Doctoral thesis

Finite-time blow-up in quasilinear chemotaxis systems with logistic source

(ロジスティック項をもつ準線形走化性方程式系における有限時刻爆発)

March, 2023

Yuya Tanaka
(田中 悠也)

Department of Mathematics
Graduate School of Science
Tokyo University of Science

Acknowledgements

First and foremost, I would like to express my deepest and sincere appreciation to my supervisor, Professor Tomomi Yokota. In the second year of the master's course I could not decide whether advancing to the doctoral course. Then he listened seriously to my worries and told me that "if you decide to go on to the next course, I will do my best so that you can become a mathematician". Thanks to this word, I determined to advance to the doctoral course. After that, he supported me many times, for which I cannot thank him enough. In fact, whenever I asked about studies and papers, he quickly replied me a lot of comments and proper advices even on busy days and holiday. Also, he taught me the importance of talking interests in my studies and behavior as a mathematician. Furthermore, he gave me many chances to have valuable experiences to participate domestic and international conferences. Through them, I could discussed with many mathematicians and obtained a lot of knowledge and results e.g. [**43**], and so my recent five years are very fruitful.

I also would like to thank Professor Masaaki Mizukami. He often encouraged me when my study was not going well. Thanks to his kind words, I could worked as hard as possible everyday. Additionally, he gave me a lot of advice on how to spend time at domestic and international conferences, and so I could enjoyed every conferences, in particular, Equadiff 15 with him. Furthermore, I would be also grateful for the collaborations with him [**33, 34**]. In my writing these papers, he taught me that it is important to create clear and easy-to-read sentences having story. That was helpful to develop my ability of making story.

I would be also deeply thankful to Professor Giuseppe Viglialoro. When I was a first year master's student, he came to Japan with Professor Monica Marras. That was the first time I met Professor Viglialoro and took care of mathematicians from abroad. His humorous talks inspired me, and thanks to that, I could get the hang of talking with foreign mathematicians. Also, Section 3.6 in Chapter 3 is based on the collaboration with him. In our writing the collaboration paper, his polite expressions impressed me, which influenced writings of my subsequent papers.

I also would like to express my gratitude to Professor Stella Vernier-Piro for giving a chance to collaborate with Professor Monica Marras. Moreover, I would be grateful to Professor Marras for holding online meetings frequently. By the collaboration with her [**5**] I could get a lot of knowledge related to my studies.

I experienced a wonderful visit to Professor Johannes Lankeit in Hannover in July 2022. He gave me an opportunity of a joint work including Dr. Mario Fuest. I could spent with them two fulfilling weeks. When I wanted to discuss our study, they made time immediately. Thanks to that, we were able to achieve a result [**14**]. I am very happy to spend a great and fruitful time and deeply thankful to them! Moreover, I was surprised by their cleverness and ideas, which made me strongly want to obtain more results by many ideas like them. Also, *Herrenhäuser Gärten* recommended by them was very beautiful and a great place for refreshment.

I would express my gratitude to Dr. Shunsuke Kurima and Dr. Hiroshi Wakui for attending my seminar in the doctoral course every time. Dr. Kurima taught me the importance of talking logically. Dr. Wakui gave me advice on the way to explain main points and essence. By virtue of those, I could make my talks better in workshops.

I want to thank my colleague Mr. Yutaro Chiyo. He is not only a good friend but also good rival. I could spent everyday supporting each other and improving ourselves through competition. I think that thanks to him, I worked harder in my study life.

I wish to express my thanks to the senior members Ms. Natsumi Yoshimiya, Mr. Tsukasa Ogawa, who have taken care of me for two years since I was fourth year student in college, and the junior member Ms. Ayako Uchida. Thanks to them, I could enjoyed my life in the university outside of my study. Moreover, the joint work with Ms. Uchida [**6**] is the paper of memories which reminds me of my study life with her.

I really appreciate my family and friends supporting me and spending a great time. Thanks to the time with them, my life became more fulfilling.

I would like to appreciate the dissertation committee, the chief of the committee Professor Tomomi Yokota and the members of the committee Professor Keiichi Kato, Professor Naoyuki Koike, Professor Masahito Ohta and Professor Hiromichi Itou using time for me in their busy schedule and reviewing this thesis.

This research is supported by JSPS KAKENHI Grant Number JP22J11193.

Yuya Tanaka
Tokyo, March 2023

Contents

Chapter 1

Introduction

The following ordinary differential equations are known as typical examples giving qualitative properties of solutions:

$$\text{(i) } u'(t) = c_1 u^{1+\alpha}(t) - c_2, \quad t > 0, \qquad \text{(ii) } u'(t) = u(t) - u^{\kappa}(t), \quad t > 0,$$

where $c_1, c_2, \alpha > 0$ and $\kappa > 1$. For suitable positive initial data the solution of (i) blows up at some finite time T (that is, $u(t) \to \infty$ as $t \nearrow T$), whereas the solution of (ii) is bounded (that is, $\sup_{t\in(0,T)} u(t) < \infty$). As in these examples, whether solutions blow up or not is one of mathematical themes also in partial differential equations, and such a theme can be considered in chemotaxis systems. The original chemotaxis system was proposed by Keller and Segel in 1970s. As a model including population dynamics or pattern formation in bacteria colonies, there is the Keller–Segel system with logistic source,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + u - u^{\kappa}, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

with homogeneous Neumann boundary conditions, where $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a smooth bounded domain and $\kappa > 1$. The term $-\nabla \cdot (u\nabla v)$ is called a chemotaxis term, which promotes blow-up at some time T, where blow-up means $\lim_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ or $\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. On the other hand, the logistic term $u - u^{\kappa}$ has a strong effect of blow-up prevention. Thus the following question arises:

Are solutions of Keller–Segel systems with logistic source always prevented?

The answer is indeed no! To explain the mechanism we observe the system

$$u_t = -\nabla \cdot (u\nabla v) - u^{\kappa}, \quad 0 = \Delta v + u.$$

Testing the first equation by u^{p-1} $(p > 1)$ and integrating by parts as well as using the second equation, we see from Hölder's and Young's inequalities that if $\kappa < 2$, then

$$\begin{aligned}\frac{1}{p} \cdot \frac{d}{dt} \int_\Omega u^p \, dx &= -\frac{p-1}{p} \int_\Omega u^p \Delta v \, dx - \int_\Omega u^{p+\kappa-1} \, dx \\ &= \frac{p-1}{p} \int_\Omega u^{p+1} \, dx - \int_\Omega u^{p+\kappa-1} \, dx \\ &\geq c_1 \left(\int_\Omega u^p \, dx \right)^{\frac{p+1}{p}} - c_2\end{aligned}$$

for all $t > 0$ with some constants $c_1, c_2 > 0$. This implies that $\lim_{t \nearrow T} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty$ with some finite time T. Therefore, it is conjectured that if $\kappa < 2$, then solutions of Keller–Segel systems with logistic source possibly blow up. As to this conjecture, Winkler [**60**] succeeded in showing finite-time blow-up under a smallness condition for $\kappa > 1$ in the Keller–Segel system with logistic source. In such a circumstance, the following question arises:

Can solutions blow up in a situation added further factors preventing blow-up?

This thesis provides some positive answers to this question in quasilinear chemotaxis systems with logistic source.

In Part I we study finite-time blow-up in parabolic–elliptic Keller–Segel systems with density-dependent sensitivity and logistic source. In Chapter 2 we consider the case of linear diffusion and sublinear sensitivity. This case means that the effect of a chemotaxis term is small, so that we try to derive finite-time blow-up of solutions under a smallness condition for logistic source. Chapter 3 gives an investigation in the case of nonlinear diffusion and super- and sub-linear sensitivity, which includes also the case that the effect of a chemotaxis term is strong in contrast to Chapter 2. Moreover, we give a related result on blow-up prevention in a fully parabolic system with nonlinear production.

In Part II we show finite-time blow-up in quasilinear Jäger–Luckhaus systems with logistic source and nonlinear production. Here, a Jäger–Luckhaus system was proposed as a simplification of a Keller–Segel system. Since the system of this type is useful to obtain more precise behavior of solutions, we deal with the aforementioned system. Chapter 4 is concerned with the case of nondegenerate diffusion. In this case we consider not only finite-time blow-up but also blow-up prevention. In Chapter 5 we show existence of blow-up solutions in the case of degenerate diffusion. In this case, taking into account the lack of regularity of solutions, we introduce a *moment solution* concept.

Part I

Finite-time blow-up in parabolic–elliptic Keller–Segel systems with density-dependent sensitivity and logistic source

Chapter 2

The case of linear diffusion and sublinear sensitivity

2.1. Introduction

This chapter is motivated by Viglialoro [**52**], and we consider finite-time blow-up in the following parabolic–elliptic Keller–Segel system with density-dependent sublinear sensitivity and logistic source:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u(u+1)^{\alpha-1}\nabla v) + \lambda u - \mu u^{\kappa}, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1.1}$$

where $\Omega = B_R(0) \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a ball with some $R > 0$; $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$ are constants; ν is the outward normal vector to $\partial\Omega$;

$$u_0 \in C^0(\overline{\Omega}) \text{ is radially symmetric and nonnegative.} \tag{2.1.2}$$

The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the density of cells and the concentration of the chemical substance at $x \in \Omega$ and $t \geq 0$, respectively. The logistic source $\lambda u - \mu u^{\kappa}$ represents the proliferation and death of the cells and the sublinear sensitivity $u(u+1)^{\alpha-1}$ with $\alpha < 1$ indicates that the chemotactic effect is small.

The Keller–Segel system was proposed as a part of the life cycle of cellular slime molds with *chemotaxis* by Keller and Segel in [**23**] and was studied extensively (for instance, global existence and boundedness can be found in [**4, 39, 54**] and blow-up can be referred to [**17, 54, 58**]); moreover, many variations of the original Keller–Segel system were proposed by Hillen and Painter [**16**]. Chemotaxis induces aggregation phenomena caused by the direct movement of cells as a response to gradients of a chemical signal substance, so that we are interested in whether the corresponding solution of these systems blows up.

From a mathematical point of view, it is meaningful to give a clear answer to the question whether solutions to (2.1.1) blow up or remain bounded. The system (2.1.1) is a special case of the following quasilinear chemotaxis systems:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + \lambda u - \mu u^{\kappa}, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{2.1.3}$$

where $\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$, $\tau \in \{0, 1\}$ and D, $S \in C^2([0, \infty))$. In (2.1.3) there are results on boundedness in [**3, 22, 50, 55, 66, 67**] and blow-up in [**60**] explained later.

Before introducing previous works about the system (2.1.1), let us recall known results about the quasilinear chemotaxis system (2.1.3) without logistic source,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{2.1.4}$$

About this system there are some results related to global existence of smooth bounded solutions and existence of blow-up solutions. In the nondegenerate chemotaxis system given by (2.1.4) with $D(u) = (u+1)^{m-1}$ and $S(u) = u(u+1)^{\alpha-1}$ with $m, \alpha \in \mathbb{R}$ and $\tau = 1$, Tao and Winkler [**50**] proved that if $\alpha < m - \frac{n-2}{n}$ and Ω is a convex smooth domain, then global bounded solutions exist; after that, the convexity condition of Ω was removed in [**18**], whereas Cieślak and Stinner [**7, 8**] showed that if $\alpha > m - \frac{n-2}{n}$ and either $m \geq 1$ or $\alpha \geq 1$, then there exists a solution which blows up in finite time; moreover, Winkler [**61**] established infinite-time blow-up in the case $m - \frac{n-2}{n} < \alpha \leq 0$. Lower bounds for the blow-up time of such solutions were obtained in [**37**]. In its parabolic–elliptic version ($\tau = 0$) Lankeit [**25**] showed that solutions are global and bounded when $\alpha < m - \frac{n-2}{n}$ and blow up in infinite time when $n \geq 3$, $\alpha \leq 0$ and $\alpha > m - \frac{n-2}{n}$; moreover, in the case that the second equation is $0 = \Delta v - M(t) + u$, where $M(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$, Winkler and Djie [**63**] proved that if $\alpha + 1 < m + \frac{2}{n}$, then all solutions are global in time and bounded, whereas if $\alpha + 1 > m + \frac{2}{n}$, $\alpha > 0$ and Ω is a ball, then there exist solutions that are unbounded in finite time. Lower bounds for the blow-up time of such solutions were recently derived by Marras, Nishino and Viglialoro [**29**]. In the degenerate chemotaxis system written as (2.1.4) with $D(u) = u^{m-1}$ and $S(u) = u^{\alpha-1}$ with $m \geq 1$, $\alpha \geq 2$ and $\tau = 1$, if $m > \alpha - \frac{2}{n}$ and Ω is a bounded domain, then existence of global weak solutions was shown in [**18**], and if $m < \alpha - \frac{2}{n}$, then finite-time blow-up was established in [**15**].

On the other hand, in the system (2.1.3), there are many results about global existence and boundedness because the logistic term $\lambda u - \mu u^\kappa$ suppresses blow-up phenomena. For instance, in the system (2.1.3) with $D(u) = 1$, $S(u) = u$, $\tau = 1$ and $\kappa = 2$, Winkler [**55**] proved that if μ is sufficient large, then global bounded solutions exist. Jin and Xiang [**22**] established global bounded solutions for all $\mu > 0$ in the two-dimensional setting. In the parabolic–elliptic case, Tello and Winkler [**51**, Corollary 2.6] asserted that the system (2.1.3) admits a global classical solution for all $\mu > \max\{0, \frac{n-2}{n}\chi\}$ in the case $\kappa = 2$ or for all $\mu > 0$ in the case $\kappa > 2$. In the system (2.1.3) with $D(u) = (u+1)^{m-1}$ and $S(u) = u(u+1)^{\alpha-1}$ with $m, \alpha \in \mathbb{R}$ and $\tau = 1$, Zheng [**67**] showed existence of global classical bounded solutions under the condition that $\lambda = \mu = 1$, $\kappa = 2$ and $0 < 1 - m + \alpha < \frac{4}{n+2}$ with $n \geq 3$. In the parabolic–elliptic case with $m \geq 1$ and $\alpha > 0$, all classical solutions are global in time and bounded when $\alpha + 1 < \max\{\kappa, m + \frac{2}{n}\}$ and when $\alpha + 1 = \kappa$ and $\mu > \mu_0 = \mu_0(m, \kappa, \chi) > 0$ (see [**66**]). In the system (2.1.3) with $D(u) = u^m$ and $S(u) = u^\alpha$ with $m \in \mathbb{R}$, $\alpha < 1$ and $\tau = 1$, Cao [**3**] proved that the solution is global in time and bounded in the case $\kappa = 2$.

From these previous studies, one might infer that logistic source always suppresses blow-up and induces boundedness in chemotaxis systems. However, in contrast to this inference, Winkler [**60**] succeeded in obtaining the condition for $\kappa > 1$ such that finite-time blow-up occurs in the system (2.1.1) with $\alpha = 1$,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{2.1.5}$$

in both low ($n = 3, 4$) and higher ($n \geq 5$) dimensional cases. In detail, if $\kappa > 1$ satisfies

$$\kappa < \begin{cases} \frac{7}{6} & \text{if} \quad n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if} \quad n \geq 5, \end{cases}$$

then an initial data leading to finite-time blow-up was found in [**60**]. We also note that Winkler [**57**] has already shown finite-time blow-up when $n \geq 5$ and the second equation in (2.1.5) is $0 = \Delta v - M(t) + u$, where $M(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)\, dx$ (cf. Zheng, Mu and Hu [**68**] for its analog with density-dependent superlinear sensitivity). Thus the results in [**57, 60, 68**] imply that the small logistic-type dampening cannot suppress blow-up phenomena. On the other hand, blow-up in the case that the chemotaxis term $-\nabla \cdot (u \nabla v)$ in (2.1.5) is generalized to $-\chi \nabla \cdot (u(u+1)^{\alpha-1} \nabla v)$ with $\alpha < 1$, that is, blow-up in the system (2.1.1) has not been studied yet, even though in the corresponding system with generalized chemotaxis terms there are a lot of results ([**7, 8, 18, 25, 50, 61, 63, 66, 67**]).

The purpose of this chapter is to determine conditions for α and κ such that the solution of (2.1.1) blows up in finite time in the case that $0 < \alpha < 1$ and $n \geq 3$. This is not trivial because not only the system (2.1.1) has the logistic source but the chemotactic effect is smaller than that in the standard case that $\alpha = 1$. In the system (2.1.1), the chemotactic sensitivity decays with α approaching to 0, so that we can expect that the smaller α is, the smaller κ has to be in order to have a blow-up scenario.

Now the main result reads as follows:

Theorem 2.1.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3, R > 0)$ and let $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Assume that α and κ satisfy that*

$$\text{if } n = 3, \qquad \tfrac{5}{6} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \tfrac{6\alpha-5}{6(3-2\alpha)}, \tag{2.1.6}$$

$$\text{if } n = 4, \qquad \tfrac{5}{6} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \tfrac{6\alpha-5}{6(4-3\alpha)}, \tag{2.1.7}$$

$$\text{if } n = 5, \qquad \begin{cases} \tfrac{10}{11} < \alpha \leq \tfrac{11}{12} \quad \text{and} \quad \kappa < 1 + \tfrac{10\alpha-9}{40(1-\alpha)}, \\ \tfrac{11}{12} < \alpha \leq \tfrac{27}{28} \quad \text{and} \quad \kappa < 1 + \tfrac{20\alpha-17}{20(5-4\alpha)}, \\ \tfrac{27}{28} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \tfrac{4\alpha-3}{8(3-2\alpha)}, \end{cases} \tag{2.1.8}$$

$$\text{if } n \geq 6, \qquad \begin{cases} 1 - \tfrac{4}{(3n-4)(n-1)} < \alpha \leq 1 - \tfrac{4}{(3n-2)(n-1)} \\ \qquad \text{and} \quad \kappa < 1 + \tfrac{4-(n-1)^2(1-\alpha)\{2+n(1-\alpha)\}}{n(n-1)^2(1-\alpha)\{2+(n-1)(1-\alpha)\}}, \\ 1 - \tfrac{4}{(3n-2)(n-1)} < \alpha < 1 \\ \qquad \text{and} \quad \kappa < 1 + \tfrac{1-(n-1)(1-\alpha)}{(n-1)\{2+(n-1)(1-\alpha)\}}. \end{cases} \tag{2.1.9}$$

Then for all $\widetilde{L} > 0$, $M_0 > 0$ and $M_1 \in (0, M_0)$, one can find a positive constant $r_\star = r_\star(R, \chi, \alpha, \lambda, \mu, \kappa, \widetilde{L}, M_0, M_1) < R$ such that if u_0 satisfies (2.1.2) and

$$u_0(x) \leq \widetilde{L}|x|^{-n(n-1)} \text{ for all } x \in \Omega, \quad \int_\Omega u_0(x)\,dx \leq M_0, \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_1, \tag{2.1.10}$$

then there exist $T^* \in (0, \infty)$ and an exactly one pair (u, v) of functions

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T^*)) \cap C^{2,1}(\overline{\Omega} \times (0, T^*)), \\ v \in \bigcap_{q>n} L^\infty_{\mathrm{loc}}([0, T^*); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T^*)), \end{cases}$$

which solves (2.1.1) classically and blows up at $t = T^*$ in the sense that

$$\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1.11}$$

Remark 2.1.1. Marras, Vernier Piro and Viglialoro [**30, 31**] obtained a lower bound for the blow-up time of solutions to a system similar to (2.1.1). It is an open question whether an explicit lower bound for the blow-up time in Theorem 2.1.1 can be obtained.

Remark 2.1.2. In order to prove Theorem 2.1.1 we will refer to a method in [**60**, Theorem 1.1]. In other applications of the method in [**60**, Theorem 1.1], we obtained a blow-up result in a parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with logistic source in [**5**].

The proof of Theorem 2.1.1 is based on that of Winkler [**60**, Theorem 1.1]. First, we define the mass accumulation function $w = w(s,t)$ and $z = z(s,t)$ as

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\, d\rho,$$
$$z(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho,t)\, d\rho,$$

where $s := r^n$ and $r \in [0,R]$. Then the system (2.1.1) is reduced to the parabolic equation

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + \chi n w_s (n w_s + 1)^{\alpha-1}(w - z) + \lambda w - n^{\kappa-1}\mu \int_0^s w_s^{\kappa}(\sigma,t)\, d\sigma. \quad (2.1.12)$$

In [**60**] the second term on the right-hand side of the above equation is $\chi n w_s(w - z)$. Next, using the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\, ds$$

with some $\gamma > 0$, we will derive a super-linear differential inequality for ϕ. However, we cannot use the same argument as in [**60**] with $\alpha = 1$ because of the factor $(nw_s+1)^{\alpha-1}$ of the second term on the right-hand side of (2.1.12) in our case $\alpha < 1$. Therefore, separately using the estimates $(nw_s+1)^{\alpha-1} \le 1$ and $(nw_s+1)^{\alpha-1} \ge (Cs^{-(n-1)-\frac{\varepsilon}{n}}+1)^{\alpha-1}$ (see (2.4.5)) on a case by case basis, we derive a super-linear differential inequality for ϕ by introducing four more conditions for γ in $\phi(t)$ than those in [**60**], and thus the arguments in this chapter are divided into many cases, whereas those in [**60**] were divided into two cases.

This chapter is organized as follows. In Section 2.2 we recall local existence and transformation of solutions to the system (2.1.1). Section 2.3 consists of elementary results including pointwise estimates for solutions. Section 2.4 is the main part of this chapter and is devoted to deriving a super-linear differential inequality for the moment-type functional ϕ. Finally, the proof of Theorem 2.1.1 is given in Section 2.5.

2.2. Local existence and mass accumulation functions

We first introduce a result on local existence of classical solutions to (2.1.1). Because the proof is similar to that in [**9, 35, 63**], we provide only the statement of the lemma.

Lemma 2.2.1. *Let $n \geq 1$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Assume that u_0 satisfies (2.1.2). Then there exist $T_{\max} \in (0, \infty]$ and an exactly one pair (u, v) of radially symmetric nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ v \in \bigcap_{q>n} L^\infty_{\mathrm{loc}}([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{\max})), \end{cases}$$

which solves (2.1.1) classically. Moreover,

$$\textit{if } T_{\max} < \infty, \ \textit{then} \lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

In the following we assume that initial data u_0 satisfies (2.1.2) and we denote by $(u, v) = (u(r, t), v(r, t))$ the radially symmetric local solution of (2.1.1) and by $T_{\max}$ the maximal existence time in Lemma 2.2.1. Based on [**21**], we set the mass accumulation functions w and z such that

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}), \tag{2.2.1}$$

$$z(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}). \tag{2.2.2}$$

Then we have

$$w_s(s, t) = \frac{1}{n} u(s^{\frac{1}{n}}, t), \quad w_{ss}(s, t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t), \tag{2.2.3}$$

$$z_s(s, t) = \frac{1}{n} v(s^{\frac{1}{n}}, t), \quad z_{ss}(s, t) = \frac{1}{n^2} s^{\frac{1}{n}-1} v_r(s^{\frac{1}{n}}, t) \tag{2.2.4}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. The second equation in (2.1.1) implies that

$$r^{n-1} v_r(r, t) = z(r^n, t) - w(r^n, t) \quad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{\max}). \tag{2.2.5}$$

Integrating the first equation in (2.1.1) over $(0, r)$ and using (2.2.5), we see that

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + \chi n w_s (n w_s + 1)^{\alpha - 1}(w - z) + \lambda w - n^{\kappa-1}\mu \int_0^s w_s^\kappa(\sigma, t)\, d\sigma \tag{2.2.6}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. Moreover, the function w defined in (2.2.1) fulfills

$$0 = w(0, t) \leq w(s, t) \leq w(R^n, t) = \frac{1}{\omega_n} \int_\Omega u(x, t)\, dx$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$, where $\omega_n := n|B_1(0)|$.

2.3. Estimates for mass and pointwise bounds for solutions

In this section we give three lemmas which were essentially proved in [**60**].

Lemma 2.3.1. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. If the initial data u_0 satisfies* (2.1.2)*, then*

$$\int_\Omega u(x,t)\,dt \leq e^{\lambda_+} \int_\Omega u_0(x)\,dx \quad \text{for all } t \in (0, \widehat{T}_{\max}),$$

where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$ and $\lambda_+ := \max\{0, \lambda\}$.

Proof. Integrating the first equation in (2.1.1) over Ω, we obtain

$$\frac{d}{dt}\int_\Omega u\,dx = \lambda \int_\Omega u\,dx - \mu \int_\Omega u^\kappa\,dx$$

for all $t \in (0, T_{\max})$. Hence an argument similar to that in [**60**, Lemma 3.1] implies the conclusion of this lemma. □

The following lemma is proved by the same argument as that in [**60**, Lemma 3.2].

Lemma 2.3.2. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Then for all $M_0 > 0$ there exists $C = C(R, \lambda, M_0) > 0$ such that if the initial data u_0 satisfies* (2.1.2) *and $\int_\Omega u_0(x)\,dx \leq M_0$, then*

$$|v_r(r,t)| \leq Cr^{1-n} \quad \text{for all } r \in (0,R) \text{ and } t \in (0, \widehat{T}_{\max})$$

and

$$v(r,t) \leq Cr^{2-n} \quad \text{for all } r \in (0,R) \text{ and } t \in (0, \widehat{T}_{\max}),$$

where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Noting that $(u+1)^{\alpha-1} \leq 1$, we can obtain the following lemma by an argument similar to that in [**60**, Lemma 3.3].

Lemma 2.3.3. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Then for all $\widetilde{L} > 0$, $M_0 > 0$ and $\varepsilon > 0$ there exists $C = C(R, \lambda, \widetilde{L}, M_0, \varepsilon) > 0$ such that if u_0 satisfies* (2.1.2)*, $\int_\Omega u_0(x)\,dx \leq M_0$ and*

$$u_0(r) \leq \widetilde{L} r^{-n(n-1)} \quad \text{for all } r \in (0,R), \tag{2.3.1}$$

then

$$u(r,t) \leq Cr^{-n(n-1)-\varepsilon} \quad \text{for all } r \in (0,R) \text{ and } t \in (0, \widehat{T}_{\max}),$$

where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

2.4. Differential inequalities for a moment-type functional ϕ

In order to show the main result we establish a key inequality (see Lemma 2.4.12). To this end, we first give a lower estimate for the derivative of ϕ defined in (2.4.1). Functions of the form in (2.4.1) have been the core of precedents where blow-up has been detected for related systems; recent examples in this regard include "pure" Keller–Segel systems [**62**], but also logistic-type chemotaxis systems [**12**] and some chemotaxis systems involving saturation effects in the signal production mechanism [**59**].

Lemma 2.4.1. *Let $n \ge 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Assume that u_0 satisfies (2.1.2). Let $\gamma \in (1 - \frac{2}{n}, 1)$ and $s_0 \in (0, R^n)$. Define*

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in [0, T_{\max}). \tag{2.4.1}$$

Then $\phi \in C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ and

$$\begin{aligned}
\phi'(t) \ge \chi n &\int_0^{s_0} s^{-\gamma}(s_0 - s)w_s(s,t)(nw_s(s,t)+1)^{\alpha-1}w(s,t)\,ds \\
&- \chi n(\gamma+1)s_0 \int_0^{s_0} s^{-\gamma-1}z(s,t)w(s,t)\,ds \\
&- n^2\left(2 - \frac{2}{n} - \gamma\right)\left(\gamma + \frac{2}{n}\right)s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}}w(s,t)\,ds \\
&- \lambda_- \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds - \frac{n^{\kappa-1}\mu}{1-\gamma}s_0^{1-\gamma}\int_0^{s_0}(s_0 - s)w_s^{\kappa}(s,t)\,ds
\end{aligned} \tag{2.4.2}$$

for all $t \in (0, T_{\max})$, where $\lambda_- := \max\{0, -\lambda\}$.

Proof. By an argument similar to that in the proof of [**60**, Lemma 4.1], we can show that $\phi \in C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$. Moreover, we see from (2.2.6) that

$$\begin{aligned}
\phi'(t) &= \int_0^{s_0} s^{-\gamma}(s_0 - s)w_t\,ds \\
&= n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)w_{ss}\,ds \\
&\quad + \chi n \int_0^{s_0} s^{-\gamma}(s_0 - s)w_s(nw_s+1)^{\alpha-1}w\,ds \\
&\quad - \chi n \int_0^{s_0} s^{-\gamma}(s_0 - s)w_s(nw_s+1)^{\alpha-1}z\,ds \\
&\quad + \lambda \int_0^{s_0} s^{-\gamma}(s_0 - s)w\,ds - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s)\left\{\int_0^s w_s^{\kappa}(\sigma,t)\,d\sigma\right\}ds
\end{aligned} \tag{2.4.3}$$

for all $t \in (0, T_{\max})$. Because $0 < \alpha < 1$ and $w_s \geq 0$, we have $(nw_s + 1)^{\alpha-1} \leq 1$. Thus, we can estimate the third term on the right-hand side of (2.4.3) as

$$-\chi n \int_0^{s_0} s^{-\gamma}(s_0 - s) w_s (nw_s + 1)^{\alpha-1} z \, ds \geq -\chi n \int_0^{s_0} s^{-\gamma}(s_0 - s) w_s z \, ds. \tag{2.4.4}$$

Estimating the right-hand side of (2.4.4) and the first, fourth and fifth terms on the right-hand side of (2.4.3) similarly in [**60**, Proof of Lemma 4.1], we obtain (2.4.2). □

2.4.1. Estimates for the three integrals in the inequality for ϕ'

In this subsection we show estimates for the first, third, fourth and fifth terms on the right-hand side of (2.4.2). First we establish an estimate for the first term on the right-hand side of (2.4.2).

Lemma 2.4.2. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$ and $\gamma \in \left(1 - \frac{2}{n}, 1\right)$. For all $\widetilde{L} > 0$, $M_0 > 0$ and $\varepsilon > 0$, there is $C = C(R, \alpha, \lambda, \widetilde{L}, M_0, \varepsilon) > 0$ such that if u_0 satisfies* (2.1.2), (2.3.1) *and $\int_\Omega u_0(x)\, dx \leq M_0$, then for each $s_0 \in (0, R)$,*

$$\begin{aligned} &\int_0^{s_0} s^{-\gamma}(s_0 - s) w_s(s,t)(nw_s(s,t) + 1)^{\alpha-1} w(s,t) \, ds \\ &\quad \geq C \int_0^{s_0} s^{-\gamma + (n-1)(1-\alpha) + \frac{\varepsilon}{n}(1-\alpha)} (s_0 - s) w_s(s,t) w(s,t) \, ds \end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. Let $\varepsilon > 0$ and u_0 satisfy (2.1.2) and (2.3.1) as well as $\int_\Omega u_0(x)\, dx \leq M_0$. Then, according to Lemma 2.3.3, we can find $c_1 = c_1(R, \lambda, \widetilde{L}, M_0, \varepsilon) > 0$ such that

$$u(r,t) \leq c_1 r^{-n(n-1)-\varepsilon} \quad \text{for all } r \in (0,R) \text{ and } t \in (0, \widehat{T}_{\max}).$$

Therefore the first equality of (2.2.3) and this inequality yield

$$nw_s(s,t) = u(s^{\frac{1}{n}}, t) \leq c_1 s^{-(n-1)-\frac{\varepsilon}{n}} \quad \text{for all } s \in (0, R^n) \text{ and } t \in (0, \widehat{T}_{\max}). \tag{2.4.5}$$

Thanks to (2.4.5), we infer from the condition $0 < \alpha < 1$ that for each $s_0 \in (0, R^n)$,

$$\begin{aligned} &\int_0^{s_0} s^{-\gamma}(s_0 - s) w_s (nw_s + 1)^{\alpha-1} w \, ds \\ &\quad \geq \int_0^{s_0} s^{-\gamma}(s_0 - s)(c_1 s^{-(n-1)-\frac{\varepsilon}{n}} + 1)^{\alpha-1} w_s w \, ds \\ &\quad = \int_0^{s_0} s^{-\gamma + (n-1)(1-\alpha) + \frac{\varepsilon}{n}(1-\alpha)} (s_0 - s)(c_1 + s^{(n-1)+\frac{\varepsilon}{n}})^{\alpha-1} w_s w \, ds \\ &\quad \geq (c_1 + R^{n(n-1)+\varepsilon})^{\alpha-1} \int_0^{s_0} s^{-\gamma + (n-1)(1-\alpha) + \frac{\varepsilon}{n}(1-\alpha)} (s_0 - s) w_s w \, ds \end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$, which concludes the proof. □

We next have the following lemma which plays an important role in obtaining an estimate for w defined in (2.2.1). The proof is based on that of [**60**, Lemma 4.2].

Lemma 2.4.3. *Let $\gamma \in (0,2)$, $0<\beta<\gamma$ and $s_0>0$. Assume that $\varphi \in C^1([0,s_0])$ is nonnegative, $\varphi(0)=0$ and $\varphi'(s)\ge 0$ for all $s\in(0,s_0)$. Then*

$$\varphi(s) \le (\beta+2)^{\frac{1}{\beta+2}} s^{\frac{\gamma-\beta}{\beta+2}} (s_0-s)^{-\frac{1}{\beta+2}} \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0-\sigma)\varphi^{\beta+1}(\sigma)\varphi'(\sigma)\,d\sigma \right\}^{\frac{1}{\beta+2}} \tag{2.4.6}$$

for all $s \in (0,s_0)$.

Proof. We set $\psi(s) := \frac{1}{\beta+2} s^{-\gamma+\beta}(s_0-s)\varphi^{\beta+2}(s)$, $s\in(0,s_0]$. Since $\gamma\in(0,2)$, $0<\beta<\gamma$, $\varphi' \in C^0([0,s_0])$ and $\varphi(0)=0$, we can regard ψ as a function in $C^0([0,s_0])\cap C^1((0,s_0))$ with $\psi(0)=0$. Therefore we see from a direct computation that for all $s\in(0,s_0)$,

$$\begin{aligned}
\psi(s) &= \int_0^s \psi'(\sigma)\,d\sigma \\
&= \int_0^s \Big\{ \sigma^{-\gamma+\beta}(s_0-\sigma)\varphi^{\beta+1}(\sigma)\varphi'(\sigma) - \frac{\gamma-\beta}{\beta+2}\sigma^{-\gamma+\beta-1}(s_0-\sigma)\varphi^{\beta+2}(\sigma) \\
&\qquad\qquad - \frac{1}{\beta+2}\sigma^{-\gamma+\beta}\varphi^{\beta+2}(\sigma) \Big\}\,d\sigma \\
&\le \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0-\sigma)\varphi^{\beta+1}(\sigma)\varphi'(\sigma)\,d\sigma,
\end{aligned}$$

which derives (2.4.6) from the definition of ψ. □

In order to prepare estimates for the third, fourth and fifth terms on the right-hand side of (2.4.2) we derive an estimate for w in terms of ww_s. Let $\varepsilon>0$, $\gamma\in(1-\frac{2}{n},1)$ and $s_0\in(0,R^n)$. We put

$$\beta := (n-1)(1-\alpha) + \frac{\varepsilon}{n}(1-\alpha) < \gamma. \tag{2.4.7}$$

Assume $M_0>0$ and $\int_\Omega u_0(x)\,dx \le M_0$. Then, by means of Lemma 2.3.1, we can take $c_1 = c_1(R,\lambda,M_0)>0$ such that

$$w(\sigma,t)\le c_1$$

for all $\sigma\in(0,s_0)$ and $t\in(0,\widehat{T}_{\max})$. Applying Lemma 2.4.3 to w, from this inequality we can find $c_2=c_2(\gamma,\beta)>0$ such that

$$\begin{aligned}
w(s,t) &\le c_2 s^{\frac{\gamma-\beta}{\beta+2}}(s_0-s)^{-\frac{1}{\beta+2}} \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0-\sigma) w^{\beta+1} w_s\,d\sigma \right\}^{\frac{1}{\beta+2}} \\
&\le c_2 c_1^{\frac{\beta}{\beta+2}} s^{\frac{\gamma-\beta}{\beta+2}}(s_0-s)^{-\frac{1}{\beta+2}} \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0-\sigma) w w_s\,d\sigma \right\}^{\frac{1}{\beta+2}}
\end{aligned} \tag{2.4.8}$$

for all $t \in (0, \widehat{T}_{\max})$. Using (2.4.8), we establish estimates for the third, fourth and fifth terms on the right-hand side of (2.4.2) in the following three lemmas.

Lemma 2.4.4. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$, and let $\gamma \in (0,2)$ and $0 < \beta < \gamma$ be such that*

$$\frac{1}{\beta+1}\left(2 - \frac{4}{n}\right) - \frac{\beta}{\beta+1} \cdot \frac{2}{n} > \gamma. \tag{2.4.9}$$

Then for all $M_0 > 0$ there exists $C = C(R, \lambda, M_0, \gamma, \beta) > 0$ such that if u_0 satisfies (2.1.2) and $\int_\Omega u_0(x)\, dx \leq M_0$, then for each $s_0 \in (0, R^n)$,

$$s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w(s,t)\, ds \leq C s_0^{2-\gamma-\frac{2}{n}+\frac{\gamma-\beta-1}{\beta+2}} \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0 - s) w(s,t) w_s(s,t)\, ds \right\}^{\frac{1}{\beta+2}}$$

for all $t \in (0, \widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. The proof of this lemma is similar to that of [**60**, Lemma 4.3]. By virtue of (2.4.8), we see that there exists $c_1 = c_1(R, \lambda, M_0, \gamma, \beta) > 0$ such that

$$\begin{aligned} &s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}} w\, ds \\ &\quad \leq c_1 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0 - \sigma) w w_s\, d\sigma \right\}^{\frac{1}{\beta+2}} s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} (s_0 - s)^{-\frac{1}{\beta+2}}\, ds \end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$. By a variable transformation as $s = s_0\sigma$, we obtain

$$\begin{aligned} &s_0 \int_0^{s_0} s^{-\gamma-\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} (s_0 - s)^{-\frac{1}{\beta+2}}\, ds \\ &\quad = s_0 \int_0^1 (s_0\sigma)^{-\gamma-\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} (s_0 - s_0\sigma)^{-\frac{1}{\beta+2}} s_0\, ds \\ &\quad = s_0^{2-\gamma-\frac{2}{n}+\frac{\gamma-\beta-1}{\beta+2}} B\left(1 - \gamma - \frac{2}{n} + \frac{\gamma-\beta}{\beta+2}, 1 - \frac{1}{\beta+2}\right), \end{aligned}$$

where B is Euler's Beta function. Now, noting from (2.4.9) that

$$\begin{aligned} 1 - \gamma - \frac{2}{n} + \frac{\gamma-\beta}{\beta+2} &= 1 - \frac{2}{n} - \frac{\beta}{\beta+2} - \frac{\beta+1}{\beta+2} \cdot \gamma \\ &> 1 - \frac{2}{n} - \frac{\beta}{\beta+2} - \frac{1}{\beta+2}\left(2 - \frac{4}{n}\right) + \frac{\beta}{\beta+2} \cdot \frac{2}{n} \\ &= 0, \end{aligned}$$

we have $B\big(1 - \gamma - \frac{2}{n} + \frac{\gamma-\beta}{\beta+2}, 1 - \frac{1}{\beta+2}\big) < \infty$. Thus we attain the proof. □

Next we derive an estimate for the fourth term.

Lemma 2.4.5. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$, and let $\gamma \in (0, 2)$ and $0 < \beta < \gamma$ be such that*

$$\frac{2}{1+\beta} > \gamma. \tag{2.4.10}$$

Then for all $M_0 > 0$ there exists $C = C(R, \lambda, M_0, \gamma, \beta) > 0$ such that if u_0 satisfies (2.1.2) and $\int_\Omega u_0(x)\, dx \leq M_0$, then for each $s_0 \in (0, R^n)$,

$$\begin{aligned} &\int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\, ds \\ &\quad \leq C s_0^{2-\gamma+\frac{\gamma-\beta-1}{\beta+2}} \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0 - s) w(s,t) w_s(s,t)\, ds \right\}^{\frac{1}{\beta+2}} \end{aligned} \tag{2.4.11}$$

for all $t \in (0, \widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. The proof is based on that of [**60**, Lemma 4.4]. From (2.4.10) we have

$$\begin{aligned} 1 - \gamma + \frac{\gamma - \beta}{\beta + 2} &= 1 - \frac{\beta}{\beta+2} - \frac{\beta+1}{\beta+2} \cdot \gamma \\ &> 1 - \frac{\beta}{\beta+2} - \frac{2}{\beta+2} \\ &= 0. \end{aligned}$$

Hence, noting that $B\big(1 - \gamma + \frac{\gamma-\beta}{\beta+2}, 1 - \frac{1}{\beta+2}\big) < \infty$, we obtain from (2.4.8) that there exists $c_1 = c_1(R, \lambda, M_0, \gamma, \beta) > 0$ such that

$$\begin{aligned} &\int_0^{s_0} s^{-\gamma}(s_0 - s) w\, ds \\ &\quad \leq s_0 \int_0^{s_0} s^{-\gamma} w\, ds \\ &\quad \leq c_1 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0 - \sigma) w w_s\, d\sigma \right\}^{\frac{1}{\beta+2}} s_0 \int_0^{s_0} s^{-\gamma+\frac{\gamma-\beta}{\beta+2}} (s_0 - s)^{-\frac{1}{\beta+2}}\, ds \\ &\quad = c_1 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0 - \sigma) w w_s\, d\sigma \right\}^{\frac{1}{\beta+2}} s_0^{2-\gamma+\frac{\gamma-\beta-1}{\beta+2}} B\left(1 - \gamma + \frac{\gamma-\beta}{\beta+2}, 1 - \frac{1}{\beta+2}\right) \end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$. Thus we attain (2.4.11). $\square$

The next lemma gives an estimate for the last term in the right-hand side of (2.4.2).

Lemma 2.4.6. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$, and let $\gamma \in (0,2)$ and $0 < \beta < \gamma$ be such that*

$$(n-1)(\kappa-1) < \frac{\gamma-\beta}{\beta+2}. \tag{2.4.12}$$

Then for all $\widetilde{L} > 0$, $M_0 > 0$ and $\varepsilon > 0$ there is $C = C(R,\lambda,\kappa,\gamma,\beta,\widetilde{L},M_0,\varepsilon) > 0$ such that if u_0 satisfies (2.1.2), (2.3.1) and $\int_\Omega u_0(x)\,dx \leq M_0$, then for each $s_0 \in (0,R^n)$,

$$\begin{aligned} &s_0^{1-\gamma}\int_0^{s_0}(s_0-s)w_s^\kappa(s,t)\,ds \\ &\quad \leq C s_0^{2-\gamma-(n-1)(\kappa-1)+\frac{\gamma-\beta-1}{\beta+2}-\varepsilon}\left\{\int_0^{s_0} s^{-\gamma+\beta}(s_0-s)w(s,t)w_s(s,t)\,ds\right\}^{\frac{1}{\beta+2}} \end{aligned}$$

for all $t \in (0,\widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. The proof of this lemma is based on arguments in the proof of [**60**, Lemma 4.5]. Let $\varepsilon > 0$. In view of (2.4.12) we can find $\eta > 0$ fulfilling

$$\frac{\eta}{n}(\kappa-1) < \min\{1,\varepsilon\} \tag{2.4.13}$$

and

$$(n-1)(\kappa-1) + \frac{\eta}{n}(\kappa-1) < \frac{\gamma-\beta}{\beta+2}. \tag{2.4.14}$$

In light of Lemma 2.3.3, there exists $c_1 = c_1(R,\lambda,\widetilde{L},M_0,\varepsilon) > 0$ such that

$$u(r,t) \leq c_1 r^{-n(n-1)-\eta}$$

for all $r \in (0,R)$ and $t \in (0,\widehat{T}_{\max})$. Thus we have from this inequality and the first identity of (2.2.3) that

$$\begin{aligned} w_s^{\kappa-1}(s,t) &= \left(\frac{u(s^{\frac{1}{n}},t)}{n}\right)^{\kappa-1} \\ &\leq c_2 s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} \end{aligned}$$

for all $s \in (0,R^n)$ and $t \in (0,\widehat{T}_{\max})$, where $c_2 := \left(\frac{c_1}{n}\right)^{\kappa-1}$. By using this inequality, we see that

$$s_0^{1-\gamma}\int_0^{s_0}(s_0-s)w_s^\kappa\,ds \leq c_2 s_0^{1-\gamma}\int_0^{s_0} s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)}(s_0-s)w_s\,ds \tag{2.4.15}$$

for all $t \in (0, \widehat{T}_{\max})$. Moreover, from integration by parts, (2.4.13) and the relation $s < s_0$, we infer that

$$
\begin{aligned}
& c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)}(s_0 - s) w_s \, ds \\
&\quad = -\liminf_{\delta \searrow 0} \left\{ c_2 s_0^{1-\gamma} \delta^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)}(s_0 - \delta) w(\delta, t) \right\} \\
&\qquad + \left[(n-1)(\kappa-1) + \frac{\eta}{n}(\kappa-1) \right] c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)}(s_0 - s) w \, ds \\
&\qquad + c_2 s_0^{1-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-\frac{\eta}{n}(\kappa-1)} w \, ds \\
&\quad \le [(n-1)(\kappa-1)+1] c_2 s_0^{2-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w \, ds \\
&\qquad + c_2 s_0^{2-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w \, ds \\
&\quad = [(n-1)(\kappa-1)+2] c_2 s_0^{2-\gamma} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w \, ds
\end{aligned}
\tag{2.4.16}
$$

for all $t \in (0, \widehat{T}_{\max})$. Recalling (2.4.8), we obtain $c_3 = c_3(R, \lambda, M_0, \gamma, \beta) > 0$ such that

$$
\begin{aligned}
& \int_0^{s_0} s^{-(n-1)(\kappa-1)-1-\frac{\eta}{n}(\kappa-1)} w \, ds \\
&\quad \le c_3 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0 - \sigma) w w_s \, d\sigma \right\}^{\frac{1}{\beta+2}} \int_0^{s_0} s^{-(n-1)(\kappa-1)-1+\frac{\gamma-\beta}{\beta+2}-\frac{\eta}{n}(\kappa-1)}(s_0 - s)^{-\frac{1}{\beta+2}} \, ds \\
&\quad = c_3 c_4 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0 - \sigma) w w_s \, d\sigma \right\}^{\frac{1}{\beta+2}} s_0^{-(n-1)(\kappa-1)+\frac{\gamma-\beta-1}{\beta+2}-\frac{\eta}{n}(\kappa-1)},
\end{aligned}
\tag{2.4.17}
$$

where $c_4 := B\big(\frac{\gamma-\beta}{\beta+2} - (n-1)(\kappa-1) - \frac{\eta}{n}(\kappa-1), 1 - \frac{1}{\beta+2}\big)$, which is finite from (2.4.14). A combination of (2.4.15)–(2.4.17) and the relation (2.4.13) imply that

$$
\begin{aligned}
& s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa \, ds \\
&\quad \le [(n-1)(\kappa-1)+2] c_5 s_0^{2-\gamma-(n-1)(\kappa-1)+\frac{\gamma-\beta-1}{\beta+2}-\frac{\eta}{n}(\kappa-1)} \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0 - s) w w_s \, ds \right\}^{\frac{1}{\beta+2}} \\
&\quad \le [(n-1)(\kappa-1)+2] c_5 c_6 s_0^{2-\gamma-(n-1)(\kappa-1)+\frac{\gamma-\beta-1}{\beta+2}-\varepsilon} \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0 - s) w w_s \, ds \right\}^{\frac{1}{\beta+2}}
\end{aligned}
$$

for all $t \in (0, \widehat{T}_{\max})$, where $c_5 := c_2 c_3 c_4$ and $c_6 := R^{n\varepsilon - \eta(\kappa-1)}$, which concludes the proof. □

2.4.2. Pointwise estimate for a math accumulation function z

We first state the following lemma which was proved in [**60**].

Lemma 2.4.7. *Let $\alpha \in (1,2)$, $\beta \in (0,1)$. Then there exists $C = C(\alpha,\beta) > 0$ such that if $s_0 > 0$, then*

$$\int_0^s \int_\sigma^{s_0} \xi^{-\alpha}(s_0-\xi)^{-\beta}\, d\xi d\sigma \le C s_0^{-\beta} s^{2-\alpha} \quad \text{for all } s \in (0,s_0).$$

Next, in order to have an estimate for the second term of the right-hand side of (2.4.2) we establish an estimate for z defined in (2.2.2).

