学位論文

Measures for Symmetry of Marginal Distribution in Contingency Table Analysis

(分割表解析における周辺分布に関する対称性を測る尺度)

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Chapter 1

Introduction

1.1 Background

Many fields use categorical data analysis. These include social science, medical, and pharmaceutical fields. Categorical data analysis such as the voting behavior or occupational choices of parents and their children often uses contingency tables to organize data. A categorical variable is a measurement scale consisting of a series of categories. For example, average temperatures consist of "below normal," "normal," and "above normal," while voting behaviors consist of "Conservative," "Labor," and "Liberal" for each voting party in the United Kingdom.

According to the measurement scale, categorical variables are distinguished into nominal and ordinal. An order cannot be assigned to nominal variables, while ordinal categories have a natural ordering relationship. Examples of nominal categories are the names of different regions, "Asia," "Europe," and "North America," and the status of cancer, "presence" and "absence." Examples of ordinal variables are responses to a questionnaire such as "too little," "about right," and "too much." Whether the contingency table involves nominal or ordinal categories influences the analysis method. Analysis methods for nominal categories are independent of the category arrangement, whereas methods for nominal categories depend on the category order. Therefore, an analysis method that suits the characteristics of each one is necessary.

First, in a two-way or multi-way contingency table, a general concern is whether row and column categories are independent. For example, given data on smoking and lung cancer, it is natural to be interested in whether smoking has a negative (or positive) effect on lung cancer or whether they are related. If two categories are independent, they are unassociated. For this reason, Goodman (1979b) called a model in which such independence holds the null association model. If independence does not hold for the row and column categories, the interest shifts to the degree of association between the row and column variables. For this reason, various measures have been proposed to measure the degree of association between a row and a column (e.g., Yule's coefficient, Cramér's coefficient, and Goodman and Kruskal's coefficient).

Now, consider a unique contingency table with the same row and column classifications. Such a contingency table is called a square contingency table. Table 1.1 (Upton, 1977) shows the change in voting parties in the United Kingdom between 1964 and 1970. The number of people whose voting party (or abstention) did not change between the two years is 63, 72, 25, and 5, in order from "Conservative," which includes about 77% of the respondents in the tabulation. Thus, a square contingency table tends to have many observations in or near the main diagonal cell. Consequently, independence analysis is almost useless for square contingency tables. For such data, symmetric (or asymmetric) structures other than the main diagonal cell are of interest (i.e., the probability structures when row and column values differ).

The concept of symmetry in square contingency tables was first introduced by McNemar (1947) in 2×2 tables. Bowker (1948) extended it to $R \times R$ tables. The marginal homogeneity model, which has looser restrictions, was proposed to address the severe restrictions of the symmetry model. In practice, contingency tables cannot be analyzed using only the symmetrical model. Therefore, models with characteristic changes to the symmetric cells have also been proposed. These include the conditional symmetry model (McCullagh, 1978), the diagonals-parameter symmetry model (Goodman, 1979a), and the linear diagonals-parameter symmetry model (Agresti, 1983a).

Table 1.2 shows the results of a panel survey of voters' party support in Erie County, Ohio, in 1940, which is the table of Bishop et al. (1975, p.270) with the $t-2$ time point as marginalized and reordering categories. In an election campaign, long-term voter support and candidates who can win the floating vote are important factors in determining the winner. In the present data, we are interested in whether the observations are symmetry with respect to the center point of the contingency table. While the symmetry model has a symmetry structure of probabilities with respect to the main diagonal of the contingency table, we can consider a model with a probability structure symmetric with respect to the center point of the contingency table. Wall and Lienert (1976) proposed such a model as the point symmetry model.

Agresti (2002) discussed model decomposition. Assuming that model M1 can be decomposed into M2 and M3, model M1 holds if and only if both M2 and M3 hold. Caussinus (1965) gave the decomposition of the symmetry model.

