamma,e_1}u| = |u|,$$

$|u|$ is a periodic function of cycle e_1. Therefore, since $u \in L^p(\mathbb{R}^N)$, we obtain $u = 0$. □

Corollary 5.10. For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $\gamma \in (-\pi, \pi]$ and $w \in \mathbb{R}^N$, if $B_{L^2}(u,\delta) \cap T_{1,\gamma,w}B_{L^2}(u,\delta) \neq \emptyset$, then $|\gamma| + |y| < \varepsilon$.

Proof. We prove by contradiction. We assume that

$$B_{L^2}\left(u, \frac{1}{n}\right) \cap T_{1,\gamma_n,w_n}B_{L^2}\left(u, \frac{1}{n}\right) \neq \emptyset, \quad |\gamma_n| + |w_n| \geq \varepsilon$$

for some $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, $\varepsilon > 0$, $\gamma_n \in (-\pi, \pi]$, and $w_n \in \mathbb{R}^N$. Similarly to the proof of Lemma 5.8, we obtain

$$\|T_{1,\gamma_n,w_n}u - u\|_2 < \frac{2}{n}.$$

Since $\frac{1}{n} < \frac{1}{2}\|u\|_2$ if n sufficiently large, $(w_n)_{n\in\mathbb{N}}$ is bounded by Lemma 5.8. Obviously, $(\gamma_n)_{n\in\mathbb{N}}$ is also bounded. Therefore, there exist convergent subsequences $(w_{n_k})_{k\in\mathbb{N}}$ and $(\gamma_{n_k})_{k\in\mathbb{N}}$ and the convergence limits of these sequences is denoted y_0 and γ_0, respectively. Then, from Proposition 5.4,

$$\begin{aligned}\|T_{1,\gamma_0,w_0}u - u\|_2 \leq& \|T_{1,\gamma_0,w_0}u - T_{1,\gamma_{n_k},w_{n_k}}u\|_2 + \|T_{1,\gamma_{n_k},w_{n_k}}u - u\|_2 \\ \to& 0\ (k \to \infty).\end{aligned}$$

Consequently, $T_{1,\gamma_0,w_0}u = u$.

On the other hand, since $|w_{n_k}| + |\gamma_{n_k}| \geq \varepsilon$, we obtain $|w_0| + |\gamma_0| \geq \varepsilon > 0$. Namely, $w_0 \neq 0$ or $\gamma_0 \neq 0$. Therefore, from Lemma 5.9, $u = 0$. It is contradiction. □

Proposition 5.11. Let $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ be $\Lambda u \in L^2(\mathbb{R}^N)$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for $\lambda > 0$, $\gamma \in (-\pi, \pi]$, and $w \in \mathbb{R}^N$, if $B_{H^1}(u,\delta) \cap T_{\lambda,\gamma,w}B_{H^1}(u,\delta) \neq \emptyset$, then $|1-\lambda| + |\gamma| + |w| < \varepsilon$.

Proof. From Corollary 5.7, there exists $\delta_1 > 0$ such that for $\gamma \in (-\pi, \pi]$ and $w \in \mathbb{R}^N$, if $B_{L^2}(u,\delta_1) \cap T_{1,\gamma,w}B_{L^2}(u,\delta_1) \neq \emptyset$, then $|\gamma| + |w| < \frac{\varepsilon}{2}$.

From Proposition 5.4, there exists $\delta_2 > 0$ such that if $|1-\lambda| < \delta_2$, then $\|u - T_{\lambda,0,0}u\|_2 < \frac{\delta_1}{2}$.

From Corollary 5.6, there exists $\delta_3 > 0$ such that for $\lambda > 0$, $\gamma \in (-\pi, \pi]$, and $w \in \mathbb{R}^N$, if $B_{H^1}(u,\delta_3) \cap T_{\lambda,\gamma,w}B_{H^1}(u,\delta_3) \neq \emptyset$, then $|1-\lambda| < \min\left\{\delta_2, \frac{\varepsilon}{2}\right\}$.

