学位論文

Studies on support τ-tilting modules and related objects for blocks of finite groups

(有限群のブロック上の台τ傾加群及び関連する対象に関する研究)

2023年3月

小塩　遼太郎

Acknowledgments

I would like to express my appreciation to my supervisor Professor Naoko Kunugi for giving me an interesting and attractive topic, and for her encouragement and support during the last 6 years including my master studies. I was able to learn many things including mathematics from her.

I would also like to thank Professor Yuta Kozakai for his continuous support and thoughtful guidance throughout the studies. Without him, I could not have completed this thesis.

I am grateful to the members of the thesis committee, Professor Naoko Kunugi, Professor Masanari Kida, Professor Katsunori Sanada, Professor Takao Satoh and Professor Hiroshi Sekigawa for sparing their precious time for my thesis defense and for helpful comments.

Finally, I am deeply thankful to my dear family and my dear wife.

Ryotaro KOSHIO

Contents

Chapter 1

Introduction

This thesis is based on [**14**] and [**13**].

The study of derived equivalences of blocks of finite groups has been motived and inspired by "Broué's conjecture", which can be conceived of as a local-global principle in the modular representation theory of finite groups. In [**16**], a solution to the problem of finding a derived equivalence of two given algebras was reduced to the problem of finding an appropriate tilting complex. Therefore, abundant constructions of tilting complexes over blocks enable us to find algebras which are derived equivalent to the blocks. Of course, it is very hard to construct appropriate tilting complexes over blocks and to determine all tilting complexes over blocks. The classes of tilting complexes called two-term tilting complexes are considered to be non-trivial and a bit easier to handle because it is showed that there exists a one-to-one correspondence between the two-term tilting complexes and the support τ-tilting modules over symmetric algebras in [**2**]. Abundant constructions of two-term tilting complexes over blocks are also useful for plenty of constructions of general tilting complexes over blocks by using the tilting mutations introduced in [**3**]. Therefore, we focus on support τ-tilting modules and consequently, we got some results which work effectively for the purpose stated above.

In order to describe these, we set notation as follows: Let k be an algebraically closed field of characteristic $p > 0$, $\tilde{G}$ a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B, that is, $1_B 1_{\tilde{B}} \neq 0$, where 1_B and $1_{\tilde{B}}$ mean the respective identity elements of B and $\tilde{B}$. In this setting, there are some useful properties about the restriction functor $\mathrm{Res}_G^{\tilde{G}}$ and the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ between the category of B-modules and the one of $\tilde{B}$-modules. We denote the inertial group of the block B in $\tilde{G}$ by $I_{\tilde{G}}(B)$. We say that a B-module U is $I_{\tilde{G}}(B)$-invariant if $xU \cong U$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Furthermore, we use the following notation: For modules or complexes X and X', we write $X \sim_{\mathrm{add}} X'$ if $\operatorname{add} X = \operatorname{add} X'$. Then the relation $\sim_{\mathrm{add}}$ is an equivalence relation.

- $\mathrm{s}\tau\text{-tilt}\, B$ (or $\mathrm{s}\tau\text{-tilt}\, \tilde{B}$) means the set of equivalence classes of support τ-tilting modules over B (or $\tilde{B}$, respectively) under the equivalence relation $\sim_{\mathrm{add}}$,

- 2-tilt B (or 2-tilt $\tilde{B}$) means the set of equivalence classes of two-term tilting complexes in $K^b(B\text{-proj})$ (or $K^b(\tilde{B}\text{-proj})$, respectively) under the equivalence relation $\sim_{\mathrm{add}}$,

The following results, which is proved in Chapter 4, contribute to classify support τ-tilting $\tilde{B}$-modules and two-term tilting complexes in $K^b(\tilde{B}\text{-proj})$.

Main Theorem 1 (see Theorem 4.2.5)**.** Assume that $\tilde{G}/G$ is a p-group and that B satisfies the following conditions:

(I) Any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant.

(II) The block B is τ-tilting finite (i.e., $\#\,\mathrm{s}\tau\text{-tilt}\,B < \infty$).

Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an isomorphism from $\mathrm{s}\tau\text{-tilt}\,B$ to $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ as partially ordered sets.

Main Theorem 2 (see Corollary 4.2.6)**.** Assume that $\tilde{G}/G$ is a p-group and that B satisfies the conditions (I) and (II) in Theorem 1. Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an isomorphism 2-tilt $B \cong$ 2-tilt $\tilde{B}$ of partially ordered sets which commutes the following diagram of partially ordered sets

$$\begin{array}{ccc} \mathrm{s}\tau\text{-tilt}\,B & \xrightarrow[\mathrm{Ind}_G^{\tilde{G}}]{\sim} & \mathrm{s}\tau\text{-tilt}\,\tilde{B} \\ \wr\downarrow & \circlearrowleft & \downarrow\wr \\ 2\text{-tilt}\,B & \xrightarrow[\sim]{\mathrm{Ind}_G^{\tilde{G}}} & 2\text{-tilt}\,\tilde{B} \end{array}$$

where the vertical maps are isomorphisms given by [**2**, Theorem 3.2].

If B has a cyclic defect group, then the conditions (I) and (II) hold for B automatically (see Lemma 4.1.3). Moreover, in that case the block B is a Brauer tree algebra or simple algebra, thus the number of elements in $\mathrm{s}\tau\text{-tilt}\,B$ is equal to $\binom{2e}{e}$, where e is the number of isomorphism classes of simple B-modules and $\binom{2e}{e}$ means the binomial coefficient ([**5**], [**6**]). Combining Main Theorem 1 with these facts, we get the following.

Main Theorem 3. Assume that $\tilde{G}/G$ is a p-group and a block B of kG has a cyclic defect group. Then $\mathrm{s}\tau\text{-tilt}\,B$ and $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ are isomorphic as partially ordered sets. In particular, we get $\#\,\mathrm{s}\tau\text{-tilt}\,\tilde{B} = \binom{2e}{e}$ where e is the number of isomorphism classes of simple B-modules.

In Chapter 5, we investigate inertial-invariant support τ-tilting modules over blocks of finite group and present the following results, which relaxes the assumptions in Main Theorem 1.

Main Theorem 4 (see Theorems 5.1.1 and 5.1.2)**.** Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a support τ-tilting B-modules satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then the induced module $\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $k\tilde{G}$-module. In particular, the module $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $\tilde{B}$-module.

We will demonstrate that there is a relation between $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules and support τ-tilting $k\tilde{G}$-modules. Now we recall that the set $\mathrm{s}\tau\text{-tilt}\, B$ of support τ-tilting module has a partially ordered set structure (see Definition-Proposition 2.2.1).

Main Theorem 5 (see Theorem 5.1.5)**.** Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a B-modules satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then M is a support τ-tilting B-module if and only if $\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $k\tilde{G}$-module. Moreover, for any two $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules M and M', $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\, B$ if and only if $\mathrm{Ind}_G^{\tilde{G}} M \geq \mathrm{Ind}_G^{\tilde{G}} M'$ in $\mathrm{s}\tau\text{-tilt}\, k\tilde{G}$.

In this paper, we use the following notation. Modules mean finitely generated left modules and complexes mean cochain complexes. For a finite dimensional algebra Λ over a field k and a Λ-module M, we denote by $\mathrm{Rad}(M)$ the Jacobson radical of M, by $P(M)$ the projective cover of M, by $\Omega(M)$ the syzygy of M and τM the Auslander–Reiten translate of M. We denote by Λ-mod the module category of Λ and by $K^b(\Lambda\text{-proj})$ the homotopy category consisting of bounded complexes of projective Λ-modules. For an object X of Λ-mod (or of $K^b(\Lambda\text{-proj})$), we denote by $\mathrm{add}\, X$ the full subcategory of Λ-mod (or of $K^b(\Lambda\text{-proj})$, respectively) whose objects are finite direct sums of direct summands of X. We say that X is basic if any two indecomposable direct summands of X are non-isomorphic.

Chapter 2

Preliminaries for τ-tilting theory

In this chapter, let k be an algebraically closed field and Λ a finite dimensional k-algebra. We denote by τ the Auslander–Reiten translation. For a Λ-module M, we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M.

2.1. Support τ-tilting modules and mutations

In this subsection, we recall some definitions and basic properties of support τ-tilting modules.

Definition 2.1.1 ([**2**, Definition 0.1])**.** Let M be a Λ-module.

(1) We say that M is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

(2) We say that M is *τ-tilting* if M is a τ-rigid module and $|M| = |\Lambda|$.

(3) We say that M is *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $\Lambda/\Lambda e\Lambda$-module.

Definition 2.1.2 ([**2**, Definition 0.3])**.** Let M be a Λ-module and P a projective Λ-module.

(1) We say that the pair (M, P) is *τ-rigid* if M is τ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$.

(2) We say that the pair (M, P) is *support τ-tilting* (or *almost complete support τ-tilting*) if the pair (M, P) is τ-rigid and $|M| + |P| = |\Lambda|$ (or $|M| + |P| = |\Lambda| - 1$, respectively).

Remark 2.1.3 ([**1**, Proposition 2.3 (a), (b)])**.** Since $e = 0$ is an idempotent of Λ and $\Lambda/\Lambda e\Lambda = \Lambda$, any τ-tilting module is a support τ-tilting module. Moreover, for any τ-rigid Λ-module M, the following conditions are equivalent:

(1) M is a support τ-tilting module.

(2) There exists a projective Λ-module P satisfying $\mathrm{Hom}_\Lambda(P, M) = 0$ and $|M|+|P| = |\Lambda|$, that is, (M, P) is a support τ-tilting pair.

Proposition 2.1.4 ([**2**, Proposition 2.3])**.** Let (M, P) be a pair with a Λ-module M and a projective Λ-module P. Let e be an idempotent of Λ such that $\mathrm{add}\, P = \mathrm{add}\, \Lambda e$.

(1) The pair (M, P) is a τ-rigid (or support τ-tilting) pair if and only if M is a τ-rigid (or τ-tilting, respectively) $\Lambda/\Lambda e\Lambda$-module.

(2) If (M, P) and (M, Q) are support τ-tilting pairs for some projective Λ-module Q, then $\mathrm{add}\, P = \mathrm{add}\, Q$.

Proposition 2.1.5 ([**2**, Corollary 2.13])**.** Let M be a τ-rigid Λ-module and P a projective Λ-module satisfying that $\mathrm{Hom}_\Lambda(P, M) = 0$. Then the following conditions are equivalent:

(1) $|M| + |P| = |\Lambda|$, that is, M is a support τ-tilting Λ-module (see Remark 2.1.3).