In contingency table analysis, that the above models are most likely not applicable. If a model does not hold, interest often shifts to the degree of departure from the model. Tomizawa (1994) proposed a measure for the degree of departure from the symmetry model using Kullback-Leibler information content and Pearson's chi-squared. Tomizawa et al. (1998) redefined the above measure using power divergence. Tomizawa (1995) and Tomizawa and Makii (2001) also devised measures of the departure from the marginal homogeneity model. These measures are defined as the weighted arithmetic mean, while Saigusa et al. (2016) expressed a measure by defining it as the weighted geometric mean. Among the structures with the largest departure from the symmetry model, two stand out: structures with all zeros in the upper triangle and structures with all zeros in the lower triangle. Tahata et al. (2009) proposed a measure to distinguish between these two. Researchers continue to develop additional models, decompositions, and measures.

This thesis aims to review previous works of symmetry on contingency tables and to propose new measures of the departure from the model associated with marginal probabilities. Various models and measures have been proposed, which are related to marginal probabilities. However, the generalization of the measure proposed by Iki and Tomizawa (2018) has not been considered. Additionally, the weighted harmonic mean type measures with respect to marginal homogeneity have not been treated. Therefore, we propose such measures in Chapters 2 and 3.

This thesis is organized as follows. The rest of Chapter 1 introduces existing models, decomposition theorems, and measures for contingency table analysis. Chapter 2, which is based on Saito et al. (2022a), proposes a measure of the departure from the marginal point symmetry model of a two-way contingency table and extends the measure to a multi-way table. Chapter 3, which is based on Saito et al. (2022b), details two measures for the local marginal homogeneity model. Chapter 4 presents the conclusions of this study.

1.2 Preliminaries

1.2.1 Joint and Marginal Probabilities

First, we present the probabilities for $R \times C$ contingency tables. Let X_1 and X_2 denote the row and column variables, respectively, and let $p_{ij} = \Pr(X_1 = i,\ X_2 = j)$ denote the probability that an observation will fall in the (i, j)th cell of the table $(i = 1, \ldots, R;\ j = 1, \ldots, C)$. Probability $\{p_{ij}\}$ is the joint distribution X_1 and X_2. These satisfy $\sum_{i=1}^{R} \sum_{j=i}^{C} p_{ij} = 1$. Let $p_{i\cdot} = \sum_{t=1}^{C} p_{it} = \Pr(X_1 = i)$ and $p_{\cdot j} = \sum_{t=1}^{R} p_{tj} = \Pr(X_2 = j)$. Then $\{p_{i\cdot}\}$ is the marginal distribution of X_1 and $\{p_{\cdot j}\}$ is the marginal distribution of X_2.

Second, we show the probabilities for multi-way contingency tables. For the

$R_1 \times \cdots \times R_k$ contingency table ($k \geq 2$), let $\boldsymbol{i} = (i_1, \ldots, i_k)$ for $i_t = 1, \ldots, R_t$ ($t = 1, \ldots, k$), and let $p_{\boldsymbol{i}}$ denote the probability that an observation will fall in the $\boldsymbol{i}$th cell of the table. Let X_t ($t = 1, \ldots, k$) denote the tth variable. The hth-order ($h = 1, \ldots, k-1$) marginal probability $\Pr(X_{s_1} = i_1, \ldots, X_{s_h} = i_h)$ is denoted as $p_{\boldsymbol{i}}^{\boldsymbol{s}}$, where $\boldsymbol{s} = (s_1, \ldots, s_h)$ and $\boldsymbol{i} = (i_1, \ldots, i_h)$ with $1 \leq s_1 < \cdots < s_h \leq k$ and $i_t = 1, \ldots, R_{s_t} (t = 1, \ldots, h)$. For example, consider an $R_1 \times R_2 \times R_3$ contingency table, $p_{\boldsymbol{i}} = \Pr(X_1 = i_1, X_2 = i_2, X_3 = i_3)$ ($\boldsymbol{i} = (i_1, i_2, i_3)$). The second-order marginal probability $p_{(t_1,t_2)}^{(1,3)} = \sum_{i_2} p_{\boldsymbol{i}} = \Pr(X_1 = t_1, X_3 = t_2)$, and the first-order marginal probability $p_{t_1}^{[2]} = \sum_{i_1} \sum_{i_3} p_{\boldsymbol{i}} = \Pr(X_2 = t_1)$.