Lemma 2.4.8. *Let $n \ge 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$, $\gamma \in \left(0, 2-\frac{4}{n}\right)$ and $0 < \beta < \gamma$. Then for all $M_0 > 0$ there is $C = C(R,\lambda,M_0,\gamma,\beta) > 0$ such that if u_0 satisfies (2.1.2) and $\int_\Omega u_0(x)\,dx \le M_0$, then for each $s_0 \in (0,R^n)$, z defined in (2.2.2) fulfills*

$$z(s,t) \le C s_0^{\frac{2}{n}-1} s + C s_0^{-\frac{1}{\beta+2}} s^{\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0-s) w(s,t) w_s(s,t)\, ds \right\}^{\frac{1}{\beta+2}}$$

for all $s \in (0,s_0)$ and $t \in (0,\widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. The proof is similar to that of [**60**, Lemma 4.7]. According to Lemma 2.3.2, there exists $c_1 = c_1(R,\lambda,M_0) > 0$ such that

$$v(r,t) \le c_1 r^{2-n} \tag{2.4.18}$$

for all $r \in (0,R)$ and $t \in (0,\widehat{T}_{\max})$. We infer from (2.2.5) that

$$r^{n-1} v_r(r,t) \ge -w(r^n,t)$$

for all $r \in (0,R)$ and $t \in (0,T_{\max})$. Thus, for all $s \in (0,R^n)$ and $t \in (0,T_{\max})$ we obtain from the second identity of (2.2.4) that

$$z_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} v_r(s^{\frac{1}{n}},t) \ge -\frac{1}{n^2} s^{\frac{2}{n}-2} w(s,t).$$

Let $s_0 \in (0,R^n)$. By making use of (2.4.18) and this inequality, we can observe that

$$\begin{aligned} z_s(s,t) &= z_s(s_0,t) - \int_s^{s_0} z_{ss}(\sigma,t)\, d\sigma \\ &= \frac{1}{n} v(s_0^{\frac{1}{n}},t) - \int_s^{s_0} z_{ss}(\sigma,t)\, d\sigma \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} + \frac{1}{n^2}\int_s^{s_0} \sigma^{\frac{2}{n}-2} w(\sigma,t)\, d\sigma \end{aligned}$$

for all $s \in (0, s_0)$ and $t \in (0, \widehat{T}_{\max})$. Therefore, z defined in (2.2.2) satisfies

$$\begin{aligned} z(s,t) &= \int_0^s z_s(\sigma, t)\, d\sigma \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{1}{n^2} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2} w(\xi, t)\, d\xi d\sigma \end{aligned}$$

for all $s \in (0, s_0)$ and $t \in (0, \widehat{T}_{\max})$. Moreover, applying (2.4.8) to w of the second term on the right-hand side of the above inequality, we find $c_2 = c_2(\lambda, M_0, \gamma, \beta) > 0$ such that

$$\begin{aligned} &z(s,t) \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{c_2}{n^2} \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta} (s_0 - \sigma) w w_s \, d\sigma \right\}^{\frac{1}{\beta+2}} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\gamma-\beta}{\beta+2}} (s_0 - \xi)^{-\frac{1}{\beta+2}}\, d\xi d\sigma \end{aligned}$$

for all $s \in (0, s_0)$ and $t \in (0, \widehat{T}_{\max})$. Since $\gamma \in \left(0, 2 - \frac{4}{n}\right)$, we see that $\frac{\gamma-\beta}{\beta+2} < \frac{\gamma}{2} < 1 - \frac{2}{n}$. Thus we have that

$$2 > -\frac{2}{n} + 2 - \frac{\gamma - \beta}{\beta + 2} > -\frac{2}{n} + 2 - 1 + \frac{2}{n} = 1.$$

By virtue of Lemma 2.4.7, it follows that there exists $c_3 = c_3(\gamma, \beta) > 0$ such that

$$\int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\gamma-\beta}{\beta+2}} (s_0 - \xi)^{-\frac{1}{\beta+2}}\, d\xi d\sigma \le c_3 s_0^{-\frac{1}{\beta+2}} s^{\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}}$$

for all $s \in (0, s_0)$, which infers the claim. □

2.4.3. Estimate for the integral involving *z*

In this subsection we establish an estimate for the second term on the right-hand side of (2.4.2). The proof of the following lemma is based on that of [**60**, Lemma 4.8].

Lemma 2.4.9. *Let $n \ge 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$, and let $\gamma \in (0,1)$ with $\gamma < 2 - \frac{4}{n}$ and $0 < \beta < \gamma$ be such that*

$$\frac{2}{n} \cdot \frac{\beta + 2}{\beta} - 2 > \gamma. \tag{2.4.19}$$

Then, for all $M_0 > 0$ there exists $C = C(R, \lambda, M_0, \gamma, \beta) > 0$ such that if u_0 satisfies (2.1.2) and $\int_\Omega u_0(x)\, dx \le M_0$, then for each $s_0 \in (0, R^n)$,

$$\begin{aligned} &n(\gamma+1) s_0 \int_0^{s_0} s^{-\gamma-1} z(s,t) w(s,t)\, ds \\ &\quad \le C s_0^{\frac{2}{n}+1-\gamma} + C s_0^{1+\frac{2}{n}+\frac{2(\gamma-\beta-1)}{\beta+2}-\gamma} \left\{ \int_0^{s_0} s^{-\gamma+\beta} (s_0 - s) w(s,t) w_s(s,t)\, ds \right\}^{\frac{2}{\beta+2}} \end{aligned} \tag{2.4.20}$$

for all $t \in (0, \widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$.

Proof. From Lemma 2.4.8 we find $c_1 = c_1(R, \lambda, M_0, \gamma) > 0$ such that

$$\begin{aligned}
s_0 \int_0^{s_0} & s^{-\gamma-1} zw\, ds \\
&\le c_1 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} w\, ds \\
&\quad + c_1 \left\{ \int_0^{s_0} \sigma^{-\gamma+\beta}(s_0-\sigma) w w_s\, d\sigma \right\}^{\frac{1}{\beta+2}} s_0^{1-\frac{1}{\beta+2}} \int_0^{s_0} s^{-\gamma-1+\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} w\, ds \qquad (2.4.21)
\end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$. Moreover, Lemma 2.3.1 ensures that there is $c_2 = c_2(\lambda, M_0) > 0$ such that

$$w(s,t) \le c_2 \quad \text{for all } s \in (0, R^n) \text{ and } t \in (0, \widehat{T}_{\max}).$$

This estimate and the condition $\gamma < 1$ imply

$$\begin{aligned}
c_1 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} w\, ds &\le c_1 c_2 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma}\, ds \\
&= \frac{c_1 c_2}{1-\gamma} s_0^{\frac{2}{n}+1-\gamma} \qquad (2.4.22)
\end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$. On the other hand, we deduce from (2.4.19) that

$$\begin{aligned}
\frac{2}{n} - \gamma + \frac{2(\gamma-\beta)}{\beta+2} &= \frac{2}{n} - \frac{2\beta}{\beta+2} - \frac{\beta}{\beta+2} \cdot \gamma \\
&> \frac{2}{n} - \frac{2\beta}{\beta+2} - \frac{2}{n} + \frac{2\beta}{\beta+2} \\
&= 0.
\end{aligned}$$

Hence we see from (2.4.8) that there exists $c_3 = c_3(\lambda, M_0, \gamma, \beta) > 0$ such that

$$\begin{aligned}
s_0^{1-\frac{1}{\beta+2}} & \int_0^{s_0} s^{-\gamma-1+\frac{2}{n}+\frac{\gamma-\beta}{\beta+2}} w\, ds \\
&\le c_3 \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0-s) w w_s\, ds \right\}^{\frac{1}{\beta+2}} s_0^{1-\frac{1}{\beta+2}} \int_0^{s_0} s^{-\gamma-1+\frac{2}{n}+\frac{2(\gamma-\beta)}{\beta+2}} (s_0-s)^{-\frac{1}{\beta+2}}\, ds \\
&= c_3 c_4 \left\{ \int_0^{s_0} s^{-\gamma+\beta}(s_0-s) w w_s\, ds \right\}^{\frac{1}{\beta+2}} s_0^{1+\frac{2}{n}-\gamma+\frac{2(\gamma-\beta-1)}{\beta+2}} \qquad (2.4.23)
\end{aligned}$$

for all $t \in (0, \widehat{T}_{\max})$, where $c_4 := B\big(\frac{2}{n} - \gamma + \frac{2(\gamma-\beta)}{\beta+2}, 1 - \frac{1}{\beta+2}\big)$. A combination of (2.4.21), (2.4.22) and (2.4.23) yields (2.4.20). □

2.4.4. Differential inequalities for ϕ

In this subsection we derive a super-linear differential inequality for ϕ defined in (2.4.1). In order to apply Lemmas 2.4.4, 2.4.5, 2.4.6 and 2.4.9 to (2.4.2) we will find $\gamma \in (1-\frac{2}{n}, 1)$ satisfying that $\gamma < 2 - \frac{4}{n}$ and

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < \gamma, \tag{2.4.24}$$

$$\frac{1}{c_{n,\alpha}+1}\left(2-\frac{4}{n}\right) - \frac{c_{n,\alpha}}{c_{n,\alpha}+1}\cdot\frac{2}{n} > \gamma, \tag{2.4.25}$$

$$\frac{2}{n}\cdot\frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2 > \gamma, \tag{2.4.26}$$

$$\frac{2}{c_{n,\alpha}+1} > \gamma, \tag{2.4.27}$$

where

$$c_{n,\alpha} := (n-1)(1-\alpha).$$

We put

$$A_{n,\alpha} := \frac{1}{c_{n,\alpha}+1}\left(2-\frac{4}{n}\right) - \frac{c_{n,\alpha}}{c_{n,\alpha}+1}\cdot\frac{2}{n} \quad \text{and} \quad B_{n,\alpha} := \frac{2}{n}\cdot\frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2.$$

In Sections 2.4.4 and 2.4.4 we treat the cases $A_{n,\alpha} < B_{n,\alpha}$ and $B_{n,\alpha} \le A_{n,\alpha}$, respectively, in order to choose $\gamma \in (1-\frac{2}{n}, 1)$ fulfilling $\gamma < 2-\frac{4}{n}$ and (2.4.24)–(2.4.27). In Section 2.4.4 we derive a super-linear differential inequality for ϕ.

Existence of γ. Case 1: $A_{n,\alpha} < B_{n,\alpha}$

We first prove that in the case $A_{n,\alpha} < B_{n,\alpha}$ there exists $\gamma \in (1-\frac{2}{n}, 1)$ with $\gamma < 2-\frac{4}{n}$ satisfying (2.4.24)–(2.4.27).

Lemma 2.4.10. *Let $n \ge 3$, $0 < \alpha < 1$ and $\kappa > 1$. Assume that α and κ satisfy the following conditions:*

$$n = 3, \qquad \tfrac{5}{6} < \alpha < 1 \quad \textit{and} \quad \kappa < 1 + \tfrac{6\alpha-5}{6(3-2\alpha)}, \tag{2.4.28}$$

$$n = 4, \qquad \tfrac{5}{6} < \alpha < 1 \quad \textit{and} \quad \kappa < 1 + \tfrac{6\alpha-5}{6(4-3\alpha)}, \tag{2.4.29}$$

$$n = 5, \qquad \begin{cases} \tfrac{11}{12} < \alpha \le \tfrac{27}{28} \quad \textit{and} \quad \kappa < 1 + \tfrac{20\alpha-17}{20(5-4\alpha)}, \\ \tfrac{27}{28} < \alpha < 1 \quad \textit{and} \quad \kappa < 1 + \tfrac{4\alpha-3}{8(3-2\alpha)}, \end{cases} \tag{2.4.30}$$

$$n \ge 6, \qquad 1 - \tfrac{1}{(n-2)(n-1)} < \alpha < 1 \quad \textit{and} \quad \kappa < 1 + \tfrac{1-(n-1)(1-\alpha)}{(n-1)\{2+(n-1)(1-\alpha)\}}. \tag{2.4.31}$$

Then there exists $\gamma \in (1-\frac{2}{n}, 1)$ such that $\gamma < 2-\frac{4}{n}$ and (2.4.24)–(2.4.27) hold.

Proof. First we have $A_{n,\alpha} < B_{n,\alpha}$. Indeed, in the case $n = 3$, noting that $c_{3,\alpha} = 2(1-\alpha)$, we obtain from (2.4.28) that

$$\begin{aligned} B_{3,\alpha} - A_{3,\alpha} &= \left(\frac{2}{3} \cdot \frac{c_{3,\alpha}+2}{c_{3,\alpha}} - 2\right) - \left(\frac{1}{c_{3,\alpha}+1} \cdot \frac{2}{3} - \frac{c_{3,\alpha}}{c_{3,\alpha}+1} \cdot \frac{2}{3}\right) \\ &= \frac{(4\alpha-2)(2-\alpha)}{3(1-\alpha)(3-2\alpha)} > 0. \end{aligned}$$

In the case $n = 4$, invoking that $c_{4,\alpha} = 3(1-\alpha)$, we use (2.4.29) to deduce that

$$\begin{aligned} B_{4,\alpha} - A_{4,\alpha} &= \left(\frac{1}{2} \cdot \frac{c_{4,\alpha}+2}{c_{4,\alpha}} - 2\right) - \left(\frac{1}{c_{4,\alpha}+1} - \frac{c_{4,\alpha}}{c_{4,\alpha}+1} \cdot \frac{1}{2}\right) \\ &= \frac{(5-3\alpha)(6\alpha-5)}{6(1-\alpha)(4-3\alpha)} > 0. \end{aligned}$$

In the case $n = 5$, by noticing that $c_{5,\alpha} = 4(1-\alpha)$, it follows from (2.4.30) that

$$\begin{aligned} B_{5,\alpha} - A_{5,\alpha} &= \left(\frac{2}{5} \cdot \frac{c_{5,\alpha}+2}{c_{5,\alpha}} - 2\right) - \left(\frac{1}{c_{5,\alpha}+1} \cdot \frac{6}{5} - \frac{c_{5,\alpha}}{c_{5,\alpha}+1} \cdot \frac{2}{5}\right) \\ &= \frac{(12\alpha-11)(3-2\alpha)}{5(5-4\alpha)(1-\alpha)} > 0. \end{aligned}$$

In the case $n \geq 6$, recalling $c_{n,\alpha} = (n-1)(1-\alpha)$, we see from (2.4.31) that

$$\begin{aligned} B_{n,\alpha} - A_{n,\alpha} &= \left(\frac{2}{n} \cdot \frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2\right) - \left(\frac{1}{c_{n,\alpha}+1}\left(2 - \frac{4}{n}\right) - \frac{c_{n,\alpha}}{c_{n,\alpha}+1} \cdot \frac{2}{n}\right) \\ &= \frac{2[1-(n-2)c_{n,\alpha}](c_{n,\alpha}+2)}{nc_{n,\alpha}(c_{n,\alpha}+1)} \\ &= \frac{2\{1-(n-2)(n-1)(1-\alpha)\}(c_{n,\alpha}+2)}{nc_{n,\alpha}(c_{n,\alpha}+1)} > 0. \end{aligned}$$

Next we show that the following conditions hold:

$$A_{n,\alpha} > 1 - \frac{2}{n}, \tag{2.4.32}$$

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < \min\left\{1, 2-\frac{4}{n}\right\}, \tag{2.4.33}$$

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < A_{n,\alpha}. \tag{2.4.34}$$

In the case $n = 3$, since $\frac{5}{6} < \alpha < 1$ and $\kappa < 1 + \frac{6\alpha-5}{6(3-2\alpha)}$, we have

$$A_{3,\alpha} = \frac{1}{c_{3,\alpha}+1} \cdot \frac{2}{3} - \frac{c_{3,\alpha}}{c_{3,\alpha}+1} \cdot \frac{2}{3} = \frac{4\alpha-2}{3(3-2\alpha)} > \frac{1}{3} = 1 - \frac{2}{3} \tag{2.4.35}$$

and

$$\begin{aligned}2(\kappa-1)(2+c_{3,\alpha})+c_{3,\alpha} &= 2(\kappa-1)(4-2\alpha)+2(1-\alpha)\\ &< \frac{4\alpha-2}{3(3-2\alpha)}\\ &< \frac{2}{3} = 2-\frac{4}{3}.\end{aligned}$$

Moreover, noting the left-hand side in (2.4.35), we obtain

$$2(\kappa-1)(2+c_{3,\alpha})+c_{3,\alpha} < \frac{4\alpha-2}{3(3-2\alpha)} = A_{3,\alpha},$$

that is, in the case $n=3$ we conclude that (2.4.32), (2.4.33) and (2.4.34) hold. Similarly, in the case $n=4$, since $\frac{5}{6}<\alpha<1$ and $\kappa<1+\frac{6\alpha-5}{6(4-3\alpha)}$, we see that

$$A_{4,\alpha} = \frac{1}{c_{4,\alpha}+1} - \frac{c_{4,\alpha}}{c_{4,\alpha}+1}\cdot\frac{1}{2} = \frac{3\alpha-1}{2(4-3\alpha)} > \frac{1}{2} = 1-\frac{2}{4}$$

and

$$\begin{aligned}3(\kappa-1)(2+c_{4,\alpha})+c_{4,\alpha} &= 3(\kappa-1)(5-3\alpha)+3(1-\alpha)\\ &< \frac{3\alpha-1}{2(4-3\alpha)}\\ &< 1\end{aligned}$$

as well as

$$3(\kappa-1)(2+c_{4,\alpha})+c_{4,\alpha} < \frac{3\alpha-1}{2(4-3\alpha)} = A_{4,\alpha},$$

that is, in the case $n=4$ we attain (2.4.32), (2.4.33) and (2.4.34). We next observe the case $n=5$. If $\frac{11}{12}<\alpha\le\frac{27}{28}$ and $\kappa<1+\frac{20\alpha-17}{20(5-4\alpha)}$, then

$$A_{5,\alpha} = \frac{1}{c_{5,\alpha}+1}\cdot\frac{6}{5} - \frac{c_{5,\alpha}}{c_{5,\alpha}+1}\cdot\frac{2}{5} = \frac{8\alpha-2}{5(5-4\alpha)} > \frac{4}{5} > \frac{3}{5} = 1-\frac{2}{5}$$

and

$$\begin{aligned}4(\kappa-1)(2+c_{5,\alpha})+c_{5,\alpha} &= 4(\kappa-1)(6-4\alpha)+4(1-\alpha)\\ &< \frac{8\alpha-2}{5(5-4\alpha)}\\ &\le 1\end{aligned}$$

as well as

$$4(\kappa-1)(2+c_{5,\alpha})+c_{5,\alpha}<\frac{8\alpha-2}{5(5-4\alpha)}=A_{5,\alpha}.$$

If $\frac{27}{28}<\alpha<1$ and $\kappa<1+\frac{4\alpha-3}{8(3-2\alpha)}$, then, noting that $\frac{11}{12}<\alpha$, we obtain

$$A_{5,\alpha}>1-\frac{2}{5}$$

and

$$\begin{aligned}4(\kappa-1)(2+c_{5,\alpha})+c_{5,\alpha}&=4(\kappa-1)(6-4\alpha)+4(1-\alpha)\\&<4\alpha-3+4-4\alpha=1\end{aligned}$$

as well as, noticing that $A_{5,\alpha}=\frac{8\alpha-2}{5(5-4\alpha)}>1$, we can verify that

$$4(\kappa-1)(2+c_{5,\alpha})+c_{5,\alpha}<1<A_{5,\alpha}.$$

Thus, in the case $n=5$ we have that (2.4.32), (2.4.33) and (2.4.34) hold. In the case $n\geq 6$, invoking $0<1-\alpha<\frac{1}{(n-2)(n-1)}$ and $\kappa<1+\frac{1-(n-1)(1-\alpha)}{(n-1)(2+c_{n,\alpha})}$, we see that

$$\begin{aligned}\frac{1}{c_{n,\alpha}+1}\left(2-\frac{4}{n}\right)-\frac{c_{n,\alpha}}{c_{n,\alpha}+1}\cdot\frac{2}{n}&=\frac{2n-4-2(n-1)(1-\alpha)}{\{(n-1)(1-\alpha)+1\}n}\\&>2-\frac{6}{n}\geq 1>1-\frac{2}{n}\end{aligned}$$

and

$$\begin{aligned}&(n-1)(\kappa-1)(2+c_{n,\alpha})+c_{n,\alpha}\\&\quad<(n-1)\cdot\frac{1-(n-1)(1-\alpha)}{(n-1)(2+c_{n,\alpha})}\cdot(2+c_{n,\alpha})+(n-1)(1-\alpha)\\&\quad=1\end{aligned}$$

as well as

$$(n-1)(\kappa-1)(2+c_{n,\alpha})+c_{n,\alpha}<1<\frac{1}{c_{n,\alpha}+1}\left(2-\frac{4}{n}\right)-\frac{c_{n,\alpha}}{c_{n,\alpha}+1}\cdot\frac{2}{n}=A_{n,\alpha},$$

that is, in the case $n\geq 6$ we infer that (2.4.32), (2.4.33) and (2.4.34) hold. Therefore we can pick $\gamma\in(1-\frac{2}{n},1)$ with $\gamma<2-\frac{4}{n}$ satisfying (2.4.24) and (2.4.25). Noting that $A_{n,\alpha}<B_{n,\alpha}$ and

$$\frac{2}{c_{n,\alpha}+1}>\frac{1}{c_{n,\alpha}+1}\left(2-\frac{4}{n}\right)-\frac{c_{n,\alpha}}{c_{n,\alpha}+1}\cdot\frac{2}{n}=A_{n,\alpha},$$

we attain (2.4.26) and (2.4.27). □

Existence of γ. Case 2: $B_{n,\alpha} \le A_{n,\alpha}$

Next we show that in the case $B_{n,\alpha} \le A_{n,\alpha}$ there is $\gamma \in (1-\frac{2}{n}, 1)$ with $\gamma < 2 - \frac{4}{n}$ satisfying (2.4.24)–(2.4.27).

Lemma 2.4.11. *Let $n \ge 5$, $0 < \alpha < 1$ and $\kappa > 1$. Assume that α and κ satisfy that*

$$n = 5, \qquad \tfrac{10}{11} < \alpha \le \tfrac{11}{12} \quad \textit{and} \quad \kappa < 1 + \tfrac{10\alpha - 9}{40(1-\alpha)}, \tag{2.4.36}$$

$$n \ge 6, \qquad \begin{cases} 1 - \frac{4}{(3n-4)(n-1)} < \alpha \le 1 - \frac{4}{(3n-2)(n-1)} \\ \quad \textit{and} \quad \kappa < 1 + \frac{4-(n-1)^2(1-\alpha)\{2+n(1-\alpha)\}}{n(n-1)^2(1-\alpha)\{2+(n-1)(1-\alpha)\}}, \\ 1 - \frac{4}{(3n-2)(n-1)} < \alpha \le 1 - \frac{1}{(n-2)(n-1)} \\ \quad \textit{and} \quad \kappa < 1 + \frac{1-(n-1)(1-\alpha)}{(n-1)\{2+(n-1)(1-\alpha)\}}. \end{cases} \tag{2.4.37}$$

Then there exists $\gamma \in (1-\frac{2}{n}, 1)$ satisfying (2.4.24)*–*(2.4.27)*.*

Proof. First we can observe that $B_{n,\alpha} \le A_{n,\alpha}$. Indeed, in the case $n = 5$, noting that $c_{5,\alpha} = 4(1-\alpha)$, we deduce from (2.4.36) that

$$\begin{aligned} B_{5,\alpha} - A_{5,\alpha} &= \left(\frac{2}{5} \cdot \frac{c_{5,\alpha}+2}{c_{5,\alpha}} - 2\right) - \left(\frac{1}{c_{5,\alpha}+1}\left(2 - \frac{4}{5}\right) - \frac{c_{5,\alpha}}{c_{5,\alpha}+1} \cdot \frac{2}{5}\right) \\ &= \frac{(12\alpha - 11)(3-2\alpha)}{5(5-4\alpha)(1-\alpha)} \\ &\le 0, \end{aligned}$$

and moreover, in the case $n \ge 6$, recalling $c_{n,\alpha} = (n-1)(1-\alpha)$, we see from (2.4.37) that

$$\begin{aligned} B_{n,\alpha} - A_{n,\alpha} &= \left(\frac{2}{n} \cdot \frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2\right) - \left(\frac{1}{c_{n,\alpha}+1}\left(2 - \frac{4}{n}\right) - \frac{c_{n,\alpha}}{c_{n,\alpha}+1} \cdot \frac{2}{n}\right) \\ &= \frac{2\{1-(n-2)(n-1)(1-\alpha)\}(c_{n,\alpha}+2)}{nc_{n,\alpha}(c_{n,\alpha}+1)} \\ &\le 0. \end{aligned}$$

Next we show that the following conditions hold:

$$B_{n,\alpha} > 1 - \frac{2}{n}, \tag{2.4.38}$$

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < 1, \tag{2.4.39}$$

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < B_{n,\alpha}. \tag{2.4.40}$$

In the case $n = 5$, since $\frac{10}{11} < \alpha \leq \frac{11}{12}$ and $\kappa < 1 + \frac{10\alpha - 9}{40(1-\alpha)}$, we have

$$B_{5,\alpha} = \frac{2}{5} \cdot \frac{c_{5,\alpha} + 2}{c_{5,\alpha}} - 2 = \frac{8\alpha - 7}{5(1-\alpha)} > \frac{3}{5} = 1 - \frac{2}{5}$$

and

$$\begin{aligned} 4(\kappa - 1)(2 + c_{5,\alpha}) + c_{5,\alpha} &= 4(\kappa - 1)(6 - 4\alpha) + 4(1 - \alpha) \\ &< \frac{8\alpha - 7}{5(1-\alpha)} \\ &< \frac{4}{5} < 1, \end{aligned}$$

as well as

$$4(\kappa - 1)(2 + c_{5,\alpha}) + c_{5,\alpha} < \frac{8\alpha - 7}{5(1-\alpha)} = B_{5,\alpha},$$

that is, (2.4.38), (2.4.39) and (2.4.40) hold in the case $n = 5$. We next consider the case $n \geq 6$. If

$$1 - \frac{4}{(3n-4)(n-1)} < \alpha \leq 1 - \frac{4}{(3n-2)(n-1)}$$

and

$$\kappa < 1 + \frac{4 - (n-1)^2(1-\alpha)\{2 + n(1-\alpha)\}}{n(n-1)^2(1-\alpha)(2 + c_{n,\alpha})},$$

then we obtain from (2.4.37) that

$$\begin{aligned} B_{n,\alpha} = \frac{2}{n} \cdot \frac{c_{n,\alpha} + 2}{c_{n,\alpha}} - 2 &= \frac{2}{n} + \frac{2}{n} \cdot \frac{2}{(n-1)(1-\alpha)} - 2 \\ &> \frac{2}{n} + 3 - \frac{4}{n} - 2 = 1 - \frac{2}{n} \end{aligned}$$

and

$$\begin{aligned} &(n-1)(\kappa - 1)(2 + c_{n,\alpha}) + c_{n,\alpha} \\ &\quad < (n-1) \cdot \frac{4 - (n-1)^2(1-\alpha)\{2 + n(1-\alpha)\}}{n(n-1)^2(1-\alpha)(2 + c_{n,\alpha})} \cdot (2 + c_{n,\alpha}) + (n-1)(1-\alpha) \\ &\quad = \frac{4}{n(n-1)(1-\alpha)} - 2 + \frac{2}{n} \\ &\quad \leq 1 \end{aligned}$$

as well as

$$(n-1)(\kappa - 1)(2 + c_{n,\alpha}) + c_{n,\alpha} < \frac{4}{n(n-1)(1-\alpha)} - 2 + \frac{2}{n} = B_{n,\alpha}.$$

If

$$1 - \frac{4}{(3n-2)(n-1)} < \alpha \le 1 - \frac{1}{(n-2)(n-1)}$$

and

$$\kappa < 1 + \frac{1-(n-1)(1-\alpha)}{(n-1)(2+c_{n,\alpha})},$$

then

$$\begin{aligned} B_{n,\alpha} &= \frac{2}{n} \cdot \frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2 \\ &= \frac{2}{n} + \frac{2}{n} \cdot \frac{2}{(n-1)(1-\alpha)} - 2 \\ &> 1 \\ &> 1 - \frac{2}{n} \end{aligned}$$

and

$$\begin{aligned} &(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} \\ &\quad < (n-1) \cdot \frac{1-(n-1)(1-\alpha)}{(n-1)(2+c_{n,\alpha})} \cdot (2+c_{n,\alpha}) + (n-1)(1-\alpha) \\ &\quad = 1 \end{aligned}$$

as well as

$$(n-1)(\kappa-1)(2+c_{n,\alpha}) + c_{n,\alpha} < 1 < \frac{2}{n} \cdot \frac{c_{n,\alpha}+2}{c_{n,\alpha}} - 2 = B_{n,\alpha},$$

that is, in the case $n \ge 6$ we can make sure that (2.4.38), (2.4.39) and (2.4.40) hold. Therefore we can take $\gamma \in (1 - \frac{2}{n}, 1)$ satisfying (2.4.24) and (2.4.26). Noting that $B_{n,\alpha} \le A_{n,\alpha}$ and

$$\begin{aligned} \frac{2}{c_{n,\alpha}+1} &> \frac{1}{c_{n,\alpha}+1}\left(2 - \frac{4}{n}\right) - \frac{c_{n,\alpha}}{c_{n,\alpha}+1} \cdot \frac{2}{n} \\ &= A_{n,\alpha}, \end{aligned}$$

we can verify that (2.4.25) and (2.4.27) hold. □

Derivation of super-linear differential inequalities for ϕ

Finally, by using Lemmas 2.4.10 and 2.4.11 we obtain a super-linear differential inequality for ϕ.

Lemma 2.4.12. *Let $n \geq 3$, $R > 0$, $\chi > 0$, $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, $\mu > 0$ and $\kappa > 1$. Assume that α and κ satisfy* (2.1.6), (2.1.7), (2.1.8) *and* (2.1.9). *Then there exist $\gamma = \gamma(\alpha, \kappa) \in (1 - \frac{2}{n}, 1)$ and $\beta \in (0, \gamma)$ satisfying* (2.4.9), (2.4.10), (2.4.12) *and* (2.4.19) *with the following property: For all $\widetilde{L} > 0$ and $M_0 > 0$ there exists $C = C(R, \chi, \alpha, \lambda, \mu, \kappa, \beta, \widetilde{L}, M_0) > 0$ such that whenever u_0 satisfies* (2.1.2) *and* (2.3.1) *as well as $\int_\Omega u_0(x)\,dx \leq M_0$, for each $s_0 \in (0, R^n)$ the function ϕ defined in* (2.4.1) *fulfills that*

$$\text{if } n = 3, \qquad \phi'(t) \geq \frac{1}{C} s_0^{(\beta+1)\gamma - (\beta+3)} \phi^{\beta+2}(t) - C s_0^{\frac{\beta+3}{\beta+1} - \gamma - \frac{2}{n} \cdot \frac{\beta+2}{\beta+1}}$$

for all $t \in (0, \widehat{T}_{\max})$ and

$$\text{if } n \geq 4, \qquad \phi'(t) \geq \frac{1}{C} s_0^{(\beta+1)\gamma - (\beta+3)} \phi^{\beta+2}(t) - C s_0^{\frac{2}{n} + 1 - \gamma}$$

for all $t \in (0, \widehat{T}_{\max})$.

Proof. From Lemmas 2.4.10 and 2.4.11 we see that there exists $\gamma \in (1 - \frac{2}{n}, 1)$ with $\gamma < 2 - \frac{4}{n}$ satisfying (2.4.24)–(2.4.27). Noting that $c_{n,\alpha} < \gamma$ in view of (2.4.24) and that $c_{n,\alpha} < \frac{2}{n}$ by the conditions for α in (2.1.6), (2.1.7), (2.1.8) and (2.1.9), we can take $\varepsilon > 0$ such that

$$\beta := c_{n,\alpha} + \frac{\varepsilon}{n}(1 - \alpha) < \gamma$$

fulfills (2.4.9), (2.4.10), (2.4.12), (2.4.19) and $\beta < \frac{2}{n}$. Moreover, we choose $\tilde{\varepsilon} > 0$ satisfying that

$$\tilde{\varepsilon} < \frac{5\beta + 1}{3(\beta + 2)} \quad \text{if } n = 3, \tag{2.4.41}$$

$$\frac{\beta + 2}{\beta + 1}\tilde{\varepsilon} < 1 - \frac{1 - \beta}{\beta + 1} \cdot \frac{2}{n} \quad \text{if } n \geq 4. \tag{2.4.42}$$

Thanks to Lemma 2.4.1, we can find $c_1 = c_1(\lambda, \mu, \kappa) > 0$ such that

$$\begin{aligned} \phi'(t) \geq \chi n \int_0^{s_0} s^{-\gamma}(s_0 - s) w_s (n w_s + 1)^{\alpha - 1} w \, ds \\ - \chi n(\gamma + 1) s_0 \int_0^{s_0} s^{-\gamma - 1} z w \, ds - c_1 s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w \, ds \\ - c_1 \int_0^{s_0} s^{-\gamma}(s_0 - s) w \, ds - c_1 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^\kappa \, ds \end{aligned} \tag{2.4.43}$$

for all $t \in (0, T_{\max})$. Now we set

$$\psi(t) := \int_0^{s_0} s^{-\gamma+\beta}(s_0 - s) w w_s \, ds \quad t \in (0, T_{\max}).$$

From Lemma 2.4.2 there exists $c_2 = c_2(R, \alpha, \lambda, \widetilde{L}, M_0, \varepsilon) > 0$ such that

$$\chi n \int_0^{s_0} s^{-\gamma}(s_0 - s) w_s (n w_s + 1)^{\alpha - 1} w \, ds \geq \chi n c_2 \psi(t) \tag{2.4.44}$$

for all $t \in (0, \widehat{T}_{\max})$. Since (2.4.9) holds, we infer from Lemma 2.4.4 and the Young inequality that there exists $c_3 = c_3(R, \alpha, \lambda, \mu, \kappa, M_0, \varepsilon) > 0$ such that

$$\begin{aligned} c_1 s_0 \int_0^{s_0} s^{-\gamma - \frac{2}{n}} w \, ds &\leq c_3 s_0^{2 - \gamma - \frac{2}{n} + \frac{\gamma - \beta - 1}{\beta + 2}} \psi^{\frac{1}{\beta + 2}}(t) \\ &\leq \chi c_2 \frac{n}{8} \psi(t) + \frac{2 c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\left(2 - \gamma - \frac{2}{n} + \frac{\gamma - \beta - 1}{\beta + 2}\right) \frac{\beta + 2}{\beta + 1}} \\ &= \chi c_2 \frac{n}{8} \psi(t) + \frac{2 c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1} - \gamma - \frac{2}{n} \cdot \frac{\beta+2}{\beta+1}} \end{aligned} \tag{2.4.45}$$

for all $t \in (0, \widehat{T}_{\max})$. Recalling that γ and β satisfy (2.4.12), we see from Lemma 2.4.6 that there exists $c_4 = c_4(R, \alpha, \lambda, \mu, \kappa, \widetilde{L}, M_0, \varepsilon) > 0$ such that

$$\begin{aligned} & c_1 s_0^{1-\gamma} \int_0^{s_0} (s_0 - s) w_s^{\kappa} \, ds \\ & \leq c_4 s_0^{2 - \gamma - (n-1)(\kappa - 1) + \frac{\gamma - \beta - 1}{\beta + 2} - \tilde{\varepsilon}} \psi^{\frac{1}{\beta+2}}(t) \\ & \leq \chi c_2 \frac{n}{8} \psi(t) + \frac{2 c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1} - \gamma - \frac{\beta+2}{\beta+1}((n-1)(\kappa-1) + \tilde{\varepsilon})} \end{aligned} \tag{2.4.46}$$

for all $t \in (0, \widehat{T}_{\max})$. Since it is assumed that γ and β satisfy (2.4.19), it follows from Lemma 2.4.9 that there exists $c_5 = c_5(R, \chi, \alpha, \lambda, \mu, \kappa, M_0, \varepsilon) > 0$ such that

$$\begin{aligned} & \chi n (\gamma + 1) s_0 \int_0^{s_0} s^{-\gamma - 1} z w \, ds \\ & \leq c_5 s_0^{\frac{2}{n} + 1 - \gamma} + c_5 s_0^{1 + \frac{2}{n} + \frac{2(\gamma - \beta - 1)}{\beta + 2} - \gamma} \psi^{\frac{2}{\beta + 2}}(t) \\ & \leq c_5 s_0^{\frac{2}{n} + 1 - \gamma} + \chi c_2 \frac{n}{8} \psi(t) + \frac{2 c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\left(1 + \frac{2}{n} - \gamma\right) \frac{\beta+2}{\beta} + \frac{2(\gamma - \beta - 1)}{\beta}} \end{aligned} \tag{2.4.47}$$

for all $t \in (0, \widehat{T}_{\max})$. By making use of (2.4.10), we can apply Lemma 2.4.5 to obtain $c_6 = c_6(R, \alpha, \lambda, \mu, \kappa, M_0, \varepsilon) > 0$ such that

$$\begin{aligned} c_1 \int_0^{s_0} & s^{-\gamma}(s_0 - s) w \, ds \\ & \le c_6 s_0^{2-\gamma+\frac{\gamma-\beta-1}{\beta+2}} \psi^{\frac{1}{\beta+2}}(t) \\ & \le \chi c_2 \frac{n}{8} \psi(t) + \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma} \end{aligned} \tag{2.4.48}$$

for all $t \in (0, \widehat{T}_{\max})$. Moreover, recalling the definition of ϕ (see (2.4.1)), we see from the first estimate in (2.4.48) that

$$\begin{aligned} \psi(t) & \ge \left(\frac{c_1}{c_6}\right)^{\beta+2} s_0^{-(2-\gamma)(\beta+2)-(\gamma-\beta-1)} \phi^{\beta+2}(t) \\ & = \left(\frac{c_1}{c_6}\right)^{\beta+2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t) \end{aligned} \tag{2.4.49}$$

for all $t \in (0, \widehat{T}_{\max})$. According to (2.4.44)–(2.4.49), we have

$$\begin{aligned} \phi'(t) \ge {} & \chi c_2 \frac{n}{2} \left(\frac{c_1}{c_6}\right)^{\beta+2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t) \\ & - c_5 s_0^{\frac{2}{n}+1-\gamma} - \frac{2c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\left(1+\frac{2}{n}-\gamma\right)\frac{\beta+2}{\beta}+\frac{2(\gamma-\beta-1)}{\beta}} - \frac{2c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \\ & - \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma} - \frac{2c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{\beta+2}{\beta+1}((n-1)(\kappa-1)+\tilde{\varepsilon})} \end{aligned} \tag{2.4.50}$$

for all $t \in (0, \widehat{T}_{\max})$. If $n = 3$, then, comparing the exponents of s_0 in the second and third terms on the right-hand side of (2.4.50) with the exponent of s_0 in the fourth term, we infer that

$$\begin{aligned} c_7 & := \left(\frac{2}{n} + 1 - \gamma\right) - \left(\frac{\beta+3}{\beta+1} - \gamma - \frac{2}{n} \cdot \frac{\beta+2}{\beta+1}\right) \\ & = \frac{2\beta+3}{\beta+1} \cdot \frac{2}{n} - \frac{2}{\beta+1} \\ & = \frac{2}{\beta+1} \cdot \frac{2\beta}{3} \\ & > 0 \end{aligned}$$

and

$$
\begin{aligned}
c_8 &:= \left[\left(1+\frac{2}{n}-\gamma\right)\frac{\beta+2}{\beta}+\frac{2(\gamma-\beta-1)}{\beta}\right]-\left(\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}\right)\\
&= (\beta+2)\left(\frac{1}{\beta}+\frac{1}{\beta+1}\right)\frac{2}{n}-1-\frac{\beta+3}{\beta+1}\\
&= \frac{2\beta+1}{\beta(\beta+1)}(\beta+2)\frac{2}{n}-\frac{2\beta+4}{\beta+1}\\
&= \frac{2(\beta+2)}{\beta+1}\left(\frac{2\beta+1}{3\beta}-1\right)>0,
\end{aligned}
$$

because $\beta < 1$. Moreover, as to the sixth term on the right-hand side of (2.4.50), noting from (2.4.12) that

$$
(n-1)(\kappa-1) < \frac{1-\beta}{\beta+2},
$$

we obtain from (2.4.41) that

$$
\begin{aligned}
c_9 &:= \left[\frac{\beta+3}{\beta+1}-\gamma-\frac{\beta+2}{\beta+1}((n-1)(\kappa-1)+\tilde{\varepsilon})\right]-\left(\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}\right)\\
&= -\frac{\beta+2}{\beta+1}(n-1)(\kappa-1)-\frac{\beta+2}{\beta+1}\tilde{\varepsilon}+\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}\\
&> -\frac{1-\beta}{\beta+1}-\frac{\beta+2}{\beta+1}\tilde{\varepsilon}+\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}\\
&= \frac{5\beta+1}{3(\beta+1)}-\frac{\beta+2}{\beta+1}\tilde{\varepsilon}>0.
\end{aligned}
$$

Thus we see that if $n = 3$, then the second through sixth terms on the right-hand side of (2.4.50) are estimated as

$$
\begin{aligned}
& c_5 s_0^{\frac{2}{n}+1-\gamma} + \frac{2c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\left(1+\frac{2}{n}-\gamma\right)\frac{\beta+2}{\beta}+\frac{2(\gamma-\beta-1)}{\beta}} + \frac{2c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}}\\
&+ \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma} + \frac{2c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{\beta+2}{\beta+1}((n-1)(\kappa-1)+\tilde{\varepsilon})}\\
&\le \left(c_5 R^{c_7 n} + \frac{2c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_8 n} + \frac{2c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} + \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{2\frac{\beta+2}{\beta+1}} + \frac{2c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_9 n}\right) s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}}. \quad (2.4.51)
\end{aligned}
$$

If $n \geq 4$, then, observing the exponents of s_0 in the third, fourth and fifth terms on the right-hand side of (2.4.50) with the exponent of s_0 in the second term, we have that

$$
\begin{aligned}
c_{10} &:= \left[\left(1 + \frac{2}{n} - \gamma\right)\frac{\beta+2}{\beta} + \frac{2(\gamma-\beta-1)}{\beta}\right] - \left(\frac{2}{n} + 1 - \gamma\right) \\
&= \left(\frac{\beta+2}{\beta} - \frac{2\beta+2}{\beta} - 1\right) + \left(\frac{\beta+2}{\beta} - 1\right)\frac{2}{n} \\
&= -2 + \frac{2}{\beta}\cdot\frac{2}{n} \\
&> 0
\end{aligned}
$$

and

$$
\begin{aligned}
c_{11} &:= \left(\frac{\beta+3}{\beta+1} - \gamma - \frac{2}{n}\cdot\frac{\beta+2}{\beta+1}\right) - \left(\frac{2}{n} + 1 - \gamma\right) \\
&= \frac{2}{\beta+1} - \frac{2\beta+3}{\beta+1}\cdot\frac{2}{n} \\
&= \frac{2}{\beta+1}\left(1 - \frac{2\beta+3}{n}\right) \\
&> 0,
\end{aligned}
$$

because $\beta < \frac{2}{n}$, as well as

$$
\begin{aligned}
c_{12} &:= \left(\frac{\beta+3}{\beta+1} - \gamma\right) - \left(\frac{2}{n} + 1 - \gamma\right) \\
&> c_{11} > 0.
\end{aligned}
$$

Furthermore, as to the sixth term on the right-hand side of (2.4.50), recalling that

$$
(n-1)(\kappa-1) < \frac{1-\beta}{\beta+2},
$$

we have from (2.4.42) that

$$
\begin{aligned}
c_{13} &:= \left[\frac{\beta+3}{\beta+1} - \gamma - \frac{\beta+2}{\beta+1}((n-1)(\kappa-1)+\tilde{\varepsilon})\right] - \left(\frac{2}{n} + 1 - \gamma\right) \\
&> \frac{2}{\beta+1} - \frac{1-\beta}{\beta+1} - \frac{\beta+2}{\beta+1}\tilde{\varepsilon} - \frac{2}{n} \\
&= 1 - \frac{\beta+2}{\beta+1}\tilde{\varepsilon} - \frac{2}{n} \\
&> 0.
\end{aligned}
$$

Thus we see that if $n \geq 4$, then the second through sixth terms on the right-hand side of (2.4.50) are estimated as

$$
\begin{aligned}
&c_5 s_0^{\frac{2}{n}+1-\gamma} + \frac{2c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\left(1+\frac{2}{n}-\gamma\right)\frac{\beta+2}{\beta}+\frac{2(\gamma-\beta-1)}{\beta}} + \frac{2c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \\
&+ \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma} + \frac{2c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{\beta+2}{\beta+1}((n-1)(\kappa-1)+\tilde{\varepsilon})} \\
&\leq \left(c_5 + \frac{2c_5^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_{10}n} + \frac{2c_3^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_{11}n} + \frac{2c_6^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_{12}n} + \frac{2c_4^{\frac{\beta+2}{\beta+1}}}{\chi c_2 n} R^{c_{13}n} \right) s_0^{\frac{2}{n}+1-\gamma}. \quad (2.4.52)
\end{aligned}
$$

Therefore a combination of (2.4.51) and (2.4.52) with (2.4.50) derives this lemma. □

2.5. Proof of Theorem 2.1.1

We are now in a position to complete the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Suppose that α and κ satisfy (2.1.6)–(2.1.9) and let $\widetilde{L} > 0$ and $M_0 > 0$. Thanks to Lemma 2.4.12, we see that there exist $\gamma = \gamma(\alpha, \kappa) \in (1 - \frac{2}{n}, 1)$, $0 < \beta < \gamma$ and $c_i = c_i(R, \chi, \alpha, \lambda, \mu, \kappa, \beta, \widetilde{L}, M_0) > 0$, $i \in \{1, 2, 3\}$ such that for each $s_0 \in (0, R^n)$ and any u_0 fulfilling the first and second conditions in (2.1.10), the function ϕ defined in (2.4.1) satisfies that

$$\text{if } n = 3, \qquad \phi'(t) \geq c_1 s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t) - c_2 s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \qquad (2.5.1)$$

for all $t \in (0, \widehat{T}_{\max})$ and that

$$\text{if } n \geq 4, \qquad \phi'(t) \geq c_1 s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t) - c_3 s_0^{\frac{2}{n}+1-\gamma} \qquad (2.5.2)$$

for all $t \in (0, \widehat{T}_{\max})$, where $\beta := (n-1)(1-\alpha) + \frac{\varepsilon}{n}(1-\alpha)$ with some $\varepsilon > 0$. Here, in the case $n = 3$, since $(n-1)(1-\alpha) < 2 \cdot \frac{1}{6} = \frac{1}{3} = 1 - \frac{2}{n}$, we can pick $0 < \beta < \gamma$ with $\beta < 1 - \frac{2}{n}$. Also, we take $M_1 \in (0, M_0)$ and $s_0 = s_0(R, \alpha, \lambda, \mu, \kappa, \widetilde{L}, M_0, M_1, \beta) \in (0, R^n)$ fulfilling that

$$\text{if } n = 3, \quad s_0^{\frac{\beta+2}{\beta+1}\left(1-\frac{2}{n}-\beta\right)} \leq \frac{c_2 \omega_n^{\beta+2}}{2^{(\beta+2)(\gamma-3)-1} c_1 M_1^{\beta+2}} \text{ and } s_0^{1-\beta} \leq \frac{(\beta+1)c_1 M_1^{\beta+1}}{2^{(3-\gamma)(\beta+1)+3} \omega_n^{\beta+1}}, \quad (2.5.3)$$

$$\text{if } n \geq 4, \quad s_0^{\frac{2}{n}-\beta} \leq \frac{c_3 \omega_n^{\beta+2}}{2^{(\beta+2)(\gamma-3)-1} c_1 M_1^{\beta+2}} \text{ and } s_0^{1-\beta} \leq \frac{(\beta+1)c_1 M_1^{\beta+1}}{2^{(3-\gamma)(\beta+1)+3} \omega_n^{\beta+1}}. \qquad (2.5.4)$$

Moreover, we let

$$r_\star := \left(\frac{s_0}{4}\right)^{\frac{1}{n}} \in (0, R).$$

Then we suppose that u_0 is a function satisfying (2.1.2) and (2.1.10). In order to show that $T_{\max} \le \frac{1}{2}$, assuming that $T_{\max} > \frac{1}{2}$, we will derive a contradiction. The function w defined in (2.2.1) satisfies

$$w(s,0) \ge w\left(\frac{s_0}{4}, 0\right) \ge \frac{M_1}{\omega_n}, \quad s \in \left(\frac{s_0}{4}, R^n\right).$$

This entails that

$$\phi(0) = \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,0)\,ds \ge \int_{\frac{s_0}{4}}^{\frac{s_0}{2}} \left(\frac{s_0}{2}\right)^{-\gamma} \cdot \frac{s_0}{2} \cdot \frac{M_1}{\omega_n}\,ds = \frac{2^{\gamma-3}M_1}{\omega_n} s_0^{2-\gamma}.$$

In the case $n = 3$, according to the above inequality, we have that

$$\frac{\frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(0)}{c_2 s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}}} \ge \frac{2^{(\beta+2)(\gamma-3)} c_1 M_1^{\beta+2}}{2c_2\omega_n^{\beta+2}} s_0^{2(\beta+2)-\frac{(\beta+2)(\beta+3)}{\beta+1}+\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \tag{2.5.5}$$

and the first inequality for s_0 in (2.5.3) yields that

$$\begin{aligned} \frac{2^{(\beta+2)(\gamma-3)} c_1 M_1^{\beta+2}}{2c_2\omega_n^{\beta+2}} s_0^{2(\beta+2)-\frac{(\beta+2)(\beta+3)}{\beta+1}+\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} &= \frac{2^{(\beta+2)(\gamma-3)-1} c_1 M_1^{\beta+2}}{c_2\omega_n^{\beta+2}} s_0^{-\frac{\beta+2}{\beta+1}\left(1-\frac{2}{n}-\beta\right)} \\ &\ge 1. \end{aligned} \tag{2.5.6}$$

Combining (2.5.5) and (2.5.6), we see that

$$\frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(0) \ge c_2 s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}}.$$

Hence it follows that

$$c_1 s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(0) - c_2 s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \ge \frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(0).$$

Therefore, by the straightforward ODE comparison argument we can show from (2.5.1) that

$$c_1 s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t) - c_2 s_0^{\frac{\beta+3}{\beta+1}-\gamma-\frac{2}{n}\cdot\frac{\beta+2}{\beta+1}} \ge \frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t)$$

for all $t \in \left(0, \frac{1}{2}\right)$. Recalling (2.5.1) again and plugging the above inequality into the right-hand side of (2.5.1), we obtain

$$\phi'(t) \ge \frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} \phi^{\beta+2}(t)$$

for all $t \in \left(0, \frac{1}{2}\right)$. This yields that

$$\frac{c_1}{2} s_0^{(\beta+1)\gamma-(\beta+3)} t \le -\frac{1}{\beta+1} \cdot \frac{1}{\phi^{\beta+1}(t)} + \frac{1}{\beta+1} \cdot \frac{1}{\phi^{\beta+1}(0)}$$
$$\le \frac{1}{\beta+1} \cdot \frac{\omega_n^{\beta+1}}{2^{(\beta+1)(\gamma-3)} M_1^{\beta+1}} s_0^{(\gamma-2)(\beta+1)}$$

for all $t \in \left(0, \frac{1}{2}\right)$, which implies from the second inequality for s_0 in (2.5.3) that

$$t \le \frac{2^{(3-\gamma)(\beta+1)+1} \omega_n^{\beta+1}}{(\beta+1) c_1 M_1^{\beta+1}} s_0^{1-\beta} \le \frac{1}{4}$$

for all $t \in \left(0, \frac{1}{2}\right)$, which leads to a contradiction. In the case $n \ge 4$, we can similarly derive a contradiction. Therefore we must have $T_{\max} \le \frac{1}{2} < \infty$ and thus from Lemma 2.2.1 we arrive at the conclusion (2.1.11). $\square$

Chapter 3

The case of nonlinear diffusion and super- and sub-linear sensitivity

3.1. Introduction

In this chapter we consider finite-time blow-up in the quasilinear parabolic–elliptic Keller–Segel system with logistic source:

$$\begin{cases} u_t = \Delta(u+1)^m - \chi\nabla\cdot(u(u+1)^{\alpha-1}\nabla v) + \lambda(|x|)u - \mu(|x|)u^\kappa, & x\in\Omega,\ t>0,\\ 0 = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{3.1.1}$$

where $\Omega = B_R(0) \subset \mathbb{R}^n$ ($n\in\mathbb{N}$) be a ball with some $R>0$; $m>0$, $\chi>0$, $\alpha>0$, $\kappa\ge 1$ are constants; λ and μ are continuous nonnegative functions and $\mu(r)\le\mu_1 r^q$ for all $r\in[0,R]$ with some $\mu_1>0$ and $q\ge 0$; ν is the outward normal vector to $\partial\Omega$; $u_0\in C^0(\overline{\Omega})$ is radially symmetric and nonnegative. The unknown functions $u=u(x,t)$ and $v=v(x,t)$ represent the density of cells and the concentration of the chemoattractant at $x\in\Omega$ and $t\ge 0$, respectively.