Let $\delta := \min\{\frac{\delta_1}{6}, \delta_3\}$. Then, for $\lambda > 0$, $\gamma \in (-\pi, \pi]$, and $w \in \mathbb{R}^N$, if $B_{H^1}(u,\delta) \cap T_{\lambda,\gamma,w}B_{H^1}(u,\delta) \neq \emptyset$, then $|1-\lambda| < \min\left\{\delta_2, \frac{\varepsilon}{2}\right\}$ since $B_{H^1}(u,\delta) \subset B_{H^1}(u,\delta_3)$. Therefore, $\|u - T_{\lambda,0,0}u\|_2 < \frac{\delta_1}{2}$. Moreover, since

$$\begin{aligned}&\|T_{1,\gamma,y}v - u\|_2 \\ =&\|T_{1,\gamma,y}v - T_{1,\gamma,y}u + T_{1,\gamma,y}u - T_{\lambda,\gamma,y}u + T_{\lambda,\gamma,y}u - T_{\lambda,\gamma,y}v + T_{\lambda,\gamma,y}v - u\|_2 \\ \leq& 2\|u - v\|_2 + \|T_{\lambda,\gamma,y}v - u\|_2 + \|u - T_{\lambda,0,0}u\|_2 \\ <& 3\delta + \frac{\delta_1}{2} \\ \leq& \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1\end{aligned}$$

for some $v \in B_{H^1}(u,\delta) \cap T_{\lambda,\gamma,w}B_{H^1}(u,\delta)$, we obtain $|\gamma| + |y| < \frac{\varepsilon}{2}$.

Consequently,

$$|1-\lambda| + |\gamma| + |y| < \min\left\{\delta_2, \frac{\varepsilon}{2}\right\} + \frac{\varepsilon}{2} \leq \varepsilon.$$

□

5.3 Proof of decomposition lemma

Definition 5.12. Let $V_{\delta,l_0} \subset H^1(\mathbb{R}^N) \times (0,\infty)$ define by

$$(u,l) \in V_{\delta,l_0} \;:\Leftrightarrow\; \exists\lambda > 0,\ \gamma \in \mathbb{R},\ w \in \mathbb{R}^N \text{ s.t. } \left\|\lambda^{\frac{N}{2}} u(\lambda\cdot - w)e^{-i\gamma} - Q\right\|_{H^1} < \delta \quad \text{and} \quad \lambda^\alpha l \in (-l_0, l_0)$$

for $\delta, l_0 > 0$.

In this section, we extend Proposition 5.2 to be on V_{δ,l_0}. As a result, we obtain Lemma 2.2.

Proposition 5.13. There exist $\delta', l_0' > 0$ such that for any $u \in B_{H^1}(Q,\delta')$, $l \in (-l_0', l_0')$, $\lambda > 0$, $\gamma \in (-\pi,\pi]$, and $w \in \mathbb{R}^N$, if $T_{\lambda,\gamma,w}u \in B_{H^1}(Q,\delta')$, then

$$\tilde{\lambda}(u,l) = \lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \qquad \tilde{b}(u,l) = \tilde{b}(T_{\lambda,\gamma,w}u, \lambda^\alpha l),$$
$$\tilde{\gamma}(u,l) = \tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma, \qquad \tilde{w}(u,l) = \lambda\tilde{w}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w$$

hold.

Proof. Let $\overline{C} > 0$ be sufficiently small and δ and $l_0 > 0$ be from Proposition 5.2.

Since $\tilde{S}(Q,0) = (1,0,0,0)$, there exist $\delta_1 \in (0,\delta)$ and $l_1 \in (0,l_0)$ such that

$$\tilde{S}(B_{H^1}(Q,\delta_1) \times (-l_1,l_1)) \subset \left(1 - \frac{1}{2}\overline{C}, 1 + \frac{1}{2}\overline{C}\right) \times (-\overline{C},\overline{C}) \times \left(-\frac{1}{3}\overline{C}, \frac{1}{3}\overline{C}\right)^N.$$

Next, let $0 < \lambda < 2^{\frac{1}{\alpha}} - 1$. In particular,

$$|\lambda^\alpha - 1| < 1$$

holds.

From Proposition 5.11, there exist $\delta_2 > 0$ such that for $\lambda > 0$ and $\gamma \in (-\pi,\pi]$, if $B_{H^1}(Q,\delta_2) \cap T_{\lambda,\gamma,w}B_{H^1}(Q,\delta_2) \neq \emptyset$, then $|1-\lambda| + |\gamma| + |w| < \min\left\{\frac{\overline{C}}{2+\overline{C}}, 2^{\frac{1}{\alpha}} - 1\right\}$.