(2) If $\mathrm{Hom}_\Lambda(M, \tau X) = 0$, $\mathrm{Hom}_\Lambda(X, \tau M) = 0$ and $\mathrm{Hom}_\Lambda(P, X) = 0$, then $X \in \mathrm{add}\, M$ for any Λ-module X.

The following proposition plays an important role in the proof of our main result.

Proposition 2.1.6 ([**11**, Proposition 2.14])**.** Let Λ be a finite dimensional k-algebra and M a τ-rigid Λ-module. Then M is a support τ-tilting Λ-module if and only if there exists an exact sequence

$$\Lambda \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

in Λ-mod with $M', M'' \in \mathrm{add}\, M$ and f a left $\mathrm{add}\, M$-approximation of Λ, that is, the map

$$\mathrm{Hom}_\Lambda(M', X) \xrightarrow{\bullet \circ f} \mathrm{Hom}_\Lambda(\Lambda, X)$$

is surjective for any $X \in \mathrm{add}\, M$.

The following proposition gives the definitions of mutations of support τ-tilting pairs and support τ-tilting modules. We recall that we say a pair (V, Q) of a Λ-module V and a projective Λ-module Q is basic if V and Q are basic.

Proposition 2.1.7 ([**2**, Theorem 2.18])**.** If (V, Q) is a basic almost complete support τ-tilting pair, then there exist exactly two basic support τ-tilting pairs containing (V, Q) as a direct summand.

Definition 2.1.8 ([**2**, Definition 2.19])**.** Let (M, P) be a basic support τ-tilting pair and X be an indecomposable summand of either M or P. Let (V, Q) be the basic almost complete support τ-tilting pair satisfying either $M \cong V \oplus X$ or $P \cong Q \oplus X$. By Proposition 2.1.7, there exist a unique basic support τ-tilting pair (M', P') distinct to (M, P) and having (V, Q) as a direct summand. We denote this support τ-tilting pair (M', P') by $\mu_X(M, P)$ and it is called a *mutation* of (M, P) with respect to X.

Definition 2.1.9 ([**2**, Definition 2.19])**.** Let M be a basic support τ-tilting module. According to Proposition 2.1.4, we can find the basic projective Λ-module P such that (M, P) is a basic support τ-tilting pair. Let X be an indecomposable summand of either M or P. Let (M', P') be the mutation of (M, P) with respect to X. We denote M' by $\mu_X(M)$ and it is called a *mutation* of M with respect to X.

2.2. Poset structures and connections with silting theory

For support τ-tilting Λ-modules M and M', we write $M \sim_{\text{add}} M'$ if $\operatorname{add} M = \operatorname{add} M'$. Then the relation $\sim_{\text{add}}$ is an equivalence relation. We denote $\text{s}\tau\text{-tilt}\,\Lambda$ the set of equivalence classes of all support τ-tilting Λ-modules under the equivalence relation $\sim_{\text{add}}$. We remark that basic support τ-tilting Λ-modules form a set of representatives of $\text{s}\tau\text{-tilt}\,\Lambda$.

Definition-Proposition 2.2.1 ([**2**, Theorem 2.7])**.** For $M, M' \in \text{s}\tau\text{-tilt}\,\Lambda$, we write $M \geq M'$ if there exist a positive integer r and an epimorphism

$$M^{\oplus r} \xrightarrow{\varphi} M'.$$

Then we get a partial order on $\text{s}\tau\text{-tilt}\,\Lambda$.

Theorem 2.2.2 ([**2**, Theorem 2.32])**.** Let M and M' be support τ-tilting Λ-modules. Then the following conditions are equivalent:

(1) M and M' are mutation of each other, and $M > M'$.

(2) $M > M'$ and there is no support τ-tilting Λ-module L such that $M > L > M'$.

We denote $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ the Hasse quiver (Hasse diagram) for the partially ordered set $\text{s}\tau\text{-tilt}\,\Lambda$. The theorem above implies that any arrow in $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ corresponds to a support τ-tilting mutation. We remark that the underlying graph of $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ is a $|\Lambda|$-regular graph because we can take $|\Lambda|$ sorts of mutations for any support τ-tilting module M. The next proposition plays an important role to prove our main theorems.

Proposition 2.2.3 ([**2**, Corollary 2.38])**.** If $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ has a connected component having finite vertices, then $\mathcal{H}(\text{s}\tau\text{-tilt}\,\Lambda)$ is connected.

Now we recall the definition of silting complexes which is a generalization of tilting complexes. The concept of silting complex is originated from [**12**], and recently there has been many papers starting with [**3**]. In particular, in [**2**], it is shown that there is a one-to-one correspondence between two-term silting complexes and support τ-tilting modules.

Definition 2.2.4. Let T be a complex in $K^b(\Lambda\text{-proj})$.

(1) We say that T is *presilting* (or *pretilting*) if $\operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T, T[i]) = 0$ for any $i > 0$ (or for any $i \neq 0$, relatively).

(2) We say that T is *silting* (or *tilting*) if it is presilting (or pretilting, respectively) and satisfies $\operatorname{thick} T = K^b(\Lambda\text{-proj})$, where $\operatorname{thick} T$ is the full subcategory of $K^b(\Lambda\text{-proj})$ generated by $\operatorname{add} T$ as a triangulated category.

For silting complexes (tilting complexes) T and T' in $K^b(\Lambda\text{-proj})$, we write $T \sim_{\text{add}} T'$ if $\operatorname{add} T = \operatorname{add} T'$. Then the relation $\sim_{\text{add}}$ is an equivalence relation. We denote $\operatorname{silt} \Lambda$ ($\operatorname{tilt} \Lambda$) the set of equivalence classes of all silting complexes (tilting complexes) in $K^b(\Lambda\text{-proj})$ under the equivalence relation $\sim_{\text{add}}$. We remark that basic silting complexes (basic tilting complexes) in $K^b(\Lambda\text{-proj})$ form a set of representatives of $\operatorname{silt} \Lambda$ ($\operatorname{tilt} \Lambda$).

Definition 2.2.5 ([**3**, Definition 2.10])**.** For $T, T' \in \operatorname{silt} \Lambda$, we write $T \geq T'$ if

$$\operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T, T'[i]) = 0,$$

for any $i > 0$. Then we get a partial order on $\operatorname{silt} \Lambda$.

Definition 2.2.6. We say that a complex $T \in K^b(\Lambda\text{-proj})$ is *two-term* if $T^i = 0$ for all $i \neq 0, -1$. We denote by $2\text{-silt}\,\Lambda$ the subset of $\operatorname{silt} \Lambda$ consisting of all equivalent classes of two-term silting complexes in $K^b(\Lambda\text{-proj})$.

Theorem 2.2.7 ([**2**, Theorem 3.2 and Corollary 3.9])**.** There is an isomorphism

$$\text{s}\tau\text{-tilt}\,\Lambda \to 2\text{-silt}\,\Lambda$$

of partially ordered sets given by $\text{s}\tau\text{-tilt}\,\Lambda \ni (M, P) \mapsto (T_1 \oplus P \xrightarrow{(f_1\ 0)} T_0) \in 2\text{-silt}\,\Lambda$, where $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$ is a minimal projective presentation of M.

We remark that the correspondence above commutes with support τ-tilting mutations and silting mutations [**2**, Corollary 3.9].

Remark 2.2.8 ([**3**, Example 2.8])**.** If Λ is a finite dimensional symmetric k-algebra, then any silting complex over Λ is in fact a tilting complex.

Proof. Let T be a silting complex over Λ. We obtain from [**10**, Lemma 3.1] that the following isomorphisms:

$$\begin{aligned} D\operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T, T[-i]) &\cong \operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T[-i], \nu T) \\ &\cong \operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T[-i], T) \\ &\cong \operatorname{Hom}_{K^b(\Lambda\text{-proj})}(T, T[i]) \\ &= 0 \end{aligned}$$

for any $i > 0$ where D is the k-duality and ν is the Nakayama functor, which is naturally isomorphic to the identity functor on Λ-proj by the assumption, that Λ is symmetric. □

By this fact, silting complexes over the group algebra and over the blocks of group algebras are tilting complexes in fact. Hence, the classifications of support τ-tilting modules over the group algebra and over the blocks mean the ones of two-term tilting complexes over them.

Chapter 3

Preliminaries for modular representation theory of finite groups

In this chapter, let k be an algebraically closed field of characteristic $p > 0$. For any finite group G, the field k can always be regarded as a kG-module by defining $gx = x$ for any $g \in G$ and $x \in k$. This module is called the trivial module and is denoted by k_G. For kG-modules U and V, the k-module $U \otimes V = U \otimes_k V$ has a kG-module structure given by $g(u \otimes v) = gu \otimes gv$ for all $g \in G$, $u \in U$ and $v \in V$.

3.1. Restriction functors and induction functors

Let G be a finite group and H a subgroup of G. We denote by Res^G_H the restriction functor from kG-mod to kH-mod and $\mathrm{Ind}^G_H := {}_{kG}kG \otimes_{kH} \bullet$ the induction functor from kH-mod to kG-mod. The functors Res^G_H and Ind^G_H are exact functors and have the following properties.

Proposition 3.1.1 (see [**4**, Lemma 8.5, Lemma 8.6])**.** Let G be a finite group, K a subgroup of G, H a subgroup of K, U a kG-module and V a kH-module. Then the following hold:

(1) $\mathrm{Res}^K_H \mathrm{Res}^G_K \cong \mathrm{Res}^G_H$.

(2) $\mathrm{Ind}^G_K \mathrm{Ind}^K_H \cong \mathrm{Ind}^G_H$.

(3) The functors Res^G_H and Ind^G_H are left and right adjoint to each other.

(4) The functors Res^G_H and Ind^G_H send projective modules to projective modules.

(5) $U \otimes k_G \cong U$.

(6) $\mathrm{Ind}^G_H(\mathrm{Res}^G_H U \otimes V) \cong U \otimes \mathrm{Ind}^G_H V$.

Let H be a subgroup of G and U a kH-module. For $g \in G$, we define a $k[gHg^{-1}]$-module gU consisting of symbols gu, where $u \in U$, as a set and its $k[gHg^{-1}]$-module structure is given by $gu + gu' := g(u + u')$, $s(gu) := g(su)$ and $ghg^{-1}(gu) := g(hu)$ for any $u, u' \in U$, $s \in k$ and $ghg^{-1} \in gHg^{-1}$. We remark that if H is a normal subgroup of G, then gU is also a kH-module.

Remark 3.1.2. For finite groups $H \subset K \subset G$ and kG-module U, we have $g\mathrm{Res}^K_H U \cong \mathrm{Res}^{gKg^{-1}}_{gHg^{-1}} gU$ by the easy calculation.