The system (3.1.1) is one of variations of the original Keller–Segel system in [**23**]. In such systems it is a fundamental theme to clarify whether solutions blow up or remain bounded. Now we recall known results related to the system (3.1.1). In the quasilinear Keller–Segel system

$$\begin{cases} u_t = \Delta(u+1)^m - \chi\nabla\cdot(u(u+1)^{\alpha-1}\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{3.1.2}$$

where m, $\alpha \in \mathbb{R}$, $\chi > 0$ and $\tau \in \{0, 1\}$, it is known that the relation between m and α determines the properties of solutions to (3.1.2); solutions of (3.1.2) blow up when $m - \alpha < \frac{n-2}{n}$ (the case $\tau = 1$ in [**7, 8, 56, 61**] and the case $\tau = 0$ in [**25**]) and remain bounded when $m - \alpha > \frac{n-2}{n}$ (the case $\tau = 1$ in [**18, 50**] and the case $\tau = 0$ in [**25**]).

In the study of the quasilinear Keller–Segel system with logistic source,

$$\begin{cases} u_t = \Delta(u+1)^m - \chi\nabla\cdot(u(u+1)^{\alpha-1}\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{3.1.3}$$

where m, $\alpha \in \mathbb{R}$, $\kappa \geq 1$, $\chi > 0$, $\lambda \geq 0$, $\mu > 0$ and $\tau \in \{0, 1\}$, which is the model of population dynamics [**16, 40**] and pattern formation in bacterial colonies [**64**], several results on boundedness were obtained due to the suppression of blow-up phenomena by the logistic term $\lambda u - \mu u^\kappa$ in [**38, 51, 55, 66, 67**].

From the above results about the system (3.1.3), one might imply that the logistic term $\lambda u - \mu u^\kappa$ suppresses blow-up. However, on the contrary, Winkler [**60**] found a condition for $\kappa > 1$ such that there exists an initial data leading to blow up in finite time in the system (3.1.3) with $m = 1$, $\alpha = 1$ and $\tau = 0$. For the details, an initial data such that finite-time blow-up occurs can be obtained under the condition that

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases}$$

Moreover, the blow-up result by Winkler [**60**] was generalized in [**2**] and Chapter 2. In the system (3.1.3) with $m = 1$ and $\tau = 0$, some conditions for κ and α such that there exist initial data leading to blow-up were found in Chapter 2 in the case of sublinear sensitivity ($\alpha < 1$). To the best of our knowledge, in the case of superlinear sensitivity ($\alpha > 1$), a blow-up result is not obtained. On the other hand, in the system (3.1.1) with $\alpha = 1$ and $\tau = 0$, Black, Fuest and Lankeit [**2**] constructed initial data such that the corresponding solution blows up under some conditions for $\kappa \geq 1$ and $m \in \left[1, \frac{2n-2}{n}\right)$. For more related works on finite-time blow-up for Keller-Segel systems with logistic source, we can refer [**2, 12, 13, 57**].

In summary, these results imply that blow-up occurs when the exponent κ of logistic source is small. In particular, in the system with nonlinear diffusion [**2**], that is, in the system (3.1.1) with $\alpha = 1$, finite-time blow-up was proved under some conditions for κ and m. However, conditions leading to blow-up have not been obtained when $m \neq 1$ and $\alpha \neq 1$ in the system (3.1.1). The purpose of this chapter is to give conditions for m, α and κ such that the solutions of (3.1.1) blow up.

Let $p \geq n$. In order to state the main theorem we give the conditions (A1)–(A4), (B1)–(B3), (C1)–(C3) and (D1)–(D2) as follows:

- In the case $n = 3$,

  - ► $$1 - \frac{1}{p} < \alpha < 1 + \frac{3}{2p}, \quad 0 < m < 1 + \frac{1}{p}, \tag{A1}$$

  - ► $$1 + \frac{3}{2p} \leq \alpha < 1 + \frac{2}{p}, \quad 0 < m < \frac{2}{p}, \quad 2\alpha - m > 2 + \frac{2}{p}, \tag{A2}$$

  - ► $$\left[1 - \frac{1}{p} < \alpha < 1 + \frac{2}{p}, \quad \frac{1}{p} \leq m < \frac{2}{p}, \quad 2\alpha - m \leq 2 + \frac{2}{p}\right],$$
    $$\text{or} \quad \left[1 - \frac{1}{p} < \alpha < 1, \quad \frac{2}{p} \leq m < \frac{3}{p}, \quad m + \alpha < 1 + \frac{2}{p}\right], \tag{A3}$$

  - ► $$1 - \frac{1}{p} < \alpha < 1 + \frac{2}{p}, \quad \frac{2}{p} \leq m < 1 + \frac{1}{p}, \quad m + \alpha \geq 1 + \frac{2}{p}, \quad m - \alpha < \frac{1}{p}. \tag{A4}$$

- In the case $n = 4$,

  - ► $$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \quad 0 < m < \frac{2}{p}, \tag{B1}$$

  - ► $$1 - \frac{2}{p} < \alpha < 1, \quad \frac{2}{p} \leq m < \frac{4}{p}, \quad m + \alpha < 1 + \frac{2}{p}, \tag{B2}$$

  - ► $$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \quad \frac{2}{p} \leq m < 1 + \frac{2}{p}, \quad m + \alpha \geq 1 + \frac{2}{p}, \quad m - \alpha < \frac{2}{p}. \tag{B3}$$

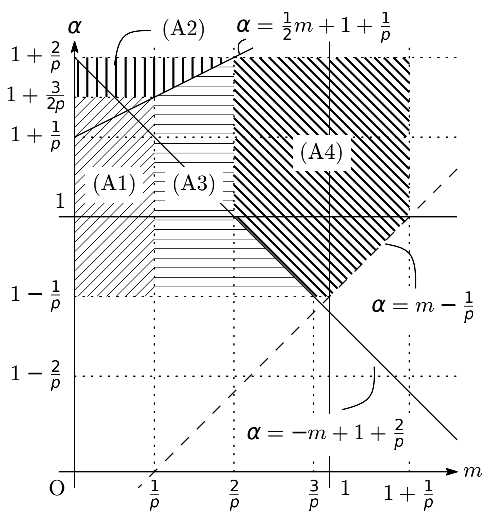

Figure 3.1: $n = 3$

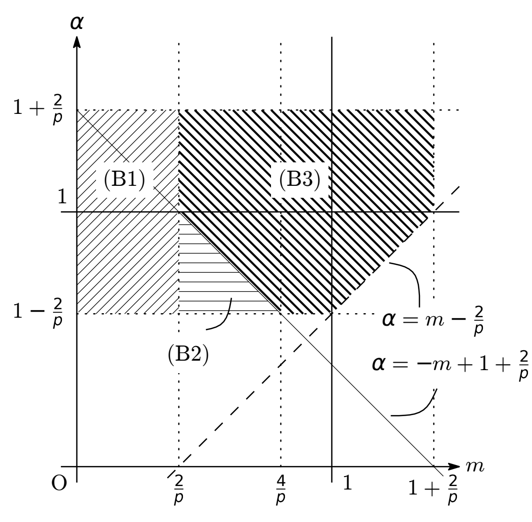

Figure 3.2: $n = 4$

- In the case $n = 5$,

  ▶ $\left[1 - \frac{2}{p} < \alpha \le 1 - \frac{1}{p}, \quad 0 < m < \frac{3}{p}\right]$,

  or $\left[1 - \frac{1}{p} < \alpha < 1 + \frac{2}{p}, \quad 0 < m < 1 + \frac{1}{2p}, \quad 2m - \alpha < 1 + \frac{1}{p}\right]$, (C1)

  ▶ $1 - \frac{2}{p} < \alpha < 1 - \frac{1}{p}, \quad \frac{3}{p} \le m < \frac{4}{p}, \quad m + \alpha < 1 + \frac{2}{p}$, (C2)

  ▶ $\left[1 - \frac{2}{p} < \alpha \le 1 - \frac{1}{p}, \quad \frac{3}{p} \le m < 1, \quad m + \alpha \ge 1 + \frac{2}{p}\right]$,

  or $\left[1 - \frac{2}{p} < \alpha < 1, \quad 1 \le m < 1 + \frac{1}{2p}, \quad 2m - \alpha \ge 1 + \frac{1}{p}\right]$,

  or $\left[1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \quad 1 + \frac{1}{2p} \le m < 1 + \frac{3}{p}, \quad m - \alpha < \frac{3}{p}\right]$. (C3)

- In the case $n \ge 6$,

  ▶ $1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \quad 0 < m < 1 + \frac{n-4}{2p}, \quad 2m - \alpha < 1 + \frac{n-4}{p}$, (D1)

  ▶ $\left[1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \ 1 + \frac{n-6}{2p} \le m < 1 + \frac{n-4}{2p}, \ 2m - \alpha \ge 1 + \frac{n-4}{p}\right]$,

  or $\left[1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \ 1 + \frac{n-4}{2p} \le m < 1 + \frac{n-2}{p}, \ m - \alpha < \frac{n-2}{p}\right]$. (D2)

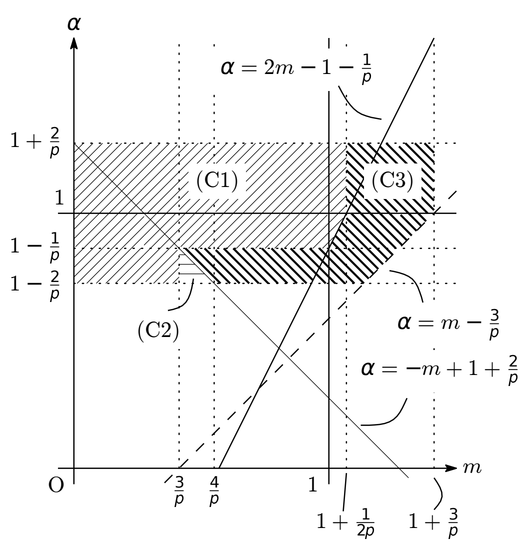

Figure 3.3: $n = 5$

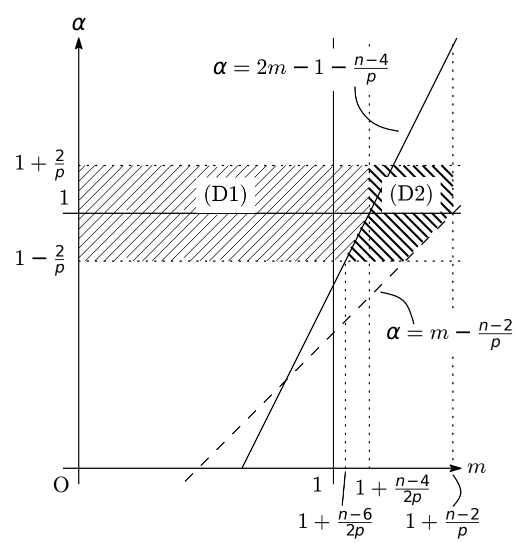

Figure 3.4: $n \ge 6$

Moreover, setting $\beta_+ := \max\{0, \beta\}$ for $\beta \in \mathbb{R}$, we assume that $\kappa \geq 1$ fulfills the following conditions:

- $\kappa < 1 + \frac{3}{p} + \frac{q}{p} - (\alpha - 1)$ if (A2) holds, (**I**)
- $\kappa < 1 + \frac{n}{2p} + \frac{q}{p} - \frac{(1-\alpha)_+}{2}$ if (A1), (B1), (C1) or (D1) hold, (**II**)
- $\kappa < 1 + \frac{n-1}{p} + \frac{q}{p} - \frac{m}{2} - \frac{(1-\alpha)_+}{2}$ if (A3), (B2) or (C2) hold, (**III**)
- $\kappa < 1 + \frac{n-2}{p} + \frac{q}{p} - (m-1)_+ - (1-\alpha)_+$ if (A4), (B3), (C3) or (D2) hold. (**IV**)

Now we state the main theorems. The first result is concerned with blow-up when we assume an upper bound of solutions.

Theorem 3.1.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ with $R > 0$ and let $m > 0$, $\alpha > 0$, $\chi > 0$, $\kappa \geq 1$, $\mu_1 > 0$, $p \geq n$, $q \geq 0$, $M_0 > 0$, $M_1 \in (0, M_0)$, $\widetilde{K} > 0$ and $T > 0$. Suppose that λ and μ satisfy that*

$$0 \leq \lambda, \mu \in C^0([0, R]) \tag{3.1.4}$$

and

$$\mu(r) \leq \mu_1 r^q \quad \text{for all } r \in [0, R] \tag{3.1.5}$$

and assume that κ fulfills (**I**)–(**IV**). *Then one can find $r_\star \in (0, R)$ with the following property*: *If*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ v \in \bigcap_{\vartheta > n} C^0([0, T_{\max}); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \end{cases}$$

is a classical solution of (3.1.1) *for some $T^* \in (0, \infty]$ with*

$$u_0 \in C^0(\overline{\Omega}) \text{ being radially symmetric and nonnegative} \tag{3.1.6}$$

and

$$\int_\Omega u_0(x)\,dx = M_0 \quad \text{but} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_1$$

as well as

$$\sup_{t \in (0, \min\{T, T^*\})} u(x, t) \leq \widetilde{K} |x|^{-p} \quad \text{for all } x \in \Omega, \tag{3.1.7}$$

then (u, v) blows up at $t = T^* < \infty$ in the sense that

$$\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{3.1.8}$$

Remark 3.1.1. As to the conditions (A1)–(D2) and (**I**)–(**IV**), if $\alpha = 1$, then we can obtain the conditions such that

$$1 \le \kappa < 1 + \frac{q}{p} + \min\left\{\frac{n}{2p}, \frac{n-2}{p} - (m-1)_+\right\}$$
$$\text{if } m \in \left[\frac{2}{p}, 1 + \frac{n-2}{p}\right)$$

and such that

$$1 \le \kappa < 1 + \frac{q}{p} + \min\left\{\frac{n}{2p}, \frac{n-1}{p} - \frac{m}{2}\right\}$$
$$\text{if } m \in \left(0, \frac{2}{p}\right).$$

The above conditions for m and κ connect with conditions in [**2**, Theorem 1.1]. Thus, Theorem 3.1.1 is a generalization of the previous work [**2**, Theorem 1.1].

By an argument similar to that in the proof of [**2**, Lemma 5.2] we can find an initial data such that the corresponding solution satisfies (3.1.7). Therefore, in view of Theorem 3.1.1 we can show that there exists an initial data such that the corresponding solution blows up in finite time. Before we introduce this result, we give the conditions (E1), (F1) and (F2) as follows:

- In the case $n \in \{3, 4\}$,

$$\blacktriangleright \quad m \ge 1, \quad \alpha < \frac{2}{n+1}m + \frac{n^2-n+2}{n(n+1)},$$
$$\alpha < -\frac{1}{n-2}m + \frac{n^2-2}{n(n-2)}, \quad m - \alpha < \frac{n-2}{n}. \tag{E1}$$

- In the case $n \ge 5$,

$$\blacktriangleright \quad m \ge 1, \quad -\frac{2}{n-3}m + \frac{n^2-n-2}{n(n-3)} < \alpha < \frac{2}{n+1}m + \frac{n^2-n+2}{n(n+1)},$$
$$\alpha < -\frac{n+2}{n-4}m + \frac{2n^2-n-4}{n(n-4)}, \quad \alpha \le \frac{n+2}{3}m - \frac{n^2-4}{3n}, \tag{F1}$$

$$\blacktriangleright \quad m \ge 1, \quad -\frac{2}{n-3}m + \frac{n^2-n-2}{n(n-3)} < \alpha < \frac{2}{n+1}m + \frac{n^2-n+2}{n(n+1)},$$
$$-\frac{n+2}{n-4}m + \frac{2n^2-n-4}{n(n-4)} \le \alpha < -\frac{1}{n-2}m + \frac{n^2-2}{n(n-2)},$$
$$m - \alpha < \frac{n-2}{n}. \tag{F2}$$

The regions of (E1), (F1) and (F2) are described as Figures 3.5–3.8.

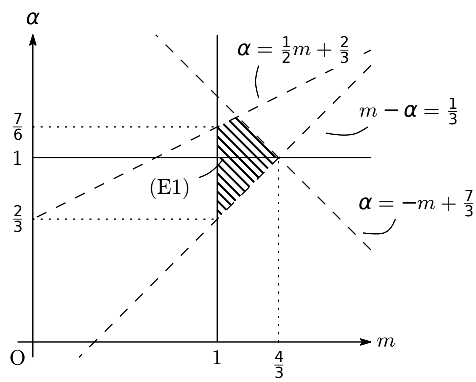

Figure 3.5: $n = 3$

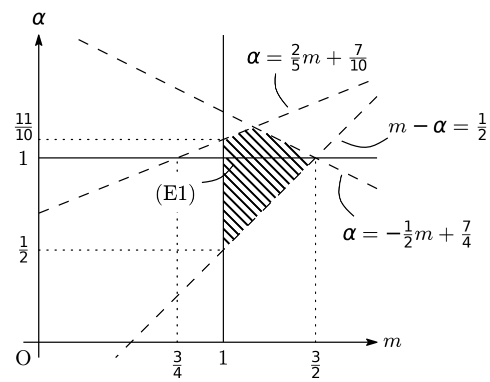

Figure 3.6: $n = 4$

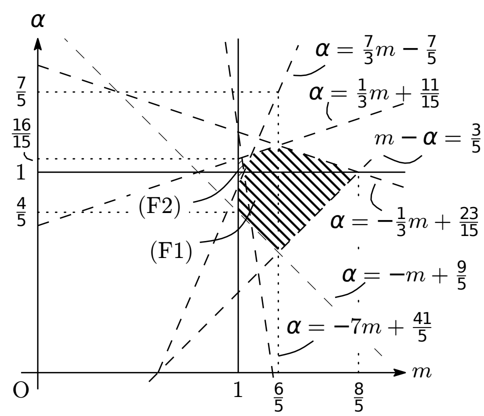

Figure 3.7: $n = 5$

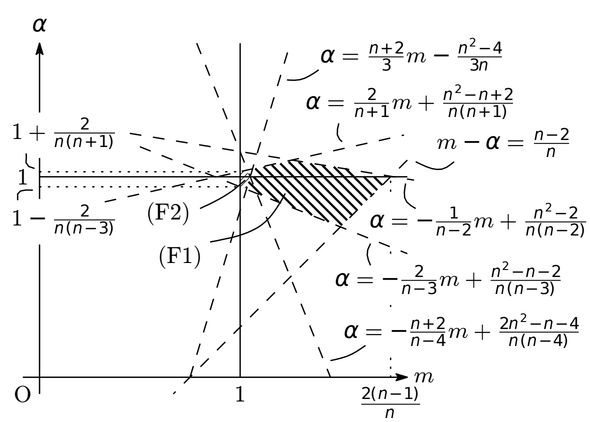

Figure 3.8: $n \geq 6$

Aided by Theorem 3.1.1, we obtain an initial data such that the corresponding solution blows up in finite time.

Theorem 3.1.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ with $R > 0$ and let $m > 0$, $\alpha > 0$, $\chi > 0$, $\kappa \geq 1$, $\mu_1 > 0$, $q \geq 0$, $M_0 > 0$, $M_1 \in (0, M_0)$ and $\widetilde{L} > 0$. Suppose that λ and μ satisfy* (3.1.4) *and* (3.1.5). *Moreover, assume that m, α and κ fulfill the following conditions*:

(i) *If* (E1) *holds, then*

$$\kappa < 1 + \frac{(n-2)[(m-\alpha)n+1]}{n(n-1)} + \frac{q[(m-\alpha)n+1]}{n(n-1)} - (m-1) - (1-\alpha)_+.$$

(ii) *If* (F1) *holds, then*

$$\kappa < 1 + \frac{(n-2)[(m-\alpha)n+1]}{n(n-1)} + \frac{q[(m-\alpha)n+1]}{n(n-1)} - (m-1) - (1-\alpha)_+.$$

(iii) *If* (F2) *holds, then*

$$\kappa < 1 + \frac{(m-\alpha)n+1}{2(n-1)} + \frac{q[(m-\alpha)n+1]}{n(n-1)} - \frac{(1-\alpha)_+}{2}.$$

Then one can find $\varepsilon_0 > 0$ and $r_\star \in (0, R)$ with the following property: *If u_0 with* (3.1.6) *satisfies $\int_\Omega u_0(x)\,dx = M_0$ and $\int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_1$ as well as $u_0(x) \leq \widetilde{L}|x|^{-p}$ for all $x \in \Omega$, where $p := \frac{n(n-1)}{(m-\alpha)n+1} + \varepsilon_0$, then the corresponding solution (u, v) of* (3.1.1) *fulfills* (3.1.8) *for some $T^* < \infty$.*

Remark 3.1.2. If $\alpha = 1$, then we have from the conditions (E1)–(F2) and (i)–(iii) that

$$\kappa < 1 + \frac{q[(m-1)n+1]}{n(n-1)} + \min\left\{\frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)}\right\} \text{ if } m \in \left[1, \frac{2n-2}{n}\right)$$

which is the condition in [**2**, Theorem 1.2]. Thus, Theorem 3.1.2 is a generalization of the previous work [**2**, Theorem 1.2]. On the other hand, if $m = 1$, $\alpha < 1$ and $q = 0$, then we can obtain that

$$\text{if } n \in \{3,4\}, \qquad \frac{2}{n} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \frac{(n-2)-(1-\alpha)n}{n(n-1)}, \tag{3.1.9}$$

$$\text{if } n = 5, \qquad \begin{cases} \frac{4}{5} < \alpha \leq \frac{14}{15} \quad \text{and} \quad \kappa < 1 + \frac{(n-2)-(1-\alpha)n}{n(n-1)}, \\ \frac{14}{15} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \frac{2-\alpha}{2(n-1)}, \end{cases} \tag{3.1.10}$$

$$\text{if } n \geq 6, \qquad 1 - \frac{2}{n(n-3)} < \alpha < 1 \quad \text{and} \quad \kappa < 1 + \frac{2-\alpha}{2(n-1)}. \tag{3.1.11}$$

The conditions (3.1.9)–(3.1.11) improve lower bounds for α and upper bounds for κ in Chapter 2. Moreover, while blow-up result was only obtained in the case $\alpha < 1$ in Chapter 2, we can see that the result is extended to the case $\alpha \geq 1$.

Remark 3.1.3. In the conditions (E1)–(F2) there are some restrictions in addition to the condition $m - \alpha < \frac{n-2}{n}$. On the one hand, in the system (3.1.2) it is known that there exist many results about blow-up under the condition $m - \alpha < \frac{n-2}{n}$ without these restrictions (see [**7, 8, 25, 56, 61**]). Thus, Theorem 3.1.2 may hold under the condition $m - \alpha < \frac{n-2}{n}$ even without these restrictions.

The proofs of Theorems 3.1.1 and 3.1.2 are based on those of [**2**]. We first introduce the mass accumulation functions $w = w(s,t)$ and $z = z(s,t)$ given by

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\, d\rho, \quad z(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho,t)\, d\rho,$$

where $s := r^n$ for $r \in [0,R]$. The system (3.1.1) is transformed to the parabolic equation

$$\begin{aligned} w_t = {} & n^2 m s^{2-\frac{2}{n}} (nw_s + 1)^{m-1} w_{ss} + \chi n w_s (nw_s + 1)^{\alpha-1} (w - z) \\ & + n \int_0^s \lambda(\sigma^{\frac{1}{n}}) w_s(\sigma,t)\, d\sigma - n^{\kappa-1} \int_0^s \mu(\sigma^{\frac{1}{n}}) w^{\kappa}(\sigma,t)\, d\sigma. \end{aligned} \tag{3.1.12}$$

Next, by making use of this equation and the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s,t)\, ds$$

with some $s_0 \in (0, R^n)$ and $\gamma \in (0,1)$, we show that the functional ϕ is a supersolution of the ordinary differential equation $\phi' = c_1 \phi^2 - c_2$ with some $c_1 > 0$ and $c_2 > 0$. Here, as to the factor $(nw_s + 1)^{m-1}$ in the first term on the right-hand side of (3.1.12) we can apply the same estimates as in [**2**]. However, in order to derive a super-linear differential inequality for ϕ we have to estimate the factor $(nw_s + 1)^{\alpha-1}$ in the second term on the right-hand side of (3.1.12). To this end, in the case $\alpha < 1$ we use the estimates $(nw_s + 1)^{\alpha-1} \leq 1$ and $(nw_s + 1)^{\alpha-1} \geq (Cs^{-\frac{p}{n}} + 1)^{\alpha-1}$ as in Chapter 2 and in the case $\alpha \geq 1$ we establish the estimates $(nw_s + 1)^{\alpha-1} \leq (Cs^{-\frac{p}{n}} + 1)^{\alpha-1}$ and $(nw_s + 1)^{\alpha-1} \geq 1$ on a case by case basis. Moreover, by taking $\gamma \in (0,1)$ satisfying some conditions, we can obtain a super-linear differential inequality for ϕ. As to the proof of Theorem 3.1.2, we can obtain initial data such that the solution fulfills (3.1.7) by the recent study of blow-up profiles in [**11**].

This chapter is organized as follows. In Section 2 we recall local existence of classical solutions in (3.1.1). In Section 3 we establish some estimates in order to construct a subsolution of a super-linear differential inequality for ϕ. In Section 4 we prove existence of $\gamma \in (0,1)$ which satisfies conditions to derive a super-linear differential inequality for ϕ and obtain a super-linear differential inequality. Finally, the proofs of the main theorems are given in Section 5.

3.2. Local existence

We first introduce a result on local existence of classical solutions to (3.1.1). We provide only the statement of the lemma since the proof is based on a standard fixed point argument (see [**9, 51**]).

Lemma 3.2.1. *Let $n \geq 1$, $R > 0$, $m > 0$, $\alpha > 0$, $\chi > 0$, $\kappa \geq 1$ and $M_0 > 0$, and assume that λ and μ comply with* (3.1.4) *and* (3.1.5)*. If u_0 satisfies* (3.1.6) *and $\int_\Omega u_0(x)\,dx = M_0$, then there exist $T_{\max} \in (0,\infty]$ and an exactly pair (u,v) of radially symmetric nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ v \in \bigcap_{\vartheta > n} C^0([0,T_{\max}); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \end{cases}$$

which solves (3.1.1) *classically. Moreover,*

$$\textit{if } T_{\max} < \infty, \ \textit{ then } \lim_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

3.3. Some inequalities related to a moment-type functional ϕ

In the following let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball with some $R > 0$ and we fix the initial data u_0 satisfying (3.1.6) and $\int_\Omega u_0(x)\,dx = M_0$. We note that all constants below are independent of u_0. Also, let (u,v) be the radially symmetric solution of (3.1.1) on $[0,T_{\max})$ as in Lemma 3.2.1. By introducing $r := |x|$, we regard $u(x,t)$ and $v(x,t)$ as $u(r,t)$ and $v(r,t)$, respectively. Based on [**21**], we define the mass accumulation functions w and z as

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\,d\rho, \quad z(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho,t)\,d\rho$$

for all $s \in [0,R^n]$ and $t \in [0,T_{\max})$. Moreover, given $s_0 \in (0,R^n)$ and $\gamma \in (0,1)$, we set

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\,ds \quad \text{for all } t \in [0,T_{\max}), \tag{3.3.1}$$

which is introduced in [**2, 60**], and

$$\psi_\alpha(t) := \int_0^{s_0} s^{-\gamma+\frac{p}{n}(1-\alpha)_+}(s_0 - s) w(s,t) w_s(s,t)\,ds \quad \text{for all } t \in [0,T_{\max}),$$

where $p \geq n$, $\alpha > 0$ and $(1-\alpha)_+ := \max\{0, 1-\alpha\}$. Now we recall the following properties for the functions w and z (see Chapter 2 and [**2**, Lemma 3.1]).

Lemma 3.3.1. *We have*

$$w \in C^{1,0}([0,R^n]\times[0,T_{\max})) \cap C^{2,1}([0,R^n]\times(0,T_{\max})) \cap C^{3,0}((0,R^n]\times(0,T_{\max})),$$
$$z \in C^{1,0}([0,R^n]\times[0,T_{\max})) \cap C^{2,0}([0,R^n]\times(0,T_{\max})) \cap C^{3,0}((0,R^n]\times(0,T_{\max}))$$

and

$$w_s(s,t) = \frac{1}{n}u(s^{\frac{1}{n}},t), \quad w_{ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_r(s^{\frac{1}{n}},t),$$
$$z_s(s,t) = \frac{1}{n}v(s^{\frac{1}{n}},t), \quad z_{ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}v_r(s^{\frac{1}{n}},t)$$

for all $s \in (0,R^n)$ and $t \in (0,T_{\max})$ as well as, with $\widetilde{K}$ and T from (3.1.7),

$$nw_s(s,t) \leq \widetilde{K}s^{-\frac{p}{n}} \quad \textit{for all } s \in (0,R^n] \textit{ and } t \in (0,T). \tag{3.3.2}$$

In order to obtain a key inequality in (3.4.15) we first prove the following lemma.

Lemma 3.3.2. *Let $\gamma \in (0,1)$ and $s_0 \in (0,R^n)$. Then the function ϕ defined as* (3.3.1) *belongs to $C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ and satisfies*

$$\begin{aligned}
\phi'(t) \geq{}& \chi n \int_0^{s_0} s^{-\gamma}(s_0-s)(nw_s(s,t)+1)^{\alpha-1}w(s,t)w_s(s,t)\,ds \\
&+ n^2 m \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)(nw_s(s,t)+1)^{m-1}w_{ss}(s,t)\,ds \\
&- \chi n \int_0^{s_0} s^{-\gamma}(s_0-s)(nw_s(s,t)+1)^{\alpha-1}z(s,t)w_s(s,t)\,ds \\
&- n^{\kappa-1}\mu_1 \int_0^{s_0} s^{-\gamma}(s_0-s)\left\{\int_0^{s_0}\sigma^{\frac{q}{n}}w_s^\kappa(\sigma,t)\,d\sigma\right\}ds \\
=:{}& I_1 + I_2 + I_3 + I_4
\end{aligned} \tag{3.3.3}$$

for all $t \in (0,T_{\max})$.

Proof. We obtain $\phi \in C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ as in the proof of [**60**, Lemma 4.1]. From the second equation in (3.1.1) we have that

$$r^{n-1}v_r(r,t) = z(r^n,t) - w(r^n,t) \tag{3.3.4}$$

for all $r \in (0,R)$ and $t \in (0,T_{\max})$. Noting that $\lambda \geq 0$ and (3.1.5), we can observe from (3.3.4) and the first equation in (3.1.1) that

$$\begin{aligned}
w_t \geq{}& n^2 m s^{2-\frac{2}{n}}(nw_s+1)^{m-1}w_{ss} \\
&+ \chi n w_s(nw_s+1)^{\alpha-1}(w-z) - n^{\kappa-1}\mu_1\int_0^s \sigma^{\frac{q}{n}}w_s^\kappa(\sigma,t)\,d\sigma
\end{aligned} \tag{3.3.5}$$

for all $s \in (0,R^n)$ and $t \in (0,T_{\max})$. Thanks to (3.3.1) and (3.3.5), we attain (3.3.3). □

Next we derive an estimate for I_1 on the right-hand side of (3.3.3).

Lemma 3.3.3. *Let $\gamma \in (0,1)$ and let $\alpha > 0$, $\chi > 0$ and $p \geq n$ and suppose that* (3.1.7) *holds with some $\widetilde{K} > 0$ and $T > 0$. Then there exists $C = C(R, \chi, \alpha, p, \widetilde{K}) > 0$ such that for any $s_0 \in (0, R^n)$*

$$I_1 \geq C\psi_\alpha(t)$$

for all $t \in (0, \min\{T, T_{\max}\})$.

Proof. In the case $0 < \alpha < 1$, using (3.3.2) and $s < R^n$, we can establish that

$$(nw_s + 1)^{\alpha-1} \geq (\widetilde{K}s^{-\frac{p}{n}} + 1)^{-(1-\alpha)} \geq (\widetilde{K} + R^p)^{-(1-\alpha)} s^{\frac{p}{n}(1-\alpha)}$$

for all $s \in (0, s_0)$. On the other hand, in the case $\alpha \geq 1$ it follows that

$$(nw_s + 1)^{\alpha-1} \geq 1$$

for all $s \in (0, s_0)$. Thus we obtain

$$\begin{aligned} I_1 &= \chi n \int_0^{s_0} s^\gamma (s_0 - s)(nw_s + 1)^{\alpha-1} w w_s \, ds \\ &\geq \chi n (\widetilde{K} + R^p)^{-(1-\alpha)_+} \int_0^{s_0} s^{-\gamma + \frac{p}{n}(1-\alpha)_+} (s_0 - s) w w_s \, ds \end{aligned}$$

for all $t \in (0, \min\{T, T_{\max}\})$, which concludes the proof. □

To show estimates for I_2, I_3 and I_4 on the right-hand side of (3.3.3) we introduce two lemmas. The following lemma has already been proved in [**2**, Lemma 3.3].

Lemma 3.3.4. *For all $a > -1$ and $b > -1$ and any $s_0 \leq 0$ we have*

$$\int_0^{s_0} s^a (s_0 - s)^b \, ds = B(a+1, b+1) s_0^{a+b+1},$$

where B is Euler's beta function.

Lemma 3.3.5. *Let $\alpha > 0$ and $p \geq n$. Assume that $\gamma \in (0,1)$ satisfies that*

$$\gamma - \frac{p}{n}(1-\alpha)_+ \in (0,1). \tag{3.3.6}$$

Then for any $s_0 \in (0, R^n)$

$$w(s,t) \leq \sqrt{2} s^{\frac{\gamma}{2} - \frac{p}{2n}(1-\alpha)_+} (s_0 - s)^{-\frac{1}{2}} \sqrt{\psi_\alpha(t)} \tag{3.3.7}$$

for all $s \in (0, s_0)$ and $t \in (0, T_{\max})$.

Proof. By using the function

$$\psi(s) := \frac{1}{2} s^{-\gamma+\frac{p}{n}(1-\alpha)_+}(s_0 - s)w^2(s,t) \quad \text{for all } t \in (0, T_{\max})$$

instead of ψ in the proof of [**60**, Lemma 4.2], we can verify that (3.3.7) holds. □

We establish an estimate for I_4.

Lemma 3.3.6. *Let $\alpha > 0$, $\mu_1 > 0$, $\kappa \geq 1$, $p \geq n$ and $q \geq 0$ and suppose that* (3.1.7) *holds with some $\widetilde{K} > 0$ and $T > 0$. Assume that $\gamma \in (0,1)$ satisfies* (3.3.6) *and*

$$\frac{p}{n}[2(\kappa - 1) + (1-\alpha)_+] - \frac{2q}{n} < \gamma. \tag{3.3.8}$$

Then there exists $C = C(\gamma, \alpha, \mu_1, \kappa, p, q, \widetilde{K}) > 0$ such that for any $s_0 \in (0, R^n)$

$$I_4 \geq -C s_0^{\frac{3-\gamma}{2}+\frac{q}{n}-\frac{p}{2n}[2(\kappa-1)+(1-\alpha)_+]} \sqrt{\psi_\alpha(t)} \tag{3.3.9}$$

for all $t \in (0, \min\{T, T_{\max}\})$.

Proof. Aided by an argument similar to that in the proof of [**2**, Lemma 3.5], we have from straightforward calculations that

$$I_4 \geq -\frac{n^{\kappa-1}\mu_1}{1-\gamma} s_0^{1-\gamma} \int_0^{s_0} s^{\frac{q}{n}}(s_0 - s) w_s^\kappa \, ds \tag{3.3.10}$$

for all $t \in (0, T_{\max})$ and

$$\int_0^{s_0} s^{\frac{q}{n}}(s_0 - s) w_s^\kappa \, ds \leq c_1 s_0 \int_0^{s_0} s^{\frac{q}{n}-\frac{p}{n}(\kappa-1)-1} w \, ds \tag{3.3.11}$$

for all $t \in (0, \min\{T, T_{\max}\})$, where $c_1 := \frac{\widetilde{K}^{\kappa-1}}{n^{\kappa-1}}\left[\left(\frac{p}{n}(\kappa-1) - \frac{q}{n}\right)_+ + 1\right]$. By virtue of Lemmas 3.3.4 and 3.3.5 it follows that

$$\begin{aligned}
c_1 s_0 &\int_0^{s_0} s^{\frac{q}{n}-\frac{p}{n}(\kappa-1)-1} w \, ds \\
&\leq \sqrt{2} c_1 s_0 \int_0^{s_0} s^{\frac{q}{n}-\frac{p}{n}(\kappa-1)+\frac{\gamma}{2}-\frac{p}{2n}(1-\alpha)_+-1}(s_0 - s)^{-\frac{1}{2}} \, ds \sqrt{\psi_\alpha(t)} \\
&= \sqrt{2} c_1 c_2 s_0^{\frac{1}{2}+\frac{\gamma}{2}+\frac{q}{n}-\frac{p}{2n}[2(\kappa-1)+(1-\alpha)_+]} \sqrt{\psi_\alpha(t)}
\end{aligned} \tag{3.3.12}$$

for all $t \in (0, \min\{T, T_{\max}\})$, where $c_2 := B\left(\frac{\gamma}{2} + \frac{q}{n} - \frac{p}{2n}[2(\kappa-1) + (1-\alpha)_+], \frac{1}{2}\right)$. Now, noting from (3.3.8) that

$$\frac{\gamma}{2} + \frac{q}{n} - \frac{p}{2n}[2(\kappa - 1) + (1-\alpha)_+] > 0,$$

we see that $c_2 < \infty$. A combination of (3.3.10), (3.3.11) and (3.3.12) yields (3.3.9). □

Next we show an estimate for I_2.

Lemma 3.3.7. *Let $\alpha > 0$, $m > 0$ and $p \geq n$ and suppose that (3.1.7) holds with some $\widetilde{K} > 0$ and $T > 0$ and that $\gamma \in (0,1)$ satisfies (3.3.6).*

(i) *Assume that*

$$0 < m < 1 + \frac{n-2}{p}$$

and

$$1 - \frac{2}{n} - \frac{p}{n}(m-1)_+ < \gamma < 2 - \frac{4}{n} - \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]. \tag{3.3.13}$$

Then there exists $C > 0$ such that for any $s_0 \in (0, R^n)$

$$I_2 \geq -Cs_0^{\frac{3-\gamma}{2} - \frac{2}{n} - \frac{p}{2n}[2(m-1)_+ + (1-\alpha)_+]} \sqrt{\psi_\alpha(t)} - Cs_0^{3 - \frac{2}{n} - \gamma}$$

for all $t \in (0, \min\{T, T_{\max}\})$.

(ii) *Assume that*

$$0 < m < \min\left\{1, \frac{2(n-1)}{p}\right\}$$

and

$$0 < \gamma < 2 - \frac{2}{n} - \frac{pm}{n}. \tag{3.3.14}$$

Then there exists $C > 0$ such that for any $s_0 \in (0, R^n)$

$$I_2 \geq -Cs_0^{3-\gamma-\frac{2}{n}-\frac{pm}{n}} - Cs_0^{3-\frac{2}{n}-\gamma}$$

for all $t \in (0, \min\{T, T_{\max}\})$.

Proof. By an argument similar to that in the proof of [**2**, Lemma 3.6], we can arrive at the conclusion of this lemma. □

The following lemma has already been proved in the proof of [**60**, Lemma 4.6]. Thus we only recall the statement of the lemma.

Lemma 3.3.8. *Let $a \in (1,2)$ and $b \in (0,1)$. Then there exists $C = C(a,b) > 0$ such that if $s_0 > 0$, then*

$$\int_0^{s_0} \int_\sigma^{s_0} \xi^{-a}(s_0 - \xi)^{-b} \, d\xi d\sigma \leq Cs_0^{-b} s^{2-a}$$

for all $s \in (0, s_0)$.

We establish an estimate for I_3. The proof of the following lemma is based on that of [**2**, Lemma 3.9].

Lemma 3.3.9. *Let $\alpha > 0$, $\chi > 0$, $p \ge n$, $M_0 > 0$, $\widetilde{K} > 0$ and $T > 0$ and suppose that $\gamma \in (0,1)$ satisfies (3.3.6). Assume that*

$$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p} \quad \text{and} \quad \gamma < 2 - \frac{2p}{n}(\alpha - 1)_+. \tag{3.3.15}$$

Then there exists $C > 0$ such that if u_0 fulfills $\int_\Omega u_0(x)\,dx = M_0$ and (3.1.7) holds, then for any $s_0 \in (0, R^n)$

$$\begin{aligned} I_3 \ge &-C s_0^{\frac{2}{n} + \frac{1-\gamma}{2} - \frac{p}{2n}[(1-\alpha)_+ + 2(\alpha-1)_+]} \sqrt{\psi_\alpha(t)} \\ &- C s_0^{\frac{2}{n} - \frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]} \psi_\alpha(t) \end{aligned} \tag{3.3.16}$$

for all $t \in (0, \min\{T, T_{\max}\})$.