Let $\delta' := \min\{\delta_1,\delta_2\}$ and $l_0' := \frac{l_1}{2}$. Then if $u \in B_{H^1}(Q,\delta')$ is $T_{\lambda,\gamma,w}u \in B_{H^1}(Q,\delta')$, then

$$0 = S(\tilde{S}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), T_{\lambda,\gamma,w}u, \lambda^\alpha l)$$

for $l < l_0'$ since $\lambda^\alpha l < l_1$. Moreover,

$$\begin{aligned}
\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &\in \left(1 - \frac{1}{2}\overline{C}, 1 + \frac{1}{2}\overline{C}\right),\\
\tilde{b}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &\in \left(-\overline{C}, \overline{C}\right),\\
\tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &\in \left(-\frac{1}{2}\overline{C}, \frac{1}{2}\overline{C}\right),\\
\tilde{w}_j(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &\in \left(-\frac{1}{3}\overline{C}, \frac{1}{3}\overline{C}\right)
\end{aligned}$$

hold. Since $\delta' \le \delta_2$, we obtain

$$\begin{aligned}
|1-\lambda| &< \frac{\overline{C}}{2+\overline{C}} \le \frac{\overline{C}}{2-\overline{C}}, \quad \text{i.e. } 1 - \frac{\overline{C}}{2-\overline{C}} < \lambda < 1 + \frac{\overline{C}}{2+\overline{C}},\\
|\gamma| &< \frac{\overline{C}}{2+\overline{C}} \le \frac{1}{2}\overline{C}\\
|w| &< \frac{\overline{C}}{2+\overline{C}} \le \frac{1}{2}\overline{C}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &< \left(1 + \frac{1}{2}\overline{C}\right)\left(1 + \frac{\overline{C}}{2+\overline{C}}\right) = \left(1 + \frac{1}{2}\overline{C}\right)\left(1 + \frac{\frac{1}{2}\overline{C}}{1 + \frac{1}{2}\overline{C}}\right) = 1 + \overline{C},\\
\lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) &> \left(1 - \frac{1}{2}\overline{C}\right)\left(1 - \frac{\overline{C}}{2-\overline{C}}\right) = \left(1 - \frac{1}{2}\overline{C}\right)\left(1 - \frac{\frac{1}{2}\overline{C}}{1 - \frac{1}{2}\overline{C}}\right) = 1 - \overline{C},\\
|\tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma| &\le |\tilde{\gamma}(T_{\lambda,\gamma,w}u)| + |\gamma| < \frac{1}{2}\overline{C} + \frac{1}{2}\overline{C} = \overline{C}\\
|\lambda\tilde{w}_j(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w_j| &\le \left(1 + \frac{\overline{C}}{2+\overline{C}}\right)\frac{1}{3}\overline{C} + \frac{1}{2}\overline{C} < \frac{3}{2}\cdot\frac{1}{3}\overline{C} + \frac{1}{2}\overline{C} = \overline{C},
\end{aligned}$$

i.e., we obtain

$$\lambda^\alpha l \in (-\bar{l}, \bar{l}), \qquad \lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) \in \left(1-\overline{C}, 1+\overline{C}\right),$$
$$\tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma \in \left(-\overline{C}, \overline{C}\right), \qquad \lambda\tilde{w}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w \in \left(-\overline{C}, \overline{C}\right)^N.$$

On the other hand, since

$$\begin{aligned} S(\tilde{S}(u,l), u, l) &= 0 = S(\tilde{S}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), T_{\lambda,\gamma,w}u, \lambda^\alpha l) \\ &= S(\lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{b}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma, \lambda\tilde{w}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w, u, l) \end{aligned}$$

and $\tilde{S}$ is unique, we obtain

$$\tilde{S}(u,l) = \left(\lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{b}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma, \lambda\tilde{w}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w\right).$$

□

Corollary 5.14. Let $\overline{C}, \delta, l_0 > 0$ be sufficiently small. Then the domain of $\tilde{S}$ can be extended to V_{δ,l_0}. In particular, the extension is unique and $\tilde{S}$ is C^1 function, where $\tilde{\gamma}$ is $\mathbb{R}/2\pi\mathbb{Z}$-valued function.