Let H and K be subgroups of G. We denote by $[G/H]$, $[K\backslash G]$ and $[K\backslash G/H]$ sets of representatives of G/H, $K\backslash G$ and $K\backslash G/H$, respectively.

Theorem 3.1.3 (Mackey's decomposition formula)**.** Let H and K be subgroups of G, and U a kH-module. Then we have

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H U \cong \bigoplus_{g \in [K\backslash G/H]} \mathrm{Ind}^K_{K\cap gHg^{-1}} \mathrm{Res}^{gHg^{-1}}_{K\cap gHg^{-1}} gU.$$

Remark 3.1.4. Let N be a normal subgroup of G and H a subgroup of G containing N. Then $N\backslash G/H = G/H$ and for any kH-module U,

$$\mathrm{Res}^G_N \mathrm{Ind}^G_H U \cong \bigoplus_{g \in [G/H]} g\mathrm{Res}^H_N U.$$

In particular, if $N = H$ then

$$\mathrm{Res}^G_N \mathrm{Ind}^G_N U \cong \bigoplus_{g \in [G/N]} gU.$$

Let G be a normal subgroup of a finite group $\tilde{G}$. For a kG-module U, we denote by $I_{\tilde{G}}(U)$ the inertial group of U in $\tilde{G}$, that is

$$I_{\tilde{G}}(U) := \left\{ x \in \tilde{G} \;\middle|\; xU \cong U \text{ as } kG\text{-modules} \right\}.$$

Theorem 3.1.5 (Clifford's Theorem for simple modules)**.** Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, S a simple $k\tilde{G}$-module and S' a simple kG-submodule of $\mathrm{Res}^G_N S$. Then we have a kG-module isomorphism

$$\mathrm{Res}^{\tilde{G}}_G S \cong \bigoplus_{x \in [\tilde{G}/I_{\tilde{G}}(S')]} xS'^{\oplus r}$$

for some integer r, which is called the ramification index of S in $\tilde{G}$.

From now on, we will consider the case where $\tilde{G}/G$ is a p-group. The following theorem makes substantial contribution in this paper.

Theorem 3.1.6 (Green's indecomposability theorem [**7**])**.** If G is a normal subgroup of $\tilde{G}$ such that $\tilde{G}/G$ is a p-group, then $\mathrm{Ind}_G^{\tilde{G}}V$ is an indecomposable $k\tilde{G}$-module for any indecomposable kG-module V.

Lemma 3.1.7. Let G be a normal subgroup of $\tilde{G}$. For indecomposable kG-modules U and U', if the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is isomorphic to $\mathrm{Ind}_G^{\tilde{G}}U'$, then U is isomorphic to xU' for some $x \in \tilde{G}$.

Proof. Let U and U' be indecomposable kG-modules with $\mathrm{Ind}_G^{\tilde{G}}U$ is isomorphic to $\mathrm{Ind}_G^{\tilde{G}}U'$. Then we have

$$\bigoplus_{x\in[\tilde{G}/G]} xU \cong \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}U \cong \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}U' \cong \bigoplus_{x\in[\tilde{G}/G]} xU'.$$

By the Krull–Schmidt Theorem, we get $U \cong xU'$ for some $x \in \tilde{G}$. □

Corollary 3.1.8. Let G be a normal subgroup of $\tilde{G}$. For indecomposable $\tilde{G}$-invariant kG-modules U and U', if the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is isomorphic to $\mathrm{Ind}_G^{\tilde{G}}U'$, then U is isomorphic to U'.

Proposition 3.1.9 (see [**4**, Exercise 19.1])**.** Let G be a normal subgroup of $\tilde{G}$ and T a simple kG-module such that $I_{\tilde{G}}(T) = \tilde{G}$. If $\tilde{G}/G$ is a p-group, then there exists a unique simple $k\tilde{G}$-module S such that $\mathrm{Res}_G^{\tilde{G}}S \cong T$.

Lemma 3.1.10. Let G be a normal subgroup of $\tilde{G}$ and S a simple $k\tilde{G}$-module. Suppose that $\tilde{G}/G$ is a p-group, then S is the only simple $k\tilde{G}$-module which can be a composition factor of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}S$.

Proof. We remark that the group algebra of any p-group over k is a local k-algebra (for example, see [**4**, Corollary 3.3]). For this reason, it is only trivial module k_G that can be composition factor of ${}_{k\tilde{G}}k(\tilde{G}/G)$. Hence, we get the following isomorphisms as $k\tilde{G}$-modules:

$$\begin{aligned}\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}S &\cong \mathrm{Ind}_G^{\tilde{G}}((\mathrm{Res}_G^{\tilde{G}}S) \otimes k_G)\\ &\cong S \otimes \mathrm{Ind}_G^{\tilde{G}}k_G\\ &\cong S \otimes k(\tilde{G}/G).\end{aligned}$$

Therefore, all composition factors of the module in the right-hand side are isomorphic to $S \otimes k_{\tilde{G}} \cong S$. □

Corollary 3.1.11. Let T be a simple kG-module. Suppose that $\tilde{G}/G$ is a p-group, then the $k\tilde{G}$-module $\mathrm{Ind}_G^{\tilde{G}}T$ has only one sort of simple module which can be a composition factor.

Proof. We can take a simple $k\tilde{G}$-module S satisfying the condition $\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}T, S) \neq 0$. Since $\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}T, S) \cong \mathrm{Hom}_{kG}(T, \mathrm{Res}_G^{\tilde{G}}S)$ by Theorem 3.1.1, the restriction module $\mathrm{Res}_G^{\tilde{G}}S$ has a submodule isomorphic to T. Hence, the induced module $\mathrm{Ind}_G^{\tilde{G}}T$ is isomorphic to a submodule of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}}S$. Therefore, by Lemma 3.1.10, the conclusion follows. □

Lemma 3.1.12. Let G be a normal subgroup of $\tilde{G}$ such that $\tilde{G}/G$ is a p-group and S a simple $k\tilde{G}$-module. Assume that $\mathrm{Res}_G^{\tilde{G}}S$ is a simple kG-module and denote this by T. Then the following hold:

(1) $I_{\tilde{G}}(P(T)) = \tilde{G}$,

(2) $\mathrm{Ind}_G^{\tilde{G}}P(T) \cong P(S)$,

(3) $\mathrm{Res}_G^{\tilde{G}}P(S) \cong P(T)^{\oplus|\tilde{G}:G|}$.

Proof. The assumption implies that $I_{\tilde{G}}(T) = \tilde{G}$ by Theorem 3.1.5, which implies $xT \cong T$. Hence, for any $x \in \tilde{G}$, we have that $xP(T) \cong P(xT) \cong P(T)$ and the first assertion is proved. Since the induced module $\mathrm{Ind}_G^{\tilde{G}}P(T)$ is an indecomposable projective module by Theorem 3.1.1 and Theorem 3.1.6, and

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}P(T), S) \cong \mathrm{Hom}_{kG}(P(T), \mathrm{Res}_G^{\tilde{G}}S) = \mathrm{Hom}_{kG}(P(T), T) \neq 0,$$

the second assertion is proved. The third assertion is trivial by previous two assertions and Theorem 3.1.3. □

Lemma 3.1.13. Let G be a normal subgroup of a finite group $\tilde{G}$ and M a kG-module satisfying $xM \cong M$ as kG-modules for any $x \in \tilde{G}$. Then the following hold:

(1) $xP(M) \cong P(M)$ for any $x \in \tilde{G}$.

(2) $x\Omega(M) \cong \Omega(M)$ for any $x \in \tilde{G}$.

(3) $\mathrm{Ind}_G^{\tilde{G}}\Omega(M) \cong \Omega(\mathrm{Ind}_G^{\tilde{G}}M)$.

(4) $\tau(\mathrm{Ind}_G^{\tilde{G}}M) \cong \mathrm{Ind}_G^{\tilde{G}}\tau M$.

Proof. For any $x \in \tilde{G}$, we have an isomorphism $\phi\colon xM \to M$ by the assumption. We consider the following commutative diagram in kG-mod with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & x\Omega(M) & \longrightarrow & xP(M) & \xrightarrow{x\pi_M} & xM & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi''} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & \Omega(M) & \longrightarrow & P(M) & \xrightarrow[\pi_M]{} & M & \longrightarrow & 0.
\end{array}$$

Since π_M is an essential epimorphism and ϕ is an isomorphism, the vertical morphisms ϕ' and ϕ'' are isomorphisms and so (1) and (2) holds.

By Proposition 3.1.1 (4), we have the following commutative diagram in $k\tilde{G}$-mod with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ind}_G^{\tilde{G}}\Omega(M) & \longrightarrow & \mathrm{Ind}_G^{\tilde{G}}P(M) & \xrightarrow{\mathrm{Ind}_G^{\tilde{G}}\pi_M} & \mathrm{Ind}_G^{\tilde{G}}M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}_{\mathrm{Ind}_G^{\tilde{G}}M}} & & \\ 0 & \longrightarrow & \Omega(\mathrm{Ind}_G^{\tilde{G}}M) & \longrightarrow & P(\mathrm{Ind}_G^{\tilde{G}}M) & \xrightarrow[\pi_{\mathrm{Ind}_G^{\tilde{G}}M}]{} & \mathrm{Ind}_G^{\tilde{G}}M & \longrightarrow & 0. \end{array}$$

Since $\pi_{\mathrm{Ind}_G^{\tilde{G}}M}$ is an essential epimorphism, we have that the vertical morphisms φ and φ' are split epimorphisms, that $\mathrm{Ker}\,\varphi \cong \mathrm{Ker}\,\varphi'$ are projective $k\tilde{G}$-modules and that $\Omega(\mathrm{Ind}_G^{\tilde{G}}M) \oplus \mathrm{Ker}\,\varphi' \cong \mathrm{Ind}_G^{\tilde{G}}\Omega(M)$. By Theorem 3.1.3 and (2), we have

$$\begin{aligned} \Omega(M)^{\oplus|\tilde{G}:G|} &\cong \bigoplus_{x\in[\tilde{G}/G]} x\Omega(M) \\ &\cong \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}\Omega(M) \\ &\cong \mathrm{Res}_G^{\tilde{G}}\Omega(\mathrm{Ind}_G^{\tilde{G}}M) \oplus \mathrm{Res}_G^{\tilde{G}}\,\mathrm{Ker}\,\varphi'. \end{aligned}$$

Since $\mathrm{Res}_G^{\tilde{G}}\,\mathrm{Ker}\,\varphi'$ is projective by Proposition 3.1.1 (4) and $\Omega(M)$ has no non-zero projective summands by the self-injectivity of the group algebra kG, we have that $\mathrm{Ker}\,\varphi \cong \mathrm{Ker}\,\varphi' = 0$. This finishes the proof of (3).