Proof. By an argument similar to that in the proof of [**60**, Lemma 4.7] we can see that there exists $c_1 = c_1(R, \lambda, M_0) > 0$ such that

$$z \le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{1}{n^2} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2} w(\xi, t)\, d\xi d\sigma \tag{3.3.17}$$

for all $s \in (0, s_0)$ and $t \in (0, \min\{T, T_{\max}\})$. First we show the estimate (3.3.16) in the case $1 - \frac{2}{n} < \alpha < 1$. Since $1 - \frac{2}{n} < \alpha$ and $\gamma < 1$, it follows that

$$\gamma - \frac{4}{n} + \frac{2p}{n}(1 - \alpha) < \gamma$$

and

$$\begin{aligned} &\left(2 - \frac{4}{n} + \frac{p}{n}(1-\alpha)\right) - \left(\gamma - \frac{4}{n} + \frac{2p}{n}(1-\alpha)\right) \\ &= 2 - \frac{p}{n}(1-\alpha) - \gamma \\ &> 2 - \frac{2}{n} - \gamma \\ &> 0. \end{aligned}$$

Moreover, we have from (3.3.6) that $\gamma > \frac{p}{n}(1 - \alpha)$. Thus we can take

$$\widetilde{\gamma} \in \left(\max\left\{ \frac{p}{n}(1-\alpha), \gamma - \frac{4}{n} + \frac{2p}{n}(1-\alpha) \right\}, \min\left\{ \gamma, 2 - \frac{4}{n} + \frac{p}{n}(1-\alpha) \right\} \right).$$

Noticing that $\widetilde{\gamma} - \frac{p}{n}(1-\alpha) \in (0,1)$, we infer from Lemma 3.3.5 that

$$z \le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{\sqrt{2}}{n^2} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\widetilde{\gamma}}{2}-\frac{p}{n}(1-\alpha)} (s_0-\xi)^{-\frac{1}{2}} \, d\xi d\sigma \left\{ \int_0^{s_0} s^{-\widetilde{\gamma}+\frac{p}{n}(1-\alpha)} (s_0-s) w w_s \, ds \right\}^{\frac{1}{2}}$$

for all $s \in (0, s_0)$ and $t \in (0, \min\{T, T_{\max}\})$. Since the conditions $1 - \frac{2}{p} < \alpha$ and $\widetilde{\gamma} < 2 - \frac{4}{n} + \frac{p}{n}(1-\alpha)$ yield

$$1 < 2 - \frac{2}{n} - \frac{\widetilde{\gamma}}{2} + \frac{p}{2n}(1-\alpha) < 2,$$

from Lemma 3.3.8 we can find $c_2 = c_2(\gamma, \alpha, p) > 0$ such that

$$\begin{aligned} z &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + c_2 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\widetilde{\gamma}}{2}-\frac{p}{2n}(1-\alpha)} \left\{ \int_0^{s_0} s^{-\widetilde{\gamma}+\frac{p}{n}(1-\alpha)} (s_0-s) w w_s \, ds \right\}^{\frac{1}{2}} \\ &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + c_2 s_0^{-\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} s^{\frac{2}{n}+\frac{\widetilde{\gamma}}{2}-\frac{p}{2n}(1-\alpha)} \sqrt{\psi_\alpha(t)} \end{aligned} \tag{3.3.18}$$

for all $s \in (0, s_0)$ and $t \in (0, \min\{T, T_{\max}\})$. Now we have from the fact $(nw_s+1)^{\alpha-1} \le 1$ that

$$\begin{aligned} I_3 &= -\chi n \int_0^{s_0} s^{-\gamma} (s_0-s)(nw_s+1)^{\alpha-1} z w_s \, ds \\ &\ge -\chi n \int_0^{s_0} s^{-\gamma}(s_0-s) z w_s \, ds \end{aligned} \tag{3.3.19}$$

for all $t \in (0, \min\{T, T_{\max}\})$. Furthermore, by an argument similar to that in the proof of [**60**, Lemma 4.1] and (3.3.18) we see that

$$\begin{aligned} &- \chi n \int_0^{s_0} s^{-\gamma}(s_0-s) z w_s \, ds \\ &\quad \ge -\chi n(\gamma+1) s_0 \int_0^{s_0} s^{-\gamma-1} z w \, ds \\ &\quad \ge -c_1 \chi(\gamma+1) s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} w \, ds \\ &\qquad - c_2 \chi n(\gamma+1) s_0^{\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-\gamma+\frac{\widetilde{\gamma}}{2}-\frac{p}{2n}(1-\alpha)-1} w \, ds \sqrt{\psi_\alpha(t)} \end{aligned} \tag{3.3.20}$$

for all $t \in (0, \min\{T, T_{\max}\})$. Since the conditions $1 - \frac{2}{p} < \alpha$ and $\widetilde{\gamma} > \gamma - \frac{4}{n} + \frac{2p}{n}(1-\alpha)$ imply that

$$1 - \frac{\gamma}{2} - \frac{p}{2n}(1-\alpha) > 1 - \frac{\gamma}{2} - \frac{1}{n} > 0$$

and

$$\frac{2}{n} - \frac{\gamma - \widetilde{\gamma}}{2} - \frac{p}{n}(1-\alpha) > \frac{2}{n} - \frac{2}{n} + \frac{p}{n}(1-\alpha) - \frac{p}{n}(1-\alpha)$$
$$= 0,$$

respectively, we infer from Lemmas 3.3.4 and 3.3.5 that

$$\begin{aligned}
&s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma} w \, ds \\
&\leq \sqrt{2} s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\frac{\gamma}{2} - \frac{p}{2n}(1-\alpha)} (s_0 - s)^{-\frac{1}{2}} \, ds \sqrt{\psi_\alpha(t)} \\
&= \sqrt{2} B \left(1 - \frac{\gamma}{2} - \frac{p}{2n}(1-\alpha), \frac{1}{2}\right) s_0^{\frac{1-\gamma}{2} + \frac{2}{n} - \frac{p}{2n}(1-\alpha)} \sqrt{\psi_\alpha(t)}
\end{aligned} \tag{3.3.21}$$

and

$$\begin{aligned}
&s_0^{\frac{1}{2} + \frac{\gamma - \widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n} - \gamma + \frac{\widetilde{\gamma}}{2} - \frac{p}{2n}(1-\alpha) - 1} w \, ds \sqrt{\psi_\alpha(t)} \\
&\leq \sqrt{2} s_0^{\frac{1}{2} + \frac{\gamma - \widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n} - \frac{\gamma - \widetilde{\gamma}}{2} - \frac{p}{n}(1-\alpha) - 1} (s_0 - s)^{-\frac{1}{2}} \, ds \cdot \psi_\alpha(t) \\
&= \sqrt{2} B \left(\frac{2}{n} - \frac{\gamma - \widetilde{\gamma}}{2} - \frac{p}{n}(1-\alpha), \frac{1}{2}\right) s_0^{\frac{2}{n} - \frac{p}{n}(1-\alpha)} \psi_\alpha(t)
\end{aligned} \tag{3.3.22}$$

for all $t \in (0, \min\{T, T_{\max}\})$. A combination of (3.3.19) and (3.3.20)–(3.3.22) yields (3.3.16). Similarly, we next establish the estimate (3.3.16) in the case $1 \leq \alpha < 1 + \frac{2}{p}$. Since $\alpha < 1 + \frac{2}{p}$ and $\gamma < 2 - \frac{2p}{n}(\alpha - 1)$, we see that

$$\gamma - \frac{4}{n} + \frac{2p}{n}(\alpha - 1) < \gamma$$

and

$$\begin{aligned}
&\left(2 - \frac{4}{n}\right) - \left(\gamma - \frac{4}{n} + \frac{2p}{n}(\alpha - 1)\right) \\
&= 2 - \frac{2p}{n}(\alpha - 1) - \gamma \\
&> 0.
\end{aligned}$$

Hence we can choose

$$\widetilde{\gamma} \in \left(\max\left\{0, \gamma - \frac{4}{n} + \frac{2p}{n}(\alpha - 1)\right\}, \min\left\{\gamma, 2 - \frac{4}{n}\right\}\right).$$

From (3.3.17), Lemmas 3.3.5 and 3.3.8 we observe that there exists $c_3 = c_3(\gamma) > 0$ such that

$$
\begin{aligned}
z &\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + \frac{\sqrt{2}}{n^2} \int_0^s \int_\sigma^{s_0} \xi^{\frac{2}{n}-2+\frac{\widetilde{\gamma}}{2}} (s_0-\xi)^{-\frac{1}{2}}\, d\xi d\sigma \left\{ \int_0^{s_0} s^{-\widetilde{\gamma}} (s_0 - s) w w_s \, ds \right\}^{\frac{1}{2}} \\
&\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + c_3 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\widetilde{\gamma}}{2}} \left\{ \int_0^{s_0} s^{-\widetilde{\gamma}} (s_0 - s) w w_s \, ds \right\}^{\frac{1}{2}} \\
&\le \frac{c_1}{n} s_0^{\frac{2}{n}-1} s + c_3 s_0^{-\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} s^{\frac{2}{n}+\frac{\widetilde{\gamma}}{2}} \sqrt{\psi_\alpha(t)} \qquad (3.3.23)
\end{aligned}
$$

for all $s \in (0, s_0)$ and $t \in (0, \min\{T, T_{\max}\})$. Thanks to (3.3.2), it follows that

$$
(nw_s + 1)^{\alpha-1} \le (\widetilde{K} s^{-\frac{p}{n}} + 1)^{\alpha-1} \le c_4 s^{-\frac{p}{n}(\alpha-1)} \qquad (3.3.24)
$$

for all $s \in (0, s_0)$ and $t \in (0, \min\{T, T_{\max}\})$, where

$$
c_4 := (\widetilde{K} + R^p)^{\alpha-1}.
$$

Applying (3.3.24) to I_3, by an argument similar to that in the proof of [**60**, Lemma 4.1] and (3.3.23) we see that

$$
\begin{aligned}
I_3 &= -\chi n \int_0^{s_0} s^{-\gamma} (s_0 - s)(nw_s + 1)^{\alpha-1} z w_s \, ds \\
&\ge -\chi n c_4 \int_0^{s_0} s^{-\gamma-\frac{p}{n}(\alpha-1)} (s_0 - s) z w_s \, ds \\
&\ge -\chi n c_4 \left(\gamma + \frac{p}{n}(\alpha-1) + 1 \right) s_0 \int_0^{s_0} s^{-\gamma-1-\frac{p}{n}(\alpha-1)} z w \, ds \\
&\ge -\chi c_1 c_5 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma-\frac{p}{n}(\alpha-1)} w \, ds \\
&\quad - \chi n c_3 c_5 s_0^{\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-\gamma+\frac{\widetilde{\gamma}}{2}-\frac{p}{n}(\alpha-1)-1} w \, ds \sqrt{\psi_\alpha(t)} \qquad (3.3.25)
\end{aligned}
$$

for all $t \in (0, \min\{T, T_{\max}\})$, where $c_5 := c_4 \left(\gamma + \frac{p}{n}(\alpha-1) + 1 \right)$. Here, noticing from $\gamma < 2 - \frac{2p}{n}(\alpha-1)$ and $\widetilde{\gamma} > \gamma - \frac{4}{n} + \frac{2p}{n}(\alpha-1)$ that

$$
1 - \frac{\gamma}{2} - \frac{p}{n}(\alpha-1) > 0
$$

and

$$
\frac{2}{n} - \frac{\gamma-\widetilde{\gamma}}{2} - \frac{p}{n}(\alpha-1) > \frac{2}{n} - \frac{2}{n} + \frac{p}{n}(\alpha-1) - \frac{p}{n}(\alpha-1) = 0,
$$

we have from Lemmas 3.3.4 and 3.3.5 that

$$\begin{aligned}
&s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\gamma-\frac{p}{n}(\alpha-1)} w\,ds \\
&\quad\le \sqrt{2} s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\frac{\gamma}{2}-\frac{p}{n}(\alpha-1)}(s_0-s)^{-\frac{1}{2}}\,ds \sqrt{\psi_\alpha(t)} \\
&\quad= \sqrt{2} B\left(1-\frac{\gamma}{2}-\frac{p}{n}(\alpha-1), \frac{1}{2}\right) s_0^{\frac{1-\gamma}{2}+\frac{2}{n}-\frac{p}{n}(\alpha-1)} \sqrt{\psi_\alpha(t)}
\end{aligned} \tag{3.3.26}$$

and

$$\begin{aligned}
&s_0^{\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-\gamma+\frac{\widetilde{\gamma}}{2}-\frac{p}{n}(\alpha-1)-1} w\,ds \sqrt{\psi_\alpha(t)} \\
&\quad\le \sqrt{2} s_0^{\frac{1}{2}+\frac{\gamma-\widetilde{\gamma}}{2}} \int_0^{s_0} s^{\frac{2}{n}-\frac{\gamma-\widetilde{\gamma}}{2}-\frac{p}{n}(\alpha-1)-1}(s_0-s)^{-\frac{1}{2}}\,ds \cdot \psi_\alpha(t) \\
&\quad= \sqrt{2} B\left(\frac{2}{n}-\frac{\gamma-\widetilde{\gamma}}{2}-\frac{p}{n}(\alpha-1), \frac{1}{2}\right) s_0^{\frac{2}{n}-\frac{p}{n}(\alpha-1)} \psi_\alpha(t)
\end{aligned} \tag{3.3.27}$$

for all $t \in (0, \min\{T, T_{\max}\})$. Hence, combining (3.3.25), (3.3.26) and (3.3.27) leads to (3.3.16). □

We finally derive an estimate for ψ_α.

Lemma 3.3.10. *Let $\alpha > 0$ and $p \ge n$. Suppose that $\gamma \in (0,1)$ satisfies that*

$$\gamma < 2 - \frac{p}{n}(1-\alpha)_+. \tag{3.3.28}$$

Then there exists $C = C(\gamma, \alpha, p) > 0$ such that for any $s_0 \in (0, R^n)$

$$\phi(t) \le C s_0^{\frac{3-\gamma}{2}-\frac{p}{2n}(1-\alpha)_+} \sqrt{\psi_\alpha(t)}$$

for all $t \in (0, T_{\max})$.

Proof. From (3.3.28) it follows that

$$1 - \frac{\gamma}{2} - \frac{p}{2n}(1-\alpha)_+ > 0.$$

Therefore we infer from Lemmas 3.3.4 and 3.3.5 that

$$\begin{aligned}
\phi(t) &= \int_0^{s_0} s^{-\gamma}(s_0-s) w\,ds \\
&\le s_0 \int_0^{s_0} s^{-\gamma} w\,ds \\
&\le \sqrt{2} s_0 \int_0^{s_0} s^{-\frac{\gamma}{2}-\frac{p}{2n}(1-\alpha)_+}(s_0-s)^{-\frac{1}{2}}\,ds \sqrt{\psi_\alpha(t)} \\
&= \sqrt{2} B\left(1-\frac{\gamma}{2}-\frac{p}{2n}(1-\alpha)_+, \frac{1}{2}\right) s_0^{\frac{3-\gamma}{2}-\frac{p}{2n}(1-\alpha)_+} \sqrt{\psi_\alpha(t)}
\end{aligned}$$

for all $t \in (0, T_{\max})$, which concludes the proof. □

3.4. Differential inequalities for ϕ

In this section we will derive a super-linear differential inequality for the moment-type functional ϕ by using the pointwise lower estimates for I_1, I_2, I_3 and I_4. To this end, we find $\gamma \in (0,1)$ which enable us to apply Lemmas 3.3.3, 3.3.6, 3.3.7, 3.3.9 and 3.3.10 to (3.3.3). We give the conditions (A3-1), (A3-2), (B1-1), (B1-2), (C1-1), (C1-2), (C3-1), (C3-2), (C3-3), (D2-1) and (D2-2) as follows:

- In the case $n = 3$,

$$1 - \frac{1}{p} < \alpha < 1 + \frac{2}{p}, \quad \frac{1}{p} \le m < \frac{2}{p}, \quad 2\alpha - m \le 2 + \frac{2}{p}, \tag{A3-1}$$

$$1 - \frac{1}{p} < \alpha < 1, \quad \frac{2}{p} \le m < \frac{3}{p}, \quad m + \alpha < 1 + \frac{2}{p}. \tag{A3-2}$$

- In the case $n = 4$,

$$1 - \frac{2}{p} < \alpha < 1, \quad 0 < m < \frac{2}{p}, \tag{B1-1}$$

$$1 \le \alpha < 1 + \frac{2}{p}, \quad 0 < m < \frac{2}{p}. \tag{B1-2}$$

- In the case $n = 5$,

$$1 - \frac{2}{p} < \alpha \le 1 - \frac{1}{p}, \quad 0 < m < \frac{3}{p}, \tag{C1-1}$$

$$1 - \frac{1}{p} < \alpha < 1 + \frac{2}{p}, \quad 0 < m < 1 + \frac{1}{2p}, \quad 2m - \alpha < 1 + \frac{1}{p}, \tag{C1-2}$$

$$1 - \frac{2}{p} < \alpha \le 1 - \frac{1}{p}, \quad \frac{3}{p} \le m < 1, \quad m + \alpha \ge 1 + \frac{2}{p}, \tag{C3-1}$$

$$1 - \frac{2}{p} < \alpha < 1, \quad 1 \le m < 1 + \frac{1}{2p}, \quad 2m - \alpha \ge 1 + \frac{1}{p}, \tag{C3-2}$$

$$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \quad 1 + \frac{1}{2p} \le m < 1 + \frac{3}{p}, \quad m - \alpha < \frac{3}{p}. \tag{C3-3}$$

- In the case $n \ge 6$,

$$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \ 1 + \frac{n-6}{2p} \le m < 1 + \frac{n-4}{2p}, \ 2m - \alpha \ge 1 + \frac{n-4}{p}, \tag{D2-1}$$

$$1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}, \ 1 + \frac{n-4}{2p} \le m < 1 + \frac{n-2}{p}, \ m - \alpha < \frac{n-2}{p}. \tag{D2-2}$$

We first show that there exists $\gamma \in (0,1)$ satisfying (3.3.6), (3.3.8), (3.3.13), the second condition of (3.3.15) and (3.3.28).

Lemma 3.4.1. *Let $m > 0$, $\alpha > 0$, $\kappa \geq 1$, $p \geq n$ and $q \geq 0$. Assume that m and α satisfy* (A4), (B1-2), (B3), (C1-2), (C3), (D1) *or* (D2). *Suppose that κ fulfills* (**II**) *and* (**IV**). *Then there exists $\gamma \in (0,1)$ such that*

$$\max\left\{\frac{p}{n}(1-\alpha)_+, \frac{p}{n}[2(\kappa-1)+(1-\alpha)_+] - \frac{2q}{n}, 1-\frac{2}{n}-\frac{p}{n}(m-1)_+\right\}$$
$$< \gamma < \min\left\{1, 2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\right\}. \tag{3.4.1}$$

Proof. We first consider the case that m and α satisfy (A4), (B3), (C3) or (D2). In the cases that $n = 3$ and the condition (A4) holds and that $n = 4$ and the condition (B3) holds we see that

$$\begin{aligned}
1 &- \left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\right)\\
&= -1+\frac{4}{n}+\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\\
&\geq -1+\frac{4}{n} \geq 0.
\end{aligned}$$

Moreover, thanks to the conditions $\alpha \leq 1-\frac{1}{p}$ in the case that $n = 5$ and (C3-1) holds, $2m-\alpha \geq 1+\frac{n-4}{p}$ in the cases that $n = 5$ and (C3-2) holds and that $n \geq 6$ and (D2-1) holds and $m \geq 1+\frac{n-4}{2p}$ in the cases that $n = 5$ and (C3-3) holds and that $n \geq 6$ and (D2-2) holds, we obtain

$$\begin{aligned}
1 &- \left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\right)\\
&= -1+\frac{4}{n}+\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\\
&\geq -1+\frac{4}{n}+\frac{n-4}{n} = 0.
\end{aligned}$$

Thus it suffices to show that the following conditions hold:

$$\frac{p}{n}(1-\alpha)_+ < 2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+], \tag{3.4.2}$$

$$\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+] - \frac{2q}{n} < 2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+], \tag{3.4.3}$$

$$1-\frac{2}{n}-\frac{p}{n}(m-1)_+ < 2-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]. \tag{3.4.4}$$

Now we note that

$$\left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_{+}+(1-\alpha)_{+}]\right)-\frac{p}{n}(1-\alpha)_{+}$$
$$=2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}].$$

In the cases that $n=3$ and (A4) holds and that $n=4$ and (B3) holds, if $m<1$ and $\alpha\geq 1$, then it follows that

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}]=2-\frac{4}{n}>0.$$

Furthermore, invoking from (A4) and (B3) that

$$\alpha>1-\frac{n-2}{p} \qquad \text{if} \quad m<1 \quad \text{and} \quad \alpha<1,$$
$$m-\alpha<\frac{n-2}{p} \qquad \text{if} \quad m\geq 1 \quad \text{and} \quad \alpha<1,$$
$$m<1+\frac{n-2}{p} \qquad \text{if} \quad m\geq 1 \quad \text{and} \quad \alpha\geq 1,$$

we can observe that

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}]>2-\frac{4}{n}-\frac{2(n-2)}{n}=0.$$

On the other hand, in the case that $n=5$ and (C3-1) holds we see from the conditions $m<1$ and $\alpha>1-\frac{2}{p}$ that

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}]>2-\frac{4}{n}-\frac{4}{n}=\frac{2}{5}>0.$$

In the cases that $n=5$ and (C3-2) holds and that $n\geq 6$ and (D2-1) holds, by virtue of the conditions $n\geq 5$, $m<1+\frac{n-4}{2p}$ and $\alpha>1-\frac{2}{p}$ we obtain

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}]>2-\frac{4}{n}-\frac{2p}{n}\left[\frac{n-4}{2p}+\frac{2}{p}\right]=1-\frac{4}{n}>0.$$

In the cases that $n=5$ and (C3-3) holds and that $n\geq 6$ and (D2-2) holds, recalling $m-\alpha<\frac{n-2}{p}$, we can establish that

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_{+}+(1-\alpha)_{+}]>2-\frac{4}{n}-\frac{2(n-2)}{n}=0.$$

Therefore we attain (3.4.2). Moreover, from the fact

$$2-\frac{4}{n}-\frac{2p}{n}[(m-1)_++(1-\alpha)_+]>0$$

we deduce that

$$\begin{aligned}&\left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_++(1-\alpha)_+]\right)-\left(1-\frac{2}{n}-\frac{p}{n}(m-1)_+\right)\\&=1-\frac{2}{n}-\frac{p}{n}[(m-1)_++(1-\alpha)_+]\\&=\frac{1}{2}\left(2-\frac{4}{n}-\frac{2p}{n}[(m-1)_++(1-\alpha)_+]\right)\\&>0,\end{aligned}$$

which implies that (3.4.4) holds. Noticing from (**IV**) that

$$\kappa<1+\frac{n-2}{p}+\frac{q}{p}-(m-1)_+-(1-\alpha)_+,$$

we have that

$$\begin{aligned}&\left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_++(1-\alpha)_+]\right)-\left(\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+]-\frac{2q}{n}\right)\\&=2-\frac{4}{n}+\frac{2q}{n}-\frac{2p}{n}[(\kappa-1)+(m-1)_++(1-\alpha)_+]\\&>2-\frac{4}{n}+\frac{2q}{n}-\frac{2p}{n}\cdot\left[\frac{n-2}{p}+\frac{q}{p}\right]\\&=0,\end{aligned}$$

which attains (3.4.3). Thus, in the cases (A4), (B3), (C3) and (D2) we can take $\gamma\in(0,1)$ with (3.4.1). Next we consider the case that m and α fulfill (B1-2). Since it follows from the conditions $n=4$, $m<1$ and $\alpha\geq 1$ that

$$1-\left(2-\frac{4}{n}-\frac{p}{n}[2(m-1)_++(1-\alpha)_+]\right)=-1+\frac{4}{n}=0,$$

we confirm that

$$\frac{p}{n}(1-\alpha)_+<1, \tag{3.4.5}$$

$$\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+]-\frac{2q}{n}<1, \tag{3.4.6}$$

$$1-\frac{2}{n}-\frac{p}{n}(m-1)_+<1. \tag{3.4.7}$$

Noting that $(m-1)_+ = 0$ and $(1-\alpha)_+ = 0$, we see that (3.4.5) and (3.4.7) hold. Furthermore, recalling from (**II**) that

$$\kappa < 1 + \frac{n}{2p} + \frac{q}{p},$$

we obtain

$$1 - \left(\frac{2p}{n}(\kappa - 1) - \frac{2q}{n}\right) > 1 - \frac{2p}{n}\left(\frac{n}{2p} + \frac{q}{p}\right) + \frac{2q}{n} = 0,$$

which infers that we can choose $\gamma \in (0,1)$ with (3.4.1). Finally we verify that there exists $\gamma \in (0,1)$ with (3.4.1) in the case that m and α satisfy the condition (C1-2) or (D1). The conditions $n \geq 5$ and $2m - \alpha < 1 + \frac{n-4}{p}$ yield that

$$\begin{aligned}
1 &- \left(2 - \frac{4}{n} - \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\right) \\
&= -1 + \frac{4}{n} + \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+] \\
&< -1 + \frac{4}{n} + \frac{n-4}{n} \\
&= 0.
\end{aligned}$$

Thus we show (3.4.5)–(3.4.7). Since $n \geq 5$ and $\alpha > 1 - \frac{2}{p}$, we have that

$$1 - \frac{p}{n}(1-\alpha)_+ > 1 - \frac{2}{n} > 0.$$

Moreover, it follows that

$$1 - \left(1 - \frac{2}{n} - \frac{p}{n}(m-1)_+\right) = \frac{2}{n} + \frac{p}{n}(m-1)_+ > 0.$$

Invoking from (**II**) that

$$\kappa < 1 + \frac{n}{2p} + \frac{q}{p} - \frac{(1-\alpha)_+}{2},$$

we see that

$$\begin{aligned}
1 &- \left(\frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n}\right) \\
&> 1 - \frac{p}{n} \cdot \left[\frac{n}{p} + \frac{2q}{p}\right] + \frac{2q}{n} \\
&= 0.
\end{aligned}$$

Therefore, since (3.4.5)–(3.4.7) hold, we can find $\gamma \in (0,1)$ with (3.4.1). $\square$

Next we prove that there exists $\gamma \in (0,1)$ such that (3.3.6), (3.3.8), (3.3.14), the second condition of (3.3.15) and (3.3.28) hold.

Lemma 3.4.2. *Let $m > 0$, $\alpha > 0$, $\kappa \ge 1$, $p \ge n$ and $q \ge 0$. Assume that m and α satisfy* (A1), (A2), (A3), (B1-1), (B2), (C1-1) *or* (C2). *Suppose that κ fulfills* (**I**),(**II**) *and* (**III**). *Then there exists $\gamma \in (0,1)$ such that*

$$\max\left\{\frac{p}{n}(1-\alpha)_+, \frac{p}{n}[2(\kappa-1)+(1-\alpha)_+] - \frac{2q}{n}\right\}$$
$$< \gamma < \min\left\{1, 2-\frac{2}{n}-\frac{pm}{n}, 2-\frac{2p}{n}(\alpha-1)_+\right\}. \tag{3.4.8}$$

Proof. First we consider the case that m and α fulfill (A1), (B1-1) or (C1-1). By virtue of the condition $m < \frac{n-2}{p}$ ($n \in \{3,4,5\}$) we obtain

$$\begin{aligned}
&\left(2-\frac{2}{n}-\frac{pm}{n}\right)-1\\
&= 1-\frac{2}{n}-\frac{pm}{n}\\
&> 1-\frac{2}{n}-\frac{n-2}{n} = 0.
\end{aligned}$$

Furthermore, in the case that $n=3$ and (A1) holds we can estimate from the condition $\alpha < 1+\frac{3}{2p}$ that

$$\left(2-\frac{2p}{n}(\alpha-1)_+\right)-1 = 1-\frac{2p}{n}(\alpha-1)_+ > 1-\frac{3}{n} = 0.$$

In the cases that $n=4$ and (B1-1) holds and that $n=5$ and (C1-1) holds, noticing that $(\alpha-1)_+ = 0$, we can verify that

$$\left(2-\frac{2p}{n}(\alpha-1)_+\right)-1 = 1 > 0.$$

Hence it suffices to show that

$$\frac{p}{n}(1-\alpha)_+ < 1, \tag{3.4.9}$$
$$\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+] - \frac{2q}{n} < 1. \tag{3.4.10}$$

In the case that $n=3$ and (A1) holds we have from the condition $\alpha > 1-\frac{1}{p}$ that

$$1-\frac{p}{n}(1-\alpha)_+ > 1-\frac{1}{n} > 0.$$

On the other hand, in the cases that $n = 4$ and (B1-1) holds and that $n = 5$ and (C1-1) holds we see from the condition $\alpha > 1 - \frac{2}{p}$ that

$$1 - \frac{p}{n}(1-\alpha)_+ > 1 - \frac{2}{n} > 0.$$

Thus the condition (3.4.9) holds. Recalling from (**II**) that

$$\kappa < 1 + \frac{n}{2p} + \frac{q}{p} - \frac{(1-\alpha)_+}{2},$$

we can show that

$$1 - \left(\frac{p}{n}[2(\kappa - 1) + (1-\alpha)_+] - \frac{2q}{n}\right) > 1 - \frac{p}{n} \cdot \left[\frac{n}{p} + \frac{2q}{p}\right] + \frac{2q}{n} = 0,$$

which implies that we attain (3.4.10). Since (3.4.9) and (3.4.10) hold, we can take $\gamma \in (0,1)$ with (3.4.8). Next we consider the case that m and α satisfy (A2). From the conditions $n = 3$ and $2\alpha - m > 2 + \frac{2}{p}$ we obtain

$$\begin{aligned} &\left(2 - \frac{2}{n} - \frac{pm}{n}\right) - \left(2 - \frac{2p}{n}(\alpha - 1)_+\right) \\ &= -\frac{2}{n} + \frac{p}{n}(2\alpha - m - 2) \\ &> -\frac{2}{n} + \frac{2}{n} \\ &= 0. \end{aligned}$$

Moreover, the condition $\alpha \geq 1 + \frac{3}{2p}$ yields that

$$1 - \left(2 - \frac{2p}{n}(\alpha - 1)_+\right) = -1 + \frac{2p}{n}(\alpha - 1) \geq -1 + \frac{3}{n} = 0.$$

Accordingly, we confirm that

$$\frac{p}{n}(1-\alpha)_+ < 2 - \frac{2p}{n}(\alpha - 1)_+, \tag{3.4.11}$$

$$\frac{p}{n}[2(\kappa - 1) + (1-\alpha)_+] - \frac{2q}{n} < 2 - \frac{2p}{n}(\alpha - 1)_+. \tag{3.4.12}$$

Since $\frac{p}{n}(1-\alpha)_+ = 0$ and it follows from the condition $\alpha < 1 + \frac{2}{p}$ that

$$2 - \frac{2p}{n}(\alpha - 1)_+ > 2 - \frac{4}{n} > 0,$$

we can verify that (3.4.11) holds. Noticing from (**I**) that

$$\kappa < 1 + \frac{3}{p} + \frac{q}{p} - (\alpha - 1),$$

we see that

$$\begin{aligned}
&\left(2 - \frac{2p}{n}(\alpha-1)_+\right) - \left(\frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n}\right) \\
&= 2 - \frac{2p}{n}[(\kappa-1) + (\alpha-1)] + \frac{2q}{n} \\
&> 2 - \frac{2p}{n} \cdot \left[\frac{3}{p} + \frac{q}{p}\right] + \frac{2q}{n} \\
&= 0,
\end{aligned}$$

which infers (3.4.12). Consequently, we can find $\gamma \in (0,1)$ with (3.4.8). Finally we consider the case that m and α fulfill (A3), (B2) or (C2). In light of the condition $2\alpha - m \le 2 + \frac{2}{p}$ in the case that $n = 3$ and (A3-1) holds and the condition $(\alpha - 1)_+ = 0$ in the cases that $n = 3$ and (A3-2) holds, that $n = 4$ and (B2) holds and that $n = 5$ and (C2) holds we have that

$$\left(2 - \frac{2p}{n}(\alpha-1)_+\right) - \left(2 - \frac{2}{n} - \frac{pm}{n}\right) = \frac{2}{n} - \frac{p}{n}[2(\alpha-1)_+ - m] \ge 0.$$

Moreover, since $m \ge \frac{n-2}{p}$ in the cases that $n = 3$ and (A3) holds, that $n = 4$ and (B2) holds and that $n = 5$ and (C2) holds, it follows that

$$1 - \left(2 - \frac{2}{n} - \frac{pm}{n}\right) = -1 + \frac{2}{n} + \frac{pm}{n} \ge -1 + \frac{2}{n} + \frac{n-2}{n} = 0.$$

Therefore it suffices to show that

$$\frac{p}{n}(1-\alpha)_+ < 2 - \frac{2}{n} - \frac{pm}{n}, \tag{3.4.13}$$

$$\frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n} < 2 - \frac{2}{n} - \frac{pm}{n}. \tag{3.4.14}$$

Now we note that

$$\left(2 - \frac{2}{n} - \frac{pm}{n}\right) - \frac{p}{n}(1-\alpha)_+ = 2 - \frac{2}{n} - \frac{p}{n}[m + (1-\alpha)_+].$$

In the case that $n = 3$ and (A3) holds, thanks to the conditions $m < \frac{3}{p}$ and $\alpha > 1 - \frac{1}{p}$, we obtain

$$2 - \frac{2}{n} - \frac{p}{n}[m + (1-\alpha)_+] > 2 - \frac{2}{n} - \frac{p}{n} \cdot \left[\frac{3}{p} + \frac{1}{p}\right] = 0.$$

On the other hand, in the cases that $n = 4$ and (B2) holds and that $n = 5$ and (C2) holds we deduce from the conditions $m < \frac{4}{p}$ and $\alpha > 1 - \frac{2}{p}$ that

$$2 - \frac{2}{n} - \frac{p}{n}[m + (1-\alpha)_+] > 2 - \frac{2}{n} - \frac{p}{n} \cdot \left[\frac{4}{p} + \frac{2}{p}\right] \geq 0.$$

Thus the condition (3.4.13) holds. Invoking from (**III**) that

$$\kappa < 1 + \frac{n-1}{p} + \frac{q}{p} - \frac{m}{2} - \frac{(1-\alpha)_+}{2},$$

we can show that

$$\begin{aligned}
&\left(2 - \frac{2}{n} - \frac{pm}{n}\right) - \left(\frac{p}{n}[2(\kappa - 1) + (1-\alpha)_+] - \frac{2q}{n}\right) \\
&= 2 - \frac{2}{n} - \frac{p}{n}[2(\kappa-1) + m + (1-\alpha)_+] + \frac{2q}{n} \\
&> 2 - \frac{2}{n} - \frac{p}{n} \cdot \left[\frac{n-1}{p} + \frac{q}{p}\right] + \frac{2q}{n} \\
&= 0,
\end{aligned}$$

which infers that (3.4.14) holds. Accordingly, we can choose $\gamma \in (0,1)$ with (3.4.8). □

Thanks to Lemmas 3.4.1 and 3.4.2, we can apply Lemmas 3.3.3, 3.3.6, 3.3.7, 3.3.9 and 3.3.10 to (3.3.3). Thus, we finally establish a super-linear differential inequality for the moment-type functional ϕ defined as (3.3.1).

Lemma 3.4.3. *Let $m > 0$, $\alpha > 0$, $\chi > 0$, $\mu_1 > 0$, $\kappa \geq 1$, $p \geq n$, $q \geq 0$, $M_0 > 0$, $\widetilde{K} > 0$ and $T > 0$.*

(i) *Assume that m and α satisfy* (A4), (B1-2), (B3), (C1-2), (C3), (D1) *or* (D2) *and suppose that κ fulfills* (**II**) *and* (**IV**). *Then one can find $C > 0$, $\gamma \in (0,1)$, $\theta \in \left(0, 2 - \frac{p}{n}(1-\alpha)_+\right)$ and $s_1 \in (0, R^n)$ such that if $\int_\Omega u_0(x)\,dx = M_0$ and* (3.1.7) *holds, then*

$$\phi'(t) \geq \frac{1}{C} s_0^{\gamma - 3 + \frac{p}{n}(1-\alpha)_+} \phi^2(t) - C s_0^{3-\gamma-\theta} \tag{3.4.15}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$.

(ii) *Assume that m and α satisfy* (A1), (A2), (A3), (B1-1), (B2), (C1-1) *or* (C2) *and suppose that κ fulfills* (**I**), (**II**) *and* (**III**). *Then one can find $C > 0$, $\gamma \in (0,1)$, $\theta \in \left(0, 2 - \frac{p}{n}(1-\alpha)_+\right)$ and $s_1 \in (0, R^n)$ such that if $\int_\Omega u_0(x)\,dx = M_0$ and* (3.1.7) *holds, then* (3.4.15) *holds for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$.*

Proof. We first prove (3.4.15) in the case that m and α satisfy (A4), (B1-2), (B3), (C1-2), (C3), (D1) or (D2). By virtue of Lemma 3.4.1 we can take $\gamma \in (0,1)$ fulfilling (3.4.1). Thus, applying Lemmas 3.3.3, 3.3.6, part (i) of Lemma 3.3.7 and 3.3.9 to (3.3.3), we can observe that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned}\phi'(t) \geq\ & c_1\psi_\alpha(t) \\ & - c_2 s_0^{\frac{3-\gamma}{2}-\frac{2}{n}-\frac{p}{2n}[2(m-1)_+ + (1-\alpha)_+]}\sqrt{\psi_\alpha(t)} - c_2 s_0^{3-\frac{2}{n}-\gamma} \\ & - c_2 s_0^{\frac{2}{n}+\frac{1-\gamma}{2}-\frac{p}{2n}[(1-\alpha)_+ + 2(\alpha-1)_+]}\sqrt{\psi_\alpha(t)} \\ & - c_2 s_0^{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}\psi_\alpha(t) \\ & - c_2 s_0^{\frac{3-\gamma}{2}+\frac{q}{n}-\frac{p}{2n}[2(\kappa-1)+(1-\alpha)_+]}\sqrt{\psi_\alpha(t)}\end{aligned}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, \min\{T, T_{\max}\})$. Aided by Young's inequality, we can verify that for any $\eta > 0$ there exists $c_3 = c_3(\eta) > 0$ such that

$$\begin{aligned}\phi'(t) \geq\ & c_1\psi_\alpha(t) - \eta\psi_\alpha(t) - c_2 s_0^{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}\psi_\alpha(t) \\ & - c_3\Big(s_0^{3-\gamma-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]} + s_0^{3-\frac{2}{n}-\gamma} \\ & + s_0^{\frac{4}{n}+1-\gamma-\frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+]} + s_0^{3-\gamma+\frac{2q}{n}-\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+]}\Big)\end{aligned} \tag{3.4.16}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, \min\{T, T_{\max}\})$. Since it follows from the condition $1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}$ that

$$\frac{2}{n} - \frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+] > \frac{2}{n} - \frac{2}{n} = 0,$$

we can take $s_1 \in (0, R^n)$ satisfying

$$s_1 \leq \left(\frac{c_1}{4c_2}\right)^{\frac{1}{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}}.$$

Therefore we see that

$$s_0^{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}\psi_\alpha(t) \leq \frac{c_1}{4c_2}\psi_\alpha(t) \tag{3.4.17}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$. Fixing $\eta = \frac{c_1}{4}$, we infer from (3.4.16) and (3.4.17) that

$$\begin{aligned}\phi'(t) \geq\ & \frac{c_1}{2}\psi_\alpha(t) \\ & - c_3\Big(s_0^{3-\gamma-\frac{4}{n}-\frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]} + s_0^{3-\frac{2}{n}-\gamma} \\ & + s_0^{\frac{4}{n}+1-\gamma-\frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+]} + s_0^{3-\gamma+\frac{2q}{n}-\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+]}\Big)\end{aligned} \tag{3.4.18}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$. Next, putting

$$\theta_1 := \max\left\{\frac{4}{n} + \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+], \frac{2}{n}, 2 - \frac{4}{n} + \frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+], \right. \\ \left. -\frac{2q}{n} + \frac{p}{n}[2(\kappa-1) + (1-\alpha)_+]\right\},$$

we show $\theta_1 \in \left(0, 2 - \frac{p}{n}(1-\alpha)_+\right)$, that is, we confirm that

$$\frac{4}{n} + \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+] < 2 - \frac{p}{n}(1-\alpha)_+, \tag{3.4.19}$$

$$\frac{2}{n} < 2 - \frac{p}{n}(1-\alpha)_+, \tag{3.4.20}$$

$$2 - \frac{4}{n} + \frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+] < 2 - \frac{p}{n}(1-\alpha)_+, \tag{3.4.21}$$

$$-\frac{2q}{n} + \frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] < 2 - \frac{p}{n}(1-\alpha)_+. \tag{3.4.22}$$

Since the inequality $2 - \frac{4}{n} - \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+] > \frac{p}{n}(1-\alpha)_+$ holds from (3.4.1), it follows that

$$\left(2 - \frac{p}{n}(1-\alpha)_+\right) - \left(\frac{4}{n} + \frac{p}{n}[2(m-1)_+ + (1-\alpha)_+]\right) > 0.$$

Moreover, we can establish from the condition $1 - \frac{2}{p} < \alpha < 1 + \frac{2}{p}$ that

$$\left(2 - \frac{p}{n}(1-\alpha)_+\right) - \frac{2}{n} > \left(2 - \frac{2}{n}\right) - \frac{2}{n} > 0$$

and

$$\begin{aligned}
&\left(2 - \frac{p}{n}(1-\alpha)_+\right) - \left(2 - \frac{4}{n} + \frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+]\right) \\
&= \frac{4}{n} - \frac{2p}{n}[(1-\alpha)_+ + (\alpha-1)_+] \\
&> \frac{4}{n} - \frac{4}{n} = 0.
\end{aligned}$$

Noticing from (3.4.1) that $1 > \frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n}$, we see from the condition $1 - \frac{2}{p} < \alpha$ that

$$\begin{aligned}
&\left(2 - \frac{p}{n}(1-\alpha)_+\right) - \left(-\frac{2q}{n} + \frac{p}{n}[2(\kappa-1) + (1-\alpha)_+]\right) \\
&> \left(2 - \frac{2}{n}\right) - \left(\frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n}\right) \\
&> 1 - \left(\frac{p}{n}[2(\kappa-1) + (1-\alpha)_+] - \frac{2q}{n}\right) > 0.
\end{aligned}$$

Thus, since we know that (3.4.19)–(3.4.22) hold, we have $\theta_1 \in \big(0, 2 - \frac{p}{n}(1-\alpha)_+\big)$. Invoking that $\gamma \in (0,1)$, $\theta_1 \in \big(0, 2 - \frac{p}{n}(1-\alpha)_+\big)$ and $s_0 < R^n$, we can deduce from (3.4.18) that there exists $c_4 = c_4(R, m, \alpha, \kappa, p) > 0$ such that

$$\phi'(t) \ge \frac{c_1}{2}\psi_\alpha(t) - c_3 c_4 s_0^{3-\gamma-\theta_1}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$. Moreover, in light of Lemma 3.3.10, we can take $c_5 > 0$ such that

$$\phi'(t) \ge \frac{c_1}{2} c_5 s_0^{\gamma-3+\frac{p}{n}(1-\alpha)_+}\phi^2(t) - c_3 c_4 s_0^{3-\gamma-\theta_1}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{T, T_{\max}\})$. Hence we attain (3.4.15). As to (ii), since m and α fulfill (A1), (A2), (A3), (B1-1), (B2), (C1-1) or (C2), we can pick $\gamma \in (0,1)$ with (3.4.8). Applying Lemmas 3.3.3, 3.3.6, part (ii) of Lemma 3.3.7 and 3.3.9 to (3.3.3), we find $c_6 > 0$ and $c_7 > 0$ such that

$$\begin{aligned}
\phi'(t) \ge\ & c_6\psi_\alpha(t) - c_7 s_0^{3-\gamma-\frac{2}{n}-\frac{pm}{n}} - c_7 s_0^{3-\frac{2}{n}-\gamma} \\
& - c_7 s_0^{\frac{2}{n}+\frac{1-\gamma}{2}-\frac{p}{2n}[(1-\alpha)_+ + 2(\alpha-1)_+]}\sqrt{\psi_\alpha(t)} \\
& - c_7 s_0^{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}\psi_\alpha(t) \\
& - c_7 s_0^{\frac{3-\gamma}{2}+\frac{q}{n}-\frac{p}{2n}[2(\kappa-1)+(1-\alpha)_+]}\sqrt{\psi_\alpha(t)}
\end{aligned}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, \min\{T, T_{\max}\})$. By an argument similar to the proof in (i), we can show that there exists $c_8 > 0$ such that

$$\begin{aligned}
\phi'(t) \ge\ & \frac{c_6}{2}\psi_\alpha(t) \\
& - c_8\Big(s_0^{3-\gamma-\frac{2}{n}-\frac{pm}{n}} + s_0^{3-\frac{2}{n}-\gamma} \\
& + s_0^{\frac{4}{n}+1-\gamma-\frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+]} + s_0^{3-\gamma+\frac{2q}{n}-\frac{p}{n}[2(\kappa-1)+(1-\alpha)_+]}\Big)
\end{aligned}$$

for all $s_0 \in (0, s_2)$ and $t \in (0, \min\{T, T_{\max}\})$, where $s_2 := \big(\frac{c_6}{4c_7}\big)^{\frac{1}{\frac{2}{n}-\frac{p}{n}[(1-\alpha)_+ + (\alpha-1)_+]}}$. We set

$$\begin{aligned}
\theta_2 := \max\Big\{ & \frac{2}{n} + \frac{pm}{n}, \frac{2}{n}, 2 - \frac{4}{n} + \frac{p}{n}[(1-\alpha)_+ + 2(\alpha-1)_+], \\
& -\frac{2q}{n} + \frac{p}{n}[2(\kappa-1) + (1-\alpha)_+]\Big\}.
\end{aligned}$$

Here, we note that (3.4.20)–(3.4.22) hold. To verify that $\theta_2 \in \big(0, 2 - \frac{p}{n}(1-\alpha)_+\big)$ we confirm that

$$\frac{2}{n} + \frac{pm}{n} < 2 - \frac{p}{n}(1-\alpha)_+. \tag{3.4.23}$$

Since it follows from (3.4.8) that $2 - \frac{2}{n} - \frac{pm}{n} > \frac{p}{n}(1-\alpha)_+$, we have that

$$\left(2 - \frac{p}{n}(1-\alpha)_+\right) - \left(\frac{2}{n} + \frac{pm}{n}\right) = \left(2 - \frac{2}{n} - \frac{pm}{n}\right) - \frac{p}{n}(1-\alpha)_+ > 0.$$

Thus, since we know that (3.4.20)–(3.4.23) hold, we attain that $\theta_2 \in \left(0, 2 - \frac{p}{n}(1-\alpha)_+\right)$. From the fact $s_0 < R^n$ and Lemma 3.3.10 we can find $c_9 > 0$ and $c_{10} > 0$ such that $\phi'(t) \geq \frac{c_6}{2} c_9 s_0^{\gamma-3+\frac{p}{n}(1-\alpha)_+} \phi^2(t) - c_{10} s_0^{3-\gamma-\theta_2}$ for all $s_0 \in (0, s_2)$ and $t \in (0, \min\{T, T_{\max}\})$, which concludes the proof. □

3.5. Proof of the main results

Now we are in a position to complete the proofs of Theorems 3.1.1 and 3.1.2. Due to a use of the same moment-type functional ϕ as in [**2**], by the argument in [**2**, Lemma 4.1] we can proved the following lemma which is needed for the proof of Theorem 3.1.1.

Lemma 3.5.1. *Let $\gamma \in (0,1)$, $s_0 \in (0, R^n)$, $M_1 \geq 0$ and $\eta \in (0,1)$. Put $s_\eta := (1-\eta)s_0$ and $r_\star := s_\eta^{\frac{1}{n}}$. If*

$$\int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_1,$$

then

$$\phi(0) \geq \frac{\eta^2 M_1}{\omega_{n-1}} \cdot s_0^{2-\gamma}.$$

Proof of Theorem 3.1.1. We have from Lemma 3.4.3 that (3.4.15) holds. By virtue of Lemma 3.5.1 and an argument similar to that in the proof of [**2**, Theorem 1.1] we see that $T_{\max} < T$. Thanks to Lemma 3.2.1, we arrive at the conclusion (3.1.8). □

To give the proof of Theorem 3.1.2 we show the pointwise upper estimate for u.