Moreover, for $(u,l) \in V_{\delta,l_0}$ such that $T_{\lambda,\gamma,w}u \in B_{H^1}(Q,\delta)$ and $\lambda^\alpha l \in (-l_0, l_0)$,

$$\left|\frac{\tilde{\lambda}(u,l)}{\lambda} - 1\right| + |\tilde{b}(u,l)| + |\tilde{\gamma}(u,l) - \gamma|_{\mathbb{R}/2\pi\mathbb{Z}} + \left|\frac{\tilde{w}(u,l) - w}{\tilde{\lambda}(u,l)}\right| < \overline{C}$$

holds.

Proof. Firstly, for $(u,l) \in V_{\delta,l_0}$,

$$T_{\lambda,\gamma,w}u \in B_{H^1}(Q,\delta), \quad \lambda^\alpha l \in (-l_0, l_0)$$

hold for some $\lambda > 0$, $\gamma \in (-\pi, \pi]$, and $w \in \mathbb{R}^N$. Then the extension is

$$\tilde{S}(u) := \left(\lambda\tilde{\lambda}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{b}(T_{\lambda,\gamma,w}u, \lambda^\alpha l), \tilde{\gamma}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + \gamma, \lambda\tilde{w}(T_{\lambda,\gamma,w}u, \lambda^\alpha l) + w\right).$$

By Proposition 5.13, this definition is well-defined and the extension is unique. □

Proposition 5.15. Let $\delta, l_0 > 0$ be sufficiently small. Then for any $(u,l) \in V_{\delta,l_0}$, $\frac{\partial\tilde{\lambda}}{\partial u}(u,l)$, $\frac{\partial\tilde{b}}{\partial u}(u,l)$, $\frac{\partial\tilde{\gamma}}{\partial u}(u,l)$, and $\frac{\partial\tilde{w}}{\partial u}(u,l)$ belong to $H^1(\mathbb{R}^N)$.

Proof. Let be $(u,l) \in V_{\delta,l_0}$. Then there exist $\lambda > 0$, $\gamma \in (-\pi, \pi]$, and $w \in \mathbb{R}^N$ such that $T_{\lambda,\gamma,w}u \in B_{H^1}(Q,\delta)$ and $\lambda^\alpha l \in (-l_0, l_0)$.

Firstly, let

$$\begin{pmatrix} S_1(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l) \\ \vdots \\ S_{N+3}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l) \end{pmatrix} := S(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l) = \begin{pmatrix} \left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), i\Lambda P(\tilde{\lambda}, \tilde{b}, l)\right)_2 \\ \left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), |y|^2 P(\tilde{\lambda}, \tilde{b}, l)\right)_2 \\ \left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), i\rho\right)_2 \\ \left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), y_1 P(\tilde{\lambda}, \tilde{b}, l)\right)_2 \\ \vdots \\ \left(\tilde{\varepsilon}(\tilde{\lambda}, \tilde{b}, \tilde{\gamma}, \tilde{w}, u, l), y_N P(\tilde{\lambda}, \tilde{b}, l)\right)_2 \end{pmatrix}.$$

Then, since

$$0 = S(\tilde{\lambda}(u,l), \tilde{b}(u,l), \tilde{\gamma}(u,l), \tilde{w}(u,l), u, l),$$

we obtain

$$0 = \frac{\partial\tilde{\lambda}}{\partial u}\frac{\partial S}{\partial\tilde{\lambda}} + \frac{\partial\tilde{b}}{\partial u}\frac{\partial S}{\partial\tilde{b}} + \frac{\partial\tilde{\gamma}}{\partial u}\frac{\partial S}{\partial\tilde{\gamma}} + \frac{\partial\tilde{w}_1}{\partial u}\frac{\partial S}{\partial\tilde{w}_1} + \cdots + \frac{\partial\tilde{w}_N}{\partial u}\frac{\partial S}{\partial\tilde{w}_N} + \frac{\partial S}{\partial u}.$$