Finally, we prove the assertion (4). Since $k\tilde{G}$ and kG are symmetric k-algebras, it holds that $\tau M \cong \Omega\Omega(M)$ and $\tau(\mathrm{Ind}_G^{\tilde{G}}M) \cong \Omega\Omega(\mathrm{Ind}_G^{\tilde{G}}M)$ for any kG-module M. Therefore, (4) immediately follows from (3). □

3.2. Blocks of group algebras

We recall the definition of blocks of group algebras. Let G be a finite group. The group algebra kG has a unique decomposition

$$kG = B_0 \times \cdots \times B_l \tag{3.2.1}$$

into the direct product of indecomposable k-algebras B_i. We call each indecomposable direct product component B_i a block of kG and the above decomposition the block decomposition. We remark that any block B_i is a two-sided ideal of kG.

For any indecomposable kG-module U, there exists a unique block B_i of kG such that $U = B_iU$ and $B_jU = 0$ for all $j \neq i$. Then we say that U lies in the block B_i or simply U is a B_i-module. We denote by $B_0(kG)$ the principal block of kG, in which the trivial kG-module lies.

Remark 3.2.1. We remark that the block decomposition (3.2.1) induces the following isomorphism of partially ordered sets:

$$\begin{aligned} \mathrm{s}\tau\text{-tilt}(kG) &\longrightarrow \mathrm{s}\tau\text{-tilt}(B_0) \times \cdots \times \mathrm{s}\tau\text{-tilt}(B_l) \\ M &\longmapsto (B_0M, \ldots, B_lM). \end{aligned}$$

Now we recall the definition and basic properties of defect groups of blocks.

Definition 3.2.2. Let B be a block of kG. A defect group D of B is a minimal subgroup of G satisfying the following condition: the B-bimodule morphism

$$\begin{aligned} B \otimes_{kD} B &\xrightarrow{\mu_D} B \\ \beta_1 \otimes \beta_2 &\longmapsto \beta_1\beta_2 \end{aligned}$$

is a split epimorphism.

Proposition 3.2.3 (see [**4**, Chapter 4, 5])**.** Let B be a block of kG and D a defect group of B. Then the following hold:

(1) D is a p-subgroup of G and the set of all defect groups of B forms the conjugacy class of D in G.

(2) D is a cyclic group if and only if the algebra B is finite representation type.

(3) If B is the principal block of kG, then D is a Sylow p-subgroup of G.

Theorem 3.2.4 (see [**4**, Corollary 14.6, Theorem 17.1 and proof of Lemma 19.3])**.** Let B be a block of kG and D a defect group of B.

(1) D is the trivial group if and only if B is a simple algebra.

(2) D is a non-trivial cyclic group if and only if B is a Brauer tree algebra with e edges and multiplicity $(|D|-1)/e$, where e is a devisor of $p-1$.

3.3. Clifford's theory for blocks of normal subgroups

Let G be a finite group, $\tilde{G}$ a finite group containing G as a normal subgroup, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$. We say that $\tilde{B}$ covers B or that B is covered by $\tilde{B}$ if $1_B 1_{\tilde{B}} \neq 0$. We denote by $I_{\tilde{G}}(B)$ the inertial group of B in $\tilde{G}$, that is $I_{\tilde{G}}(B) := \left\{ x \in \tilde{G} \,\middle|\, xBx^{-1} = B \right\}$.

Remark 3.3.1 (see [**4**, Theorem 15.1, Lemma 15.3])**.** With the above notation, the following are equivalent:

(1) The block $\tilde{B}$ covers B.

(2) There exists a non-zero $\tilde{B}$-module U such that $\mathrm{Res}^{\tilde{G}}_{G}U$ has a non-zero direct summand lying in B.

(3) For any non-zero $\tilde{B}$-module U, there exists a non-zero direct summand of $\mathrm{Res}^{\tilde{G}}_{G}U$ lying in B.

Remark 3.3.2. The principal block $B_0(kG)$ of kG is covered by the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ and $I_{\tilde{G}}(B_0(kG)) = \tilde{G}$ since the trivial kG-module k_G is $\tilde{G}$-invariant and $\mathrm{Res}^{\tilde{G}}_{G}k_{\tilde{G}} \cong k_G$.

Remark 3.3.3. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and M a B-module. Then xM is a B-module for $x \in \tilde{G}$ if and only if $x \in I_{\tilde{G}}(B)$.

Theorem 3.3.4 (Clifford's Theorem for blocks [**4**, Theorem 15.1, Lemma 15.3])**.** Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and U a $\tilde{B}$-module. Then the following hold:

(1) The set of blocks of kG covered by $\tilde{B}$ equals to the conjugacy class of B in $\tilde{G}$:
$$\left\{B' \,\middle|\, B' \text{ is a block of } kG \text{ covered by } \tilde{B}\right\} = \left\{xBx^{-1} \,\middle|\, x \in \tilde{G}\right\}.$$

(2) We get the following isomorphism of kG-modules:
$$\mathrm{Res}^{\tilde{G}}_{G}U \cong \bigoplus_{x \in [\tilde{G}/I_{\tilde{G}}(B)]} xBU.$$

Proposition 3.3.5 (see [**15**, Theorem 5.5.10, Theorem 5.5.12])**.** Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and β a block of $kI_{\tilde{G}}(B)$ covering B. Then the following hold:

(1) For any B-module V, the induced module $\mathrm{Ind}^{I_{\tilde{G}}(B)}_{G}V$ is a direct sum of $kI_{\tilde{G}}(B)$-module lying blocks covering B.

(2) There exists a unique block $\tilde{B}$ of $k\tilde{G}$ covering B such that the induction functor
$$\mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\colon kI_{\tilde{G}}(B)\text{-mod} \to k\tilde{G}\text{-mod}$$
restricts to a Morita equivalence
$$\mathrm{Ind}^{\tilde{G}}_{I_{\tilde{G}}(B)}\colon \beta\text{-mod} \longrightarrow \tilde{B}\text{-mod}$$
and the mapping β to $\tilde{B}$ is a bijection between the set of blocks of $kI_{\tilde{G}(B)}$ covering B and the one of $k\tilde{G}$ covering B.

Proposition 3.3.6 ([**15**, Corollary 5.5.6, Theorem 5.5.13, Lemma 5.5.14])**.** Let G be a normal subgroup of $\tilde{G}$ and B a block of kG, then the following conditions hold:

(1) If $\tilde{G}/G$ is a p-group, then there exists a unique block of $k\tilde{G}$ covering B.

(2) If a defect group D of B satisfies $C_{\tilde{G}}(D) \subset G$, then there exists a unique block of $k\tilde{G}$ covering B.

Chapter 4

Poset isomorphism of support τ-tilting modules for blocks of finite groups

In this chapter, we provide one of our main result Theorem 4.2.5. In this chapter, the factor group $\tilde{G}/G$ is a p-group, where G is a normal subgroup of a finite group $\tilde{G}$.

4.1. Normal subgroups with p-power index and their blocks

In this section, we consider properties of block-covering in the case, where $\tilde{G}/G$ is a p-group. The following lemmas make substantial contribution in this paper.

Lemma 4.1.1 ([**9**, Lemma 2.2])**.** Let G be a normal subgroup of $\tilde{G}$ and B a block of kG. If $\tilde{G}/G$ is a p-group and the number of simple B-modules is strictly smaller than p, then for any simple B-module S, it holds that $I_{\tilde{G}}(S) = I_{\tilde{G}}(B)$.

Proof. Let S be a simple B-module. We consider the orbit $\{xS \mid x \in I_{\tilde{G}}(B)\} / \cong$ of S under the action of $I_{\tilde{G}}(B)$ on the set of isomorphism classes of simple B-modules. By the assumption, we have $|I_{\tilde{G}}(B) : I_{\tilde{G}}(S)| = \# \{xS \mid x \in I_{\tilde{G}}(B)\} / \cong < p$. Moreover, the natural number $|I_{\tilde{G}}(B) : I_{\tilde{G}}(S)|$ is a p-power integer since $|\tilde{G} : G| = |\tilde{G} : I_{\tilde{G}}(B)||I_{\tilde{G}}(B) : I_{\tilde{G}}(S)||I_{\tilde{G}}(S) : G|$. Therefore, we get $|I_{\tilde{G}}(B) : I_{\tilde{G}}(S)| = 1$. □

Lemma 4.1.2. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. If $\tilde{G}/G$ is a p-group and $I_{\tilde{G}}(B) = \tilde{G}$, then the following conditions are equivalent.

(1) For any simple B-module T, the inertial group $I_{\tilde{G}}(T)$ of T in $\tilde{G}$ is equal to $\tilde{G}$.

(2) For any simple $\tilde{B}$-module S, the restriction module $\mathrm{Res}_G^{\tilde{G}} S$ is a simple B-module.

In addition, if the conditions above hold, then the restriction functor $\mathrm{Res}_G^{\tilde{G}}$ induces a bijection between the set of isomorphism classes of simple $\tilde{B}$-modules and the one of simple B-modules.

Proof. First, we prove that the first condition implies the second one. Let S be a simple $\tilde{B}$-module. By Theorem 3.1.5 and the assumption, there exists a simple B-module T such that $\mathrm{Res}_G^{\tilde{G}} S \cong T^{\oplus r}$ for some $r \in \mathbb{N}$. Since $\tilde{G}/G$ is a p-group, by Proposition 3.1.9, there exists a simple $k\tilde{G}$-module S' such that $\mathrm{Res}_G^{\tilde{G}} S'$ is isomorphic to T. Since $\tilde{G}/G$ is a p-group again, by Lemma 3.1.10, all composition factors of $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}} S$ and $\mathrm{Ind}_G^{\tilde{G}}\mathrm{Res}_G^{\tilde{G}} S'$ are isomorphic to one simple module. It implies that $S \cong S'$ by the Jordan–Hölder theorem and we conclude that the first assertion implies the second one.

We next show that the second condition implies the first one. Let T be a simple B-module. By Propositions 3.3.5 and 3.3.6 the induced module $\mathrm{Ind}_G^{\tilde{G}} T$ is a $\tilde{B}$-module and there exist a simple $\tilde{B}$-module S such that $\mathrm{Hom}_{\tilde{B}}(\mathrm{Ind}_G^{\tilde{G}} T, S) \neq 0$. By the assumption and Proposition 3.1.1, we have $T \cong \mathrm{Res}_G^{\tilde{G}} S$. Hence, by Theorem 3.1.5, we have $I_{\tilde{G}}(T) = \tilde{G}$. Therefore, we have proven that the second assertion implies the first one.