Lemma 3.5.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ with $R > 0$ and let $\chi > 0$, $\kappa \geq 1$, $\mu_1 > 0$, $q \geq 0$, $M_0 > 0$, $T > 0$ and $\widetilde{L} > 0$. Suppose that λ and μ satisfy* (3.1.4) *and* (3.1.5). *Assume that $m > 0$ and $\alpha > 0$ fulfill that*

$$m \geq 1 \quad \text{and} \quad m - \alpha \in \left(-\frac{1}{n}, \frac{n-2}{n}\right].$$

For all $\varepsilon > 0$ set $p := \frac{n(n-1)}{(m-\alpha)n+1} + \varepsilon$. Then there exists $C > 0$ such that the following property holds: If u_0 satisfies (3.1.6) *and $\int_\Omega u_0 = M_0$ as well as*

$$u_0(x) \leq \widetilde{L}|x|^{-p} \quad \text{for all } x \in \Omega \tag{3.5.1}$$

and $(u,v) \in \left(C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))\right)^2$ is a classical solution to (3.1.1), *then*

$$u(x,t) \leq C|x|^{-p} \quad \text{for all } x \in \Omega \text{ and } t \in (0,T). \tag{3.5.2}$$

Proof. In view of (3.1.4) we can find a positive constant λ_1 such that $\lambda(|x|) \le \lambda_1$ for all $x \in \overline{\Omega}$. Putting $\tilde{u}(x,t) := e^{-\lambda_1 t}u(x,t)$, $D(x,t,\rho) := m(e^{\lambda_1 t}\rho + 1)^{m-1}$ and $S(x,t,\rho) := -\chi(e^{\lambda_1 t}\rho + 1)^{\alpha-1}\rho$, we deduce from (3.1.1) that

$$\begin{cases} \tilde{u}_t \le \nabla \cdot (D(x,t,\tilde{u})\nabla\tilde{u} + S(x,t,\tilde{u})\nabla v) & \text{in } \Omega \times (0,T), \\ (D(x,t,\tilde{u})\nabla\tilde{u} + S(x,t,\tilde{u})\nabla v) \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ \tilde{u}(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{3.5.3}$$

Moreover, we can verify that

$$\begin{aligned} D(x,t,\rho) &\ge m\rho^{m-1}, \\ D(x,t,\rho) &\le m(e^{\lambda_1 T} + 1)^{m-1} \max\{\rho, 1\}^{m-1}, \\ |S(x,t,\rho)| &\le \chi(e^{\lambda_1 T}\rho + 1)^{\alpha} \le \chi(e^{\lambda_1 T} + 1)^{\alpha} \max\{\rho, 1\}^{\alpha} \end{aligned}$$

for all $x \in \Omega$, $t \in (0,T)$ and $\rho \in (0,\infty)$ and

$$\int_\Omega \tilde{u}(\cdot,0) = M_0.$$

Now we take $\theta > n$ fulfilling that

$$m - \alpha \in \left(\frac{1}{\theta} - \frac{1}{n}, \frac{1}{\theta} + \frac{n-2}{n}\right]$$

and

$$\begin{aligned} p &= \frac{n(n-1)}{(m-\alpha)n + 1} + \varepsilon \\ &> \frac{n(n-1)}{(m-\alpha)n + 1 - \frac{n}{\theta}} \\ &= \frac{(n-1)}{m - \alpha + \frac{1}{n} - \frac{1}{\theta}}. \end{aligned}$$

By an argument similar to that in the proof of [**2**, Lemma 5.2], we have that

$$\int_\Omega |x|^{(n-1)\theta} |\nabla v(x,t)|^\theta \, dx \le \left(\frac{2e^{\lambda_1 T} M_0}{\omega_{n-1}}\right)^\theta |\Omega|$$

for all $t \in (0,T)$. Thus, from [**11**, Theorem 1.1] we see that there exists $c_1 > 0$ such that $\tilde{u}(x,t) \le c_1 |x|^{-p}$ for all $x \in \Omega$ and $t \in (0,T)$, which implies (3.5.2). □

We next complete the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. We set $p_0 := \frac{n(n-1)}{(m-\alpha)n+1}$. Here, we note from (E1), (F1) and (F2) that $m - \alpha < \frac{n-2}{n}$ and $p_0 > n$. In the case $n \ge 3$, by a direct computation we infer from the conditions $\alpha < \frac{2}{n+1}m + \frac{n^2-n+2}{n(n+1)}$, $\alpha < -\frac{1}{n-2}m + \frac{n^2-2}{n(n-2)}$ and $m - \alpha < \frac{n-2}{n}$ that

$$\alpha < 1 + \frac{2}{p_0}, \quad m < 1 + \frac{n-2}{p_0} \quad \text{and} \quad m - \alpha < \frac{n-2}{p_0}. \tag{3.5.4}$$

In the case (i), noting that (E1) yields (3.5.4), we can pick $\varepsilon_1 > 0$ so small that

$$\alpha < 1 + \frac{2}{p_1}, \quad 1 \le m < 1 + \frac{n-2}{p_1}, \quad m - \alpha < \frac{n-2}{p_1} \tag{3.5.5}$$

and

$$\kappa < 1 + \frac{n-2}{p_1} + \frac{q}{p_1} - (m-1) - (1-\alpha)_+, \tag{3.5.6}$$

where $p_1 := p_0 + \varepsilon_1$. We take $\widetilde{L} > 0$ and $T > 0$. Moreover, we choose $r_\star \in (0, R)$ and $u_0 \in C^0(\overline{\Omega})$ with (3.1.6) satisfying $\int_\Omega u_0 = M_0$ and $\int_{B_{r_\star}(0)} u_0 \ge M_1$ as well as $u_0(x) \le \widetilde{L}|x|^{-p_1}$ for all $x \in \Omega$. By virtue of Lemma 3.5.2 we can find $C > 0$ complying with (3.5.2). Thus, noticing from (3.5.5) and (3.5.6) that (A4), (B3) and (**IV**) hold, we see from Theorem 3.1.1 that the solution (u, v) blows up in finite time. On the other hand, in the case (ii) we observe from the conditions $-\frac{2}{n-3}m + \frac{n^2-n-2}{n(n-3)} < \alpha$, $\alpha < -\frac{n+2}{n-4}m + \frac{2n^2-n-4}{n(n-4)}$, $\alpha \le \frac{n+2}{3}m - \frac{n^2-4}{3n}$ and $-\frac{n+2}{n-4}m + \frac{2n^2-n-4}{n(n-4)} \le \alpha$ that

$$1 - \frac{2}{p_0} < \alpha, \quad m < 1 + \frac{n-4}{2p_0}, \quad 2m - \alpha \le 1 + \frac{n-4}{p_0}$$

and

$$1 + \frac{n-4}{2p_0} \le m.$$

Therefore, by picking $\varepsilon_2 > 0$ small enough, we have from (F1) and (ii) that m and α fulfill that

$$1 - \frac{2}{p_2} < \alpha < 1 + \frac{2}{p_2}, \quad 1 \le m < 1 + \frac{n-4}{2p_2}, \quad 2m - \alpha \ge 1 + \frac{n-4}{p_2} \tag{3.5.7}$$

or

$$1 - \frac{2}{p_2} < \alpha < 1 + \frac{2}{p_2}, \quad 1 + \frac{n-4}{2p_2} \le m < 1 + \frac{n-2}{p_2}, \quad m - \alpha < \frac{n-2}{p_2} \tag{3.5.8}$$

and κ satisfies that

$$\kappa < 1 + \frac{n-2}{p_2} + \frac{q}{p_2} - (m-1) - (1-\alpha)_+, \tag{3.5.9}$$

where $p_2 := p_0 + \varepsilon_2$. Moreover, in the case (iii) we obtain

$$1 - \frac{2}{p_3} < \alpha < 1 + \frac{2}{p_3}, \quad 1 \le m < 1 + \frac{n-4}{2p_3}, \quad 2m - \alpha < 1 + \frac{n-4}{p_3} \tag{3.5.10}$$

and

$$\kappa < 1 + \frac{n}{2p_3} + \frac{q}{p_3} - \frac{(1-\alpha)_+}{2}, \tag{3.5.11}$$

where $p_3 := p_0 + \varepsilon_3$ with some $\varepsilon_3 > 0$. Accordingly, since we can verify from (3.5.7), (3.5.8) and (3.5.9) that (C3), (D2) and (**IV**) hold and we can confirm from (3.5.10) and (3.5.11) that (C1), (D1) and (**II**) hold, we arrive at the conclusion by an argument similar to that in the proof of the case (i). □

3.6. Related results: Blow-up prevention in a fully parabolic system with nonlinear production

3.6.1. Motivations and main result

In this short section we focus on [**49**, Theorem 1.1] and [**10**, Theorem 2.1] where chemotaxis systems for two coupled parabolic equations are so formulated:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + f(u), & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega,\ t>0. \end{cases} \tag{3.6.1}$$

Herein, $\Omega\subset\mathbb{R}^n$ $(n\ge 2)$ is a bounded domain with smooth boundary, and ν denotes the outward normal vector to $\partial\Omega$. Also, the initial data (u_0, v_0) is assumed to satisfy

$$\begin{cases} u_0\in C^0(\overline{\Omega}) & \text{is nonnegative with } u_0\not\equiv 0,\\ v_0\in C^1(\overline{\Omega}) & \text{is nonnegative,} \end{cases} \tag{3.6.2}$$

and, for all $u\ge 0$ and appropriate real numbers $C_D, \widetilde{C_D}, C_S, \widehat{m}, \widehat{m}_1, \alpha, L, \ell$, the diffusion and sensitivity laws $D, S\in C^2([0,\infty))$ and the production growth $f\in C^1([0,\infty))$ are such that

$$C_D(1+u)^{-\widehat{m}} \le D(u) \le \widetilde{C_D}(1+u)^{-\widehat{m}_1}, \quad 0\le S(u)\le C_S u(1+u)^{\alpha-1}, \tag{3.6.3}$$

and

$$0\le f(u)\le Lu^{\ell}. \tag{3.6.4}$$

The aforementioned results in [**49**] and [**10**] are collected as follows.

Theorem 3.6.1. *Let $n \geq 2$ and (u_0, v_0) satisfy* (3.6.2)*. Suppose that D, S and f fulfill* (3.6.3) *and* (3.6.4)*. Then the system* (3.6.1) *admits a unique nonnegative classical solution (u, v) which is globally bounded provided that:*

*I) [**49**, Theorem 1.1] $0 < \ell \leq 1$ and*

$$\widehat{m} + \alpha + \ell < 1 + \frac{2}{n}; \tag{3.6.5}$$

*II) [**10**, Theorem 2.1] $\widehat{m} = \widehat{m}_1 = 0$, $0 < \ell < \frac{2}{n}$, $\alpha \geq \frac{2}{n}$ and*

$$\alpha + \frac{\ell}{2} < 1 + \frac{1}{n}. \tag{3.6.6}$$

These two theorems have been proved, in an independent way the one from the other, recently. Moreover, when investigating a variant of Keller–Segel systems like those in (3.6.1), we can realize that:

- for $0 < \ell < \frac{1}{n}$, the proof leading to (3.6.5) has a mathematical inconsistency; in this same range, even for the linear diffusion case $\widehat{m} = \widehat{m}_1 = 0$, the condition cannot hold true and has to be replaced by (3.6.6);
- for $\frac{1}{n} \leq \ell < \frac{2}{n}$ and $\widehat{m} = \widehat{m}_1 = 0$, assumption (3.6.6) is less accurate than (3.6.5).

Since this gap leaves the general theory about models (3.6.1) somehow incomplete and fragmented, we understand that it is of primary importance giving a revised and unified conclusion. Precisely, the role behind the forthcoming theorem is twofold: correcting [**49**, Theorem 1.1] and improving [**10**, Theorem 2.1].

Theorem 3.6.2. *Let $n \geq 2$ and (u_0, v_0) satisfy* (3.6.2)*. Suppose that D, S and f fulfill* (3.6.3) *and* (3.6.4)*. If $0 < \ell \leq 1$ and*

$$\begin{cases} \widehat{m} + \alpha + \ell < 1 + \frac{2}{n} & \text{if } \ell \in \left[\frac{1}{n}, 1\right], \\ \widehat{m} + \alpha < 1 + \frac{1}{n} & \text{if } \ell \in \left(0, \frac{1}{n}\right), \end{cases} \tag{3.6.7}$$

then the system (3.6.1) *admits a unique nonnegative classical solution (u, v) which is globally bounded.*

3.6.2. Identification of the gap

Once combined with well-known extensibility criteria, global boundedness for local classical solutions of the system (3.6.1), defined in $\Omega \times (0, T_{\max})$, is achieved by controlling $\|u(\cdot, t)\|_{L^p(\Omega)}$ and $\|\nabla v(\cdot, t)\|_{L^q(\Omega)}$ on $(0, T_{\max})$, and for p, q large enough. In particular, if we refer to [**49**], such boundedness relies on the ensuing

Proposition 3.6.3 ([**49**, Proposition 3.1])**.** *Let $n \geq 2$ and (u_0, v_0) satisfy* (3.6.2)*. Suppose that D, S and f fulfill* (3.6.3) *and* (3.6.4)*. If $0 < \ell \leq 1$, $\widehat{m}$ and α are constrained by assumption* (3.6.5)*, then for all $p \in [1, \infty)$ and each $q \in [1, \infty)$, there exists $C = C(p, q, \widehat{m}, \widehat{m}_1, \alpha, \ell) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \textit{and} \quad \|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}).$$

Unfortunately, the proof of this proposition contains an error in the case $\ell \in \left(0, \frac{1}{n}\right)$: specifically, in [**49**, (3.1) in Section 3] it is claimed that for any $0 < \ell \leq 1$ it is possible to find $s \in \left[1, \frac{n}{(n\ell-1)_+}\right)$ such that

$$\ell - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \widehat{m} - \alpha. \tag{3.6.8}$$

If from the one hand for $\ell \in \left[\frac{1}{n}, 1\right]$ such a relation and (3.6.5) fit, from the other hand they do not when $\ell \in \left(0, \frac{1}{n}\right)$, and some counterexamples of (3.6.8) can be encountered. For instance, the triplet $(\widehat{m}, \alpha, \ell) = \left(1, \frac{1}{n}, \frac{1}{2n}\right)$ is adjusted to (3.6.5), but oppositely it implies that (3.6.8) is rewritten as $-\frac{1}{2n} < \frac{1}{s} < 0$, not satisfied for any $s \geq 1$. Since relation (3.6.8) is crucial in the derivation of Proposition 3.6.3, the machinery to show [**49**, Theorem 1.1], of the item (I) above, misses its validity for $\ell \in (0, \frac{1}{n})$.

3.6.3. Correction of the gap and proof of Theorem 3.6.2

As specified, we can only confine to the case $\ell \in \left(0, \frac{1}{n}\right)$. By putting $\ell_0 := \frac{1}{n}$, we note from (3.6.7) that

$$\widehat{m} + \alpha + \ell_0 < 1 + \frac{2}{n}. \tag{3.6.9}$$

Hence we can fix $s \in [1, \infty)$, rigorously $s \in \left(\frac{1}{\ell_0}, \infty\right)$ (see Remark 3.6.1 below), such that

$$0 = \ell_0 - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \widehat{m} - \alpha. \tag{3.6.10}$$

We next pick $p \geq \overline{p}$ and $q \geq \overline{q}$, where $\overline{p}$ and $\overline{q}$ are defined as in [**49**, Section 3], and set

$$\phi(z) := \int_0^z \int_0^\rho \frac{(1+\sigma)^{p-\widehat{m}-2}}{D(\sigma)} \, d\sigma d\rho \quad \text{for } z \geq 0.$$

We can derive [**49**, (3.9)] unconditionally, that is, we can find $C_1 = C_1(q) > 0$ such that on $(0, T_{\max})$ the local solution of problem (3.6.1) complies with

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} \, dx + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \, dx \\ &\quad \leq L^2\left(2(q-1) + \frac{n}{2}\right)\int_\Omega u^{2\ell}|\nabla v|^{2(q-1)} \, dx + (C_1 - 2)\int_\Omega |\nabla v|^{2q} \, dx. \end{aligned} \tag{3.6.11}$$

From the condition $\ell < \ell_0$ and Young's inequality it follows that for all $t \in (0, T_{\max})$

$$\begin{aligned}
&\int_\Omega u^{2\ell} |\nabla v|^{2(q-1)}\,dx \\
&\quad\le \frac{\ell}{\ell_0}\int_\Omega u^{2\ell_0}|\nabla v|^{2(q-1)}\,dx + \left(1-\frac{\ell}{\ell_0}\right)\int_\Omega |\nabla v|^{2(q-1)}\,dx \\
&\quad\le \frac{\ell}{\ell_0}\int_\Omega u^{2\ell_0}|\nabla v|^{2(q-1)}\,dx + \left(1-\frac{\ell}{\ell_0}\right)\left[\left(1-\frac{1}{q}\right)\int_\Omega |\nabla v|^{2q}\,dx + \frac{|\Omega|}{q}\right].
\end{aligned}$$

Therefore, by plugging this inequality into (3.6.11), we obtain $C_2 = C_2(q) > 0$ and $C_3 = C_3(q, |\Omega|) > 0$ providing

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q}\,dx + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2\,dx \\
&\quad\le C_2\int_\Omega u^{2\ell_0}|\nabla v|^{2(q-1)}\,dx + C_2\int_\Omega |\nabla v|^{2q}\,dx + C_3 \quad \text{on} \quad (0, T_{\max}).
\end{aligned} \tag{3.6.12}$$

Since $\ell_0 = \frac{1}{n} \in \left[\frac{1}{n}, 1\right]$ and (3.6.9) holds, we can estimate the first term on the right-hand side of (3.6.12) as in the proof of [**49**], so arriving at [**49**, (3.19)], with C_{11} involving also the constant C_3. Finally, thanks to relation (3.6.10), we complete the proof by similar arguments to those employed in [**49**, Proposition 3.1]. □

Remark 3.6.1 (Comparison between [**49**, Theorem 1.1] and [**10**, Theorem 2.1])**.** The proof of [**49**, Proposition 3.1] relies, *inter alia*, on the conservation of mass property $\|u(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega u_0(x)\,dx = M_0$ for all $t \in (0, T_{\max})$, as well as on the bound $\|v(\cdot,t)\|_{W^{1,s}(\Omega)} \le C$, valid for any $s \in \left(\frac{1}{\ell}, \frac{n}{(n\ell-1)_+}\right)$, throughout all $t \in (0, T_{\max})$ and for some $C = C(s,\ell) > 0$. The first is obtainable by integrating over Ω the equation for u in (3.6.1). For the second, Neumann semigroup estimates, in conjunction with $\int_\Omega f(u)^{\frac{1}{\ell}} \le L^{\frac{1}{\ell}} M_0$, entail for some $C_0 > 0$, $\mu > 0$, and all $t \in (0, T_{\max})$ and $\frac{1}{2} < \rho < 1$

$$\|v(\cdot,t)\|_{W^{1,s}(\Omega)} \le C_0\|v_0\|_{W^{1,s}(\Omega)} + C_0\int_0^t (t-r)^{-\rho-\frac{n}{2}\left(\ell-\frac{1}{s}\right)} e^{-\mu(t-r)}\|u^\ell(\cdot,r)\|_{L^{\frac{1}{\ell}}(\Omega)}\,dr.$$

Conversely, in [**10**, Lemma 3.1] only a uniform bound for $v(\cdot,t)$ in $W^{1,n}(\Omega)$ and for any $0 < \ell < \frac{2}{n}$ is derived. Subsequently, since $\frac{n}{(n\ell-1)_+} > n$, one concludes that for s close enough to $\frac{n}{(n\ell-1)_+}$, the succeeding $W^{1,s}$-estimates involving v, have to play a sharper role on the final result than the $W^{1,n}$-estimates do. This is reflected on condition (3.6.5), milder than (3.6.6).

Part II

Finite-time blow-up in quasilinear Jäger–Luckhaus systems with logistic source and nonlinear production

Chapter 4

The case of nondegenerate diffusion

4.1. Introduction

In this chapter we consider the following quasilinear Jäger–Luckhaus system with logistic source and nonlinear production:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v) + \lambda u - \mu u^\kappa, & x\in\Omega,\ t>0,\\ 0 = \Delta v - \overline{M_f}(t) + f(u), & x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{4.1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $\lambda > 0$, $\mu > 0$ and $\kappa > 1$; $D, S \in C^2([0,\infty))$ and $D(0) > 0$; $f \in \bigcup_{\beta\in(0,1)} C^\beta_{\mathrm{loc}}([0,\infty)) \cap C^1((0,\infty))$;

$$\overline{M_f}(t) := \frac{1}{|\Omega|}\int_\Omega f(u(x,t))\,dx;$$

ν is the outward normal vector to $\partial\Omega$; $u_0 \in \bigcup_{\beta\in(0,1)} C^\beta(\overline{\Omega})$ is nonnegative. Here, D, S and f are functions generalizing the prototypes

$$D(u) = (u+1)^{m-1}, \quad S(u) = u(u+1)^{\alpha-1} \quad \text{and} \quad f(u) = u^\ell$$

with $m \in \mathbb{R}$, $\alpha > 0$ and $\ell > 0$.

The system (4.1.1) describes a motion of cellular slime molds with chemotaxis, and the unknown function $u = u(x,t)$ denotes the density of cells and the unknown function $v = v(x,t)$ represents the concentration of the chemical substance at place $x \in \Omega$ and time $t > 0$. This system is one of many types of the Keller–Segel system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{4.1.2}$$

which was proposed by Keller and Segel [**23**]. A number of variations of the original system (4.1.2) and related results for blow-up (in the radial setting) and boundedness are introduced in [**1, 16, 26**]:

- We first focus on the quasilinear Keller–Segel system,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + f(u), & x \in \Omega,\ t > 0, \end{cases}$$

  where $\tau \in \{0,1\}$. When $f(u) = u$, in the parabolic–parabolic setting ($\tau = 1$), Tao and Winkler [**50**] showed that solutions are global and bounded under the conditions that $\frac{S(u)}{D(u)} \le cu^q$ with $q < \frac{2}{n}$ and $c > 0$ and that Ω is a convex domain; Ishida, Seki and Yokota [**18**] removed the convexity of Ω; whereas Winkler [**56**] proved that solutions blow up in either finite or infinite time when $\frac{S(u)}{D(u)} \ge cu^q$ with $q > \frac{2}{n}$ and $c > 0$; in the parabolic–elliptic setting ($\tau = 0$), Lankeit [**25**] proved that solutions remain bounded in the case $q < \frac{2}{n}$ and that unbounded solutions are constructed in the case $q > \frac{2}{n}$. When $\tau = 1$ and $D(u) = 1$, $S(u) = u$ and $f(u) = u^\ell$ with $\ell > 0$, Liu and Tao [**28**] established global existence and boundedness under the condition that $0 < \ell < \frac{2}{n}$; in the case that $D(u) = (u+1)^{m-1}$ and $S(u) = u(1+u)^{\alpha-1}$ with $m \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, it was shown that solutions are bounded under the condition $\alpha - m + \max\left\{\ell, \frac{1}{n}\right\} < \frac{2}{n}$ in Chapter 3.

- We next review the quasilinear Keller–Segel system with logistic source,

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + f(u), & x \in \Omega,\ t > 0, \end{cases}$$

  where $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\tau \in \{0,1\}$. Blow-up phenomena in this system are suppressed when $\kappa \ge 2$ and $f(u) = u$. Indeed, in the parabolic–parabolic setting ($\tau = 1$), when $D(u) = 1$ and $S(u) = u$, Winkler [**55**] derived that solutions exist globally and are bounded if $\mu > 0$ is so large and $\kappa = 2$; When $D(u) = (u+1)^{m-1}$ and $S(u) = u(u+1)^{\alpha-1}$ with $m \in \mathbb{R}$ and $\alpha \in \mathbb{R}$, global existence and boundedness were obtained if $\lambda = \mu = 1$, $\kappa = 2$ and $0 < \alpha - m + 1 < \frac{4}{4+n}$ by Zheng [**67**]. In the parabolic–elliptic setting ($\tau = 0$), when $D(u) = 1$ and $S(u) = u$, Tello and Winker [**51**] showed that solutions exist globally and are bounded in the cases that $\kappa = 2$ and $\mu > \max\left\{0, \frac{n-2}{n}\right\}$ and that $\kappa > 2$ and $\mu > 0$; when $D(u) = u^{m-1}$ and $S(u) = u^\alpha$ for all $u \ge 1$ with $m \ge 1$ and $\alpha > 0$, Zheng [**66**] proved global existence and boundedness in the cases that $\kappa > 1$ and $\alpha + 1 < \max\left\{m + \frac{2}{n}, \kappa\right\}$ and that $\kappa > 1$, $\alpha + 1 = \kappa$ and $\mu > \mu_0$ for some $\mu_0 > 0$. On the other hands, in the parabolic–elliptic setting, it is known that blow-up occurs under the some conditions for $\kappa > 1$ when $f(u) = u$. When $D(u) = 1$ and $S(u) = u$, Winkler [**60**] presented that if $1 < \kappa < \frac{7}{6}$ ($n \in \{3, 4\}$) and $1 < \kappa < 1 + \frac{1}{2(n-1)}$ ($n \geq 5$), then solutions blow up in finite time; similar blow-up results were obtained in the case that $D(u) = (u+1)^{m-1}$ and $S(u) = u(u+1)^{\alpha-1}$ with $m \geq 1$ and $\alpha > 0$ (see [**2**] and Chapters 2 and 3).

- We turn our eyes into the quasilinear Jäger–Luckhaus system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_f}(t) + f(u), & x \in \Omega,\ t > 0. \end{cases}$$

A simplification of this system was introduced by Jäger and Luckhaus [**21**]. When $D(u) = (u+1)^{m-1}$ with $m \in \mathbb{R}$, $S(u) = u$ and $f(u) = u$, Cieślak and Winkler [**9**] derived global existence and boundedness in the case $2 - m < \frac{2}{n}$ and finite-time blow-up in the case $2 - m > \frac{2}{n}$; when $D(u) = (u+1)^{m-1}$ and $S(u) = u(u+1)^{\alpha-1}$ with $m \leq 1$ and $\alpha \in \mathbb{R}$ as well as $f(u) = u$, Winkler and Djie [**63**] proved that solutions are global and bounded if $\alpha - m + 1 < \frac{2}{n}$, whereas finite-time blow-up occurs if $\alpha - m + 1 > \frac{2}{n}$; when $D(u) = 1$, $S(u) = u$ and $f(u) = u^\ell$ with $\ell > 0$, Winkler [**59**] showed that solutions exist globally and remain bounded in the case $\ell < \frac{2}{n}$ and that there exist solutions which are unbounded in finite time in the case $\ell > \frac{2}{n}$; when $D(u) = (u+1)^{m-1}$, $S(u) = u$ and $f(u) = u^\ell$ with $m \in \mathbb{R}$ and $\ell > 0$, global existence and boundedness were established if $\ell - m + 1 < \frac{2}{n}$ by Li [**27**]. Moreover, in [**27**] it was asserted that finite-time blow-up occurs under the condition that $\ell - m + 1 > \frac{2}{n}$. However, this condition should be repaired because from assumptions of [**27**, Lemma 3.5] we can obtain the condition that

$$\ell - (m-1)_+ > \frac{2}{n}, \quad \text{where} \quad (m-1)_+ := \max\{0, m-1\}; \tag{4.1.3}$$

when $D(u) = 1$, $S(u) = u(u+1)^{\alpha-1}$ and $f(u) = u^\ell$ with $\alpha > 0$ and $\ell > 0$, Wang and Li [**53**] derived the critical value $\alpha + \ell - 1 = \frac{2}{n}$.

- In the system (4.1.1), when $D(u) = 1$, $S(u) = u$ and $f(u) = u$, Winkler [**57**] showed that if $1 < \kappa < \frac{3}{2} + \frac{1}{2n-2}$ ($n \geq 5$), then there exists a solution blowing up in finite time; a similar blow-up result was obtained in the case that $D(u) = (u+1)^{m-1}$ with $m \geq 1$ in [**2**]; furthermore, Fuest [**13**] showed that solutions blow up in finite time under the conditions that $1 < \kappa < \min\left\{2, \frac{n}{2}\right\}$ and $\mu > 0$ ($n \geq 3$) and that $\kappa = 2$ and $\mu \in \left(0, \frac{n-4}{n}\right)$ ($n \geq 5$); in the two dimensional setting and $\kappa = 2$, global existence and boundedness were established when $\int_\Omega u_0 < 8\pi$, whereas finite-time blow-up occurs when $\int_\Omega u_0 < m_0$ with $m_0 > 8\pi$ in [**12**].

In summary, in [**2, 12, 13, 57**], blow-up results were derived in the chemotaxis system with logistic source and *linear* production. However, boundedness and blow-up results were not obtained in the quasilinear Jäger–Luckhaus system with logistic source and *nonlinear* production (when $D(u) = 1$ and $S(u) = u$, recently, Yi, Mu, Xu and Dai [**65**] derived the blow-up result under the condition that $\ell + 1 > \kappa\left(1 + \frac{2}{n}\right)$).

Our aim of this chapter is to present conditions that solutions of (4.1.1) are bounded or blow up. Before we state the main results, we give conditions for the functions D, S and f as follows:

$$D \in C^2([0,\infty)) \text{ is positive,} \tag{4.1.4}$$

$$S \in C^2([0,\infty)) \text{ is nonnegative and nondecreasing} \tag{4.1.5}$$

and

$$f \in \bigcup_{\beta\in(0,1)} C^\beta_{\text{loc}}([0,\infty)) \cap C^1((0,\infty)) \text{ is nonnegative and nondecreasing.} \tag{4.1.6}$$

We now state the main theorems. The first one asserts boundedness of solutions.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ be a smooth bounded domain, and let $\delta \in (0,1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Assume that $u_0 \in \bigcup_{\beta\in(0,1)} C^\beta(\overline{\Omega})$ is nonnegative and D, S and f satisfy (4.1.4), (4.1.5) and (4.1.6) as well as*

$$D(\xi) \ge C_D(\xi+\delta)^{m-1}, \quad S(\xi) \le C_S\xi(\xi+\delta)^{\alpha-1} \quad \text{for all } \xi \ge 0 \tag{4.1.7}$$

and

$$f(\xi) \le L\xi^\ell \quad \text{for all } \xi \ge 0 \tag{4.1.8}$$

with $C_D > 0$, $C_S > 0$ and $L > 0$. If one of the following cases holds:

$$\alpha + \ell < \max\left\{m + \frac{2}{n}, \kappa\right\} \quad \text{and} \quad \mu > 0, \tag{4.1.9}$$

$$\alpha + \ell = \kappa \quad \text{and} \quad \mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_S L, \tag{4.1.10}$$

then there exists an exactly one pair (u,v) of functions

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ v \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,\infty)) \end{cases}$$

which solves (4.1.1) classically. Moreover, the solution (u,v) is bounded in the sense that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t \in (0,\infty)$.

We next state a result such that solutions blow up in finite time.

Theorem 4.1.2. *Let* $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \geq 1)$ *be a ball with some* $R > 0$, *and let* $\delta \in (0,1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ *and* $\ell > 0$. *Assume that* D, S *and* f *satisfy* (4.1.4), (4.1.5) *and* (4.1.6) *as well as*

$$D(\xi) \leq C_D(\xi+\delta)^{m-1}, \quad S(\xi) \geq C_S\xi(\xi+\delta)^{\alpha-1} \quad \text{for all } \xi \geq 0 \tag{4.1.11}$$

and

$$f(\xi) \geq L\xi^{\ell} \quad \text{for all } \xi \geq 0 \tag{4.1.12}$$

with $C_D > 0$, $C_S > 0$ *and* $L > 0$. *Suppose that*

$$\alpha + \ell > \max\left\{m + \frac{2}{n}\kappa, \kappa\right\}, \quad \text{if } m \geq 0, \tag{4.1.13}$$

$$\text{or} \quad \alpha + \ell > \max\left\{\frac{2}{n}\kappa, \kappa\right\}, \quad \text{if } m < 0. \tag{4.1.14}$$

Then for all $M_0 > 0$ *there exist* $\varepsilon_0 \in (0, M_0)$ *and* $r_\star \in (0,R)$ *with the following property: If*

$$0 \leq u_0 \in \bigcup_{\beta \in (0,1)} C^{\beta}(\overline{\Omega}) \text{ is radially symmetric, nonincreasing with respect to } |x| \tag{4.1.15}$$

and

$$\int_{\Omega} u_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\,dx \geq M_0 - \varepsilon_0, \tag{4.1.16}$$

then there exist $T^* \in (0,\infty)$ *and an exactly one pair* (u,v) *of functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\overline{\Omega} \times (0,T^*)), \\ v \in \bigcap_{q>n} C^0([0,T^*); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0,T^*)) \end{cases}$$

which solves (4.1.1) *classically and blows up in the sense that*

$$\lim_{t \nearrow T^*} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Remark 4.1.1. As to Theorem 4.1.1, letting $\kappa \to 1$ implies that the condition (4.1.9) is reduced the condition

$$\alpha + \ell < \max\left\{m + \frac{2}{n}, 1\right\},$$

which is a generalized condition such that solutions remain bounded in [**27, 53, 59**]. Also, as to Theorem 4.1.2, we see that the condition (4.1.13) with $m = 1$ and $\kappa \to 1$ is a generalized condition such that solutions blow up in finite time in [**53, 59**].

Remark 4.1.2. When $\alpha = 1$, letting $\kappa \to 1$ entails from (4.1.13) and (4.1.14) that

$$\ell > \max\left\{m - 1 + \frac{2}{n}, 0\right\}, \quad \text{if } m \geq 0, \tag{4.1.17}$$

$$\ell > \max\left\{-1 + \frac{2}{n}, 0\right\}, \qquad \text{if } m < 0. \tag{4.1.18}$$

For instance, when $m \leq 1 - \frac{2}{n}$, we see from (4.1.3) that $\ell > \frac{2}{n}$, whereas we can observe from (4.1.17) and (4.1.18) that $\ell > \left\{\frac{2}{n} - 1, 0\right\}$. Thus the conditions (4.1.17) and (4.1.18) improve the condition in [**27**]. (See Figures 4.1 and 4.2.)

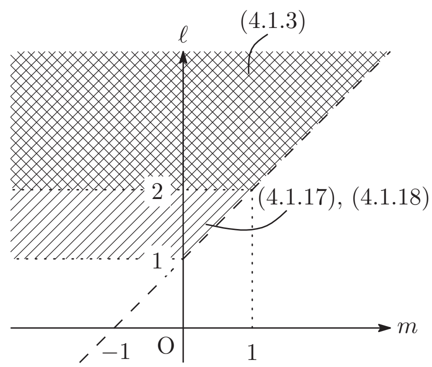

Figure 4.1: $n = 1$, $\alpha = 1$ and $\kappa \to 1$

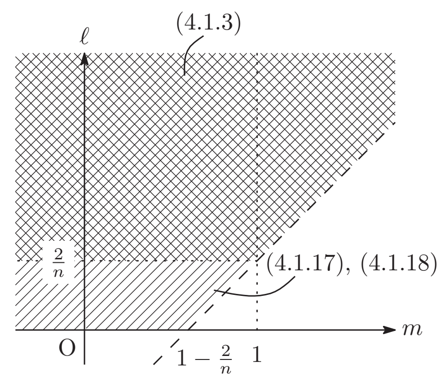

Figure 4.2: $n \geq 2$, $\alpha = 1$ and $\kappa \to 1$

Moreover, in the case that $m = 1$ and $\alpha = 1$, we can establish that

$$1 + \ell > \max\left\{1 + \frac{2}{n}\kappa, \kappa\right\}. \tag{4.1.19}$$

Because $\left(1 + \frac{2}{n}\right)\kappa > \max\left\{1 + \frac{2}{n}\kappa, \kappa\right\}$, we can make sure that the condition (4.1.19) is an improvement on the condition in [**65**]. (See Figures 4.3 and 4.4.)

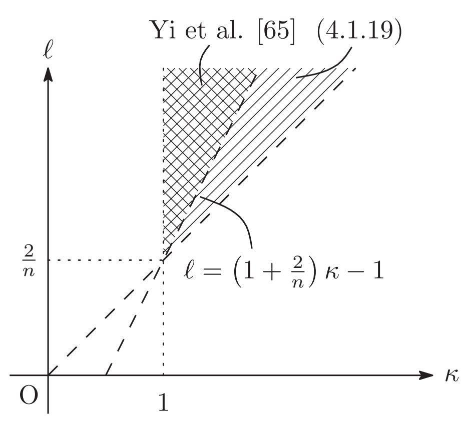

Figure 4.3: $n \in \{1, 2\}$, $m = 1$ and $\alpha = 1$

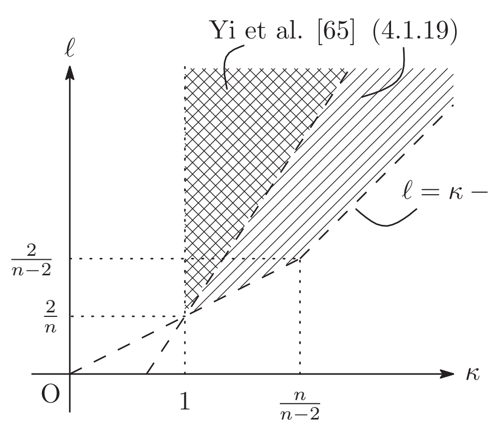

Figure 4.4: $n \geq 3$, $m = 1$ and $\alpha = 1$

The proofs of Theorems 4.1.1 and 4.1.2 are based on those in [**59**]. As to the proof of Theorem 4.1.1, our purpose is to establish an L^p-estimate for u. In order to obtain an L^p-estimate we consider three cases. With regard to the proof of Theorem 4.1.2, we first define the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}),$$

where $s := r^n$ for $r \in [0, R]$, and transform the system (4.1.1) to the parabolic equation

$$\begin{aligned} w_t = {} & n^2 s^{2-\frac{2}{n}} D(nw_s) w_{ss} - \frac{1}{n} s S(nw_s) \overline{M_f}(t) \\ & + \frac{1}{n} S(nw_s) \int_0^s f(nw_s(\sigma,t))\, d\sigma \\ & + \lambda w - n^{\kappa-1} \mu \int_0^s w_s^{\kappa}(\sigma,t)\, d\sigma. \end{aligned}$$

Next, we introduce the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\, ds$$

and the functional

$$\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0 - s) w_s^{\alpha+\ell}(s,t)\, ds$$

with some $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$. Using the above functionals and monotonicity of $w_s(\cdot, t)$, we will deduce the super-linear differential inequality

$$\phi' \geq c_1 \phi^{\alpha+\ell} - c_2,$$

where the monotonicity is derived as in [**2, 59**] by making use of a structural advantage of the second equation in (4.1.1). Also, in order to attain the inequality we will apply the inequality

$$\psi \geq c_3 \phi^{\alpha+\ell}$$

(in [**65**] the inequality $\psi \geq c_4 \phi^{\frac{1+\ell}{\kappa}}$ with some $c_4 > 0$ was obtained). Moreover, in the case $m = 0$, by using the estimate

$$\log(a + \delta) \leq \frac{1}{\varepsilon} a^{\varepsilon} + c_5$$

for all $\varepsilon > 0$ with some $c_5 > 0$, we can improve the condition (4.1.3) to the conditions (4.1.17) and (4.1.18).

This chapter is organized as follows. In Section 4.2 we recall local existence and show Theorem 4.1.1. In Section 4.3 we prove Theorem 4.1.2 and give open problems.

4.2. Blow-up prevention

In this section we derive global existence and boundedness in (4.1.1). We first introduce a result on local existence of classical solutions to (4.1.1). This lemma can be proved by a standard fixed point argument (see e.g. [**63**]).

Lemma 4.2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a smooth bounded domain, and let $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that*

$$u_0 \in \bigcup_{\beta \in (0,1)} C^\beta(\overline{\Omega}) \text{ is nonnegative}$$

and D, S and f fulfill (4.1.4), (4.1.5) and (4.1.6). Then there exist $T_{\max} \in (0, \infty]$ and a unique classical solution (u, v) of (4.1.1) satisfying

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ v \in \bigcap_{q>n} C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{\max})). \end{cases}$$

Moreover, $u \geq 0$ in $\Omega \times (0, T_{\max})$ and

$$\text{if } T_{\max} < \infty, \quad \text{then} \quad \lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

If u_0 is radially symmetric, then so are $u(\cdot, t)$ and $v(\cdot, t)$ for all $t \in (0, T_{\max})$.

In the following we assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a smooth bounded domain and $\delta \in (0, 1]$, $m \in \mathbb{R}$, $\alpha > 0$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Also, we suppose that D, S and f satisfy (4.1.7) and (4.1.8). Moreover, let (u, v) be the solution of (4.1.1) on $[0, T_{\max})$ as in Lemma 4.2.1. We next recall the following lemma which is obtained from the first equation in (4.1.1).

Lemma 4.2.2. *The classical solution u satisfies that*

$$\int_\Omega u(x,t)\,dx \leq M_* := \max\left\{ \int_\Omega u_0(x)\,dx, \left(\frac{\lambda}{\mu}|\Omega|^{\kappa-1}\right)^{\frac{1}{\kappa-1}} \right\} \tag{4.2.1}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation in (4.1.1) and using Hölder's inequality, we have

$$\frac{d}{dt}\int_\Omega u\,dx \leq \lambda \int_\Omega u\,dx - \mu|\Omega|^{1-\kappa}\left(\int_\Omega u\,dx\right)^\kappa$$

for all $t \in (0, T_{\max})$. By an ODE comparison argument we attain (4.2.1). □

In order to see global existence and boundedness of solutions it is sufficient to make sure that for each nonnegative initial data $u_0 \in \bigcup_{\beta\in(0,1)} C^\beta(\overline{\Omega})$ and for any $p > 1$ we can take $C = C(p) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \quad \text{for all } t \in (0, T_{\max}). \tag{4.2.2}$$

In the following subsections we will prove (4.2.2) in three cases as follows:

- Case 1. $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.
- Case 2. $\alpha + \ell < \kappa$ and $\mu > 0$.
- Case 3. $\alpha + \ell = \kappa$ and $\mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)} C_S L$.

4.2.1. Case 1. $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.

In this subsection we derive (4.2.2) under the condition that $\alpha + \ell < m + \frac{2}{n}$ and $\mu > 0$.

Lemma 4.2.3. *Let $\mu > 0$ and assume that $m \in \mathbb{R}$, $\alpha > 0$ and $\ell > 0$ satisfy*

$$\alpha + \ell < m + \frac{2}{n}. \tag{4.2.3}$$

Then for any $p > \max\left\{1, 2-m, 2-(\alpha+\ell), \frac{n}{2}(1-m) + \left(\frac{n}{2}-1\right)(\alpha+\ell-1)\right\}$ there is $C = C(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \tag{4.2.4}$$

for all $t \in (0, T_{\max})$.

Proof. By virtue of the first equation in (4.1.1) and $D(u) \ge C_D(u+\delta)^{m-1}$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx &\le -p(p-1)C_D\int_\Omega (u+\delta)^{p+m-3}|\nabla u|^2\,dx \\
&\quad + p(p-1)\int_\Omega (u+\delta)^{p-2}S(u)\nabla u\cdot\nabla v\,dx \\
&\quad + p\lambda\int_\Omega u(u+\delta)^{p-1}\,dx - p\mu\int_\Omega u^\kappa(u+\delta)^{p-1}\,dx \\
&= -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx \\
&\quad + p(p-1)\int_\Omega \nabla\left(\int_0^u (\xi+\delta)^{p-2}S(\xi)\,d\xi\right)\cdot\nabla v\,dx \\
&\quad + p\lambda\int_\Omega u(u+\delta)^{p-1}\,dx - p\mu\int_\Omega u^\kappa(u+\delta)^{p-1}\,dx \\
&=: I_1 + I_2 + I_3 + I_4
\end{aligned} \tag{4.2.5}$$

for all $t \in (0, T_{\max})$. Noting from $S(\xi) \le C_S(\xi+\delta)^\alpha$ and $p > 1-\alpha$ that

$$\int_0^u (\xi+\delta)^{p-2} S(\xi)\, d\xi \le C_S \int_0^u (\xi+\delta)^{p+\alpha-2}\, d\xi \le \frac{C_S}{p+\alpha-1}(u+\delta)^{p+\alpha-1},$$

from (4.1.8) and the second equation in (4.1.1) we can obtain

$$\begin{aligned} I_2 &= -p(p-1)\int_\Omega \left(\int_0^u (\xi+\delta)^{p-2} S(\xi)\, d\xi\right) \Delta v\, dx \\ &\le \frac{p(p-1)C_S}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha-1} f(u)\, dx \\ &\le \frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\, dx \end{aligned} \tag{4.2.6}$$

for all $t \in (0, T_{\max})$. As to I_3 and I_4, since we see from elementary calculations that there is $\varepsilon > 0$ so small such that $(u+\delta)^\kappa \le (1+\varepsilon)u^\kappa + C_\varepsilon \delta$, where

$$C_\varepsilon := \left(\frac{\delta}{1-(1+\varepsilon)^{-\frac{1}{\kappa-1}}}\right)^{\kappa-1} > 0,$$

we can observe

$$\begin{aligned} &I_3 + I_4 \\ &\quad \le p\lambda \int_\Omega u(u+\delta)^{p-1}\, dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\, dx + \frac{p\mu C_\varepsilon}{1+\varepsilon}\int_\Omega \delta(u+\delta)^{p-1}\, dx \\ &\quad \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\, dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\, dx \end{aligned} \tag{4.2.7}$$

for all $t \in (0, T_{\max})$, where $\widetilde{C}_\varepsilon := \max\left\{p\lambda, \frac{p\mu C_\varepsilon}{1+\varepsilon}\right\} > 0$. From (4.2.5)–(4.2.7) we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u+\delta)^p\, dx \\ &\quad \le -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla (u+\delta)^{\frac{p+m-1}{2}}|^2\, dx + \frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\, dx \\ &\qquad + \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\, dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\, dx \end{aligned} \tag{4.2.8}$$

for all $t \in (0, T_{\max})$. Here, let

$$\theta := \frac{\frac{p+m-1}{2} - \frac{p+m-1}{2(p+\alpha+\ell-1)}}{\frac{p+m-1}{2} + \frac{1}{n} - \frac{1}{2}}.$$

By means of

$$p > \max\left\{1, 2 - m - \frac{2}{n}, 2 - (\alpha + \ell), \frac{n}{2}(1 - m) + \left(\frac{n}{2} - 1\right)(\alpha + \ell - 1)\right\},$$

we see that $\theta \in (0, 1)$. Thus we can apply the Gagliardo–Nirenberg inequality to find $c_1 = c_1(\Omega, m, \alpha, \ell, p) > 0$ such that

$$\begin{aligned}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx &= \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\ &\le c_1\|\nabla(u+\delta)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}\theta}\|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}(1-\theta)} \\ &\quad + c_1\|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \qquad (4.2.9)\end{aligned}$$

for all $t \in (0, T_{\max})$. Moreover, thanks to (4.2.3), we obtain

$$\begin{aligned}\frac{2(p+\alpha+\ell-1)}{p+m-1}\theta &= \frac{p+\alpha+\ell-2}{\frac{1}{2}\left(p+m-2+\frac{2}{n}\right)} \\ &< 2.\end{aligned}$$

Therefore, noticing from Lemma 4.2.2 that $\int_\Omega u\,dx \le M_*$, from (4.2.9) and Young's inequality we can take $c_2 = c_2(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\begin{aligned}&\frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \\ &\quad \le \frac{2p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + c_2 \qquad (4.2.10)\end{aligned}$$

for all $t \in (0, T_{\max})$. A combination of (4.2.8) and (4.2.10) yields that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{2(1+\varepsilon)}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_2$$

for all $t \in (0, T_{\max})$. By Hölder's inequality there exists $c_3 = c_3(\Omega, m, \alpha, \mu, \kappa, \ell, p) > 0$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon\int_\Omega (u+\delta)^p\,dx - c_3\left(\int_\Omega (u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}} + c_2$$

for all $t \in (0, T_{\max})$. Noting the fact that $\frac{p+\kappa-1}{p} > 1$, this inequality yields (4.2.4) by an ODE comparison argument. □

4.2.2. Case 2. $\alpha + \ell < \kappa$ and $\mu > 0$.

In this subsection we show (4.2.2) under the condition that $\alpha + \ell < \kappa$ and $\mu > 0$.

Lemma 4.2.4. *Let $\mu > 0$ and assume that $\alpha > 0$, $\kappa > 1$ and $\ell > 0$ satisfy*

$$\alpha + \ell < \kappa. \tag{4.2.11}$$

Then for any $p > 1$ there exists a positive constant $C = C(\Omega, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_S)$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \tag{4.2.12}$$

for all $t \in (0, T_{\max})$.

Proof. We know from (4.2.8) that there exist $\varepsilon > 0$ and $\widetilde{C}_\varepsilon > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \\
&\quad\le -\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla (u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + \frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \\
&\qquad + \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx
\end{aligned} \tag{4.2.13}$$

for all $t \in (0, T_{\max})$. By virtue of (4.2.11), we have

$$p + \alpha + \ell - 1 < p + \kappa - 1.$$

Thus, by using Young's inequality, we can find $c_1 = c_1(\Omega, \alpha, \mu, \kappa, \ell, L, \delta, p, C_S) > 0$ such that

$$\frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \le \frac{p\mu}{4(1+\varepsilon)}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_1 \tag{4.2.14}$$

for all $t \in (0, T_{\max})$. Therefore, combining (4.2.14) with (4.2.13) and applying Hölder's inequality, we can make sure that there exists a positive constant $c_2 = c_2(\Omega, \mu, \kappa, p)$ such that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - c_2\left(\int_\Omega (u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}} + c_1$$

for all $t \in (0, T_{\max})$. Accordingly, we see that (4.2.12) holds. □

4.2.3. Case 3. $\alpha+\ell=\kappa$ and $\mu>\frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_SL$.

To prove (4.2.2) under the condition that $\alpha+\ell=\kappa$ and $\mu>\frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)}C_SL$ we first derive the L^p-estimate for some $p<1+\frac{\alpha\mu}{(C_SL-\mu)_+}$.