Namely,

$$-\frac{\partial S}{\partial u} = A\begin{pmatrix} \frac{\partial\tilde{\lambda}}{\partial u} \\ \frac{\partial\tilde{b}}{\partial u} \\ \frac{\partial\tilde{\gamma}}{\partial u} \\ \frac{\partial\tilde{w}_1}{\partial u} \\ \vdots \\ \frac{\partial\tilde{w}_N}{\partial u} \end{pmatrix},$$

where

$$A := \begin{pmatrix} \frac{\partial S_1}{\partial \tilde{\lambda}} & \frac{\partial S_1}{\partial \tilde{b}} & \frac{\partial S_1}{\partial \tilde{\gamma}} & \frac{\partial S}{\partial \tilde{w}_1} & \cdots & \frac{\partial S_1}{\partial \tilde{w}_N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial S_{N+3}}{\partial \tilde{\lambda}} & \frac{\partial S_{N+3}}{\partial \tilde{b}} & \frac{\partial S_{N+3}}{\partial \tilde{\gamma}} & \frac{\partial S}{\partial \tilde{w}_1} & \cdots & \frac{\partial S_{N+3}}{\partial \tilde{w}_N} \end{pmatrix}.$$

Since ${T_{\lambda,\gamma,w}}^{-1} = T_{\frac{1}{\lambda},-\gamma,-\frac{w}{\lambda}}$, each element of the $\frac{\partial S}{\partial u}(\tilde{S}(u,l),u,l)$ belongs to $H^1(\mathbb{R}^N)$. Therefore, it is sufficient to show that $\det A \neq 0$ on V_{δ,l_0}.

Let ϵ be sufficiently small and $\overline{C}$, δ, and l_0 be recast as sufficiently small according to ϵ. Then, by directly calculation, we obtain

$$\begin{aligned} \frac{\partial S_j}{\partial \tilde{\lambda}}(\tilde{\lambda}(u,l),\tilde{b}(u,l),\tilde{\gamma}(u,l),\tilde{w}(u,l),u,l) &= -\frac{1}{\tilde{\lambda}(u,l)}\||y|Q\|_2^2\delta_{2,j} + O\left(\frac{\epsilon}{\tilde{\lambda}(u,l)}\right), \\ \frac{\partial S_j}{\partial \tilde{b}}(\tilde{\lambda}(u,l),\tilde{b}(u,l),\tilde{\gamma}(u,l),\tilde{w}(u,l),u,l) &= -\frac{1}{4}\||y|Q\|_2^2\delta_{1,j} + O(\epsilon), \\ \frac{\partial S_j}{\partial \tilde{\gamma}}(\tilde{\lambda}(u,l),\tilde{b}(u,l),\tilde{\gamma}(u,l),\tilde{w}(u,l),u,l) &= -\frac{1}{2}\||y|Q\|_2^2\delta_{3,j} + O(\epsilon), \\ \frac{\partial S_j}{\partial \tilde{w}_k}(\tilde{\lambda}(u,l),\tilde{b}(u,l),\tilde{\gamma}(u,l),\tilde{w}(u,l),u,l) &= \frac{1}{2\tilde{\lambda}(u,l)}\|Q\|_2^2\delta_{k,j-3} + O\left(\frac{\epsilon}{\tilde{\lambda}(u,l)}\right). \end{aligned}$$

Therefore, we obtain

$$\tilde{\lambda}(u,l)^{N+1}\det A(u,l) = -\frac{1}{2^{N+3}}\||y|Q\|_2^6\|Q\|_2^N + O(\epsilon),$$

i.e., A is regular on V_{δ,l_0}. □

proof of Lemma 2.2. Let $\bar{l} > 0$ be sufficiently small. Then $\lambda(t)^\alpha < \bar{l}^\alpha < l_0$. We define

$$\tilde{\lambda}(t) := \tilde{\lambda}(u(t),1), \quad \tilde{b}(t) := \tilde{b}(u(t),1), \quad \tilde{\gamma}(t) := \tilde{\gamma}(u(t),1), \quad \tilde{w}(t) := \tilde{w}(u(t),1).$$

Then, by Corollary 5.14, the existence and uniqueness of the decomposition follows.

In addition, by Proposition 5.15, $\tilde{\lambda}$, $\tilde{b}$, $\tilde{\gamma}$, and $\tilde{w}$ are C^1 functions. □

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