The remaining deduction is immediate from the fact that the above two conditions are equivalent and from Proposition 3.1.9. □

Lemma 4.1.3. Let G be a normal subgroup of a finite group $\tilde{G}$ and B a $\tilde{G}$-invariant block of kG with a cyclic defect group. Then the following hold:

(1) If any simple B-module is $\tilde{G}$-invariant, then any indecomposable B-module is also $\tilde{G}$-invariant.

(2) If $\tilde{G}/G$ is a p-group, then any indecomposable B-module is $\tilde{G}$-invariant.

Proof. We prove that $I_{\tilde{G}}(V) = \tilde{G}$ for indecomposable B-module V by induction on the composition length of V. If V is simple or indecomposable projective, there is nothing to show by the assumption and Lemma 3.1.13. We assume that the composition length of V is two or more and that V is not projective. We remark that any indecomposable non-projective B-module is a string module (for example, see [**17**]). Hence, we can take a simple B-module S and an indecomposable B-module V' which satisfy at least one of the following conditions:

- There exists an exact sequence

$$0 \longrightarrow S \xrightarrow{\mu} U \xrightarrow{\nu} V \longrightarrow 0.$$

- There exists an exact sequence

$$0 \longrightarrow V \xrightarrow{\mu'} U \xrightarrow{\nu'} S \longrightarrow 0.$$

It suffices to prove $I_{\tilde{G}}(U) = I_{\tilde{G}}(B)$ under the assumption that there exists the first exact sequence, the other case being proved similarly. For any $x \in \tilde{G}$, we take kG-module isomorphisms $\varphi\colon xS \to S$ and $\psi\colon V \to xV$ by the induction hypothesis. We obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & xS & \xrightarrow{x\mu x^{-1}} & xU & \xrightarrow{x\nu x^{-1}} & xV & \longrightarrow & 0 \\
 & & \varphi\downarrow & \text{P.O.} & \downarrow\varphi' & & \| \,\mathrm{id} & & \\
0 & \longrightarrow & S & \xrightarrow{\varepsilon_1} & X & \xrightarrow{\sigma_1} & xV & \longrightarrow & 0 \\
 & & \mathrm{id}\,\| & & \uparrow\psi' & \text{P.B.} & \uparrow\psi & & \\
0 & \longrightarrow & S & \xrightarrow{\varepsilon_2} & Y & \xrightarrow{\sigma_2} & V & \longrightarrow & 0 \\
 & & t\downarrow & \text{P.O.} & \downarrow t' & & \| \,\mathrm{id} & & \\
0 & \longrightarrow & S & \xrightarrow{\varepsilon} & U & \xrightarrow{\sigma} & V & \longrightarrow & 0
\end{array}$$

where t is a scalar map since $\dim_k \mathrm{Ext}_b^1(V,S) = 1$ (see [**4**, Proposition 21.7]). Therefore, we get $xU \cong U$.

By Theorem 3.2.4 and Lemma 4.1.1, any simple B-module S is $\tilde{G}$-invariant, we have the assertion (2) from the first one. □

Proposition 4.1.4. Let G be a normal subgroup of a finite group $\tilde{G}$, $\tilde{B}$ a block of $k\tilde{G}$ and B a cyclic defect block of kG covered by $\tilde{B}$ satisfying one of the following conditions:

(1) There is an $I_{\tilde{G}}(B)$-invariant simple B-module S whose corresponding edge is a terminal edge of the Brauer tree of B.

(2) There is a simple B-module S whose corresponding edge of the Brauer tree of B is a terminal edge and the dimension of S whose dimension of S is distinct to that of any other simple B-module.

(3) Any two simple B-modules have distinct dimensions.

Then any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant.

Proof. We can assume that $I_{\tilde{G}}(B) = \tilde{G}$ by Proposition 3.3.5. Assume that the block B of kG satisfies the condition (1) and let S be an $I_{\tilde{G}}(B)$-invariant simple B-module whose corresponding edge is a terminal edge of the Brauer tree of B. Then, since there exists a unique simple B-module T such that $\mathrm{Ext}_B^1(S,T) \cong k$ and that $\mathrm{Ext}_B^1(S,T') = 0$ for any distinct simple B-module T' to T, we have that

$$\mathrm{Ext}_B^1(S,xT) \cong \mathrm{Ext}_B^1(xS,xT) \cong \mathrm{Ext}_B^1(S,T) \cong k.$$

Hence, we have $xT \cong T$ as B-modules for any $x \in I_{\tilde{G}}(B)$ by the uniqueness of T again. Also, since there exists a unique simple B-module U distinct to S such that $\mathrm{Ext}_B^1(T,U) \cong k$ and $\mathrm{Ext}_B^1(T,U') = 0$ for any distinct simple B-module U' to U and S, we have that

$$\mathrm{Ext}_B^1(T,xU) \cong \mathrm{Ext}_B^1(xT,xU) \cong \mathrm{Ext}_B^1(T,U) \cong k,$$

which implies that $xU \cong U$ as B-modules for any $x \in I_{\tilde{G}}(B)$. By repeating this argument, we have that any simple B-module is $I_{\tilde{G}}(B)$-invariant. Therefore, we have that any B-module is $I_{\tilde{G}}(B)$-invariant by Lemma 4.1.3 (1). In particular, any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant.

Next, assume that the block B of kG satisfies the condition (2). Then a simple B-module S whose corresponding edge is a terminal edge of the Brauer tree of B is $I_{\tilde{G}}(B)$-invariant because xS is a simple B-module with the same dimension as S for any $x \in I_{\tilde{G}}(B)$. Therefore, by (1), any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant. The statement for (3) follows from that for (2) immediately □

Corollary 4.1.5. Let G be a finite group with a cyclic Sylow p-group and $\tilde{G}$ a finite group having G as a normal subgroup. Then any indecomposable $B_0(kG)$-module is $\tilde{G}$-invariant.

Proof. The trivial kG-module k_G is $\tilde{G}$-invariant. Moreover, the trivial kG-module corresponds to the terminal edge in the Brauer tree of the principal block $B_0(kG)$ (for example, see [**8**, section 1.1]). Hence, it concludes the proof by Proposition 4.1.4. □

4.2. Induced modules of support τ-tilting modules

In this section, we give proofs of our main theorems. The next lemmas have key roles.

Lemma 4.2.1. Let G be a normal subgroup of $\tilde{G}$ such that $\tilde{G}/G$ is a p-group, B a block of kG satisfying $I_{\tilde{G}}(B) = \tilde{G}$ and $\tilde{G}$ the block of $k\tilde{G}$ covering B. Assume that the condition in Proposition 4.1.2 holds. For a τ-rigid B-module U, the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is τ-rigid if and only if $\mathrm{Hom}_B(gU, \tau U) = 0$ for all $x \in \tilde{G}$.

Proof. Let U be a τ-rigid B-module. By Lemma 3.1.13, Theorem 3.1.1 and Theorem 3.1.3, we get the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\tilde{B}}(\mathrm{Ind}_G^{\tilde{G}}U, \tau\mathrm{Ind}_G^{\tilde{G}}U) &\cong \mathrm{Hom}_{\tilde{B}}(\mathrm{Ind}_G^{\tilde{G}}U, \mathrm{Ind}_G^{\tilde{G}}\tau U)\\ &\cong \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}U, \tau U)\\ &\cong \bigoplus_{x\in[\tilde{G}/G]} \mathrm{Hom}_B(xU, \tau U).\end{aligned}$$

It concludes the proof. □

Corollary 4.2.2. With the same notations in Lemma 4.2.1, assume that any indecomposable B-module is $\tilde{G}$-invariant. For a τ-rigid B-module U, the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is a τ-rigid B-module.

Lemma 4.2.3. Let N be a normal subgroup of G such that G/N is a p-group, b a block of kN satisfying $I_G(b) = G$ and B the block of kG covering b. Assume that the condition in Proposition 4.1.2 holds. Let U be a B-module and P a projective B-module. If the pair (U, P) satisfies $\mathrm{Hom}_B(P, U) = 0$, then we have $\mathrm{Hom}_{\tilde{B}}(\mathrm{Ind}_G^{\tilde{G}}P, \mathrm{Ind}_G^{\tilde{G}}U) = 0$.

Proof. By Theorem 3.1.1, Theorem 3.1.3 and Lemma 3.1.12, we get the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\tilde{B}}(\mathrm{Ind}_G^{\tilde{G}}P, \mathrm{Ind}_G^{\tilde{G}}U) &\cong \mathrm{Hom}_B(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}P, U)\\ &\cong \mathrm{Hom}_B(\bigoplus_{x\in[\tilde{G}/G]} xP, U)\\ &\cong \bigoplus_{x\in[\tilde{G}/G]} \mathrm{Hom}_b(P, U)\\ &= 0.\end{aligned}$$

□

Lemma 4.2.4. Let Λ and Γ be finite dimensional k-algebras with the same numbers of isomorphism classes of the simple modules. Assume an exact functor F from Λ-mod to Γ-mod satisfies the following conditions:

(1) The functor F preserves indecomposability, projectivity and τ-rigidity.

(2) If $\mathrm{Hom}_\Lambda(P, M) = 0$ then $\mathrm{Hom}_\Gamma(F(P), F(M)) = 0$ for any projective Λ-module P and Λ-module U.

(3) The functor F induces an injection from the set of isomorphism classes of indecomposable modules over Λ to the one over Γ.

Then F induces an embedding of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ into $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$ which sends any connected component of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ into $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$ as a connected component. Furthermore, if Λ is a support τ-tilting finite algebra, then F induces an isomorphism from $\mathrm{s}\tau\text{-tilt}\,\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$ as partially ordered sets.