Lemma 4.2.5. *Let $\mu>0$ and assume that $\alpha>0$, $\kappa>1$ and $\ell>0$ satisfy $\alpha+\ell=\kappa$. Then for any*

$$p\in\left(1,1+\frac{\alpha\mu}{(C_SL-\mu)_+}\right)$$

there exists $C=C(\Omega,\alpha,\lambda,\mu,\kappa,L,p,C_S)>0$ such that

$$\int_\Omega u^p(x,t)\,dx\le C$$

for all $t\in(0,T_{\max})$.

Proof. Since the condition $p<1+\frac{\alpha\mu}{(C_SL-\mu)_+}$ implies that

$$\frac{p(p-1)C_SL}{p+\alpha-1}-p\mu<0,$$

we can take $\varepsilon>0$ small enough such that

$$\frac{p(p-1)C_SL}{p+\alpha-1}-\frac{p\mu}{1+\varepsilon}<0.$$

Thus we see from (4.2.8) that there exists $\widetilde{C}_\varepsilon>0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(u+\delta)^p\,dx\\&\quad\le-\frac{4p(p-1)C_D}{(p+m-1)^2}\int_\Omega|\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx+\widetilde{C}_\varepsilon\int_\Omega(u+\delta)^p\,dx\\&\qquad-\left(\frac{p\mu}{1+\varepsilon}-\frac{p(p-1)C_SL}{p+\alpha-1}\right)\int_\Omega(u+\delta)^{p+\kappa-1}\,dx\end{aligned}$$

for all $t\in(0,T_{\max})$. By Hölder's inequality, we obtain $c_1=c_1(\Omega,\alpha,\mu,\kappa,L,p,C_S)>0$ such that

$$\frac{d}{dt}\int_\Omega(u+\delta)^p\,dx\le\widetilde{C}_\varepsilon\int_\Omega(u+\delta)^p\,dx-c_1\left(\int_\Omega(u+\delta)^p\,dx\right)^{\frac{p+\kappa-1}{p}}$$

for all $t\in(0,T_{\max})$, and thereby we can arrive at the conclusion. □

Next we establish the L^p-estimate for any $p>1$.

Lemma 4.2.6. *Assume that $m \in \mathbb{R}$, $\alpha > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$ satisfy*

$$\alpha + \ell = \kappa \quad \text{and} \quad \mu > \frac{n(\alpha+\ell-m)-2}{2(\alpha-1)+n(\alpha+\ell-m)} C_S L. \tag{4.2.15}$$

Then for any $p > 1$ there exists $C = C(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\int_\Omega u^p(x,t)\,dx \le C \tag{4.2.16}$$

for all $t \in (0, T_{\max})$.

Proof. The second condition of (4.2.15) yields that

$$\left(1 + \frac{\alpha\mu}{(C_S L - \mu)_+}\right) - \frac{n}{2}(\alpha+\ell-m) > 0.$$

Hence we can pick some $p_0 \in \left(\frac{n}{2}(\alpha+\ell-m), 1 + \frac{\alpha\mu}{(C_S L-\mu)_+}\right)$. Thanks to Lemma 4.2.5, we see that there exists $c_1 = c_1(\Omega, \alpha, \lambda, \mu, \kappa, \ell, L, p, C_S) > 0$ such that

$$\int_\Omega u^{p_0}\,dx \le c_1 \tag{4.2.17}$$

for all $t \in (0, T_{\max})$. Moreover, we choose

$$p > \max\left\{p_0, p_0+1-m, p_0+1-(\alpha+\ell), \frac{n}{2}(1-m) + \left(\frac{n}{2}-1\right)(\alpha+\ell-1)\right\}$$

and take $\varepsilon > 0$ and $\widetilde{C}_\varepsilon > 0$ such that (4.2.8) holds. Applying the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx &= \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\
&\le c_2 \|\nabla(u+\delta)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}\tilde\theta} \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p_0}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}(1-\tilde\theta)} \\
&\quad + c_2 \|(u+\delta)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p_0}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}
\end{aligned}$$

for all $t \in (0, T_{\max})$ with some $c_2 = c_2(\Omega, m, \alpha, \ell, p) > 0$, where

$$\tilde\theta := \frac{\frac{p+m-1}{2p_0} - \frac{p+m-1}{2(p+\alpha+\ell-1)}}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

Here, we note from $p_0 > \frac{n}{2}(\alpha + \ell - m)$ that

$$\begin{aligned}
&\frac{2(p+\alpha+\ell-1)}{p+m-1}\tilde{\theta} - 2 \\
&\quad = \frac{\frac{p+\alpha+\ell-1}{p_0} - 1 - \left(\frac{p+m-1}{p_0} + \frac{2}{n} - 1\right)}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} \\
&\quad = \frac{\frac{\alpha+\ell-m}{p_0} - \frac{2}{n}}{\frac{p+m-1}{2p_0} + \frac{1}{n} - \frac{1}{2}} \\
&\quad < 0.
\end{aligned}$$

Therefore, by making use of (4.2.17) and Young's inequality, we can find a constant $c_3 = c_3(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\begin{aligned}
&\frac{p(p-1)C_S L}{p+\alpha-1}\int_\Omega (u+\delta)^{p+\alpha+\ell-1}\,dx \\
&\quad \le \frac{2p(p-1)C_D}{(p+m-1)^2}\int_\Omega |\nabla(u+\delta)^{\frac{p+m-1}{2}}|^2\,dx + c_3
\end{aligned} \tag{4.2.18}$$

for all $t \in (0, T_{\max})$. From (4.2.8) and (4.2.18) we infer that

$$\frac{d}{dt}\int_\Omega (u+\delta)^p\,dx \le \widetilde{C}_\varepsilon \int_\Omega (u+\delta)^p\,dx - \frac{p\mu}{1+\varepsilon}\int_\Omega (u+\delta)^{p+\kappa-1}\,dx + c_3$$

for all $t \in (0, T_{\max})$, which implies that (4.2.16) holds. □

4.2.4. Proof of Theorem 4.1.1

In this subsection we complete the proof of boundedness.

Proof of Theorem 4.1.1. Due to (4.1.9) and (4.1.10), we can apply Lemmas 4.2.3, 4.2.4 and 4.2.6. Hence, for any $p > 1$ we find $c_1 = c_1(\Omega, m, \alpha, \lambda, \mu, \kappa, \ell, L, \delta, p, C_D, C_S) > 0$ such that

$$\int_\Omega u^p\,dx \le c_1$$

for all $t \in (0, T_{\max})$. By the Moser iteration (see [**50**, Lemma A.1]), we obtain

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$$

for all $t \in (0, T_{\max})$, which concludes the proof. □

4.3. Finite-time blow-up

In this section we show Theorem 4.1.2. In the following let $\Omega := B_R(0) \subset \mathbb{R}^n$ ($n \geq 1$) be a ball with some $R > 0$ and let $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Also, we suppose that D, S and f fulfill (4.1.4), (4.1.5) and (4.1.6), respectively, and u_0 satisfies (4.1.15). Moreover, introducing $r := |x|$, we denote by

$$(u, v) = (u(r,t), v(r,t))$$

the radially symmetric local solution of (4.1.1) on $[0, T_{\max})$. Based on [**21**], we define the mass accumulation function w such that

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max}). \tag{4.3.1}$$

This implies that

$$w_s(s,t) = \frac{1}{n} u(s^{\frac{1}{n}}, t)$$

and

$$w_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t)$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. Thus we have from the first equation in (4.1.1) that

$$w_t = n^2 s^{2-\frac{2}{n}} D(nw_s) w_{ss} - s^{1-\frac{1}{n}} S(nw_s) v_r + \lambda w - n^{\kappa-1} \mu \int_0^s w_s^{\kappa}(\sigma, t)\, d\sigma \tag{4.3.2}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$, and see from the second equation in (4.1.1) that

$$s^{1-\frac{1}{n}} v_r = \overline{M_f}(t) \frac{s}{n} - \frac{1}{n} \int_0^s f(nw_s(\sigma, t))\, d\sigma \tag{4.3.3}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$. From (4.3.2) and (4.3.3) it follows that

$$\begin{aligned} w_t \geq{} & n^2 s^{2-\frac{2}{n}} D(nw_s) w_{ss} - \frac{1}{n} s S(nw_s) \overline{M_f}(t) \\ & + \frac{1}{n} S(nw_s) \int_0^s f(nw_s(\sigma, t))\, d\sigma - n^{\kappa-1} \mu \int_0^s w_s^{\kappa}(\sigma, t)\, d\sigma \end{aligned} \tag{4.3.4}$$

for all $s \in (0, R^n)$ and $t \in (0, T_{\max})$.

In Subsection 4.3.1 we give a lemma on radial monotonicity of $w_s(\cdot, t)$, which is derived by making use of a structural advantage of the second equation in (4.1.1), and recall some lemmas to obtain inequalities for a derivative of a moment-type functional. In Subsection 4.3.2 we establish some estimates which lead to a super-linear differential inequality for the moment-type functional. The proof of Theorem 4.3.3 is shown in Subsection 4.3.3. Finally, we give open problems in Subsection 4.3.4.

4.3.1. Radial monotonicity of solutions and some inequalities related to a moment-type functional ϕ

We first derive the concavity of w.

Lemma 4.3.1. *Assume that u_0 satisfies* (4.1.15). *Then*

$$u_r(r,t) \le 0 \quad \textit{for all } r \in (0,R) \textit{ and } t \in (0,T_{\max}),$$

that is, for w as in (4.3.1)

$$w_{ss}(s,t) \le 0 \quad \textit{for all } s \in (0,R^n) \textit{ and } t \in (0,T_{\max}).$$

Proof. By an argument similar to that in the proof of [**59**, Lemma 2.2] or [**2**, Lemma 5.1], we can prove this lemma. □

Given $s_0 \in (0,R^n)$ and $\gamma \in (-\infty,1)$, we set the moment-type functional

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0-s)w(s,t)\,ds \quad \text{for } t \in [0,T_{\max}).$$

Here, we note that

$$\phi \in C^0([0,T_{\max})) \cap C^1((0,T_{\max})).$$

Moreover, we introduce the functional

$$\psi(t) := \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}(s,t)\,ds \quad \text{for } t \in (0,T_{\max})$$

and

$$S_\phi := \left\{ t \in (0,T_{\max}) \;\middle|\; \phi(t) \ge \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} \cdot s_0^{2-\gamma} \right\},$$

where $M_0 > 0$. The choices of ϕ, ψ and S_ϕ as well as the underlying overall strategy closely follow the approach in [**59**]. However, in our method we do not use a set S_ψ defined in [**59**]. Next we state the following two lemmas which can be shown as in [**59**].

Lemma 4.3.2. *Assume that u_0 satisfies* (4.1.15) *and let $s_0 \in (0,R^n)$ and $\gamma \in (-\infty,1)$. Then*

$$w\left(\frac{s_0}{2},t\right) \ge \frac{1}{\omega_n} \cdot \left(M_* - \frac{4(M_* - M_0 + s_0)}{2^\gamma(3-\gamma)} \right) \quad \textit{for all } t \in S_\phi,$$

where M_* is defined in (4.2.1).

The following lemma is obtained from Lemmas 4.3.1 and 4.3.2 (for details, see [**59**, Lemma 3.2]).

Lemma 4.3.3. *Assume that u_0 satisfies* (4.1.15) *and let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$. Then*

$$\overline{M_f}(t) \le f_\gamma + \frac{1}{2s}\int_0^s f(nw_s(\sigma,t))\,d\sigma \quad \text{for all } s \in (0, s_0) \text{ and } t \in S_\phi, \tag{4.3.5}$$

where

$$f_\gamma := f\left(\frac{8n(M_* - M_0 + s_0)}{2^\gamma(3-\gamma)\omega_n s_0}\right) > 0. \tag{4.3.6}$$

In order to derive a super-linear differential inequality for ϕ we establish an estimate for ϕ'. This method has been developed in [**59**].

Lemma 4.3.4. *Let f fulfill* (4.1.12) *and let u_0 satisfy* (4.1.15). *Let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$ as well as*

$$\gamma < 2 - \frac{2}{n}. \tag{4.3.7}$$

Then

$$\begin{aligned}
\phi'(t) \ge{}& \frac{n^{\ell-1}}{2}L\int_0^{s_0} s^{1-\gamma}(s_0 - s)S(nw_s(s,t))w_s^\ell(s,t)\,ds \\
&- \frac{f_\gamma}{n}\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s(s,t))\,ds \\
&+ n^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)D(nw_s(s,t))w_{ss}(s,t)\,ds \\
&- n^{\kappa-1}\mu\int_0^{s_0} s^{-\gamma}(s_0-s)\left\{\int_0^s w_s^\kappa(\sigma,t)\,d\sigma\right\}ds \\
=:{}& I_1 + I_2 + I_3 + I_4
\end{aligned} \tag{4.3.8}$$

for all $t \in S_\phi$, where $f_\gamma > 0$ is defined as (4.3.6).

Proof. Invoking (4.3.4) and (4.3.5), we have

$$\begin{aligned}
w_t \ge{}& n^2 s^{2-\frac{2}{n}} D(nw_s)w_{ss} - \frac{f_\gamma}{n} sS(nw_s) \\
&+ \frac{1}{2n}S(nw_s)\int_0^s f(nw_s(\sigma,t))\,d\sigma - n^{\kappa-1}\mu\int_0^s w_s^\kappa(\sigma,t)\,d\sigma
\end{aligned} \tag{4.3.9}$$

for all $s \in \left(0, \frac{R^n}{4}\right]$ and $t \in S_\phi$. Here, we note from Lemma 4.3.1 that

$$w_s(\sigma,t) \ge w_s(s,t) \quad (\sigma \le s).$$

Thanks to this inequality and (4.1.12), we see that

$$S(sw_s)\int_0^s f(nw_s(\sigma,t))\,d\sigma \ge LS(nw_s)\int_0^s (nw_s(\sigma,t))^\ell\,d\sigma \ge n^\ell LsS(nw_s)w_s^\ell \tag{4.3.10}$$

for all $s \in \left(0, \frac{R^n}{4}\right]$ and $t \in S_\phi$. By virtue of (4.3.9) and (4.3.10), we attain (4.3.8). □

4.3.2. Estimates for the four integrals in the inequality for ϕ'

In this subsection, in order to derive different inequalities for ϕ we show estimates for the four integrals in (4.3.8) by using lower bound for ψ. We first provide the estimate for $I_1 + I_2$ in the following lemma.

Lemma 4.3.5. *Assume that S and f fulfill* (4.1.11) *and* (4.1.12), *and u_0 satisfies* (4.1.15). *Let $\gamma \in (-\infty, 1)$. Suppose that $\alpha > 0$ and $\ell > 0$ satisfy*

$$\alpha + \ell > 1. \tag{4.3.11}$$

Then there exist $C_1 = C_1(\alpha, \ell, L, C_S) > 0$ and $C_2 = C_2(R, \alpha, \ell, L, \gamma) > 0$ such that for any choices of $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$I_1 + I_2 \geq C_1 \psi(t) - C_2 s_0^{3-\gamma} \tag{4.3.12}$$

for all $t \in S_\phi$.

Proof. We define the function χ_A as the characteristic function of the set A and put

$$\overline{C} := \left(\frac{4 f_\gamma}{L} \right)^{\frac{1}{\ell}} > 0.$$

As to I_2, noticing that S is nondecreasing, we see that

$$\begin{aligned} I_2 &= -\frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma} (s_0 - s) S(nw_s)\, ds \\ &\quad - \frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) < \overline{C}\}} s^{1-\gamma} (s_0 - s) S(nw_s)\, ds \\ &\geq -\frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma} (s_0 - s) S(nw_s)\, ds \\ &\quad - \frac{f_\gamma}{n} S(\overline{C}) \int_0^{s_0} \chi_{\{nw_s(\cdot,t) < \overline{C}\}} s^{1-\gamma} (s_0 - s)\, ds \end{aligned} \tag{4.3.13}$$

for all $t \in S_\phi$. Moreover, we have

$$\begin{aligned} &- \frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma} (s_0 - s) S(nw_s)\, ds \\ &\geq -\frac{f_\gamma}{n} \int_0^{s_0} \chi_{\{nw_s(\cdot,t) \geq \overline{C}\}} s^{1-\gamma} (s_0 - s) S(nw_s) \left(\frac{nw_s}{\overline{C}} \right)^{\ell} ds \\ &\geq -\frac{n^{\ell-1}}{4} L \int_0^{s_0} s^{1-\gamma} (s_0 - s) S(nw_s) w_s^{\ell}\, ds \end{aligned} \tag{4.3.14}$$

and

$$-\frac{f_\gamma}{n}S(\overline{C})\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\overline{C}\}}s^{1-\gamma}(s_0-s)\,ds \geq -\frac{f_\gamma S(\overline{C})}{(2-\gamma)(3-\gamma)n}s_0^{3-\gamma} \tag{4.3.15}$$

for all $t \in S_\phi$. In light of (4.3.13)–(4.3.15), we observe that

$$\begin{aligned} I_1 + I_2 &\geq \frac{1}{2}I_1 - \frac{f_\gamma S(\overline{C})}{(2-\gamma)(3-\gamma)n}s_0^{3-\gamma} \\ &= \frac{n^{\ell-1}}{4}L\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s)w_s^\ell\,ds - \frac{f_\gamma S(\overline{C})}{(2-\gamma)(3-\gamma)n}s_0^{3-\gamma} \end{aligned} \tag{4.3.16}$$

for all $t \in S_\phi$. Recalling (4.1.11), we can obtain

$$\int_0^{s_0} s^{1-\gamma}(s_0-s)S(nw_s)w_s^\ell\,ds \geq nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \tag{4.3.17}$$

for all $t \in S_\phi$. If $\alpha \geq 1$, then it follows from $(nw_s+\delta)^{\alpha-1} \geq (nw_s)^{\alpha-1}$ that

$$nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \geq n^\alpha C_S\psi(t) \tag{4.3.18}$$

for all $t \in S_\phi$. Hence, in the case $\alpha \geq 1$ a combination of (4.3.16), (4.3.17) and (4.3.18) yields (4.3.12). On the other hand, if $\alpha < 1$, then we can show from the identity $w_s^{\ell+1} = \frac{1}{n}w_s^\ell(nw_s+\delta-\delta)$ that

$$\begin{aligned} &nC_S\int_0^{s_0} s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \\ &= nC_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \\ &\quad + nC_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^{\ell+1}\,ds \\ &\geq \frac{n^\alpha}{2^{1-\alpha}}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds \\ &\quad + C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha}w_s^\ell\,ds \\ &\quad - \delta C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^\ell\,ds \\ &\geq \frac{n^\alpha}{2^{1-\alpha}}C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)\geq\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds \\ &\quad + n^\alpha C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\,ds \\ &\quad - \delta C_S\int_0^{s_0}\chi_{\{nw_s(\cdot,t)<\delta\}}s^{1-\gamma}(s_0-s)(nw_s+\delta)^{\alpha-1}w_s^\ell\,ds \end{aligned} \tag{4.3.19}$$

for all $t \in S_\phi$. Noting from $\alpha < 1$ that

$$(nw_s + \delta)^{\alpha-1} w_s^\ell = \left(\frac{nw_s}{nw_s + \delta} \right)^{1-\alpha} n^{\alpha-1} w_s^{\alpha+\ell-1} \le n^{\alpha-1} w_s^{\alpha+\ell-1},$$

we establish that

$$\begin{aligned}
&- \delta C_S \int_0^{s_0} \chi_{\{nw_s(\cdot,t)<\delta\}} s^{1-\gamma}(s_0 - s)(nw_s + \delta)^{\alpha-1} w_s^\ell \, ds \\
&\ge -n^{\alpha-1} C_S \int_0^{s_0} \chi_{\{nw_s(\cdot,t)<\delta\}} s^{1-\gamma}(s_0 - s) w_s^{\alpha+\ell-1} \, ds \\
&\ge -n^{\alpha-1} C_S \int_0^{s_0} \chi_{\{nw_s(\cdot,t)<\delta\}} s^{1-\gamma}(s_0 - s) \, ds \\
&\ge -\frac{n^{\alpha-1} C_S}{(2-\gamma)(3-\gamma)} s_0^{3-\gamma}
\end{aligned}$$

for all $t \in S_\phi$. From this inequality and (4.3.19) we see that for all $t \in S_\phi$,

$$\begin{aligned}
&nC_S \int_0^{s_0} s^{1-\gamma}(s_0 - s)(nw_s + \delta)^{\alpha-1} w_s^{\ell+1} \, ds \\
&\ge \frac{n^\alpha}{2^{1-\alpha}} C_S \psi(t) - \frac{n^{\alpha-1} C_S}{(2-\gamma)(3-\gamma)} s_0^{3-\gamma}. \qquad (4.3.20)
\end{aligned}$$

Thus, in the case $\alpha < 1$, from (4.3.16), (4.3.17) and (4.3.20) we attain (4.3.12). □

Next, we show the estimate for I_3. In the case $m \neq 0$ the proof of the following lemma is based on that of [**27**]. However, in the case $m = 0$ we use a different estimate for $\log(x + 1)$ for any $x \ge 0$ than the one used in the proof of [**27**, Corollary 3.4].

Lemma 4.3.6. *Assume that D fulfills (4.1.11) and u_0 satisfies (4.1.15). Suppose that $m \in \mathbb{R}$, $\alpha > 0$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ satisfy*

$$\text{if } m \ge 0, \quad \text{then} \quad \alpha + \ell > m \quad \text{and} \quad 2 - \frac{2}{n} \cdot \frac{\alpha + \ell}{\alpha + \ell - m} > \gamma, \qquad (4.3.21)$$

$$\text{if } m < 0, \quad \text{then} \qquad 2 - \frac{2}{n} > \gamma. \qquad (4.3.22)$$

Then there exist $\varepsilon > 0$, $C_1 = C_1(m, \alpha, \ell, \delta, \gamma, C_D) > 0$, $C_2 = C_2(m, \delta, \gamma, C_D) > 0$, $C_3 = C_3(m, \alpha, \ell, \delta, \gamma, \varepsilon, C_D) > 0$ and $C_4 = C_4(m, \delta, \gamma, \varepsilon, C_D) > 0$ such that for any $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$I_3 \ge \begin{cases} -C_1 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) - C_2 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m > 0, \\ -C_3 s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{\varepsilon}{\alpha+\ell}}(t) - C_4 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m = 0, \\ -C_2 s_0^{3-\gamma-\frac{2}{n}} & \text{if } m < 0 \end{cases} \qquad (4.3.23)$$

for all $t \in S_\phi$.

Remark 4.3.1. In this lemma, the constants $C_1 > 0$ and $C_2 > 0$ depend on δ. However, in the case $m > 0$, we can take them which are independent of δ.

Proof. We have from (4.1.11) that

$$\begin{aligned} I_3 &\geq n^2 C_D \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)(nw_s + \delta)^{m-1} w_{ss}\, ds \\ &= nC_D \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)\frac{d}{ds}\left\{\int_0^{nw_s} (\xi + \delta)^{m-1}\, d\xi\right\} ds \end{aligned}$$

for all $t \in S_\phi$. Since it follows that

$$\int_0^{nw_s} (\xi + \delta)^{m-1}\, d\xi \leq \begin{cases} \dfrac{1}{m}(nw_s + \delta)^m & \text{if } m > 0, \\ \log(nw_s + \delta) - \log \delta & \text{if } m = 0, \\ -\dfrac{1}{m}\delta^m & \text{if } m < 0, \end{cases}$$

we obtain from integrating by parts that

$$I_3 \geq \begin{cases} -\dfrac{n}{m} C_D \left(2 - \dfrac{2}{n} - \gamma\right) \displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)(nw_s + \delta)^m\, ds & \text{if } m > 0, \\ -nC_D \left(2 - \dfrac{2}{n} - \gamma\right) \displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)\log\left(\dfrac{nw_s}{\delta} + 1\right) ds & \text{if } m = 0, \\ \dfrac{n}{m}\delta^m C_D \left(2 - \dfrac{2}{n} - \gamma\right) \displaystyle\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)\, ds & \text{if } m < 0 \end{cases} \tag{4.3.24}$$

for all $t \in S_\phi$. First, we show the estimate (4.3.23) in the case $m > 0$. By applying the inequality

$$(nw_s + \delta)^m \leq 2^m((nw_s)^m + \delta^m),$$

we know that

$$\begin{aligned} \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)(nw_s + \delta)^m\, ds &\leq 2^m n^m \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s) w_s^m\, ds \\ &\quad + 2^m \delta^m \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0 - s)\, ds \\ &=: J_1 + J_2 \end{aligned} \tag{4.3.25}$$

for all $t \in S_\phi$. Invoking from (4.3.21) that $\frac{m}{\alpha+\ell} < 1$, we see from Hölder's inequality that

$$
\begin{aligned}
J_1 &= 2^m n^m \int_0^{s_0} \left[s^{1-\gamma}(s_0 - s) w_s^{\alpha+\ell}\right]^{\frac{m}{\alpha+\ell}} \cdot s^{(1-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} (s_0 - s)^{\frac{\alpha+\ell-m}{\alpha+\ell}} \, ds \\
&\le 2^m n^m \psi^{\frac{m}{\alpha+\ell}}(t) \cdot \left(\int_0^{s_0} s^{1-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} (s_0 - s) \, ds \right)^{\frac{\alpha+\ell-m}{\alpha+\ell}}
\end{aligned}
$$

for all $t \in S_\phi$. Moreover, thanks to the condition $2 - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-m} > \gamma$, we can observe

$$
\int_0^{s_0} s^{1-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} (s_0 - s) \, ds = c_1 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}},
$$

where

$$
c_1 := \frac{1}{\left(2 - \gamma - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-m}\right)\left(3 - \gamma - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-m}\right)} > 0.
$$

Thus we establish that

$$
J_1 \le 2^m n^m c_1^{\frac{\alpha+\ell-m}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) \tag{4.3.26}
$$

for all $t \in S_\phi$. Also, since

$$
2 - \gamma - \frac{2}{n} > 2 - \gamma - \frac{2}{n} \cdot \frac{\alpha + \ell}{\alpha + \ell - m} > 0
$$

and $\delta \le 1$, it follows that

$$
J_2 = \frac{2^m \delta^m}{\left(2 - \gamma - \frac{2}{n}\right)\left(3 - \gamma - \frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}} \le \frac{2^m}{\left(2 - \gamma - \frac{2}{n}\right)\left(3 - \gamma - \frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}}. \tag{4.3.27}
$$

In the case $m > 0$, from (4.3.24)–(4.3.27) we can deduce that

$$
\begin{aligned}
I_3 \ge &- \frac{2^m n^{m+1} C_D}{m} \left(2 - \frac{2}{n} - \gamma\right) c_1^{\frac{\alpha+\ell-m}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) \\
&- \frac{2^m n C_D}{m\left(3 - \gamma - \frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}}
\end{aligned}
$$

for all $t \in S_\phi$, which implies (4.3.23). Next, we confirm that the estimate (4.3.23) holds in the case $m = 0$. Due to (4.3.21) with $m = 0$, we can take $\varepsilon > 0$ small enough such that $\alpha + \ell > \varepsilon$ and $2 - \frac{2}{n} \cdot \frac{\alpha+\ell}{\alpha+\ell-\varepsilon} > \gamma$. Furthermore, we have that

$$
\begin{aligned}
\log\left(\frac{n w_s}{\delta} + 1\right) &\le \frac{1}{\varepsilon}\left(\frac{n w_s}{\delta} + 1\right)^\varepsilon - \frac{1}{\varepsilon} \\
&= \frac{1}{\varepsilon \delta^\varepsilon} (n w_s + \delta)^\varepsilon - \frac{1}{\varepsilon}.
\end{aligned}
$$

In light of (4.3.24), we obtain

$$I_3 \geq -\frac{nC_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(nw_s+\delta)^\varepsilon\,ds \\ + \frac{nC_D}{\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\,ds \tag{4.3.28}$$

for all $t \in S_\phi$. As in the case $m>0$, we can verify that

$$-\frac{nC_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)(nw_s+\delta)^\varepsilon\,ds \\ \geq -\frac{2^\varepsilon n^{\varepsilon+1}C_D}{\varepsilon\delta^\varepsilon}\left(2-\frac{2}{n}-\gamma\right) c_2^{\frac{\alpha+\ell-\varepsilon}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell}-\frac{2}{n}} \psi^{\frac{\varepsilon}{\alpha+\ell}}(t) \\ -\frac{2^\varepsilon nC_D}{\varepsilon\delta^\varepsilon\left(3-\gamma-\frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}} \tag{4.3.29}$$

for all $t \in S_\phi$, where

$$c_2 := \frac{1}{\left(2-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}\right)\left(3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}\right)} > 0.$$

Accordingly, a combination of (4.3.28) and (4.3.29) yields (4.3.23). Finally, in the case $m<0$, we can show from (4.3.24) that

$$\frac{n}{m}\delta^m C_D\left(2-\frac{2}{n}-\gamma\right)\int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)\,ds = \frac{n\delta^m C_D}{m\left(3-\gamma-\frac{2}{n}\right)} s_0^{3-\gamma-\frac{2}{n}},$$

which concludes the proof. □

In the following lemma we derive the estimate for I_4.

Lemma 4.3.7. *Assume that u_0 satisfies* (4.1.15). *Suppose that $\alpha>0$, $\kappa>1$, $\ell>0$ and $\gamma \in (-\infty,1)$ fulfill*

$$\alpha+\ell>\kappa \quad \textit{and} \quad 2-\frac{\alpha+\ell}{\kappa}<\gamma<1. \tag{4.3.30}$$

Then there exists $C_1 = C_1(\alpha,\mu,\kappa,\ell,\gamma)>0$ such that for any choices of $s_0 \in \left(0,\frac{R^n}{4}\right]$,

$$I_4 \geq -C_1 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) \tag{4.3.31}$$

for all $t \in S_\phi$.

Proof. We apply the Fubini theorem to obtain

$$\int_0^{s_0} s^{-\gamma}(s_0 - s)\left\{\int_0^s w_s^\kappa(\sigma,t)\,d\sigma\right\} ds = \int_0^{s_0}\left\{\int_\sigma^{s_0} s^{-\gamma}(s_0-s)\,ds\right\} w_s^\kappa(\sigma,t)\,d\sigma$$
$$\le \frac{1}{1-\gamma}s_0^{1-\gamma}\int_0^{s_0}(s_0-\sigma)w_s^\kappa(\sigma,t)\,d\sigma$$

for all $t \in S_\phi$. Thus we have

$$I_4 \ge -\frac{n^{\kappa-1}\mu}{1-\gamma}s_0^{1-\gamma}\int_0^{s_0}(s_0-s)w_s^\kappa\,ds \tag{4.3.32}$$

for all $t \in S_\phi$. Owing to the first condition of (4.3.30), we see from Hölder's inequality that

$$\int_0^{s_0}(s_0-s)w_s^\kappa\,ds = \int_0^{s_0}\left[s^{1-\gamma}(s_0-s)w_s^{\alpha+\ell}\right]^{\frac{\kappa}{\alpha+\ell}}\cdot s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell}}(s_0-s)^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}}\,ds$$
$$\le \psi^{\frac{\kappa}{\alpha+\ell}}(t)\cdot\left(\int_0^{s_0} s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}}(s_0-s)\,ds\right)^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \tag{4.3.33}$$

for all $t \in S_\phi$. Here, noting from the second condition of (4.3.30) that

$$1-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa} > 1-\left(\frac{\alpha+\ell}{\kappa}-1\right)\frac{\kappa}{\alpha+\ell-\kappa} = 0,$$

we can verify that

$$\int_0^{s_0} s^{-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}}(s_0-s)\,ds = c_1 s_0^{2-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}}, \tag{4.3.34}$$

where

$$c_1 := \frac{1}{\left(1-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}\right)\left(2-(1-\gamma)\frac{\kappa}{\alpha+\ell-\kappa}\right)} > 0.$$

Thanks to (4.3.32)–(4.3.34), it follows that

$$I_4 \ge -\frac{n^{\kappa-1}\mu}{1-\gamma}c_1^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} s_0^{1-\gamma+\frac{2(\alpha+\ell-\kappa)}{\alpha+\ell}-(1-\gamma)\frac{\kappa}{\alpha+\ell}}\psi^{\frac{\kappa}{\alpha+\ell}}(t)$$
$$= -\frac{n^{\kappa-1}\mu}{1-\gamma}c_1^{\frac{\alpha+\ell-\kappa}{\alpha+\ell}} s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}}\psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$, which implies (4.3.31). □

In the next lemma we establish the estimate for w which is used later.

Lemma 4.3.8. *Assume that u_0 satisfies* (4.1.15). *Suppose that $\alpha > 0$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ fulfill*

$$\alpha + \ell > 1 \quad \textit{and} \quad 2 - (\alpha + \ell) < \gamma < 1. \tag{4.3.35}$$

Then there exists $C_1 = C_1(\alpha, \ell, \gamma) > 0$ such that for any $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$w(s,t) \leq C_1 s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell}} (s_0 - s)^{-\frac{1}{\alpha+\ell}} \psi^{\frac{1}{\alpha+\ell}}(t)$$

for all $s \in (0, s_0)$ and $t \in S_\phi$.

Proof. According to the condition $\alpha + \ell > 1$, we have from Hölder's inequality that

$$\begin{aligned} w(s,t) &= \int_0^s w_s(\sigma, t)\, d\sigma \\ &= \int_0^s [\sigma^{1-\gamma}(s_0 - \sigma)]^{\frac{1}{\alpha+\ell}} w_s(\sigma, t) \cdot [\sigma^{1-\gamma}(s_0 - \sigma)]^{-\frac{1}{\alpha+\ell}}\, d\sigma \\ &\leq \psi^{\frac{1}{\alpha+\ell}}(t) \cdot \left(\int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}} (s_0 - \sigma)^{-\frac{1}{\alpha+\ell-1}}\, d\sigma \right)^{\frac{\alpha+\ell-1}{\alpha+\ell}} \end{aligned}$$

for all $s \in (0, s_0)$ and $t \in S_\phi$. Moreover, thanks to the condition $2 - (\alpha + \ell) < \gamma < 1$, we see that

$$\begin{aligned} \int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}} (s_0 - \sigma)^{-\frac{1}{\alpha+\ell-1}}\, d\sigma &\leq (s_0 - s)^{-\frac{1}{\alpha+\ell-1}} \int_0^s \sigma^{-\frac{1-\gamma}{\alpha+\ell-1}}\, d\sigma \\ &= \left(\frac{\alpha + \ell - 1}{\alpha + \ell + \gamma - 2} \right) s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell-1}} (s_0 - s)^{-\frac{1}{\alpha+\ell-1}}. \end{aligned}$$

Thus we can obtain

$$w(s,t) \leq \left(\frac{\alpha + \ell - 1}{\alpha + \ell + \gamma - 2} \right)^{\frac{\alpha+\ell-1}{\alpha+\ell}} s^{\frac{\alpha+\ell+\gamma-2}{\alpha+\ell}} (s_0 - s)^{-\frac{1}{\alpha+\ell}} \psi^{\frac{1}{\alpha+\ell}}(t)$$

for all $s \in (0, s_0)$ and $t \in S_\phi$, which concludes the proof. □

From Lemma 4.3.8 we derive the estimate for ψ.

Lemma 4.3.9. *Assume that u_0 satisfies* (4.1.15). *Suppose that $\alpha > 0$, $\ell > 0$ and $\gamma \in (-\infty, 1)$ fulfill*

$$\alpha + \ell > 1 \quad \textit{and} \quad 2 - (\alpha + \ell) < \gamma < 1.$$

Then there exists $C_1 = C_1(\alpha, \ell, \gamma) > 0$ such that for any choices of $s_0 \in \left(0, \frac{R^n}{4}\right]$,

$$\psi(t) \geq C_1 s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) \tag{4.3.36}$$

for all $t \in S_\phi$.

Proof. By an argument similar to that in the proof of [**53**, Lemma 3.7], we can show that (4.3.36) holds. □

4.3.3. Differential inequalities for ϕ. Proof of Theorem 4.1.2

In this subsection we will prove Theorem 4.1.2. To this end, we first derive the differential inequalities for the moment-type functional ϕ in the following lemma. The proof is similar to that in [**59**].

Lemma 4.3.10. *Assume that D, S and f fulfill* (4.1.11) *and* (4.1.12). *Suppose that $m \in \mathbb{R}$, $\alpha > 0$, $\kappa > 1$ and $\ell > 0$ satisfy that*

$$\text{if } m \geq 0, \quad \text{then} \quad \alpha + \ell > \max\left\{m + \frac{2}{n}\kappa, \kappa\right\}, \tag{4.3.37}$$

$$\text{if } m < 0, \quad \text{then} \quad \alpha + \ell > \max\left\{\frac{2}{n}\kappa, \kappa\right\}. \tag{4.3.38}$$

Then there exists $\varepsilon > 0$ small enough and one can find $\gamma = \gamma(m, \alpha, \kappa, \ell) \in (-\infty, 1)$ and $C = C(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that if u_0 satisfies (4.1.15) *and $s_0 \in \left(0, \frac{R^n}{4}\right]$, then*

$$\phi'(t) \geq \begin{cases} \dfrac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} & \text{if } m > 0, \\ \dfrac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}} & \text{if } m = 0, \\ \dfrac{1}{C} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - C s_0^{3-\gamma-\frac{2}{n}} & \text{if } m < 0 \end{cases} \tag{4.3.39}$$

for all $t \in S_\phi$.

Proof. By virtue of (4.3.37), it follows that if $m \geq 0$, then

$$\begin{aligned} \left(2 - \frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}\right) - \left(2 - \frac{\alpha+\ell}{\kappa}\right) &= (\alpha+\ell)\left(\frac{1}{\kappa} - \frac{2}{n}\cdot\frac{1}{\alpha+\ell-m}\right) \\ &> (\alpha+\ell)\left(\frac{1}{\kappa} - \frac{2}{n}\cdot\frac{n}{2\kappa}\right) = 0. \end{aligned} \tag{4.3.40}$$

Thus, in the case $m \geq 0$ we can find $\gamma \in (-\infty, 1)$ such that

$$2 - \frac{\alpha+\ell}{\kappa} < \gamma < 2 - \frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}. \tag{4.3.41}$$

Thanks to the relations (4.3.37) and (4.3.41), we know that (4.3.7), (4.3.11), (4.3.21), (4.3.30) and (4.3.35) hold. In the case $m > 0$, applying Lemmas 4.3.4–4.3.7, we see that there exist $c_1 = c_1(\alpha, \ell, L, C_S) > 0$ and $c_2 = c_2(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\begin{aligned} \phi'(t) \geq\ & c_1\psi(t) - c_2 s_0^{3-\gamma} - c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell}-\frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}} \\ & - c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) \end{aligned} \tag{4.3.42}$$

for all $t \in S_\phi$. Here, noting that $\alpha+\ell > m$ and $\alpha+\ell > \kappa$, from Young's inequality we can obtain $c_i = c_i(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ ($i \in \{3,4\}$) such that

$$c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-m}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{m}{\alpha+\ell}}(t) \le \frac{c_1}{4}\psi(t) + c_3 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

and

$$c_2 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t) \le \frac{c_1}{4}\psi(t) + c_4 s_0^{3-\gamma}.$$

In light of (4.3.42), we established that

$$\phi'(t) \ge \frac{c_1}{2}\psi(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \left(s_0^{\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} + \frac{c_3}{c_2} + s_0^{\frac{2}{n}\cdot\frac{m}{\alpha+\ell-m}} + \frac{c_4}{c_2} s_0^{\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \right)$$

for all $t \in S_\phi$. Since $s_0 \le \frac{R^n}{4}$, there exists $c_5 = c_5(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge \frac{c_1}{2}\psi(t) - c_5 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

for all $t \in S_\phi$. Moreover, we infer from Lemma 4.3.9 that there exists $c_6 = c_6(\alpha, \ell, \gamma) > 0$ such that

$$\phi'(t) \ge \frac{c_1 c_6}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - c_5 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}$$

for all $t \in S_\phi$, which implies (4.3.39) in the case $m > 0$. As to the case $m = 0$, due to (4.3.40), we can pick $\varepsilon > 0$ small enough and $\gamma \in (-\infty, 1)$ such that

$$2 - \frac{\alpha+\ell}{\kappa} < \gamma < 2 - \frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-\varepsilon}.$$

Hence, using Lemmas 4.3.4–4.3.7, we can observe that there are $c_7 = c_7(\alpha, \ell, L, C_S) > 0$ and $c_8 = c_8(R, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \ge c_7\psi(t) - c_8 s_0^{3-\gamma} - c_8 s_0^{(3-\gamma)\frac{\alpha+\ell-\varepsilon}{\alpha+\ell} - \frac{2}{n}} \psi^{\frac{\varepsilon}{\alpha+\ell}}(t) - c_8 s_0^{3-\gamma-\frac{2}{n}} - c_8 s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$. As in the case $m > 0$, from this inequality we can attain (4.3.39). Finally, in the case $m < 0$ we see from (4.3.38) that

$$\begin{aligned} \left(2 - \frac{2}{n}\right) - \left(2 - \frac{\alpha+\ell}{\kappa}\right) &= \frac{\alpha+\ell}{\kappa} - \frac{2}{n} \\ &> \frac{1}{\kappa}\cdot\frac{2\kappa}{n} - \frac{2}{n} \\ &= 0. \end{aligned}$$

Thus we can take $\gamma \in (-\infty, 1)$ satisfying

$$2 - \frac{\alpha + \ell}{\kappa} < \gamma < 2 - \frac{2}{n}.$$

By virtue of Lemmas 4.3.4–4.3.7, we can show that there exist $c_9 = c_9(\alpha, \ell, L, C_S) > 0$ and $c_{10} = c_{10}(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \geq c_9 \psi(t) - c_{10} s_0^{3-\gamma} - c_{10} s_0^{3-\gamma-\frac{2}{n}} - c_{10} s_0^{(3-\gamma)\frac{\alpha+\ell-\kappa}{\alpha+\ell}} \psi^{\frac{\kappa}{\alpha+\ell}}(t)$$

for all $t \in S_\phi$. By an argument similar to that in the case $m > 0$, from Young's inequality and the relation $s_0 \leq \frac{R^n}{4}$ we obtain $c_{11} = c_{11}(R, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that

$$\phi'(t) \geq \frac{c_9}{2} \psi(t) - c_{11} s_0^{3-\gamma-\frac{2}{n}}$$

for all $t \in S_\phi$. Thanks to Lemma 4.3.9, we can verify that (4.3.39) holds in the case $m < 0$. □

We are in a position to complete the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. We first consider the case $m > 0$. Due to (4.1.13), from Lemma 4.3.10 we can find $\gamma \in (-\infty, 1)$, $c_1 = c_1(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ and $c_2 = c_2(R, m, \alpha, \mu, \kappa, \ell, L, \delta, \gamma, C_D, C_S) > 0$ such that for each u_0 satisfying (4.1.15) and $s_0 \leq \frac{R^n}{4}$, it follows that

$$\phi'(t) \geq c_1 s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \tag{4.3.43}$$

for all $t \in S_\phi$. Next we choose $s_0 \leq \frac{R^n}{4}$ small enough such that

$$s_0 \leq \frac{M_0}{2} \tag{4.3.44}$$

and

$$s_0^{(\alpha+\ell)\left(1-\frac{2}{n}\cdot\frac{1}{\alpha+\ell-m}\right)} \leq \frac{c_1}{2c_2} \left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n} \right)^{\alpha+\ell}. \tag{4.3.45}$$

Furthermore, we fix $\varepsilon_0 \in \left(0, \frac{s_0}{2}\right)$ so small and take $s_\star \in (0, s_0)$ fulfilling

$$\frac{M_0 - \varepsilon_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}. \tag{4.3.46}$$

We define

$$r_\star := s_\star^{\frac{1}{n}} \in (0, R)$$

and suppose that u_0 satisfies (4.1.15) and (4.1.16). In order to show $T_{\max} < \infty$, assuming that $T_{\max} = \infty$, we will derive a contradiction. We set

$$\widetilde{S} := \left\{ T \in (0,\infty) \,\middle|\, \phi(t) > \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \quad \text{for all } t \in [0,T] \right\}. \tag{4.3.47}$$

Here, we note that $\widetilde{S}$ is not empty. Indeed, since we have that for any $s \in (s_\star, R^n)$

$$w(s,0) \geq w(s_\star, 0) = \frac{1}{\omega_n} \int_{B_{r_\star}(0)} u_0 \, dx \geq \frac{M_0 - \varepsilon_0}{\omega_n},$$

we see from (4.3.46) that

$$\begin{aligned}
\phi(0) &\geq \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s) w(s,0)\, ds \\
&\geq \frac{M_0 - \varepsilon_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds \\
&> \frac{M_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}.
\end{aligned}$$

Thus we can put $\widetilde{T} := \sup \widetilde{S} \in (0, \infty]$. Moreover, we can confirm that $(0, \widetilde{T}) \subset S_\phi$. Owing to (4.3.47) and (4.3.44), we establish that

$$\phi(t) \geq \frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}$$

for all $t \in (0, \widetilde{T})$. From (4.3.45) it follows that

$$\frac{\frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t)}{c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}}} \geq \frac{c_1}{2c_2} \left(\frac{M_0}{2(1-\gamma)(2-\gamma)\omega_n} \right)^{\alpha+\ell} s_0^{-(\alpha+\ell)+\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \geq 1$$

for all $t \in (0, \widetilde{T})$, which implies from (4.3.43) that

$$\phi'(t) \geq \frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi^{\alpha+\ell}(t) \geq 0 \tag{4.3.48}$$

for all $t \in (0, \widetilde{T})$. This inequality yields that $\widetilde{T} = \infty$. However, from (4.3.48) and $\alpha + \ell - 1 > 0$ we can show that

$$\widetilde{T} \leq \frac{2}{(\alpha+\ell-1) c_1 \phi^{\alpha+\ell-1}(0)} s_0^{(3-\gamma)(\alpha+\ell-1)}.$$

As a consequence, we attain that $T_{\max}$ must be finite. In the cases $m = 0$ and $m < 0$, we can prove that $T_{\max} < \infty$ by an argument similar to that in the case $m > 0$. □

4.3.4. Open problems

In [**13, 53, 59**] the critical values such that solutions are bounded or blow up in finite time were derived. As to the conditions (4.1.9), (4.1.13) and (4.1.14), we see that if $n \geq 3$ and $m \geq 0$ as well as $\frac{n}{n-2}m \leq \kappa$, then $\max\left\{m + \frac{2}{n}, \kappa\right\} = \max\left\{m + \frac{2}{n}\kappa, \kappa\right\} = \kappa$. Thus we know that the critical value is $\alpha + \ell = \kappa$ in this case. However, in the cases that $n \in \{1, 2\}$ and that $n \geq 3$ and $m \geq 0$ as well as $\frac{n}{n-2}m > \kappa$, the conditions (4.1.9), (4.1.13) and (4.1.14) are not optimal. Moreover, the special cases are as follows:

- In the case that $m = \alpha = 1$, behavior of solutions is an open problem when $\max\left\{\frac{2}{n}, \kappa - 1\right\} \leq \ell \leq \frac{2}{n}\kappa$ (see Figures 4.5 and 4.6).

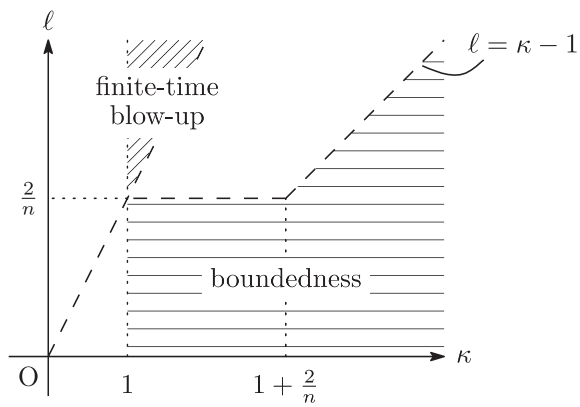

Figure 4.5: $n \in \{1, 2\}$ and $m = \alpha = 1$

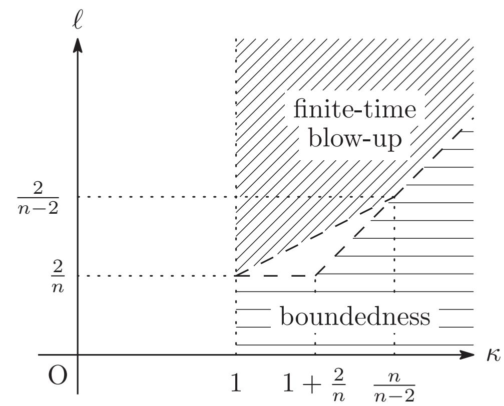

Figure 4.6: $n \geq 3$ and $m = \alpha = 1$

- When $m = 1$ and $\kappa < \frac{n}{(n-2)_+}$, we have an open question of whether solutions are bounded or blow up when $\max\left\{1 + \frac{2}{n}, \kappa\right\} \leq \alpha + \ell \leq 1 + \frac{2}{n}\kappa$ (see Figure 4.7).