Proof. We can easily see that (FM, FP) is a support τ-tilting pair (or almost complete support τ-tilting pair) over Γ for any support τ-tilting pair (or almost complete support τ-tilting pair, respectively) (M, P) over Λ. Hence, the functor F sends any support τ-tilting Λ-module to a support τ-tilting Γ-module. Now assume support τ-tilting Λ-modules M_1 and M_2 satisfy the condition $M_2 \geq M_1$. Then by the definition of partial order, there exist $r \in \mathbb{N}$ and an epimorphism

$$M_2^{\oplus r} \xrightarrow{f} M_1 \to 0.$$

Since F is an exact functor, we get an epimorphism

$$F(M_2)^{\oplus r} \xrightarrow{F(f)} F(M_1) \to 0$$

which implies $F(M_2) \geq F(M_1)$. Let (M_1, P_1) and (M_2, P_2) be distinct basic support τ-tilting pairs over Λ and (L, Q) a basic almost complete support τ-tilting pair appearing as a direct summand of both (M_1, P_1) and (M_2, P_2). Then the pairs $(F(M_1), F(P_1))$ and $(F(M_2), F(P_2))$ are distinct by the third assumption on F and have an almost complete support τ-tilting pair $(F(L), F(Q))$ as a direct summand. Therefore, $(F(M_1), F(P_1))$ and $(F(M_2), F(P_2))$ are support τ-tilting mutation of each other. Hence, we have that the functor F embeds $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ into $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$. Now we remark that $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is a $|\Lambda|$-regular quiver, so any connected component C in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is a $|\Lambda|$-regular quiver too. Hence, the image of C under the embedding above is some connected $|\Gamma|$-regular subquiver in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$ because $|\Lambda| = |\Gamma|$ and so is some connected component in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$. If Λ is a support τ-tilting finite algebra, then the image of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ under the embedding above is a finite connected component in $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$, which coincides with $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\Gamma)$ by Proposition 2.2.3. □

Now we give proofs of the main theorems. First, we state the main result again, which are stated in the introduction.

Theorem 4.2.5. Let G be a normal subgroup of a finite group $\tilde{G}$ having p-power index. Let B a block of kG and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Assume that any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant. Then we have the following:

(1) The induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is a support τ-tilting $\tilde{B}$-module for any support τ-tilting B-module U.

(2) The induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces an embedding of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,B)$ into $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\tilde{B})$ as quivers and any connected component of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,B)$ is embedded as a connected component of $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,\tilde{B})$.

(3) If B is a support τ-tilting finite block, then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces isomorphisms from $\mathrm{s}\tau\text{-tilt}\,B$ to $\mathrm{s}\tau\text{-tilt}\,\tilde{B}$ as partially ordered sets.

Proof. Let β be the block of $kI_{\tilde{G}}(B)$ covering B. By Theorem 3.3.5, the functors induced by induction functors $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}\colon B\text{-mod} \to \beta\text{-mod}$ and $\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\colon \beta\text{-mod} \to \tilde{B}\text{-mod}$ are exact functors and the latter induces a Morita equivalence. We remark that the number of isomorphism classes of the simple B-modules is equal to the one of the simple β-modules from the assumption of Theorem 4.2.5 and Lemma 4.1.2. In order to prove Theorem 4.2.5, it is enough to show that $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}$ satisfies the three conditions in Lemma 4.2.4. The functor $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}$ preserves indecomposability, projectivity and τ-rigidity by Theorem 3.1.6, Theorem 3.1.1 and Corollary 4.2.2. By Lemma 4.2.3 and Corollary 3.1.8, the functor $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}$ satisfies the second and third condition in Lemma 4.2.4. Therefore, we have completed the proof of Theorem 4.2.5. □

Corollary 4.2.6. Let G, $\tilde{G}$, B and $\tilde{B}$ be the same as in Theorem 4.2.5. With the same assumption in Theorem 4.2.5, the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces a partially ordered set morphism from 2-tilt B to 2-tilt $\tilde{B}$ commutes the following diagram of partially ordered sets

$$\begin{array}{ccc} \mathrm{s}\tau\text{-tilt}\, B & \longrightarrow & \mathrm{s}\tau\text{-tilt}\, \tilde{B} \\ \wr\downarrow & \circlearrowleft & \downarrow\wr \\ 2\text{-tilt}\, B & \longrightarrow & 2\text{-tilt}\, \tilde{B} \end{array}$$

is commutative where the vertical isomorphisms are given by [**2**] and the upper horizontal morphism is given by Theorem 4.2.5.

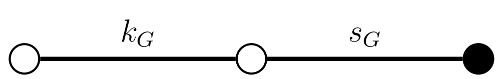
Proof. Let $(M, P) \in \mathrm{s}\tau\text{-tilt}\, B$ and $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ the minimal projective presentation of M. By Proposition 3.1.13, the sequence $\mathrm{Ind}_G^{\tilde{G}} P_1 \xrightarrow{\mathrm{Ind}_G^{\tilde{G}} f_1} \mathrm{Ind}_G^{\tilde{G}} P_0 \xrightarrow{\mathrm{Ind}_G^{\tilde{G}} f_0} \mathrm{Ind}_G^{\tilde{G}} M \to 0$ is also the minimal projective presentation of $\mathrm{Ind}_G^{\tilde{G}} M$. □

Corollary 4.2.7. Let $\tilde{G}$ be a finite group and G be a normal subgroup with cyclic Sylow p-subgroup such that the quotient group $\tilde{G}/G$ is a p-group. Then the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ induces the following isomorphism as partially ordered sets:

$$\begin{array}{rcl} \mathrm{s}\tau\text{-tilt}\, kG & \longrightarrow & \mathrm{s}\tau\text{-tilt}\, k\tilde{G} \\ M & \longmapsto & \mathrm{Ind}_G^{\tilde{G}} M. \end{array}$$

Proof. Since any defect group of a block of kG is contained in a Sylow p-subgroup of G, any block has a cyclic defect group. Hence, any block of kG is τ-tilting finite. Thus, the conclusion follows from Theorem 4.2.5 for all blocks of kG. □

Example 4.2.8. Let k be an algebraically closed field of characteristic 3 and $G := \mathrm{SL}(2, 8)$ the special linear group degree 2 over the field $\mathbb{F}_8$ of order 8. The Galois group $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2) \cong C_3$ acts G naturally. Thus, we can define the semidirect product $\tilde{G} = G \rtimes \mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$ of G and $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$. The group algebra kG is decomposed into 4 blocks $B_0(kG), B_1, B_2, B_3$ where B_1, B_2, B_3 are simple blocks and $B_0(kG)$ has a cyclic defect group. There are two simple $B_0(kG)$-modules k_G and s_G where k_G is the trivial module and s_G is the 7-dimensional kG-module. The block $B_0(kG)$ is a Brauer tree algebra for the follwing Brauer tree:



: multiplicity 4

We can calculate $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, B_0(kG))$.

$\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, B_0(kG))$:

$$\begin{array}{ccccc} & & kG & & \\ & \swarrow & & \searrow & \\ P(k_G) \oplus k_G & & & & P(s_G) \oplus \begin{smallmatrix} s_G \\ s_G \\ s_G \\ s_G \end{smallmatrix} \\ \downarrow & & & & \downarrow \\ k_G & & & & \begin{smallmatrix} s_G \\ s_G \\ s_G \\ s_G \end{smallmatrix} \\ & \searrow & & \swarrow & \\ & & 0 & & \end{array}$$

The principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ covers $B_0(kG)$. Since $\mathrm{s}\tau\text{-tilt}\, B_0(kG) \cong \mathrm{s}\tau\text{-tilt}\, B_0(k\tilde{G})$ induced by induction functor $\mathrm{Ind}_G^{\tilde{G}}$, although $B_0(k\tilde{G})$ is of wild representation type, we can make $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, B_0(k\tilde{G}))$ explicit.

$\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, B_0(k\tilde{G}))$:

$$\begin{array}{ccccc} & & k\tilde{G} & & \\ & \swarrow & & \searrow & \\ P(k_{\tilde{G}}) \oplus \mathrm{Ind}_G^{\tilde{G}} k_G & & & & P(s_{\tilde{G}}) \oplus \mathrm{Ind}_G^{\tilde{G}} \begin{smallmatrix} s_G \\ s_G \\ s_G \\ s_G \end{smallmatrix} \\ \downarrow & & & & \downarrow \\ \mathrm{Ind}_G^{\tilde{G}} k_G & & & & \mathrm{Ind}_G^{\tilde{G}} \begin{smallmatrix} s_G \\ s_G \\ s_G \\ s_G \end{smallmatrix} \\ & \searrow & & \swarrow & \\ & & 0 & & \end{array}$$

Chapter 5

Inertial-invariant support τ-tilting modules

In this chapter, we prove that induced modules of support τ-tilting modules over blocks of finite groups satisfying inertial-invariant condition are also support τ-tilting modules.

5.1. Induced module of inertial-invariant support τ-tilting modules

The following theorem is the group algebra version of Main Theorem 4.

Theorem 5.1.1. *Let G be a normal subgroup of a finite group $\tilde{G}$ and M a support τ-tilting kG-module satisfying $xM \cong M$ as kG-modules for any $x \in \tilde{G}$. Then the induced module $\mathrm{Ind}_G^{\tilde{G}}M$ of M is a support τ-tilting $k\tilde{G}$-module.*

Proof. A similar proof of [**18**, Theorem 4.2] works in this setting. By Lemma 3.1.13 (3), Proposition 3.1.1 (3), Theorem 3.1.3, the $I_{\tilde{G}}(B)$-invariance of and the τ-rigidity of M, we have the following:

$$\begin{aligned}\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}M) &\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \mathrm{Ind}_G^{\tilde{G}}\tau M)\\ &\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M, \tau M)\\ &\cong \mathrm{Hom}_{kG}(\bigoplus_{x\in[\tilde{G}/G]} xM, \tau M)\\ &\cong \bigoplus_{x\in[\tilde{G}/G]} \mathrm{Hom}_{kG}(M, \tau M)\\ &= 0.\end{aligned}$$

Therefore, we have that $\mathrm{Ind}_G^{\tilde{G}}M$ is τ-rigid. By Proposition 2.1.6, there exists an exact sequence

$$kG \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0 \tag{5.1.1}$$

with $M', M'' \in \operatorname{add} M$ and f a left $\operatorname{add} M$-approximation of kG. Applying the functor $\operatorname{Ind}_G^{\tilde{G}}$ to the exact sequence (5.1.1), we get the exact sequence

$$k\tilde{G} \cong \operatorname{Ind}_G^{\tilde{G}} kG \xrightarrow{\operatorname{Ind}_G^{\tilde{G}} f} \operatorname{Ind}_G^{\tilde{G}} M' \xrightarrow{\operatorname{Ind}_G^{\tilde{G}} f'} \operatorname{Ind}_G^{\tilde{G}} M'' \longrightarrow 0$$

satisfying that $\operatorname{Ind}_G^{\tilde{G}} M', \operatorname{Ind}_G^{\tilde{G}} M'' \in \operatorname{add} \operatorname{Ind}_G^{\tilde{G}} M$. Then by Proposition 2.1.6, we only have to prove that $\operatorname{Ind}_G^{\tilde{G}} f$ is a left $\operatorname{add} \operatorname{Ind}_G^{\tilde{G}} M$-approximation of $k\tilde{G}$, that is, the map

$$\operatorname{Hom}_{k\tilde{G}}(\operatorname{Ind}_G^{\tilde{G}} M', X) \xrightarrow{\bullet \circ \operatorname{Ind}_G^{\tilde{G}} f} \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, X) \tag{5.1.2}$$

is surjective for any $X \in \operatorname{add} \operatorname{Ind}_G^{\tilde{G}} M$. First, we prove that the map

$$\operatorname{Hom}_{k\tilde{G}}(\operatorname{Ind}_G^{\tilde{G}} M', \operatorname{Ind}_G^{\tilde{G}} M) \xrightarrow{\bullet \circ \operatorname{Ind}_G^{\tilde{G}} f} \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, \operatorname{Ind}_G^{\tilde{G}} M) \tag{5.1.3}$$

is surjective. By Proposition 3.1.1 (3), Theorem 3.1.3 and the assumption, we get the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_{k\tilde{G}}(\operatorname{Ind}_G^{\tilde{G}} M', \operatorname{Ind}_G^{\tilde{G}} M) & \xrightarrow{\bullet \circ \operatorname{Ind}_G^{\tilde{G}} f} & \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, \operatorname{Ind}_G^{\tilde{G}} M) \\
\wr\downarrow & & \downarrow\wr \\
\operatorname{Hom}_{kG}(M', \operatorname{Res}_G^{\tilde{G}} \operatorname{Ind}_G^{\tilde{G}} M) & \xrightarrow{\bullet \circ f} & \operatorname{Hom}_{kG}(kG, \operatorname{Res}_G^{\tilde{G}} \operatorname{Ind}_G^{\tilde{G}} M) \\
\wr\downarrow & & \downarrow\wr \\
\operatorname{Hom}_{kG}(M', \bigoplus_{x \in [\tilde{G}/G]} xM) & \xrightarrow{\bullet \circ f} & \operatorname{Hom}_{kG}(kG, \bigoplus_{x \in [\tilde{G}/G]} xM) \\
\wr\downarrow & & \downarrow\wr \\
\operatorname{Hom}_{kG}(M', M^{\oplus |\tilde{G}:G|}) & \xrightarrow{\bullet \circ f} & \operatorname{Hom}_{kG}(kG, M^{\oplus |\tilde{G}:G|}).
\end{array}$$

The map in the last row is surjective since f is left $\operatorname{add} M$-approximation of kG, which implies that the map in the first row, which is the map (5.1.3), is surjective. Hence, we get that

$$\operatorname{Hom}_{k\tilde{G}}(\operatorname{Ind}_G^{\tilde{G}} M', \operatorname{Ind}_G^{\tilde{G}} M^{\oplus m}) \xrightarrow{\bullet \circ \operatorname{Ind}_G^{\tilde{G}} f} \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, \operatorname{Ind}_G^{\tilde{G}} M^{\oplus m}) \tag{5.1.4}$$

is surjective for any $m \in \mathbb{N}$. Now take $X \in \operatorname{add} \operatorname{Ind}_G^{\tilde{G}} M$ and $h \in \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, X)$ arbitrarily. Then there exists $m \in \mathbb{N}$ and a split exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} \operatorname{Ind}_G^{\tilde{G}} M^{\oplus m} \xrightarrow{\beta} Y \longrightarrow 0$$

in $k\tilde{G}$-mod. Let $\gamma\colon \operatorname{Ind}_G^{\tilde{G}} M^{\oplus m} \to X$ be a retraction of α, that is, a $k\tilde{G}$-homomorphism satisfying $\gamma \circ \alpha = \operatorname{Id}_X$. Since the map (5.1.4) is surjective and $\alpha \circ h \in \operatorname{Hom}_{k\tilde{G}}(k\tilde{G}, \operatorname{Ind}_G^{\tilde{G}} M^{\oplus m})$, there exists $h' \in \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m})$ such that $h' \circ \mathrm{Ind}_G^{\tilde{G}}f = \alpha \circ h$. Hence, we have that

$$h = \mathrm{Id}_X \circ h = \gamma \circ \alpha \circ h = \gamma \circ h' \circ \mathrm{Ind}_G^{\tilde{G}}f.$$

Therefore, the map (5.1.2) is surjective. □

The following result makes the assumption in Theorem 5.1.1 weaker not only in case where the module M is a kG-module but also in case where M is a B-module.

Theorem 5.1.2. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a support τ-tilting B-module satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module. In particular, $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $\tilde{B}$-module.

Proof. Let $\tilde{B}_1 = \tilde{B}, \ldots, \tilde{B}_e$ be all blocks of $k\tilde{G}$ covering B. By Proposition 3.3.5 (2), we can take $\beta_1, \ldots, \beta_e$ the blocks of $kI_{\tilde{G}}(B)$ satisfying the induction functor $\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}$ restricts to a Morita equivalence

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon \beta_i\text{-mod} \longrightarrow \tilde{B}_i\text{-mod}$$

for any $i = 1, \ldots, e$. By Theorem 5.1.1, the induced module $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$ is a support τ-tilting $kI_{\tilde{G}}(B)$-module and hence $\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$ is a support τ-tilting β_i-module for any $i = 1, \ldots, e$. Therefore, we have that $\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$ is a support τ-tilting $\tilde{B}_i$-module. By Proposition 3.3.5 (1) and Proposition 3.1.1 (2), we have

$$\begin{aligned}\bigoplus_{i=1}^{e} \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M &\cong \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\bigoplus_{i=1}^{e}\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M \\ &\cong \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M \\ &\cong \mathrm{Ind}_G^{\tilde{G}}M.\end{aligned}$$

Hence, $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module. Therefore, we get that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$ be a support τ-tilting $\tilde{B}$-module. □

Corollary 5.1.3. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. If $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\, B$ for $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules M and M', then $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M \geq \tilde{B}\mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\, \tilde{B}$.

Proof. By the exactness of the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ and Theorem 5.1.2, the statement is obvious. □

We will demonstrate that there is an interrelation between the orders of $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules and support τ-tilting $k\tilde{G}$-modules.

Proposition 5.1.4. Let M be an $I_{\tilde{G}}(B)$-invariant B-module. If the induced module $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module, then M is a support τ-tilting B-module.

Proof. By Remark 2.1.3, we can take a projective $k\tilde{G}$-module $\tilde{P}$ satisfying that $|\tilde{P}| + |\mathrm{Ind}_G^{\tilde{G}}M| = |\tilde{B}|$ and $\mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}M) = 0$. By Proposition 2.1.5 and Proposition 3.1.1 (4), we enough to show the following:

(1) M is a τ-rigid B-module.

(2) $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, M) = 0$.

(3) If $\mathrm{Hom}_B(M, \tau X) = 0$, $\mathrm{Hom}_B(X, \tau M) = 0$ and $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) = 0$, then $X \in \mathrm{add}\, M$ for any B-module X.

First, we have the following:

$$\begin{aligned}
&\mathrm{Hom}_B(M, \tau M)^{\oplus |I_{\tilde{G}}(B):G|} \\
&\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{x\in[I_{\tilde{G}}(B)/G]} xM, \tau M\Big) && \text{(the } I_{\tilde{G}}(B)\text{-invariance of } M) \\
&\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{x\in[I_{\tilde{G}}(B)/G]} xM \oplus \bigoplus_{\substack{x\in[\tilde{G}/G]\\ x\notin I_{\tilde{G}}(B)}} xM, \tau M\Big) && \text{(Remark 3.3.3)} \\
&\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{x\in[\tilde{G}/G]} xM, \tau M\Big) \\
&\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M, \tau M) && \text{(Theorem 3.1.3)} \\
&\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \mathrm{Ind}_G^{\tilde{G}}\tau M) && \text{(Proposition 3.1.1 (3))} \\
&\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}M) && \text{(Lemma 3.1.13)} \\
&= 0. && \text{(the } \tau\text{-rigidity of } \mathrm{Ind}_G^{\tilde{G}}M)
\end{aligned}$$

Hence, we have that the B-module M is a τ-rigid B-module. Also, we have that

$$\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, M) \cong \mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}M) = 0.$$

For a B-module X, we assume that $\mathrm{Hom}_B(M, \tau X) = 0$, $\mathrm{Hom}_B(X, \tau M) = 0$ and $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) = 0$. By the assumptions and similar arguments as above, we have that $\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}X) = \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}X, \tau\mathrm{Ind}_G^{\tilde{G}}M) = 0$. Also, we have that

$$\begin{aligned}
\mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}X) &\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) \\
&\cong \mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) \\
&= 0.
\end{aligned}$$

Hence, we have that $\mathrm{Ind}_G^{\tilde{G}}X \in \mathrm{add}\,\mathrm{Ind}_G^{\tilde{G}}M$ by Proposition 2.1.5. Therefore, we have that $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}X \in \mathrm{add}\,\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M$. In particular, we have that $X \in \mathrm{add}\,M$ since $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}X \cong \bigoplus_{x\in[\tilde{G}/G]} xX$ and $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M \cong \bigoplus_{x\in[\tilde{G}/G]} M$, which implies that M is a support τ-tilting B-module by Proposition 2.1.5. □

Theorem 5.1.5. Let M and M' be $I_{\tilde{G}}(B)$-invariant B-modules. Then the following hold:

(1) M is a support τ-tilting B-module if and only if $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module.

(2) Assume that M and M' are support τ-tilting B-modules. Then $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\,B$ if and only if $\mathrm{Ind}_G^{\tilde{G}}M \geq \mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$.

Proof. (1) is clear by Theorem 5.1.2 and Proposition 5.1.4. In order to prove (2), we only show that if $\mathrm{Ind}_G^{\tilde{G}}M \geq \mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$ then $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\,B$ by Corollary 5.1.3, but it follows from the fact the restriction functor $\mathrm{Res}_G^{\tilde{G}}$ is an exact functor, the $I_{\tilde{G}}(B)$-invariance of M and Theorem 3.1.3. □

5.2. Some applications of main theorems

We give some applications and examples of main theorems in this chapter.