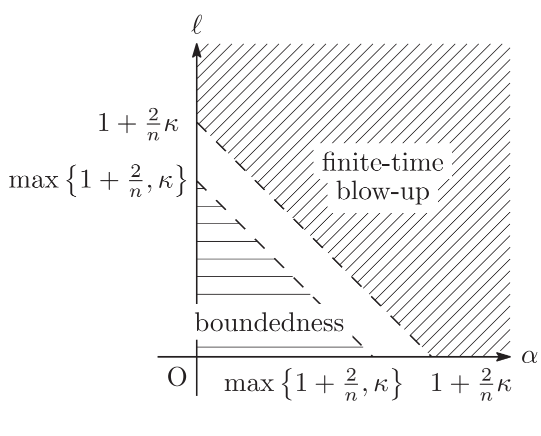

Figure 4.7: $m = 1$ and $\kappa < \frac{n}{(n-2)_+}$

- In the case that $\alpha = 1$ and $\ell > 0$, there is an open problem for behavior of solutions when $n = 1$ and $\max\{\kappa - 1, m + 1\} \le \ell \le \max\{2\kappa - 1, m + 2\kappa - 1\}$. Also, when $n \ge 2$ and $\max\left\{\kappa - 1, m - \left(1 - \frac{2}{n}\right)\right\} \le \ell \le m - \left(1 - \frac{2}{n}\kappa\right)$, the same question arises. Moreover, in the case that $\alpha > 0$ and $\ell = 1$, we obtain regions that ℓ is replaced by α in Figures 4.8 and 4.9.

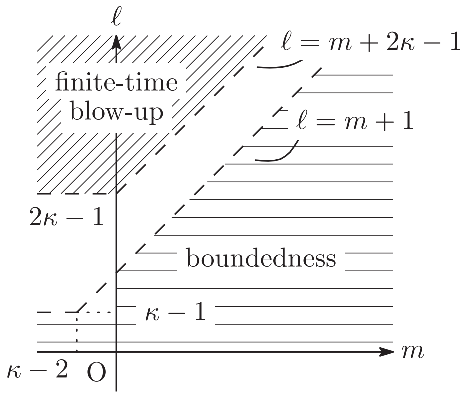

Figure 4.8: $n = 1$ and $\alpha = 1$

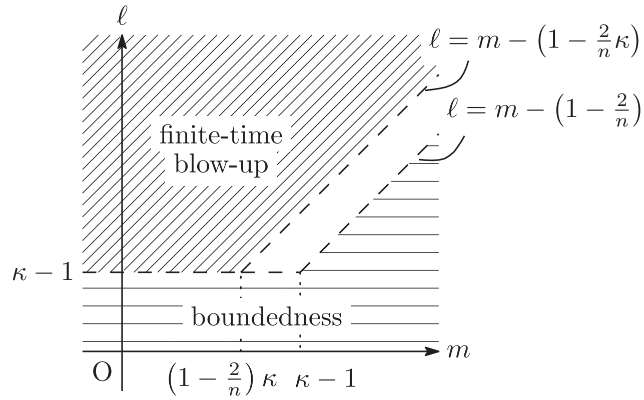

Figure 4.9: $n \ge 2$ and $\alpha = 1$

Chapter 5

The case of degenerate diffusion

5.1. Introduction

In this chapter we study occurrence of finite-time blow-up (it will be called *blow-up* throughout for short this chapter) in quasilinear *degenerate* Jäger–Luckhaus systems with logistic source and nonlinear production of the form

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u^\alpha \nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_\ell}(t) + u^\ell, & x \in \Omega,\ t > 0, \end{cases}$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a ball with some $R > 0$; $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$;

$$\overline{M_\ell}(t) := \frac{1}{|\Omega|} \int_\Omega u^\ell(x,t)\, dx;$$

the function $u = u(x,t)$ denotes the density of cells, and $v = v(x,t)$ shows the concentration of the chemical substance. The powers m and α describe the strengths of the diffusive and chemotactic effects, respectively. A quasilinear chemotaxis system with such porous medium-type diffusion was motivated from a biological point of view (see Szymańska, Morales-Rodrigo, Lachowicz and Chaplain [**42**]) and such quasilinear generalizations were introduced by Hillen and Painter [**16**] and studied by e.g. Tao and Winkler [**50**]. Also the logistic source $\lambda u - \mu u^\kappa$ represents the proliferation and death of the cells, and the damping force is given by the power κ, and this term appears in models including population dynamics [**16, 40**] and pattern formation in bacterial colonies [**64**]. Moreover, the power ℓ means the production rate of signal, and with respect to this term, the linear case ($\ell = 1$) are usually treated, whereas the actual mechanism of signal production might be complex, so that the nonlinear production term was introduced in [**28**], and investigated in [**10, 59**] and in Chapter 3, for instance.

The original model, called Keller–Segel system, was proposed by Keller and Segel [**23**] in 1970, and it is written as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u. \end{cases}$$

Moreover, a number of variations of this system were introduced and investigated in [**1, 16, 26**] and the quasilinear Keller–Segel system with degenerate diffusion

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u^\alpha \nabla v), \\ \tau v_t = \Delta v - v + u, \end{cases}$$

where $\tau \in \{0, 1\}$, is one of these systems. In systems of this type, there are a lot of results on behavior of solutions. In the case that $m = \alpha = 1$ it is known that the size of initial data determines the behavior of solutions (for instance, boundedness in [**4, 36, 39**] and blow-up in [**17, 32, 58**]). In the case that $m \neq 1$ and $\alpha \neq 1$ the relation between m and α affects behavior of solutions. In the case of nondegenerate diffusion, that is, the case that Δu^m and $u^\alpha \nabla v$ are replaced by $\Delta(u+1)^m$ and $u(u+1)^{\alpha-1}\nabla v$, respectively, a number of results on blow-up and boundedness of solutions were obtained (see e.g. [**7, 8, 18, 25, 61**]). In particular, the borderline between boundedness and blow-up is the critical value $m - \alpha = 1 - \frac{2}{n}$, and blow-up results are obtained under the condition that $m - \alpha < 1 - \frac{2}{n}$ in the literature. Furthermore, as to the above quasilinear degenerate Keller–Segel systems, blow-up was derived under the condition that $m - \alpha < 1 - \frac{2}{n}$ in [**15, 20**] (cf. [**18, 19, 41**] for boundedness). Through these results, we can understand that solutions blow up when the chemotactic effect is stronger than the diffusive effect suppressing blow-up. Here, in addition, the logistic source is able to be considered as the other suppressing effect. Therefore the following question is raised:

Does blow-up occur in chemotaxis systems
even if the systems have logistic source?

A positive answer to this question has been firstly given for a minimal Keller–Segel system with logistic source by Winkler [**60**]. After that, for a nondegenerate version of the system some finite-time blow-up results were obtained in [**2**] and Chapters 2 and 3. However, to the best of our knowledge there are no results on blow-up in the case of *degenerate* diffusion. Therefore we focus on the chemotaxis system mentioned at the beginning, which has a structural advantage that radial monotonicity of solutions is derived from the one of initial data.

Review of blow-up results for the nondegenerate system. We recall some known results in the following nondegenerate Jäger–Luckhaus system corresponding to the system

$$\begin{cases} u_t = \nabla \cdot ((u+\delta)^{m-1}\nabla u) - \chi\nabla \cdot (u(u+\delta)^{\alpha-1}\nabla v) + \lambda u - \mu u^\kappa, \\ 0 = \Delta v - \overline{M_\ell}(t) + u^\ell. \end{cases}$$

With regard to systems of this type, there are a lot of results on boundedness due to the damping force of the logistic source $\lambda u - \mu u^\kappa$ (see e.g. [**51, 55, 66, 67**] and Chapter 4). However, as to the above system, if the damping force is weak, then finite-time blow-up occurs; in the case $m = \alpha = \ell = 1$ Winkler [**57**] first derived a condition such that solutions blow up in finite time in the higher dimensional setting; also, conditions for finite-time blow-up in the three- and four-dimensional settings were obtained by Black, Fuest and Lankeit [**2**]; after that, Fuest [**13**] established finite-time blow-up under the conditions that $1 < \kappa < \min\left\{\frac{n}{2}, 2\right\}$ and $\mu > 0$ $(n \geq 3)$ and that $\kappa = 2$ and $\mu \in \left(0, \frac{n-4}{n}\right)$ $(n \geq 5)$, which conditions tell us that the optimal exponent is $\kappa = 2$ in the four- and higher dimensional settings when $m = \alpha = \ell = 1$. In the case that $m = \alpha = 1$ and $\ell > 0$ it was shown by Yi, Mu, Xu and Dai [**65**] that solutions blow up in finite time when $\ell + 1 > \kappa\left(1 + \frac{2}{n}\right)$. Moreover, in Chapter 4 finite-time blow-up result was established under the condition that $\alpha - \ell > \max\left\{\overline{m} + \frac{2}{n}\kappa, \kappa\right\}$, where $\overline{m} := \max\{m, 0\}$, which generalizes the condition in [**13**].

In summary, as to the *nondegenerate* system, finite-time blow-up in the case that $m \neq 1$, $\alpha \neq 1$ and $\ell \neq 1$ was obtained in Chapter 4.

Toward blow-up in the degenerate system. We consider finite-time blow-up in the quasilinear *degenerate* Jäger–Luckhaus system

$$\begin{cases} u_t = \Delta u^m - \chi\nabla \cdot (u^\alpha \nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_\ell}(t) + u^\ell, & x \in \Omega,\ t > 0, \\ \nabla u^m \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.1.1}$$

with

$$\overline{M_\ell}(t) := \frac{1}{|\Omega|}\int_\Omega u^\ell(x,t)\,dx,$$

where

$$\Omega := B_R(0) \subset \mathbb{R}^n\ (n \in \mathbb{N})$$

is a ball with some $R > 0$; $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$; ν is the outward normal vector to $\partial\Omega$; $u_0 \in L^\infty(\Omega)$ is nonnegative, radially symmetric and nonincreasing with respect to $|x|$.

Recalling the method in the literature, we proved finite-time blow-up by deriving the inequality $\phi'(t) \ge C\phi^{\alpha+\ell}(t)$ with some $C > 0$, where ϕ is a moment-type functional. However, since the system (5.1.1) has the degenerate diffusion term Δu^m and possibly the initial data vanishing on some open subset of Ω, we deal with the system (5.1.1) in a framework of weak solutions, and thereby we cannot directly obtain the inequality $\phi'(t) \ge C\phi^{\alpha+\ell}(t)$. Hence we will derive an integral inequality of ϕ to show finite-time blow-up. To this end, we define *moment solutions* to the system (5.1.1) in the following before giving the main theorem.

The purpose of this chapter is to establish finite-time blow-up to the system (5.1.1).

Main result. Before we state the main theorem, we define *moment solutions*, *maximal moment solutions* and *blow-up* for (5.1.1), and so we introduce two symbols w and ϕ as follows. For a pair (u, v) of nonnegative and radially symmetric functions, we regard (u, v) as $(u(r, t), v(r, t))$ with $r := |x|$ if necessary. Given $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$, we set

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \ge 0$$

and we define the moment-type functional ϕ as

$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\, ds \quad \text{for } t \ge 0.$$

Definition 5.1.1 (moment solutions)**.** Let $T \in (0, \infty]$. A pair (u, v) of nonnegative and radially symmetric functions defined on $\Omega \times (0, T)$ is called a *moment solution* of (5.1.1) on $[0, T)$ if

(i) $u \in L^\infty_{\mathrm{loc}}([0,T); L^\infty(\Omega))$ and,
$u^m \in L^2(0,T; H^1(\Omega))$ if $T < \infty$; $u^m \in L^2_{\mathrm{loc}}([0,T); H^1(\Omega))$ if $T = \infty$,

(ii) $v \in L^\infty_{\mathrm{loc}}([0,T); H^1(\Omega))$,

(iii) $u \in C^0_{\mathrm{w}-\star}([0,T); L^\infty(\Omega))$,

(iv) for all $\varphi \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$ with $\operatorname{supp}\varphi \subset [0,T)$,

$$\int_0^T \int_\Omega (\nabla u^m \cdot \nabla \varphi - \chi u^\alpha \nabla v \cdot \nabla \varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\, dxdt = \int_\Omega u_0 \varphi(0)\, dx,$$

$$\int_0^T \int_\Omega \nabla v \cdot \nabla \varphi\, dxdt + \int_0^T \left(\overline{M_\ell}(t) \int_\Omega \varphi\, dx \right) dt - \int_0^T \int_\Omega u^\ell \varphi\, dxdt = 0,$$

(v) (u,v) satisfies the following moment inequality:

$$\phi(t) - \phi(0) \geq K \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in (0,T) \tag{5.1.2}$$

for some constant $K = K(R, m, \chi, \alpha, \mu, \kappa, \ell, \gamma, s_0) > 0$.

We next define *maximal moment solutions*, which are guaranteed by Zorn's lemma as in the proof of [**24**, Lemma 2.4].

Definition 5.1.2 (maximal moment solutions)**.** Define the set $\mathcal{S}$ as

$$\mathcal{S} := \{(T,u,v) \mid T \in (0,\infty],\ (u,v) \text{ is a moment solution of (5.1.1) on } [0,T)\},$$

which is not empty by Proposition 5.2.1, with the order relation $\preceq$ given by

$$(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \ :\Longleftrightarrow\ T_1 \leq T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1.$$

Then Zorn's lemma assures some maximal element $(T_{\max}, u, v) \in \mathcal{S}$, and (u,v) is called a *maximal moment solution* of (5.1.1) on $[0, T_{\max})$.

Definition 5.1.3 (blow-up)**.** Let (u,v) be a maximal moment solution of (5.1.1) on $[0,T_{\max})$. If u satisfies

$$\limsup_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty,$$

then we say that (u,v) *blows up* at $T_{\max}$.

Now the main theorem reads as follows.

Theorem 5.1.1. *Let* $n \in \mathbb{N}$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ *and* $\ell > 0$. *Assume that*

$$\alpha + \ell > \max\left\{ m + \frac{2}{n}\kappa, \kappa \right\}. \tag{5.1.3}$$

Then for all $M_0 > 0$ *there exist* $\eta_0 \in (0, M_0)$ *and* $r_\star \in (0,R)$ *which satisfy the following property: If*

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \tag{5.1.4}$$

and

$$u_0 \text{ is radially symmetric, nonincreasing with respect to } |x| \tag{5.1.5}$$

as well as

$$\int_\Omega u_0(x)\, dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\, dx \geq M_0 - \eta_0, \tag{5.1.6}$$

then a maximal moment solution of (5.1.1) *on* $[0,T_{\max})$ *blows up at* $T_{\max} < \infty$.

Remark 5.1.1. The parameter values appearing in Theorem 5.1.1 are basically the same as in Theorem 4.1.2, in which finite-time blow-up has been obtained for the nondegenerate system. In particular, the condition (5.1.3) coincides with (4.1.13) in Theorem 4.1.2 in the case that $m \geq 1$ and $\alpha \geq 1$.

Idea of the proof. The strategy of the proof of Theorem 5.1.1 is explained as follows. Since we consider the system (5.1.1) in the framework of weak solutions, we deal with a problem approximate to the system (5.1.1) (see (5.2.1)), which is a nondegenerate system. Therefore in Section 5.2, by making use of an argument similar to that of Theorem 4.1.2, we will derive a moment inequality for an approximate solution

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^{\alpha+\ell}(\tau)\, d\tau,$$

where ϕ_ε is a moment-type functional for an approximate solution. Noticing that $u_\varepsilon(t) \to u(t)$ in $L^1(\Omega)$ as $\varepsilon \to 0$ for all $t > 0$, we see that

$$\phi_\varepsilon(t) \to \phi(t) \quad \text{as } \varepsilon \to 0$$

for all $t > 0$, and so in light of convergences a moment inequality (5.1.2) can be obtained. Since the maximal existence time T_ε for approximate solutions depends on ε, we have to make sure that T_ε is uniformly bounded below in the passage to the limit as $\varepsilon \to 0$. The proof of lower boundedness of T_ε is based on [**20**, Lemma 2.4]. We first derive that

$$\frac{d}{dt}\|u_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p \leq C$$

with some $C > 0$ on $(0, \tau_\varepsilon)$, where τ_ε is a time such that $\|u_\varepsilon(\tau_\varepsilon)\|_{L^p(\Omega)}^p = c$ with some $c = c(|\Omega|, \|u_0\|_{L^p(\Omega)}) > 0$, and next, integrating this inequality over $(0, \tau_\varepsilon)$ and using $\|u_\varepsilon(\tau_\varepsilon)\|_{L^p(\Omega)}^p = c$, we observe lower boundedness of T_ε. In Section 5.3 we will prove that existence time $T_{\max}$ is finite and a maximal moment solution blows up at $T_{\max}$. As to the proof that $T_{\max} < \infty$, we assume that $T_{\max} = \infty$ and then derive a contradiction from a moment inequality. Next, we show finite-time blow-up again by contradiction. To this end, we suppose that a maximal moment solution of (5.1.1) on $[0, T_{\max})$ does not blow up at $T_{\max}$. Then a weak solution of (5.1.1) on $[0, T_{\max}+\sigma_1)$ with some $\sigma_1 > 0$ is constructed. We will establish that the weak solution satisfies a moment inequality on $[0, T_{\max} + \sigma_1)$ by making use of the continuity of ϕ and $\int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau$, and hence obtain a moment solution of (5.1.1) on $[0, T_{\max} + \sigma_1)$, which is a contradiction.

5.2. Local existence of moment solutions

The goal of this section is to show local existence of moment solutions to (5.1.1) as in the following key proposition, which plays an important role in the proof of blow-up.

Proposition 5.2.1 (local existence of moment solutions)**.** *Let $n \in \mathbb{N}$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Assume that* (5.1.3) *is satisfied. Then for all $M_0 > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If u_0 satisfies* (5.1.4)–(5.1.6)*, then there exists $T > 0$ such that* (5.1.1) *admits a moment solution (u, v) on $[0, T)$, i.e.* (5.1.1) *has a weak solution (u, v) satisfying the moment inequality* (5.1.2)*.*

The strategy of the proof of Proposition 2.1 is displayed as follows.

- **5.2.1.** Uniform lower bound of existence time for approximate solutions
- **5.2.2.** Convergence of approximate solutions
- **5.2.3.** Moment inequality for approximate solutions
- **5.2.4.** Proof of Proposition 5.2.1

The key to the proof of blow-up is to construct the moment inequality (5.1.2), which is usually shown via the corresponding super-linear differential inequality as in [**13, 59**] and in Chapter 4. However, we cannot derive it for *weak* solutions of (5.1.1) due to the lack of the smoothness. Therefore we will obtain it for approximate *smooth* solutions, denoted by u_ε with parameter $\varepsilon > 0$. Here the maximal existence time T_ε depends on ε, and so there is a possibility that T_ε vanishes in the passage to the limit as $\varepsilon \to 0$. This explains the reason for proving *uniform* lower bound of T_ε in Section 5.2.1.

5.2.1. Uniform lower bound of existence time for approximate solutions

We recall that the system (5.1.1) includes the degenerate diffusion term Δu^m. Hence, in order to compensate for the lack of regularity of solutions to (5.1.1) we consider the following approximate problem:

$$\begin{cases} (u_\varepsilon)_t = \Delta(u_\varepsilon + \varepsilon)^m - \chi \nabla \cdot (u_\varepsilon (u_\varepsilon + \varepsilon)^{\alpha - 1} \nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon - \overline{M_{\ell,\varepsilon}}(t) + u_\varepsilon^\ell, & x \in \Omega,\ t > 0, \\ \nabla u_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{5.2.1}$$

where $\varepsilon \in (0, 1)$ and

$$\overline{M_{\ell,\varepsilon}}(t) := \frac{1}{|\Omega|} \int_\Omega u_\varepsilon^\ell(x, t)\, dx$$

as well as $u_{0\varepsilon} \in C^\infty(\overline{\Omega})$ is given by

$$u_{0\varepsilon} := (\rho_\varepsilon * \overline{u_0})|_{\overline{\Omega}},$$

where $\overline{u_0}$ denotes the zero extension of $u_0 \in L^\infty(\Omega)$, that is,

$$\overline{u_0}(x) := \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

and $\rho_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is the mollifier defined as $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\big(\int_{\mathbb{R}^n} \rho(y)\,dy\big)^{-1}\rho\big(\frac{x}{\varepsilon}\big)$, where

$$\rho(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then ρ_ε satisfies that $0 \leq \rho_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho_\varepsilon \subset \overline{B_\varepsilon(0)}$, and $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = 1$. We know that ρ_ε is nonnegative, radially symmetric and nonincreasing with respect to $|x|$. Additionally, if u_0 is nonnegative, radially symmetric and nonincreasing with respect to $|x|$, then so is $u_{0\varepsilon}$ from the definition of $u_{0\varepsilon}$.

We first recall a well-known result about local existence of classical solutions to (5.2.1). The proof is based on a standard fixed point argument (see e.g. [**63**]).

Lemma 5.2.2. *Let $\varepsilon \in (0,1)$ and let $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Then there exist $T_\varepsilon \in (0,\infty]$ and a unique classical solution $(u_\varepsilon, v_\varepsilon)$ of* (5.2.1) *satisfying*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,T_\varepsilon)) \cap C^{2,1}(\overline{\Omega} \times (0,T_\varepsilon)), \\ v_\varepsilon \in \bigcap_{q>n} C^0([0,T_\varepsilon); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0,T_\varepsilon)). \end{cases}$$

Moreover, u_ε and v_ε are nonnegative and radially symmetric.

In the following let $(u_\varepsilon, v_\varepsilon)$ be the solution of (5.2.1) on $[0, T_\varepsilon)$ as in Lemma 5.2.2. Next, in order to guarantee that the existence time T_ε does not vanish after the passage to the limit as $\varepsilon \to 0$, we confirm uniform lower bound of T_ε, that is, we find $T_0 \in (0,\infty)$ such that for any $\varepsilon \in (0,1)$,

$$T_0 \leq T_\varepsilon \quad \text{and} \quad \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq K_0 \quad \text{for all } t \in [0,T_0), \tag{5.2.2}$$

where $K_0 > 0$ is a constant independent of ε. Before we prove (5.2.2), we show the following lemma. The proof is based on that of [**20**, Lemma 2.4]. However, there are two differences from the literature. One is that the first equation in (5.2.1) has the logistic source, and the other is that the second equation in (5.2.1) is elliptic. So we give a full proof for confirmation.

Lemma 5.2.3. *Let $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$, $\ell > 0$ and*

$$p > \max\left\{1, m - 2(\alpha + \ell) + 1, \frac{n}{2}(\alpha + \ell - m)\right\}.$$

Then there exists $T_p \in (0, \infty]$ such that for any $\varepsilon \in (0, 1)$,

$$T_p \leq T_\varepsilon \quad \text{and} \quad \|u_\varepsilon(t)\|^p_{L^p(\Omega)} \leq (\|u_0\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p + 1 \quad \text{for all } t \in [0, T_p). \tag{5.2.3}$$

Proof. The proof is similar to that of [**20**, Lemma 2.4]. We put

$$\tau_\varepsilon := \sup\{\tau \in (0, T_\varepsilon) \mid \|u_\varepsilon(t)\|^p_{L^p(\Omega)} \leq c_1 \quad \text{for all } t \in (0, \tau)\}$$

with

$$c_1 := (\|u_0\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p + 1.$$

Noting that

$$u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_\varepsilon)) \subset C^0([0, T_\varepsilon); L^p(\Omega)),$$

we see that $\tau_\varepsilon > 0$. It suffices to consider the cases that $\tau_\varepsilon = T_\varepsilon = \infty$ and that $\tau_\varepsilon < T_\varepsilon$ with

$$\|u_\varepsilon(\tau_\varepsilon)\|^p_{L^p(\Omega)} = c_1. \tag{5.2.4}$$

In the case that $\tau_\varepsilon = T_\varepsilon = \infty$, by the definition of τ_ε we have

$$\|u_\varepsilon(t)\|^p_{L^p(\Omega)} \leq c_1$$

for all $t \in (0, \infty)$, which implies that $T_p = \infty$.

In the case that $\tau_\varepsilon < T_\varepsilon$ with (5.2.4), from the first equation in (5.2.1), we obtain

$$\begin{aligned}
\frac{1}{p} \cdot \frac{d}{dt}\|u_\varepsilon + \varepsilon\|^p_{L^p(\Omega)} &= -m(p-1)\int_\Omega (u_\varepsilon + \varepsilon)^{p+m-3}|\nabla u_\varepsilon|^2\,dx \\
&\quad + (p-1)\chi \int_\Omega u_\varepsilon(u_\varepsilon + \varepsilon)^{p+\alpha-3}\nabla u_\varepsilon \cdot \nabla v_\varepsilon\,dx \\
&\quad + \lambda \int_\Omega u_\varepsilon(u_\varepsilon + \varepsilon)^{p-1}\,dx - \mu \int_\Omega u_\varepsilon^\kappa (u_\varepsilon + \varepsilon)^{p-1}\,dx \\
&\leq -\frac{4m(p-1)}{(m+p-1)^2}\|\nabla(u_\varepsilon + \varepsilon)^{\frac{p+m-1}{2}}\|^2_{L^2(\Omega)} \\
&\quad + (p-1)\chi \int_\Omega \nabla\left(\int_0^{u_\varepsilon} \xi(\xi + \varepsilon)^{p+\alpha-3}\,d\xi\right) \cdot \nabla v_\varepsilon\,dx \\
&\quad + \lambda \int_\Omega u_\varepsilon(u_\varepsilon + \varepsilon)^{p-1}\,dx \\
&=: -I_1 + I_2 + I_3
\end{aligned} \tag{5.2.5}$$

for all $t \in (0, \tau_\varepsilon)$. Thanks to the second equation in (5.2.1), it follows that

$$\begin{aligned}
I_2 &= -(p-1)\chi \int_\Omega \left(\int_0^{u_\varepsilon} \xi(\xi+\varepsilon)^{p+\alpha-3}\, d\xi \right) \Delta v_\varepsilon \, dx \\
&\le (p-1)\chi \int_\Omega \left(\int_0^{u_\varepsilon} \xi(\xi+\varepsilon)^{p+\alpha-3}\, d\xi \right) u_\varepsilon^\ell \, dx \\
&\le \frac{(p-1)\chi}{p+\alpha-1} \int_\Omega (u_\varepsilon+\varepsilon)^{p+\alpha+\ell-1}\, dx \qquad (5.2.6)
\end{aligned}$$

for all $t \in (0, \tau_\varepsilon)$. We now set

$$\beta := \frac{\frac{p+m-1}{2p} - \frac{p+m-1}{2(p+\alpha+\ell-1)}}{\frac{p+m-1}{2p} + \frac{1}{n} - \frac{1}{2}}.$$

Taking $p > \max\left\{1, m-2(\alpha+\ell)+1, \frac{n}{2}(\alpha+\ell-m)\right\}$, we know that $\beta \in (0,1)$ and $\frac{2(p+\alpha+\ell-1)}{p+m-1} > 1$. Thus, applying the Gagliardo–Nirenberg inequality, we see that

$$\begin{aligned}
&\int_\Omega (u_\varepsilon+\varepsilon)^{p+\alpha+\ell-1}\, dx \\
&\quad = \|(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\
&\quad \le c_2 \|(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{(1-\beta)\frac{2(p+\alpha+\ell-1)}{p+m-1}} \|\nabla (u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\beta\frac{2(p+\alpha+\ell-1)}{p+m-1}} \\
&\qquad + c_3 \|(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}}
\end{aligned}$$

for all $t \in (0, \tau_\varepsilon)$ with some $c_2 = c_2(\Omega, m, \alpha, \ell) > 0$ and $c_3 = c_3(\Omega, m, \alpha, \ell) > 0$. Moreover, we note that $\frac{\beta(p+\alpha+\ell-1)}{p+m-1} < 1$. Combining the above inequality with (5.2.6) and using Young's inequality, we have that

$$\begin{aligned}
I_2 &\le I_1 + c_4 \left\{ \|(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{(1-\beta)\frac{2(p+\alpha+\ell-1)}{p+m-1}} \right\}^\theta \\
&\quad + c_5 \|(u_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2(p+\alpha+\ell-1)}{p+m-1}} \qquad (5.2.7)
\end{aligned}$$

for all $t \in (0, \tau_\varepsilon)$, where

$$c_4 := \frac{1}{\theta} \left[\left(\frac{\beta(p+\alpha+\ell-1)}{p+m-1} \right)^{\frac{\beta(p+\alpha+\ell-1)}{p+m-1}} \cdot \frac{(p-1)\chi}{p+\alpha-1} c_2 \right]^\theta$$

and

$$c_5 := \frac{(p-1)\chi}{p+\alpha-1} c_3$$

as well as

$$\theta := \left(1 - \frac{\beta(p+\alpha+\ell-1)}{p+m-1}\right)^{-1}.$$

The inequalities (5.2.5) and (5.2.7) yields

$$\begin{aligned}
&\frac{d}{dt}\|u_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \\
&\quad \le p\left(c_4\|u_\varepsilon+\varepsilon\|^{(1-\beta)(p+\alpha+\ell-1)\theta}_{L^p(\Omega)} + c_5\|u_\varepsilon+\varepsilon\|^{p+\alpha+\ell-1}_{L^p(\Omega)} + \lambda\|u_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)
\end{aligned}$$

for all $t \in (0,\tau_\varepsilon)$. Here the definition of τ_ε implies that for any $t \in (0,\tau_\varepsilon)$,

$$\|u_\varepsilon(t)+\varepsilon\|_{L^p(\Omega)} \le \|u_\varepsilon(t)\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}} \le c_1^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}} =: C_p,$$

and hence we have

$$\frac{d}{dt}\|u_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \le p\left(c_4 C_p^{(1-\beta)(p+\alpha+\ell-1)\theta} + c_5 C_p^{p+\alpha+\ell-1} + \lambda C_p^p\right) =: \widetilde{C}_p \tag{5.2.8}$$

for all $t \in (0,\tau_\varepsilon)$. Integrating (5.2.8) over $(0,\tau_\varepsilon)$, we obtain

$$\|u_\varepsilon(\tau_\varepsilon)+\varepsilon\|^p_{L^p(\Omega)} - \|u_{0\varepsilon}+\varepsilon\|^p_{L^p(\Omega)} \le \widetilde{C}_p\tau_\varepsilon.$$

Aided by $\|u_{0\varepsilon}+\varepsilon\|^p_{L^p(\Omega)} \le (\|u_{0\varepsilon}\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p \le (\|u_0\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p$ and $\varepsilon > 0$, we see from (5.2.4) that

$$c_1 - (\|u_0\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p \le \widetilde{C}_p\tau_\varepsilon,$$

which together with the definition of c_1 implies that

$$\frac{1}{\widetilde{C}_p} \le \tau_\varepsilon.$$

Consequently, we attain (5.2.3) with $T_p = \frac{1}{\widetilde{C}_p}$. □

Next, we give an interval ensuring L^∞-estimate for u_ε uniformly with respect to ε by using Lemma 5.2.3.

Lemma 5.2.4. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Then there exist $T_0 \in (0,\infty)$ and $K_0 = K_0(|\Omega|, \|u_0\|_{L^{p_0}(\Omega)}, \|u_0\|_{L^\infty(\Omega)}, m, \chi, \alpha, \lambda, \mu, \kappa, \ell) > 0$ with some large constant $p_0 = p_0(m,\alpha,\ell) > 1$ such that for any $\varepsilon \in (0,1)$,*

$$T_0 \le T_\varepsilon \quad \text{and} \quad \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le K_0 \quad \text{for all } t \in (0,T_0). \tag{5.2.9}$$

Proof. By making use of Lemma 5.2.3 in conjunction with the Moser iteration (see [**50**, Lemma A.1]) we can arrive at (5.2.9). □

5.2.2. Convergence of approximate solutions

In this subsection we discuss convergence of approximate solutions $(u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0$. To this end, we first show some estimates for approximate solutions u_ε.

Lemma 5.2.5. *Let $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0,\infty)$ and $K_0 > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leq K_0 \quad \text{for all } t \in (0, T_0). \tag{5.2.10}$$

Then there exists $C = C(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \ell, T_0, K_0) > 0$ such that

$$\|\nabla(u_\varepsilon+\varepsilon)^m\|_{L^2(0,T_0;L^2(\Omega))} \leq C.$$

Proof. Multiplying the first equation in (5.2.1) by u_ε and integrating it over Ω, we have

$$\begin{aligned} \frac{1}{2}\cdot\frac{d}{dt}\|u_\varepsilon\|^2_{L^2(\Omega)} \leq& -\frac{4m}{(m+1)^2}\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)} \\ &+ \chi\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\nabla v_\varepsilon\cdot\nabla u_\varepsilon\,dx + \lambda\|u_\varepsilon\|^2_{L^2(\Omega)} \end{aligned} \tag{5.2.11}$$

for all $t \in (0, T_0)$. By a computation as in (5.2.6), it follows from (5.2.10) that

$$\chi\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\nabla v_\varepsilon\cdot\nabla u_\varepsilon\,dx \leq \frac{\chi}{\alpha+1}\int_\Omega(u_\varepsilon+\varepsilon)^{\alpha+\ell+1}\,dx \leq \frac{\chi}{\alpha+1}(K_0+1)^{\alpha+\ell+1}|\Omega|.$$

Combining this inequality with (5.2.11) and integrating it over $(0, T_0)$, we obtain

$$\begin{aligned} \frac{1}{2}\|u_\varepsilon(T_0)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_{0\varepsilon}\|^2_{L^2(\Omega)} \leq& -\frac{4m}{(m+1)^2}\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \\ &+ \frac{\chi}{\alpha+1}(K_0+1)^{\alpha+\ell+1}|\Omega|T_0 + \lambda\|u_\varepsilon\|^2_{L^2(0,T_0;L^2(\Omega))}. \end{aligned}$$

Hence, noting $\|u_{0\varepsilon}\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)}$ and (5.2.10), we can show that

$$\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \leq c_1,$$

where

$$c_1 := \frac{(m+1)^2}{4m}\left(\frac{1}{2}\|u_0\|^2_{L^2(\Omega)} + \frac{\chi}{\alpha+1}(K_0+1)^{\alpha+\ell+1}|\Omega|T_0 + \lambda K_0^2|\Omega|T_0\right) > 0.$$

This entails that for any $\varepsilon \in (0,1)$,

$$\begin{aligned} \|\nabla(u_\varepsilon+\varepsilon)^m\|^2_{L^2(0,T_0;L^2(\Omega))} &= \frac{4m^2}{(m+1)^2}\|(u_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \\ &\leq \frac{4m^2}{(m+1)^2}(K_0+1)^{m-1}c_1, \end{aligned}$$

which implies the end of the proof. □

We next estimate $\|\sqrt{t}(u_\varepsilon^m)_t\|^2_{L^2(0,T_0;L^2(\Omega))}$ and $\sup_{t\in(0,T_0)}\|\sqrt{t}\nabla u_\varepsilon^m\|^2_{L^2(\Omega)}$. The proof is based on [**19**, Lemma 5.2].

Lemma 5.2.6. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0,\infty)$ and $K_0 > 0$ such that (5.2.10) holds for any $\varepsilon \in (0,1)$. Then there is $C = C(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \mu, \kappa, \ell, T_0, K_0) > 0$ such that*

$$\left\|\sqrt{t}\frac{\partial}{\partial t}u_\varepsilon^m\right\|^2_{L^2(0,T_0;L^2(\Omega))} + \sup_{t\in(0,T_0)}\|\sqrt{t}\nabla u_\varepsilon^m(t)\|^2_{L^2(\Omega)} \le C.$$

Proof. Multiplying the first equation in (5.2.1) by $\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^m$ and integrating it over Ω, we can observe that

$$\begin{aligned}
&\frac{4m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|^2_{L^2(\Omega)} \\
&\le -\frac{1}{2}\cdot\frac{d}{dt}\|\nabla(u_\varepsilon+\varepsilon)^m\|^2_{L^2(\Omega)} \\
&\quad -\frac{2m}{m+1}\int_\Omega \nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon(u_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\,dx \\
&\quad +\frac{2m}{m+1}\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon(u_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\,dx \\
&\quad +\int_\Omega \lambda u_\varepsilon\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^m\,dx - \int_\Omega \mu u_\varepsilon^\kappa\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^m\,dx
\end{aligned}$$

for all $t \in (0,T_0)$. Due to Young's inequality, we infer that

$$\begin{aligned}
&-\frac{2m}{m+1}\int_\Omega \nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon(u_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\,dx \\
&\quad +\frac{2m}{m+1}\int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon(u_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\,dx \\
&\quad \le \frac{m}{2}\int_\Omega \left|\nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon + u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon\right|^2(u_\varepsilon+\varepsilon)^{m-1}\,dx \\
&\qquad +\frac{2m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|^2_{L^2(\Omega)} \\
&\quad \le m\int_\Omega\left(\left|\nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon\right|^2 + \left|u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon\right|^2\right)(u_\varepsilon+\varepsilon)^{m-1}\,dx \\
&\qquad +\frac{2m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus it follows that

$$
\begin{aligned}
&\frac{2m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\cdot\frac{d}{dt}\|\nabla(u_\varepsilon+\varepsilon)^m\|^2_{L^2(\Omega)}\\
&\quad\le m\int_\Omega\left(\left|\nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon\right|^2+\left|u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon\right|^2\right)(u_\varepsilon+\varepsilon)^{m-1}\,dx\\
&\quad\quad+\lambda m\frac{d}{dt}\int_\Omega\left(\int_0^{u_\varepsilon}\xi(\xi+\varepsilon)^{m-1}\,d\xi\right)dx\\
&\quad\quad-\mu m\frac{d}{dt}\int_\Omega\left(\int_0^{u_\varepsilon}\xi^\kappa(\xi+\varepsilon)^{m-1}\,d\xi\right)dx
\end{aligned}
\tag{5.2.12}
$$

for all $t\in(0,T_0)$. Noticing that (5.2.10) and the elliptic regularity theory applied to the second equation in (5.2.1) lead to the inequality $\|v_\varepsilon(t)\|_{W^{2,p}(\Omega)}\le c_1(p,K_0)$ for all $p>1$ and $t\in(0,T_0)$, we can confirm from the Sobolev embedding theorem that $\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)}\le c_2(p,K_0)$ for all $t\in(0,T_0)$, and hence establish that

$$
\begin{aligned}
&m\int_\Omega\left|\nabla(u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1})\cdot\nabla v_\varepsilon\right|^2(u_\varepsilon+\varepsilon)^{m-1}\,dx\\
&\quad\le m\alpha^2\int_\Omega\left|(u_\varepsilon+\varepsilon)^{\alpha-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\right|^2(u_\varepsilon+\varepsilon)^{m-1}\,dx\\
&\quad=\frac{4m\alpha^2}{(m+1)^2}\int_\Omega\left|(u_\varepsilon+\varepsilon)^{\alpha-1}\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\cdot\nabla v_\varepsilon\right|^2dx\\
&\quad\le c_3\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)}
\end{aligned}
\tag{5.2.13}
$$

for all $t\in(0,T_0)$, where $c_3:=\frac{4m\alpha^2}{(m+1)^2}(K_0+1)^{2(\alpha-1)}c_2^2$. On the other hand, in light of the second equation in (5.2.1) and (5.2.10), we see that

$$
|\Delta v_\varepsilon|=\left|\frac{1}{|\Omega|}\int_\Omega u_\varepsilon^\ell\,dx-u_\varepsilon^\ell\right|\le\frac{1}{|\Omega|}\left(\int_\Omega u_\varepsilon^\ell\,dx\right)+u_\varepsilon^\ell\le 2K_0^\ell,
$$

which implies that

$$
\begin{aligned}
&m\int_\Omega\left|u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\Delta v_\varepsilon\right|^2(u_\varepsilon+\varepsilon)^{m-1}\,dx\\
&\quad\le 4mK_0^{2\ell+2}(K_0+1)^{m+2\alpha-3}|\Omega|=:c_4
\end{aligned}
\tag{5.2.14}
$$

for all $t\in(0,T_0)$. A combination of (5.2.13) and (5.2.14) with (5.2.12) yields

$$
\begin{aligned}
&\frac{2m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\cdot\frac{d}{dt}\|\nabla(u_\varepsilon+\varepsilon)^m\|^2_{L^2(\Omega)}\\
&\quad\le c_3\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)}+c_4+\lambda m\frac{d}{dt}\int_\Omega\left(\int_0^{u_\varepsilon}\xi(\xi+\varepsilon)^{m-1}\,d\xi\right)dx\\
&\quad\quad-\mu m\frac{d}{dt}\int_\Omega\left(\int_0^{u_\varepsilon}\xi^\kappa(\xi+\varepsilon)^{m-1}\,d\xi\right)dx
\end{aligned}
\tag{5.2.15}
$$

for all $t \in (0, T_0)$. Multiplying (5.2.15) by t and changing the variable t with s, we integrate it over $(0, t)$ to obtain

$$\begin{aligned}
&\frac{2m}{(m+1)^2} \left\| \sqrt{s} \frac{\partial}{\partial t} (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}} \right\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2} t \|\nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2} \|\nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(0,t;L^2(\Omega))} + c_3 \|\sqrt{s} \nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,t;L^2(\Omega))} + c_4 t \\
&\quad + \lambda m t \int_\Omega \left(\int_0^{u_\varepsilon} \xi (\xi + \varepsilon)^{m-1} \, d\xi \right) dx \\
&\quad + \mu m \int_0^t \left[\int_\Omega \left(\int_0^{u_\varepsilon} \xi^\kappa (\xi + \varepsilon)^{m-1} \, d\xi \right) dx \right] dt.
\end{aligned}$$

Here, as in the proof of Lemma 5.2.5, we see that

$$\|\nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,t;L^2(\Omega))} \le c_5 \quad \text{and} \quad \|\nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(0,t;L^2(\Omega))} \le c_6$$

for all $t \in (0, T_0)$ with some $c_5 = c_5(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \ell, T_0, K_0) > 0$ and $c_6 = c_6(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \ell, T_0, K_0) > 0$. Therefore, observing from (5.2.10) that

$$\begin{aligned}
\int_\Omega \left(\int_0^{u_\varepsilon} \xi (\xi + \varepsilon)^{m-1} \, d\xi \right) dx &\le \frac{1}{m+1} \int_\Omega (u_\varepsilon + \varepsilon)^{m+1} \, dx \\
&\le \frac{1}{m+1} (K_0 + 1)^{m+1} |\Omega|
\end{aligned}$$

and similarly

$$\int_0^t \left[\int_\Omega \left(\int_0^{u_\varepsilon} \xi^\kappa (\xi + \varepsilon)^{m-1} \, d\xi \right) dx \right] dt \le \frac{1}{m+\kappa} (K_0 + 1)^{m+\kappa} |\Omega| T_0,$$

we can show that

$$\frac{2m}{(m+1)^2} \left\| \sqrt{s} \frac{\partial}{\partial t} (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}} \right\|^2_{L^2(0,t;L^2(\Omega))} + \frac{1}{2} t \|\nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(\Omega)} \le c_7$$

for all $t \in (0, T_0)$, where

$$c_7 := \frac{1}{2} c_6 + c_3 c_5 T_0 + c_4 T_0 + \frac{\lambda m}{m+1} (K_0 + 1)^{m+1} |\Omega| T_0 + \frac{\mu m}{m+\kappa} (K_0 + 1)^{m+\kappa} |\Omega| T_0.$$

Thus we have that

$$\frac{2m}{(m+1)^2} \left\| \sqrt{s} \frac{\partial}{\partial t} (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}} \right\|^2_{L^2(0,T_0;L^2(\Omega))} + \frac{1}{2} \sup_{t \in (0,T_0)} \|\sqrt{t} \nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(\Omega)} \le c_7.$$

From this inequality it follows that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\left\| \sqrt{t}\frac{\partial}{\partial t} u_\varepsilon^m \right\|^2_{L^2(0,T_0;L^2(\Omega))} + \sup_{t\in(0,T_0)} \|\sqrt{t}\nabla u_\varepsilon^m(t)\|^2_{L^2(\Omega)} \\
&\quad \le \left\| \sqrt{s}\frac{\partial}{\partial t} (u_\varepsilon+\varepsilon)^{\frac{m+1}{2}} \right\|^2_{L^2(0,T_0;L^2(\Omega))} + \sup_{t\in(0,T_0)} \|\sqrt{t}\nabla (u_\varepsilon+\varepsilon)^m\|^2_{L^2(\Omega)} \\
&\quad \le \left(\frac{(m+1)^2}{2m} + 2 \right) c_7,
\end{aligned}$$

which concludes the proof. □

Finally we shall establish convergence of approximate solutions $(u_\varepsilon, v_\varepsilon)$.

Lemma 5.2.7. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0,\infty)$ and $K_0 > 0$ such that (5.2.10) holds for any $\varepsilon \in (0,1)$. Then there exist subsequences $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that*

- $u \in L^\infty(0,T_0;L^\infty(\Omega))$, $u^m \in L^2(0,T_0;H^1(\Omega))$,
- $v \in L^\infty(0,T_0;W^{1,\infty}(\Omega))$,

and as $k \to \infty$,

$$u_{\varepsilon_k} \to u \quad \text{weakly}^\star \text{ in } L^\infty(0,T_0;L^\infty(\Omega)), \tag{5.2.16}$$

$$u_{\varepsilon_k} \to u \quad \text{strongly in } C^0([\delta,T_0];L^p(\Omega)) \quad \text{for all } \delta \in (0,T_0) \text{ and } p \in [1,\infty), \tag{5.2.17}$$

$$\nabla(u_{\varepsilon_k}+\varepsilon)^m \to \nabla u^m \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)), \tag{5.2.18}$$

$$\nabla v_{\varepsilon_k} \to \nabla v \quad \text{weakly}^\star \text{ in } L^\infty(0,T_0;L^\infty(\Omega)). \tag{5.2.19}$$

Proof. Applying the elliptic regularity theory to the second equation in (5.2.1), from (5.2.10) and the Sobolev embedding theorem we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\|v_\varepsilon(t)\|_{L^\infty(\Omega)} \le c_1 \quad \text{and} \quad \|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)} \le c_2$$

for all $t \in (0,T_0)$. Therefore we can show that there exist a subsequence $\{v_{\varepsilon_k}\}$ and a function $v \in L^\infty(0,T_0;W^{1,\infty}(\Omega))$ satisfying (5.2.19). Moreover, thanks to Lemmas 5.2.5 and 5.2.6, as in the proof of [**19**, Lemma 5.3] we can extract a subsequence $\{u_{\varepsilon_k}\}$ and a function $u \in L^\infty(0,T_0;L^\infty(\Omega))$ with $u^m \in L^2(0,T_0;H^1(\Omega))$ such that (5.2.16)–(5.2.18) holds. □

5.2.3. Moment inequality for approximate solutions

In this subsection we will derive the moment inequality for $(u_\varepsilon, v_\varepsilon)$. To this end, introducing $r := |x|$, we denote by $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(r,t), v_\varepsilon(r,t))$ the radially symmetric local solution of (5.2.1) on $[0, T_\varepsilon)$. Also, we define the function w_ε and the moment-type functional ϕ_ε for the approximate solution u_ε as

$$w_\varepsilon(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_\varepsilon)$$

and

$$\phi_\varepsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s,t)\, ds \quad \text{for } t \in [0, T_\varepsilon).$$

Here we know that $\phi_\varepsilon \in C^0([0, T_\varepsilon)) \cap C^1((0, T_\varepsilon))$.

Now we state a proposition on the moment inequality for approximate solutions.