Example 5.2.1. Let k be an algebraically closed field of characteristic $p = 2$, G the alternating group A_4 of degree 4 and $\tilde{G}$ the symmetric group S_4 of degree 4. The principal blocks of kA_4 and kS_4 are themselves, respectively. Moreover, the block kA_4 is covered by kS_4. The algebras kA_4 and kS_4 are Brauer graph algebras associated to the Brauer graphs in Figure 5.1(a) and Figure 5.1(b), respectively:

(a) The Brauer graph of kA_4

(b) The Brauer graph of kS_4

Figure 5.1: Brauer graphs

Now we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,kA_4)$ of the partially ordered set $\mathrm{s}\tau\text{-tilt}\,kA_4$ as follows:

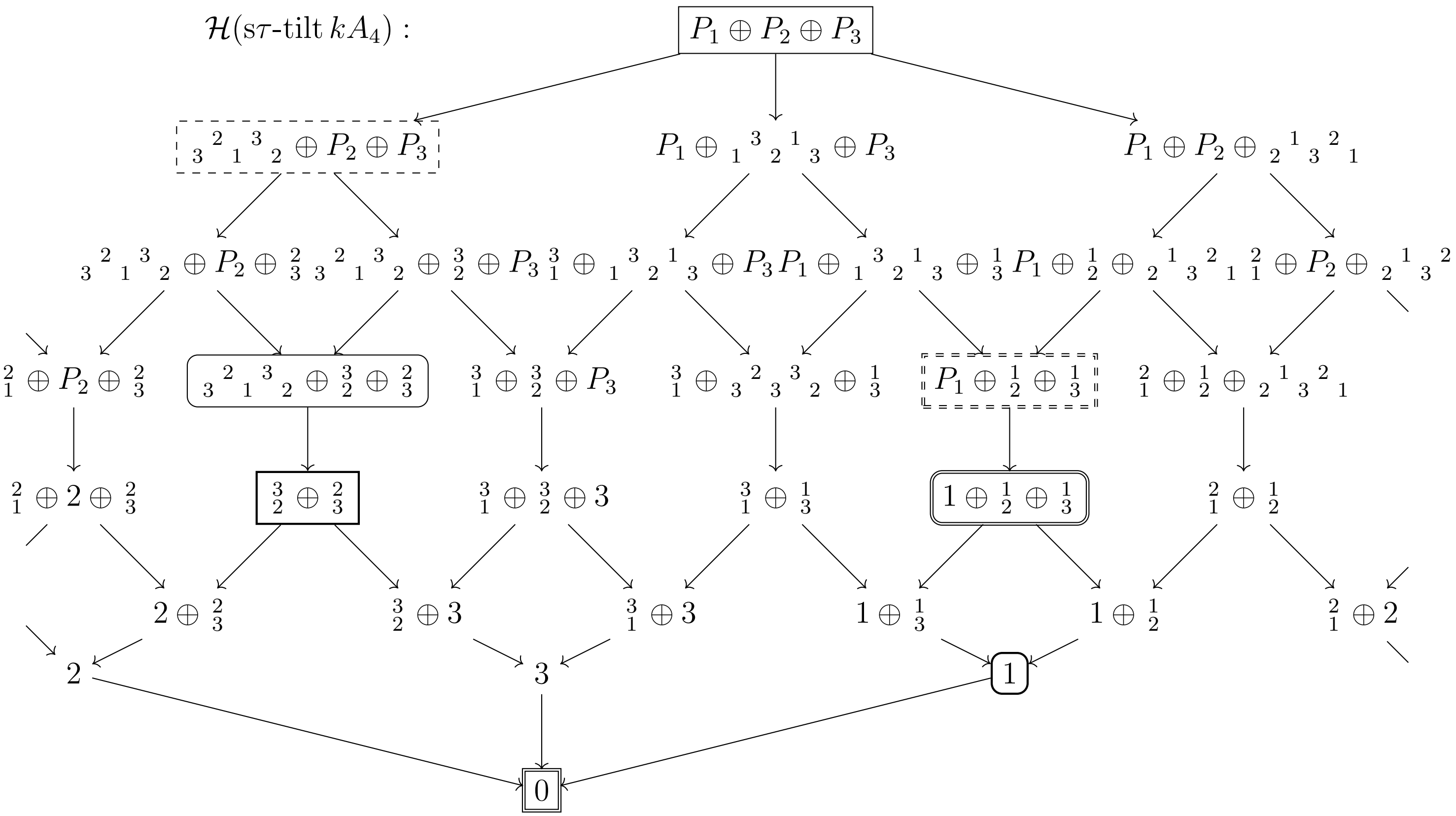


Figure 5.2: The Hasse diagram of $\mathrm{s}\tau\text{-tilt}\, kA_4$

The enclosed support τ-tilting modules in Figure 5.2 are all the invariant support τ-tilting modules under the action of S_4. Next, we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\,kS_4)$ of partially ordered set $\mathrm{s}\tau\text{-tilt}\,kS_4$ as follows:

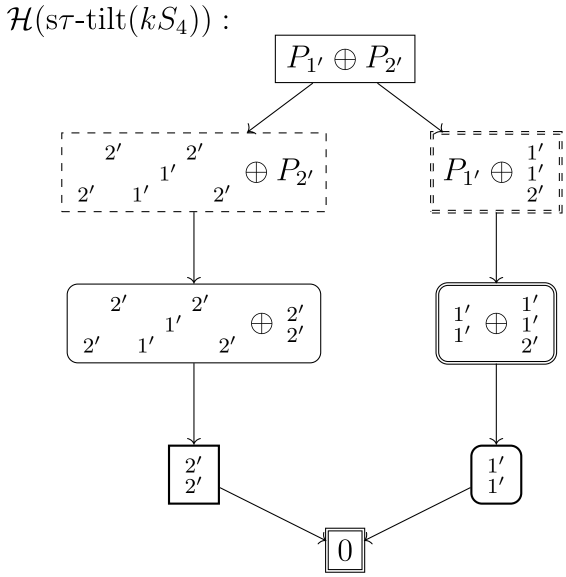


Figure 5.3: The Hasse diagram of $\mathrm{s}\tau\text{-tilt}\,kS_4$

The induction functor $\mathrm{Ind}_{A_4}^{S_4}$ takes each enclosed S_4-invariant support τ-tilting kA_4-module in Figure 5.2 to the enclosed kS_4-module in Figure 5.3 with the same square. We remark that even if a support τ-tilting kA_4-module M is basic, its induction $\mathrm{Ind}_{A_4}^{S_4} M$ is not necessarily basic. For example, the induced module $\mathrm{Ind}_{A_4}^{S_4}(1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}) \cong \begin{smallmatrix}1'\\1'\end{smallmatrix} \oplus \begin{smallmatrix}1'\\1'\\2'\end{smallmatrix} \oplus \begin{smallmatrix}1'\\1'\\2'\end{smallmatrix}$ is not basic.

Example 5.2.2. Let G_1 and G_2 be arbitrary finite groups and M a support τ-tilting kG_1-module. Then the group G_1 is a normal subgroup of the direct product group $G_1 \times G_2$, and it is clear that $M \cong xM$ for any $x \in G_1 \times G_2$. Therefore, the induced module $\mathrm{Ind}_{G_1}^{G_1 \times G_2} M \cong kG_2 \otimes_k M$ is support τ-tilting $k[G_1 \times G_2]$-module by Theorem 5.1.1.

Example 5.2.3. Let G be a normal subgroup of a finite group $\tilde{G}$ having a cyclic Sylow p-subgroup. Then the principal block $B_0(kG)$ of kG satisfying $xM \cong M$ for any support τ-tilting module M and $x \in \tilde{G}$ by Corollary 4.1.5. Therefore, $\mathrm{Ind}_G^{\tilde{G}} M$ and $B_0(k\tilde{G})\mathrm{Ind}_G^{\tilde{G}} M$ are support τ-tilting modules.

Bibliography

[1] T. Adachi, T. Aihara, and A. Chan. Classification of two-term tilting complexes over Brauer graph algebras. *Math. Z.*, 290(1-2) pages 1–36, 2018. DOI 10.1007/s00209-017-2006-9.

[2] T. Adachi, O. Iyama, and I. Reiten. τ-tilting theory. *Compos. Math.*, 150(3) pages 415–452, 2014. DOI 10.1112/S0010437X13007422.

[3] T. Aihara and O. Iyama. Silting mutation in triangulated categories. *J. London. Math. Soc. (2)*, 85(3) pages 633–668, 2012. DOI 10.1112/jlms/jdr055.

[4] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. DOI 10.1017/CBO9780511623592.

[5] T. Aoki. Brauer tree algebras have $\binom{2n}{n}$ two-term tilting complexes. *Preprint*.

[6] H. Asashiba, Y. Mizuno, and K. Nakashima. Simplicial complexes and tilting theory for Brauer tree algebras. *Journal of Algebra*, v.551 pages 119–153, 2020. DOI 10.1016/j.jalgebra.2019.12.020.

[7] J. A. Green. On the indecomposable representations of a finite group. *Math. Z.*, v.70 pages 430–445, 1958/59. DOI 10.1007/BF01558601.

[8] G. Hiss and K. Lux. *Brauer trees of sporadic groups*. Oxford: Clarendon Press, 1989.

[9] M. Holloway, S. Koshitani, and N. Kunugi. Blocks with nonabelian defect groups which have cyclic subgroups of index p. *Arch. Math. (Basel)*, 94(2) pages 101–116, 2010. DOI 10.1007/s00013-009-0075-7.

[10] M. Hoshino and Y. Kato. Tilting complexes defined by idempotents. *Comm. Algebra*, 30(1) pages 83–100, 2002. DOI 10.1081/AGB-120006480.

[11] G. Jasso. Reduction of τ-tilting modules and torsion pairs. *Int. Math. Res. Not. IMRN*, pages 7190–7237, 2015. DOI 10.1093/imrn/rnu163.

[12] B. Keller and D. Vossieck. Aisles in derived categories. *Bull. Soc. Math. Belg. Sér. A*, 40(2) pages 239–253, 1988.

[13] R. Koshio. On induced modules of inertial-invariant support τ-tilting modules over blocks of finite groups. *SUT Journal of Mathematics*, 58(2) pages 157–171, 2022. DOI 10.55937/sut/1670501315.

[14] R. Koshio and Y. Kozakai. On support τ-tilting modules over blocks covering cyclic blocks. *J. Algebra*, v.580 pages 84–103, 2021. DOI 10.1016/j.jalgebra.2021.03.021.

[15] H. Nagao and Y. Tsushima. *Representations of finite groups.* Academic Press, Inc., Boston, MA, 1989.

[16] J. Rickard. Morita theory for derived categories. *J. London Math. Soc. (2)*, 39(3) pages 436–456, 1989. DOI 10.1112/jlms/s2-39.3.436.

[17] S. Schroll. Brauer graph algebras: a survey on Brauer graph algebras, associated gentle algebras and their connections to cluster theory. In *Homological methods, representation theory, and cluster algebras*, CRM Short Courses, pages 177–223. Springer, Cham, 2018.

[18] Y. Zhang and Z. Huang. G-stable support τ-tilting modules. *Front. Math. China*, 11(4) pages 1057–1077, 2016. DOI 10.1007/s11464-016-0560-9.