Proposition 5.2.8. *Let $n \in \mathbb{N}$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa > 1$ and $\ell > 0$. Assume that* (5.1.3) *is satisfied. Then for all $M_0 > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If u_0 satisfies* (5.1.4)*–*(5.1.6)*, then there exist $T_0 \in (0, \infty)$ and $K_0 > 0$ such that* (5.2.9) *holds. Moreover, one can find $K = K(R, m, \chi, \alpha, \mu, \kappa, \ell) > 0$ and $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^{\alpha+\ell}(\tau)\, d\tau \tag{5.2.20}$$

for all $t \in (0, T_0)$.

As to the proof of Proposition 5.2.8, we apply arguments of Lemmas 4.3.4–4.3.10 to the approximate solution. To this end, we first confirm that $\int_\Omega u_\varepsilon\, dx$ is bounded and that u_ε is nonincreasing with respect to $|x|$.

Lemma 5.2.9. *Assume that u_0 satisfies* (5.1.4)*. Then for any $\varepsilon \in (0,1)$,*

$$\int_\Omega u_\varepsilon(x,t)\, dx \leq M_* := \max\left\{ \int_\Omega u_0(x)\, dx, \left(\frac{\lambda}{\mu}|\Omega|^{\kappa-1}\right)^{\frac{1}{\kappa-1}} \right\}$$

for all $t \in (0, T_\varepsilon)$.

Proof. As in the proof of Lemma 4.2.2 we have $\int_\Omega u_\varepsilon \leq \max\big\{ \int_\Omega u_{0\varepsilon}, \big(\frac{\lambda}{\mu}|\Omega|^{\kappa-1}\big)^{\frac{1}{\kappa-1}} \big\}$, which together with the relation $\int_\Omega u_{0\varepsilon} \leq \int_\Omega u_0$ implies this lemma. □

Lemma 5.2.10. *Assume that u_0 satisfies* (5.1.4)*. Then for any $\varepsilon \in (0,1)$,*

$$(u_\varepsilon)_r(r,t) \le 0$$

for all $r \in (0,R)$ and $t \in (0,T_\varepsilon)$, that is,

$$(w_\varepsilon)_{ss} \le 0$$

for all $s \in (0,R^n)$ and $t \in (0,T_\varepsilon)$.

Proof. By virtue of (5.1.4) and the definition of $u_{0\varepsilon}$, we see that $u_{0\varepsilon}$ is also nonincreasing with respect to $|x|$. Therefore the claim can be proved by an argument similar to that in the proof of [**59**, Lemma 2.2] or [**2**, Lemma 5.1]. □

Invoking that

$$\int_\Omega u_{0\varepsilon} \le \int_\Omega u_0$$

and $u_{0\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, we can pick $\xi_0 > 0$ so small and find some $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0,\varepsilon_0)$,

$$\int_\Omega u_0 - \xi_0 \le \int_\Omega u_{0\varepsilon} \le \int_\Omega u_0. \tag{5.2.21}$$

Next we take $T_0 \in (0,\infty)$ and $K_0 > 0$ fulfilling (5.2.9) and define the set S_{ϕ_ε} as

$$S_{\phi_\varepsilon} := \left\{ t \in (0,T_0) \;\middle|\; \phi_\varepsilon(t) \ge \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \right\}, \tag{5.2.22}$$

where $M_0 > 0$. We next show the lower estimate for $w_\varepsilon\big(\frac{s_0}{2},t\big)$.

Lemma 5.2.11. *Assume that u_0 satisfies* (5.1.4) *and let $s_0 \in (0,R^n)$ and $\gamma \in (-\infty,1)$. Then for any $\varepsilon \in (0,\varepsilon_0)$,*

$$w_\varepsilon\left(\frac{s_0}{2},t\right) \ge \frac{M_* - \delta_0}{\omega_n} \quad \text{for all } t \in S_{\phi_\varepsilon},$$

where

$$\delta_0 := \frac{4(M_* - M_0 + \xi_0 + s_0)}{2^\gamma(3-\gamma)}.$$

Proof. The proof of this lemma is based on that of [**59**, Lemma 3.1]. We only consider the case that $M_0 > \delta_0$. Assuming that there exists $t \in S_{\phi_\varepsilon}$ such that

$$w_\varepsilon\left(\frac{s_0}{2},t\right) < \frac{M_* - \delta_0}{\omega_n},$$

we will derive a contradiction. Thanks to the monotonicity of $w_\varepsilon(\cdot,t)$, we see that $w_\varepsilon(s,t) < \frac{M_*-\delta_0}{\omega_n}$ for all $s \in \left(0, \frac{s_0}{2}\right)$. Moreover, Lemma 5.2.9 yields

$$w_\varepsilon(s,t) \le \frac{M_*}{\omega_n}$$

for all $s \in (0, R^n)$. Thus we obtain

$$\begin{aligned}
\phi_\varepsilon(t) &< \frac{M_*-\delta_0}{\omega_n}\int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0-s)\,ds + \frac{M_*}{\omega_n}\int_{\frac{s_0}{2}}^{s_0} s^{-\gamma}(s_0-s)\,ds \\
&= \frac{M_*}{\omega_n}\int_0^{s_0} s^{-\gamma}(s_0-s)\,ds - \frac{\delta_0}{\omega_n}\int_0^{\frac{s_0}{2}} s^{-\gamma}(s_0-s)\,ds \\
&= \frac{M_*}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma} - \frac{2^\gamma(3-\gamma)\delta_0}{4(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma} \\
&= \frac{M_0-\xi_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}.
\end{aligned}$$

By virtue of the definition of S_{ϕ_ε}, this inequality leads to the contradiction. Thus we complete the proof. □

We next establish the estimate for $\overline{M_{\ell,\varepsilon}}(t)$.

Lemma 5.2.12. *Assume that u_0 satisfies (5.1.4) and let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$. Then for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\overline{M_{\ell,\varepsilon}}(t) \le \overline{L} + \frac{1}{2s}\int_0^s \left[n(w_\varepsilon)_s(\sigma,t)\right]^\ell d\sigma \quad \textit{for all } s \in (0, s_0) \textit{ and } t \in S_{\phi_\varepsilon}, \tag{5.2.23}$$

where

$$\overline{L} := \left(\frac{2n\delta_0}{\omega_n s_0}\right)^\ell.$$

Proof. The proof is similar to that of [**59**, Lemma 3.2]. By means of Lemma 5.2.11, we have

$$w_\varepsilon\left(\frac{s_0}{2}, t\right) \ge \frac{M_*-\delta_0}{\omega_n}$$

for all $t \in S_{\phi_\varepsilon}$. Moreover, we recall that

$$w_\varepsilon(s,t) \le \frac{M_*}{\omega_n}$$

for all $s \in (0, R^n)$ and $t \in S_{\phi_\varepsilon}$. Therefore it follows that

$$\frac{w_\varepsilon(s_0,t) - w_\varepsilon\left(\frac{s_0}{2},t\right)}{\frac{s_0}{2}} \le \frac{2\delta_0}{\omega_n s_0}.$$

On the other hand, aided by Lemma 5.2.10, we can observe from the concavity of $w(\cdot,t)$ that

$$\frac{w_\varepsilon(s_0,t)-w_\varepsilon\left(\frac{s_0}{2},t\right)}{\frac{s_0}{2}} \geq (w_\varepsilon)_s(s_0,t) \geq (w_\varepsilon)_s(s,t)$$

for all $s \in (s_0, R^n)$. Hence we infer that for all $s \in (s_0, R^n)$,

$$(w_\varepsilon)_s(s,t) \leq \frac{2\delta_0}{\omega_n s_0}. \tag{5.2.24}$$

Now we note that

$$\overline{M_{\ell,\varepsilon}}(t) = \frac{1}{R^n}\int_0^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma + \frac{n^\ell}{R^n}\int_{s_0}^{R^n} \big[(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma. \tag{5.2.25}$$

As to the second term on the right-hand side of (5.2.25), the relation (5.2.24) and $\ell > 0$ imply that

$$\frac{n^\ell}{R^n}\int_{s_0}^{R^n} \big[(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma \leq \frac{R^n - s_0}{R^n}\cdot\left(\frac{2n\delta_0}{\omega_n s_0}\right)^\ell \leq \left(\frac{2n\delta_0}{\omega_n s_0}\right)^\ell = \overline{L}. \tag{5.2.26}$$

Regarding the first term on the right-hand side of (5.2.25), we see that

$$\frac{1}{R^n}\int_0^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma = \frac{1}{R^n}\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma + \frac{1}{R^n}\int_s^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma$$

for all $s \in (0, s_0)$. Invoking that $(w_\varepsilon)_s(\cdot,t)$ is nonincreasing, we derive that

$$\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma \geq s\big[n(w_\varepsilon)_s(s,t)\big]^\ell$$

and

$$\frac{1}{R^n}\int_s^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma \leq \frac{s_0}{R^n}\big[n(w_\varepsilon)_s(s,t)\big]^\ell$$

for all $s \in (0, s_0)$. These two inequalities ensure that

$$\begin{aligned}\frac{1}{R^n}\int_0^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma &\leq \frac{1}{R^n}\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma + \frac{s_0}{R^n}\big[n(w_\varepsilon)_s(s,t)\big]^\ell \\ &\leq \frac{1}{R^n}\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma + \frac{s_0}{R^n s}\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma\end{aligned}$$

for all $s \in (0, s_0)$. In light of $s_0 \in \left(0, \frac{R^n}{4}\right]$, we can estimate that $\frac{1}{R^n} \leq \frac{1}{4s}$ and $\frac{s_0}{R^n s} \leq \frac{1}{4s}$ for all $s \in (0, s_0)$, which lead to obtain

$$\frac{1}{R^n}\int_0^{s_0} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma \leq \frac{1}{2s}\int_0^{s} \big[n(w_\varepsilon)_s(\sigma,t)\big]^\ell \, d\sigma \tag{5.2.27}$$

for all $s \in (0, s_0)$. A combination of (5.2.26) and (5.2.27) with (5.2.25) yields (5.2.23), which concludes the proof of this lemma. □

Now we prove Proposition 5.2.8.

Proof of Proposition 5.2.8. We first show (5.2.20). By means of Lemma 5.2.4, for any initial data u_0 with the properties (5.1.4) and (5.1.5), we can find $T_0 \in (0,\infty)$ and $K_0 > 0$ satisfying (5.2.9). Now let $\xi_0 > 0$ and $\varepsilon_0 \in (0,1)$ fulfill (5.2.21). In view of Lemmas 5.2.10–5.2.12, we can observe from an argument similar to that in the proof of Lemma 4.3.4 that

$$\begin{aligned}\phi_\varepsilon'(t) \geq \frac{n^\ell}{2}&\int_0^{s_0} s^{1-\gamma}(s_0-s)\,(n(w_\varepsilon)_s+\varepsilon)^{\alpha-1}\,(w_\varepsilon)_s^{\ell+1}\,ds\\ &-\overline{L}\int_0^{s_0} s^{1-\gamma}(s_0-s)\,(n(w_\varepsilon)_s+\varepsilon)^{\alpha-1}\,(w_\varepsilon)_s\,ds\\ &+mn^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)\,(n(w_\varepsilon)_s+\varepsilon)^{m-1}\,(w_\varepsilon)_{ss}\,ds\\ &-n^{\kappa-1}\mu\int_0^{s_0} s^{-\gamma}(s_0-s)\left\{\int_0^{s_0}(w_\varepsilon)_s^\kappa\,d\sigma\right\}ds \qquad (5.2.28)\end{aligned}$$

for all $s_0 \in \left(0,\frac{R^n}{4}\right]$ and $t \in S_{\phi_\varepsilon}$. Since we can apply Lemmas 4.3.4–4.3.10 with S_ϕ replaced by S_{ϕ_ε} to (5.2.28), there are $\gamma \in (-\infty,1)$ and $c_1 = c_1(R,m,\chi,\alpha,\mu,\kappa,\ell,\gamma) > 0$ as well as $c_2 = c_2(R,m,\chi,\alpha,\mu,\kappa,\ell,\gamma) > 0$ such that

$$\phi_\varepsilon'(t) \geq c_1 s_0^{-(3-\gamma)(\alpha+\ell-1)}\phi_\varepsilon^{\alpha+\ell}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \qquad (5.2.29)$$

for all $s_0 \in \left(0,\frac{R^n}{4}\right]$ and $t \in S_{\phi_\varepsilon}$. Here we note from Remark 4.3.1 that c_1 and c_2 are independent of ε. We fix $s_0 > 0$ such that

$$s_0 \leq \min\left\{\frac{R^n}{4},\frac{M_0-\xi_0}{2}\right\} \qquad (5.2.30)$$

and

$$s_0^{(\alpha+\ell)\left(1-\frac{1}{\alpha+\ell-m}\right)} \leq \frac{c_1}{2c_2}\left(\frac{M_0-\xi_0}{2(1-\gamma)(2-\gamma)\omega_n}\right)^{\alpha+\ell}. \qquad (5.2.31)$$

We additionally pick $\eta_0 \in \left(0,\frac{s_0}{4}\right)$ so small and take $s_\star \in (0,s_0)$ satisfying

$$\frac{M_0-\xi_0-\eta_0}{\omega_n}\int_{s_\star}^{s_0} s^{-\gamma}(s_0-s)\,ds > \frac{M_0-\xi_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}.$$

Moreover, in the following we suppose that u_0 fulfills (5.1.4)–(5.1.6) with $r_\star := s_\star^{\frac{1}{n}}$. In order to derive (5.2.20) we define the set

$$\widetilde{S}_\varepsilon := \left\{\tau \in (0,T_0)\ \middle|\ \phi_\varepsilon(t) > \frac{M_0-\xi_0-s_0}{(1-\gamma)(2-\gamma)\omega_n}s_0^{2-\gamma}\quad \text{for all } t \in [0,\tau]\right\}.$$

Here we can see that $\widetilde{S}_\varepsilon$ is not empty for sufficiently small ε. Indeed, from the second condition of (5.1.6) and $u_{0\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, we can observe that

$$\int_{B_{r_\star}(0)} u_{0\varepsilon}\, dx \geq M_0 - \xi_0 - \eta_0$$

for all $\varepsilon \in (0, \varepsilon_0)$. This inequality yields that $w_\varepsilon(s, 0) \geq w_\varepsilon(s_\star, 0) \geq \frac{M_0 - \xi_0 - \eta_0}{\omega_n}$ for all $s \in (s_\star, s_0)$. Hence we obtain

$$\begin{aligned}\phi_\varepsilon(0) &\geq \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s, 0)\, ds \\ &\geq \frac{M_0 - \xi_0 - \eta_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds \\ &> \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma},\end{aligned}$$

which together with the continuity of ϕ_ε implies that $\widetilde{S}_\varepsilon$ is not empty for any $\varepsilon \in (0, \varepsilon_0)$. Now let $\widetilde{T}_\varepsilon := \sup \widetilde{S}_\varepsilon \in (0, T_0]$. Then from (5.2.22) we can confirm that $(0, \widetilde{T}_\varepsilon) \subset S_{\phi_\varepsilon}$. Thanks to (5.2.29)–(5.2.31), as in the proof of Theorem 4.1.2, we have

$$\phi_\varepsilon'(t) \geq \frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi_\varepsilon^{\alpha+\ell}(t) > 0$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in (0, \widetilde{T}_\varepsilon)$. This ensures that $\widetilde{T}_\varepsilon = T_0$. Choosing an arbitrary $t \in (0, T_0)$ and integrating the above inequality over $(0, t)$, we attain (5.2.20). □

5.2.4. Proof of Proposition 5.2.1

We establish local existence of moment solutions to the system (5.1.1) by virtue of the passage to the limit as $\varepsilon \to 0$ in (5.2.20).

Proof of Proposition 5.2.1. Let $M_0 > 0$ and let $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ given by Proposition 5.2.8. Also, we pick u_0 fulfilling (5.1.4)–(5.1.6). Then, thanks to Lemma 5.2.2 and Proposition 5.2.8, we can obtain the approximate solution $(u_\varepsilon, v_\varepsilon)$ of (5.2.1) and find $T_0 \in (0, \infty)$ and $K_0 > 0$ such that (5.2.9) holds, and we have

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^{\alpha+\ell}(\tau)\, d\tau \tag{5.2.32}$$

for all $t \in (0, T_0)$ with some $K > 0$. By virtue of (5.2.9), we can apply Lemma 5.2.7. Hence there exist $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that $(u, v) = \lim_{k\to\infty}(u_{\varepsilon_k}, v_{\varepsilon_k})$ satisfies (i), (ii) and (iv) in Definition 5.1.1. We next show (iii) in Definition 5.1.1. Let us pick $\psi \in L^1(\Omega)$. Then for all $\xi > 0$ there is $\psi_0 \in C_c(\Omega)$ such that

$$\|\psi - \psi_0\|_{L^1(\Omega)} < \xi. \tag{5.2.33}$$

Moreover, noting from (5.2.17) that $u \in C^0([\delta, T_0]; L^1(\Omega))$ for all $\delta \in (0, T_0)$, we see that for all $t_0 > 0$,

$$\int_\Omega u(t)\psi_0\, dx \to \int_\Omega u(t_0)\psi_0\, dx \quad \text{as } t \to t_0, \tag{5.2.34}$$

and from (5.2.16) it follows that

$$\|u\|_{L^\infty(0,T_0;L^\infty(\Omega))} \le \liminf_{k\to\infty} \|u_{\varepsilon_k}\|_{L^\infty(0,T_0;L^\infty(\Omega))} < \infty. \tag{5.2.35}$$

In light of (5.2.33)–(5.2.35) we can verify that $u \in C^0_{\mathrm{w}-\star}((0, T_0); L^\infty(\Omega))$. Furthermore, by relying on the fact that $u_{\varepsilon_k} \in C^0(\overline{\Omega} \times [0, T_0))$ and $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, it follows that $u \in C^0_{\mathrm{w}-\star}([0, T_0); L^\infty(\Omega))$, that is, (iii) holds. Next we make sure that the moment inequality (5.1.2) holds. Invoking $u_{0\varepsilon_k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, we can confirm that

$$\phi_{\varepsilon_k}(0) \to \phi(0) \quad \text{as } k \to \infty.$$

Furthermore, due to (5.2.17) it follows that $u_{\varepsilon_k} \to u$ in $C^0((0, T_0]; L^1(\Omega))$ as $k \to \infty$, which ensures that

$$\phi_{\varepsilon_k}(t) \to \phi(t) \quad \text{as } k \to \infty$$

for all $t \in (0, T_0)$. Additionally, noticing that

$$w_{\varepsilon_k}(s, t) \le \frac{K_0|\Omega|}{\omega_n},$$

we can observe that

$$\phi^{\alpha+\ell}_{\varepsilon_k}(t) \le \left(\frac{K_0|\Omega|}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \right)^{\alpha+\ell}$$

for all $t \in (0, T_0)$. In view of the Lebesgue dominated convergence theorem, we infer that

$$\int_0^t \phi^{\alpha+\ell}_{\varepsilon_k}(\tau)\, d\tau \to \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{as } k \to \infty$$

for all $t \in (0, T_0)$, and so letting $k \to \infty$ in (5.2.32), we see that (v) in Definition 5.1.1 holds. This implies the end of the proof. □

5.3. Finite-time blow-up

In this section we prove finite-time blow-up of maximal moment solutions to (5.1.1). Before proceeding to the proof, we confirm the following equivalence.

Lemma 5.3.1. *Let $T \in (0, \infty)$. Assume that a pair (u, v) of nonnegative functions defined on $\Omega \times (0, T)$ satisfies*

$$u \in L^\infty(0,T;L^\infty(\Omega)), \quad u^m, v \in L^2(0,T;H^1(\Omega)), \quad u \in C^0_{\mathrm{w}-\star}([0,T);L^\infty(\Omega)). \tag{5.3.1}$$

Then the following two conditions are equivalent.

(a) *For all $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ with $\operatorname{supp}\varphi \subset [0,T)$,*

$$\begin{aligned}
&\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\varphi - \chi u^\alpha \nabla v \cdot \nabla\varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\,dxdt \\
&\quad = \int_\Omega u_0\varphi(0)\,dx, \\
&\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi\,dxdt + \int_0^T \left(\overline{M_\ell}(t)\int_\Omega \varphi\,dx\right)dt - \int_0^T \int_\Omega u^\ell \varphi\,dxdt = 0;
\end{aligned}$$

(b) $u_t \in L^2(0,T;(H^1(\Omega))^\star)$*, and for all $\psi \in H^1(\Omega)$,*

$$\int_\Omega u_t\psi\,dx = -\int_\Omega (\nabla u^m \cdot \nabla\psi - \chi u^\alpha \nabla v \cdot \nabla\psi - (\lambda u - \mu u^\kappa)\psi)\,dx, \tag{5.3.2}$$

$$\int_\Omega \nabla v \cdot \nabla\psi\,dx + \overline{M_\ell}(t)\int_\Omega \psi\,dx - \int_\Omega u^\ell\psi\,dx = 0 \tag{5.3.3}$$

for a.a. $t \in [0,T)$ with $u(0) = u_0$.

Proof. Let (u, v) satisfy (a). Then, (5.3.1) implies that for all $\varphi \in C^\infty_{\mathrm{c}}(\overline{\Omega} \times (0,T))$,

$$\begin{aligned}
&\left|\int_0^T \int_\Omega u\varphi_t\,dxdt\right| \\
&\quad\le \left|\int_0^T \int_\Omega (\nabla u^m - \chi u^\alpha \nabla v)\cdot\nabla\varphi\,dxdt\right| + \left|\int_0^T \int_\Omega (\lambda u - \mu u^\kappa)\varphi\,dxdt\right| \\
&\quad\le \Big[\|\nabla u^m\|_{L^2(0,T;L^2(\Omega))} + \chi\|u\|^\alpha_{L^\infty(0,T;L^\infty(\Omega))}\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + (\lambda\|u\|_{L^\infty(0,T;L^\infty(\Omega))} + \mu\|u\|^\kappa_{L^\infty(0,T;L^\infty(\Omega))})|\Omega|^{\frac12}T^{\frac12}\Big]\|\varphi\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

which implies that $u_t \in L^2(0,T;(H^1(\Omega))^\star)$. Also, choosing φ in (a) as $\varphi = \widetilde{\varphi}\cdot\psi$ with $\widetilde{\varphi} \in C^0_{\mathrm{c}}([0,T))$ and $\psi \in H^1(\Omega)$, we have

$$\int_0^T \left[\int_\Omega (\nabla u^m \cdot \nabla\psi - \chi u^\alpha \nabla v \cdot \nabla\psi - (\lambda u - \mu u^\kappa)\psi)\,dx\right]\widetilde{\varphi}\,dt$$
$$= \int_0^T \left[\int_\Omega u\psi\,dx\right]\widetilde{\varphi}_t\,dt + \int_\Omega u_0\psi\,dx \cdot \widetilde{\varphi}(0).$$

By taking $\widetilde{\varphi}$ with $\widetilde{\varphi}(0) = 0$, this yields (5.3.2). Moreover, from this identity and (5.3.2) we can confirm that $\int_\Omega u(0)\psi\,dx = \int_\Omega u_0\psi\,dx$ for all $\psi \in H^1(\Omega)$, which entails that $u(0) = u_0$. Similarly, (5.3.3) can be obtained. Thus (b) holds. Conversely, if (b) is satisfied, then for a.a. $t \in [0,T)$, $u_t = \Delta u^m - \chi\nabla\cdot(u^\alpha\nabla v) + \lambda u - \mu u^\kappa$ and $0 = \Delta v - \overline{M_\ell}(t) + u^\ell$ in $(H^1(\Omega))^\star$, and thereby from these identities together with (5.3.1) and $u_t \in L^2(0,T;(H^1(\Omega))^\star)$, we infer that (a) holds. □

We finally prove Theorem 5.1.1.

Proof of Theorem 5.1.1. Let $M_0 > 0$ and let $\eta_0 \in (0, M_0)$ and $r_\star \in (0,R)$ given by Proposition 5.2.8. We pick u_0 as in (5.1.4)–(5.1.6). Thanks to Proposition 5.2.1 and Definition 5.1.2, there is a maximal moment solution (u,v) of (5.1.1) on $[0, T_{\max})$. We first show that $T_{\max} < \infty$ by contradiction. To this end, we assume that $T_{\max} = \infty$. Then we have

$$\phi(t) - \phi(0) \ge K\int_0^t \phi^{\alpha+\ell}(\tau)\,d\tau \tag{5.3.4}$$

for all $t \in (0,\infty)$ with some $K > 0$, and put the function Φ as

$$\Phi(t) := \int_0^t \phi^{\alpha+\ell}(\tau)\,d\tau + \frac{\phi(0)}{K} \quad \text{for } t \in (0,\infty).$$

Also, we infer that ϕ is bounded on $[0,T')$ for all $T' < \infty$ and continuous on $[0,\infty)$ because u belongs to $L^\infty_{\mathrm{loc}}(0,\infty;L^\infty(\Omega))$ and $C^0_{\mathrm{w}-\star}([0,\infty);L^\infty(\Omega))$ due to (i) and (iii). Hence we note that $\Phi \in C^0([0,\infty)) \cap C^1((0,\infty))$. From (5.3.4) we obtain

$$\Phi'(t) \ge K^{\alpha+\ell}\Phi^{\alpha+\ell}(t) \quad \text{for all } t \in (0,\infty),$$

and thereby we can derive that

$$-\frac{1}{(\alpha+\ell-1)\Phi^{\alpha+\ell-1}(t)} + \frac{1}{(\alpha+\ell-1)\Phi^{\alpha+\ell-1}(0)} \ge K^{\alpha+\ell}t$$

for all $t \in (0,\infty)$. Thus it follows that $t \le \frac{1}{(\alpha+\ell-1)K^{\alpha+\ell}\Phi^{\alpha+\ell-1}(0)}$ for all $t \in (0,\infty)$, which is a contradiction. Therefore we see that $T_{\max} < \infty$.

Next, we prove that

$$\limsup_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} = \infty \tag{5.3.5}$$

by contradiction. To this end, we assume that $\limsup_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} < \infty$, that is, $u \in L^\infty(0, T_{\max}; L^\infty(\Omega))$. By this assumption and (i)–(iv) in Definition 5.1.1, it follows that (5.3.1) and (a) in Lemma 5.3.1 with $T = T_{\max}$ hold. Hence, noting from (b) in Lemma 5.3.1 that $u_t \in L^2(0, T_{\max}; (H^1(\Omega))^\star)$, we have

$$\|u(t) - u(s)\|_{(H^1(\Omega))^\star} \le \|u_t\|_{L^2(0,T_{\max};(H^1(\Omega))^\star)} |t - s|^{\frac{1}{2}}$$

for all $t, s \in [0, T_{\max})$, so that u is uniformly continuous on $[0, T_{\max})$ in $(H^1(\Omega))^\star$. This continuity provides $\widetilde{u}_{T_{\max}} \in (H^1(\Omega))^\star$ such that

$$\widetilde{u}_{T_{\max}} = \lim_{t \nearrow T_{\max}} u(t) \quad \text{in } (H^1(\Omega))^\star.$$

Moreover, the condition (iii) in Definition 5.1.1 with $T = T_{\max}$ guarantees that $\widetilde{u}_{T_{\max}}$ belongs to $L^\infty(\Omega)$. Indeed, by virtue of the condition (iii) in Definition 5.1.1 and the assumption $\limsup_{t \nearrow T_{\max}} \|u(t)\|_{L^\infty(\Omega)} < \infty$, we see that there exist $\{t_n\} \subset [0, T_{\max})$ and $g \in L^\infty(\Omega)$ such that $t_n \nearrow T_{\max}$ and $u(t_n) \to g$ weakly* in $L^\infty(\Omega)$ as $n \to \infty$. Since we observe that $u(t_n) \to \widetilde{u}_{T_{\max}}$ in $(H^1(\Omega))^\star$ as $n \to \infty$, it follows that $g = \widetilde{u}_{T_{\max}}$ in $(H^1(\Omega))^\star$. Noting that $L^\infty(\Omega) \subset L^2(\Omega) = (L^2(\Omega))^\star \subset (H^1(\Omega))^\star$, we arrive at the desired fact that $\widetilde{u}_{T_{\max}} \in L^\infty(\Omega)$. Choosing the initial data as $\widetilde{u}_{T_{\max}}$, by an argument similar to those in the proofs of Lemmas 5.2.3–5.2.7, we can find $T_1 > 0$ and construct a weak solution $(\widetilde{u}, \widetilde{v})$ on $[T_{\max}, T_{\max} + T_1)$. Now, we put

$$(\overline{u}, \overline{v}) := \begin{cases} (u, v) & \text{for a.a. } t \in [0, T_{\max}), \\ (\widetilde{u}, \widetilde{v}) & \text{for a.a. } t \in [T_{\max}, T_{\max} + T_1), \end{cases}$$

and confirm that $(\overline{u}, \overline{v})$ is a weak solution of (5.1.1) on $[0, T_{\max} + T_1)$. The definition of $\widetilde{u}_{T_{\max}}$ implies that $\int_\Omega u(t)\psi_0 \, dx \to \int_\Omega \widetilde{u}_{T_{\max}} \psi_0 \, dx$ as $t \nearrow T_{\max}$ for all $\psi_0 \in C_{\mathrm{c}}^\infty(\Omega)$, and $u \in L^\infty(0, T_{\max}; L^\infty(\Omega))$, and hence we see that $u \in C^0_{\mathrm{w}-\star}([0, T_{\max}]; L^\infty(\Omega))$. On the other hand, the condition corresponding to (iii) in Definition 5.1.1 tells us that $\widetilde{u} \in C^0_{\mathrm{w}-\star}([T_{\max}, T_{\max} + T_1); L^\infty(\Omega))$. Consequently, we deduce that

$$\overline{u} \in C^0_{\mathrm{w}-\star}([0, T_{\max} + T_1); L^\infty(\Omega)). \tag{5.3.6}$$

Recalling that $u_t \in L^2(0, T_{\max}; (H^1(\Omega))^\star)$ and $\widetilde{u}_t \in L^2([T_{\max}, T_{\max} + T_1); (H^1(\Omega))^\star)$ with $\widetilde{u}(T_{\max}) = \widetilde{u}_{T_{\max}}$, we can show that $\overline{u}_t \in L^2([0, T_{\max} + T_1); (H^1(\Omega))^\star)$. Indeed, it follows from (5.3.6) that for any $\varphi \in H^1([0, T_{\max} + T_1); H^1(\Omega))$,

$$\begin{aligned}
&- \int_0^{T_{\max}+T_1} \langle \overline{u}(t), \varphi_t(t) \rangle_{(H^1(\Omega))^\star, H^1(\Omega)} \, dt \\
&\quad = - \int_0^{T_{\max}} \int_\Omega u \varphi_t \, dxdt - \int_{T_{\max}}^{T_{\max}+T_1} \int_\Omega \widetilde{u} \varphi_t \, dxdt \\
&\quad = \int_\Omega \overline{u}(T_{\max}) \varphi(T_{\max}) \, dx + \int_0^{T_{\max}} \langle u_t(t), \varphi(t) \rangle_{(H^1(\Omega))^\star, H^1(\Omega)} \, dt \\
&\qquad - \int_\Omega \overline{u}(T_{\max}) \varphi(T_{\max}) \, dx + \int_{T_{\max}}^{T_{\max}+T_1} \langle \widetilde{u}_t(t), \varphi(t) \rangle_{(H^1(\Omega))^\star, H^1(\Omega)} \, dt \\
&\quad = \int_0^{T_{\max}+T_1} \langle g(t), \varphi(t) \rangle_{(H^1(\Omega))^\star, H^1(\Omega)} \, dt,
\end{aligned}$$

where

$$g := \begin{cases} u_t & \text{for a.a. } t \in [0, T_{\max}), \\ \widetilde{u}_t & \text{for a.a. } t \in [T_{\max}, T_{\max} + T_1), \end{cases}$$

which means that $\overline{u}_t = g \in L^2([0, T_{\max} + T_1); (H^1(\Omega))^\star)$. Moreover, since (u, v) and $(\widetilde{u}, \widetilde{v})$ satisfy (5.3.2), (5.3.3) for a.a. $t \in [0, T_{\max})$ and for a.a. $t \in [T_{\max}, T_{\max} + T_1)$, respectively, $(\overline{u}, \overline{v})$ fulfills (5.3.2), (5.3.3) for a.a. $t \in [0, T_{\max} + T_1)$, and hence, by means of Lemma 5.3.1, $(\overline{u}, \overline{v})$ is a weak solution of (5.1.1) on $[0, T_{\max} + T_1)$. We shall show that the weak solution $(\overline{u}, \overline{v})$ fulfills the moment inequality on $[0, T_{\max} + \sigma_1)$ with some $\sigma_1 > 0$. For this purpose, defining $\overline{w}$ and $\overline{\phi}$ as

$$\overline{w}(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} \overline{u}(\rho, t) \, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max} + T_1)$$

and

$$\overline{\phi}(t) := \int_0^{s_0} s^{-\gamma} (s_0 - s) \overline{w}(s, t) \, ds \quad \text{for } t \in [0, T_{\max} + T_1),$$

we have only to prove that there exists $\overline{K} > 0$ such that

$$\overline{\phi}(t) - \overline{\phi}(0) \geq \overline{K} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau) \, d\tau \quad \text{for all } t \in [0, T_{\max} + \sigma_1). \tag{5.3.7}$$

We know that

$$\phi(t) - \phi(0) \geq K \int_0^t \phi^{\alpha+\ell}(\tau) \, d\tau \quad \text{for all } t \in [0, T_{\max}). \tag{5.3.8}$$

In order to construct the moment inequality beyond $T_{\max}$ we make sure that

$$\overline{\phi}(T_{\max}) - \overline{\phi}(0) \geq K \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau) \, d\tau. \tag{5.3.9}$$

To this end, we confirm that $\overline{\phi} \in C^0([0, T_{\max} + T_1))$. Letting $t \to t_0 \in [0, T_{\max} + T_1)$ and noting from (5.3.6) that for any $s \in (0, R]$, $\overline{w}(s, \cdot)$ is continuous on $[0, T_{\max} + T_1)$ and $s^{-\gamma}(s_0 - s)\overline{w}(s, t) \le c_1 s^{-\gamma}(s_0 - s)$ with some $c_1 > 0$, we see from the Lebesgue dominated convergence theorem that $\overline{\phi}(t) \to \overline{\phi}(t_0)$, and so $\overline{\phi} \in C^0([0, T_{\max} + T_1))$. Thus the inequality (5.3.9) is derived by the passage to the limit in (5.3.8) as $t \nearrow T_{\max}$. Next, by setting

$$\varepsilon_K := \frac{K}{2} \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau > 0, \tag{5.3.10}$$

the continuity of $\overline{\phi}$ and $\int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau$ at $t = T_{\max}$ provides $\sigma_1 \in (0, T_1)$ such that for all $t \in [T_{\max}, T_{\max} + \sigma_1)$,

$$\left| \overline{\phi}(t) - \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau - \left(\overline{\phi}(T_{\max}) - \frac{K}{2} \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \right) \right| \le \varepsilon_K,$$

which together with (5.3.10) implies

$$\overline{\phi}(t) - \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \ge \overline{\phi}(T_{\max}) - K \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \ge \overline{\phi}(0)$$

for all $t \in [T_{\max}, T_{\max} + \sigma_1)$, that is,

$$\overline{\phi}(t) - \overline{\phi}(0) \ge \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [T_{\max}, T_{\max} + \sigma_1). \tag{5.3.11}$$

On the other hand, in light of (5.3.8), (u, v) satisfies that

$$\phi(t) - \phi(0) \ge \frac{K}{2} \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [0, T_{\max}).$$

Noting that $\overline{\phi} = \phi$ on $[0, T_{\max})$ and combining this inequality and (5.3.11), we obtain the moment inequality (5.3.7) on $[0, T_{\max} + \sigma_1)$ with $\overline{K} = \frac{K}{2}$, which contradicts the definition of maximal moment solutions. Therefore we conclude that the maximal moment solution (u, v) of (5.1.1) on $[0, T_{\max})$ satisfies (5.3.5). $\square$

References

[1] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.*, 25(9):1663–1763, 2015.

[2] T. Black, M. Fuest, and J. Lankeit. Relaxed parameter conditions for chemotactic collapse in logistic-type parabolic–elliptic Keller–Segel systems. *Z. Angew. Math. Phys.*, 72(9):Paper No. 96, 23 pp., 2021.

[3] X. Cao. Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with logistic source. *J. Math. Anal. Appl.*, 412(1):181–188, 2014.

[4] X. Cao. Global bounded solutions of the higher-dimensional Keller–Segel system under smallness conditions in optimal spaces. *Discrete Contin. Dyn. Syst.*, 35(5):1891–1904, 2015.

[5] Y. Chiyo, M. Marras, Y. Tanaka, and T. Yokota. Blow-up phenomena in a parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation. *Nonlinear Anal.*, 212:Paper No. 112550, 14 pp., 2021.

[6] Y. Chiyo, Y. Tanaka, A. Uchida, and T. Yokota. Global asymptotic stability of endemic equilibria for a diffusive SIR epidemic model with saturated incidence and logistic growth. *Discrete Contin. Dyn. Syst. Ser. B*, 28(3):2184–2210, 2023.

[7] T. Cieślak and C. Stinner. Finite-time blowup and global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions. *J. Differential Equations*, 252(10):5832–5851, 2012.

[8] T. Cieślak and C. Stinner. New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models. *J. Differential Equations*, 258(6):2080–2113, 2015.

[9] T. Cieślak and M. Winkler. Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity*, 21(5):1057–1076, 2008.

[10] S. Frassu and G. Viglialoro. Boundedness for a fully parabolic Keller–Segel model with sublinear segregation and superlinear aggregation. *Acta Appl. Math.*, 171:Paper No. 19, 20 pp., 2021.

[11] M. Fuest. Blow-up profiles in quasilinear fully parabolic Keller–Segel systems. *Nonlinearity*, 33(5):2306–2334, 2020.

[12] M. Fuest. Finite-time blow-up in a two-dimensional Keller–Segel system with an environmental dependent logistic source. *Nonlinear Anal. Real World Appl.*, 52:103022, 14 pp., 2020.

[13] M. Fuest. Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening. *NoDEA Nonlinear Differential Equations Appl.*, 28(2):Paper No. 16, 17 pp., 2021.

[14] M. Fuest, J. Lankeit, and Y. Tanaka. Critical mass phenomena in higher dimensional quasilinear Keller–Segel systems with indirect signal production. *preprint*.

[15] T. Hashira, S. Ishida, and T. Yokota. Finite-time blow-up for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type. *J. Differential Equations*, 264(10):6459–6485, 2018.

[16] T. Hillen and K.J. Painter. A user's guide to PDE models for chemotaxis. *J. Math. Biol.*, 58(1–2):183–217, 2009.

[17] D. Horstmann and G. Wang. Blow-up in a chemotaxis model without symmetry assumptions. *European J. Appl. Math.*, 12(2):159–177, 2001.

[18] S. Ishida, K. Seki, and T. Yokota. Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains. *J. Differential Equations*, 256(8):2993–3010, 2014.

[19] S. Ishida and T. Yokota. Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type. *J. Differential Equations*, 252(2):1421–1440, 2012.

[20] S. Ishida and T. Yokota. Blow-up in finite or infinite time for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type. *Discrete Contin. Dyn. Syst. Ser. B*, 18(10):2569–2596, 2013.

[21] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.*, 329(2):819–824, 1992.

[22] H. Jin and T. Xiang. Chemotaxis effect vs. logistic damping on boundedness in the 2-D minimal Keller–Segel model. *C. R. Math. Acad. Sci. Paris*, 356(8):875–885, 2018.

[23] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theoret. Biol.*, 26(3):399–415, 1970.

[24] J. Lankeit. Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion. *J. Differential Equations*, 262(7):4052–4084, 2017.

[25] J. Lankeit. Infinite time blow-up of many solutions to a general quasilinear parabolic–elliptic Keller–Segel system. *Discrete Contin. Dyn. Syst. Ser. S*, 13(2):233–255, 2020.

[26] J. Lankeit and M. Winkler. Facing low regularity in chemotaxis systems. *Jahresber. Dtsch. Math.-Ver.*, 122(1):35–64, 2020.

[27] Y. Li. Finite-time blow-up in quasilinear parabolic–elliptic chemotaxis system with nonlinear signal production. *J. Math. Anal. Appl.*, 480:123376, 18 pp., 2019.

[28] D. Liu and Y. Tao. Boundedness in a chemotaxis system with nonlinear signal production. *Appl. Math. J. Chinese Univ. Ser. B*, 31(4):379–388, 2016.

[29] M. Marras, T. Nishino, and G. Viglialoro. A refined criterion and lower bounds for the blow-up time in a parabolic–elliptic chemotaxis system with nonlinear diffusion. *Nonlinear Anal.*, 195:111725, 15 pp., 2020.

[30] M. Marras, S. Vernier Piro, and G. Viglialoro. Lower bounds for blow-up in a parabolic–parabolic Keller–Segel system. *Discrete Contin. Dyn. Syst.*, Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl.:809–816, 2015.

[31] M. Marras, S. Vernier Piro, and G. Viglialoro. Blow-up phenomena in chemotaxis systems with a source term. *Math. Methods Appl. Sci.*, 39(11):2787–2798, 2016.

[32] N. Mizoguchi and M. Winkler. Blow-up in the two-dimensional parabolic Keller–Segel system. *preprint*.

[33] M. Mizukami and Y. Tanaka. Finite-time blow-up in a two-species chemotaxis-competition model with single production. *preprint*.

[34] M. Mizukami, Y. Tanaka, and T. Yokota. Can chemotactic effects lead to blow-up or not in two-species chemotaxis-competition models? *Z. Angew. Math. Phys.*, 73(6):Paper No. 239, 2022.

[35] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.*, 5(2):581–601, 1995.

[36] T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40(3):411–433, 1997.

[37] T. Nishino and T. Yokota. Effect of nonlinear diffusion on a lower bound for the blow-up time in a fully parabolic chemotaxis system. *J. Math. Anal. Appl.*, 479(1):1078–1098, 2019.

[38] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlinear Anal.*, 51(1, Ser. A: Theory Methods):119–144, 2002.

[39] K. Osaki and A. Yagi. Finite dimensional attractor for one-dimensional Keller–Segel equations. *Funkcial. Ekvac.*, 44(3):441–469, 2001.

[40] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theoret. Biol.*, 79(1):83–99, 1979.

[41] Y. Sugiyama and H. Kunii. Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term. *J. Differential Equations*, 227(1):333–364, 2006.

[42] Z. Szymańska, C. Morales-Rodrigo, M. Lachowicz, and M. A. J. Chaplain. Mathematical modelling of cancer invasion of tissue: the role and effect of nonlocal interactions. *Math. Models Methods Appl. Sci.*, 19(2):257–281, 2009.

[43] Y. Tanaka. Existence of blow-up solutions for a degenerate parabolic–elliptic Keller–Segel system with logistic source. *preprint*.

[44] Y. Tanaka. Blow-up in a quasilinear parabolic–elliptic Keller–Segel system with logistic source. *Nonlinear Anal. Real World Appl.*, 63:Paper No. 103396, 29 pp., 2022.

[45] Y. Tanaka. Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production. *J. Math. Anal. Appl.*, 506:Paper No. 125654, 29 pp., 2022.

[46] Y. Tanaka, G. Viglialoro, and T. Yokota. Remarks on two connected papers about Keller–Segel systems with nonlinear production. *J. Math. Anal. Appl.*, 501(2):Paper No. 125188, 5 pp., 2021.

[47] Y. Tanaka and T. Yokota. Blow-up in a parabolic–elliptic Keller–Segel system with density-dependent sublinear sensitivity and logistic source. *Math. Methods Appl. Sci.*, 43(12):7372–7396, 2020.

[48] Y. Tanaka and T. Yokota. Finite-time blow-up in a quasilinear degenerate parabolic–elliptic chemotaxis system with logistic source and nonlinear production. *Discrete Contin. Dyn. Syst. Ser. B*, 28(1):262–286, 2023.

[49] X. Tao, S. Zhou, and M. Ding. Boundedness of solutions to a quasilinear parabolic–parabolic chemotaxis model with nonlinear signal production. *J. Math. Anal. Appl.*, 474(1):733–747, 2019.

[50] Y. Tao and M. Winkler. Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity. *J. Differential Equations*, 252(1):692–715, 2012.

[51] J.I. Tello and M. Winkler. A chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 32(4–6):849–877, 2007.

[52] G. Viglialoro. Private discussion about blow-up in a chemotaxis system suggested by Michael Winkler. 2019.

[53] W. Wang and Y. Li. Boundedness and finite-time blow-up in a chemotaxis system with nonlinear signal production. *Nonlinear Anal. Real World Appl.*, 59:Paper No. 103237, 21 pp., 2021.

[54] M. Winkler. Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. *J. Differential Equations*, 248(12):2889–2905, 2010.

[55] M. Winkler. Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source. *Comm. Partial Differential Equations*, 35(8):1516–1537, 2010.

[56] M. Winkler. Does a 'volume-filling effect' always prevent chemotactic collapse? *Math. Methods Appl. Sci.*, 33(1):12–24, 2010.

[57] M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Anal. Appl.*, 384(2):261–272, 2011.

[58] M. Winkler. Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system. *J. Math. Pures Appl. (9)*, 100(5):748–767, 2013.

[59] M. Winkler. A critical blow-up exponent in a chemotaxis system with nonlinear signal production. *Nonlinearity*, 31(5):2031–2056, 2018.

[60] M. Winkler. Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation. *Z. Angew. Math. Phys.*, 69(2):Paper No. 69, 40 pp., 2018.

[61] M. Winkler. Global classical solvability and generic infinite-time blow-up in quasilinear Keller–Segel systems with bounded sensitivities. *J. Differential Equations*, 266(12):8034–8066, 2019.

[62] M. Winkler. How unstable is spatial homogeneity in Keller–Segel systems? A new critical mass phenomenon in two- and higher-dimensional parabolic–elliptic cases. *Math. Ann.*, 373(3–4):1237–1282, 2019.

[63] M. Winkler and K.C. Djie. Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect. *Nonlinear Anal.*, 72(2):1044–1064, 2010.

[64] D.E. Woodward, R. Tyson, M.R. Myerscough, J.D. Murray, E.O. Budrene, and H.C. Berg. Spatio-temporal patterns generated by salmonella typhimurium. *Biophys. J.*, 68(5):2181–2189, 1995.

[65] H. Yi, C. Mu, G. Xu, and P. Dai. A blow-up result for the chemotaxis system with nonlinear signal production and logistic source. *Discrete Contin. Dyn. Syst. Ser. B*, 26(5):2537–2559, 2021.

[66] J. Zheng. Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source. *J. Differential Equations*, 259(1):120–140, 2015.

[67] J. Zheng. A note on boundedness of solutions to a higher-dimensional quasilinear chemotaxis system with logistic source. *ZAMM Z. Angew. Math. Mech.*, 97(4):414–421, 2017.

[68] P. Zheng, C. Mu, and X. Hu. Boundedness and blow-up for a chemotaxis system with generalized volume-filling effect and logistic source. *Discrete Contin. Dyn. Syst.*, 35(5):2299–2323, 2015.

List of original papers

The chapters of this thesis are based on the following papers.

Chapter 2:

⧫ [**47**] Y. Tanaka and T. Yokota. Blow-up in a parabolic–elliptic Keller–Segel system with density-dependent sublinear sensitivity and logistic source. *Math. Methods Appl. Sci.*, 43(12):7372–7396, 2020.

Chapter 3:

⧫ [**44**] Y. Tanaka. Blow-up in a quasilinear parabolic–elliptic Keller–Segel system with logistic source. *Nonlinear Anal. Real World Appl.*, 63:Paper No. 103396, 29 pp., 2022.

⧫ [**46**] Y. Tanaka, G. Viglialoro, and T. Yokota. Remarks on two connected papers about Keller–Segel systems with nonlinear production. *J. Math. Anal. Appl.*, 501:Paper No. 125188, 5 pp., 2021.

Chapter 4:

⧫ [**45**] Y. Tanaka. Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic chemotaxis system with logistic source and nonlinear production. *J. Math. Anal. Appl.*, 506:Paper No. 125654, 29 pp., 2022.

Chapter 5:

⧫ [**48**] Y. Tanaka, T. Yokota. Finite-time blow-up in a quasilinear degenerate parabolic–elliptic chemotaxis system with logistic source and nonlinear production. *Discrete Contin. Dyn. Syst. Ser. B*, 28(1):262–286, 2023.

YUYA TANAKA
Department of Mathematics
Graduate School of Science
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN
E-mail address: yuya.tns.6308@gmail.com