1.2.2 Various Symmetry Models

(a) $R \times R$ tables

Consider $R \times R$ contingency tables with the same row and column classifications. The symmetry (S) model (Bowker, 1948) is defined as

$$p_{ij} = p_{ji} \text{ for all } (i,j) \quad (i = 1, \ldots, R;\ j = 1, \ldots, R),$$

(see also Agresti, 2002; Bishop et al., 1975; Kateri, 2014). The S model indicates that the probability of an observation falling in the (i,j)th cell is equal to the probability of the observation falling in the (j,i)th cell. That is, this model has a symmetric structure of $\{p_{ij}\}$ with respect to the main diagonal of the table. The cumulative probability is defined as

$$C_{ij} = \begin{cases} \Pr(X_1 \leq i, X_2 \geq j) = \displaystyle\sum_{s=1}^{i} \sum_{t=j}^{R} p_{st} & \text{when } i < j, \\ \Pr(X_1 \geq i, X_2 \leq j) = \displaystyle\sum_{s=i}^{R} \sum_{t=1}^{j} p_{st} & \text{when } i > j. \end{cases}$$

Then the S model can also be expressed as

$$C_{ij} = C_{ji} \text{ for all } (i,j) \quad (i = 1, \ldots, R;\ j = 1, \ldots, R;\ i \neq j).$$

The marginal homogeneity (MH) model (Stuart, 1955) is given by

$$p_{i\cdot} = p_{\cdot i} \text{ for all } i \quad (i = 1, \ldots, R).$$

See also Bishop et al. (1975). The MH model indicates that the row marginal distribution and the column marginal distribution are identical. Note that the S model is more restrictive than the MH model. The cumulative probability is defined as

$$G_{1(i)} = \Pr(X_1 \leq i, X_2 \geq i+1) = \sum_{s=1}^{i}\sum_{t=i+1}^{R} p_{st},$$

$$G_{2(i)} = \Pr(X_1 \geq i+1, X_2 \leq i) = \sum_{s=i+1}^{R}\sum_{t=1}^{i} p_{st}.$$

The MH model can also be expressed as

$$G_{1(i)} = G_{2(i)} \text{ for all } i \quad (i = 1, \dots, R-1).$$

Caussinus (1965) proposed the quasi-symmetry (QS) model, which is defined as

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \quad (i = 1, \dots, R;\ j = 1, \dots, R),$$

where $\{\psi_{ij} = \psi_{ji}\}$. A special case of the QS model with $\{\alpha_i = \beta_i\}$ is the S model. Let the odds ratio for rows i and j $(> i)$ and columns s and t $(> s)$ be noted by $\theta_{i<j;s<t}$. Namely, $\theta_{i<j;s<t} = (p_{is}p_{jt})/(p_{it}p_{js})$. Using the odds ratio, the QS model can also be expressed as

$$\theta_{i<j;s<t} = \theta_{s<t;i<j} \quad (i < j;\ s < t).$$

Thus, the QS model indicates the symmetry structure with respect to the odds ratio. Caussinus (1965) gave the following for the decomposition of the S model.

Theorem 1.1. *The S model holds if and only if both the QS and MH models hold.*

The S model rarely holds in contingency table analysis because it is highly constrained. Therefore, when the S model does not hold, it can be decomposed into multiple models with looser constraints. It is possible to determine which structures do not hold based on the fit of the decomposed models. In Theorem 1.1, for example, if the QS and S models fit the data poorly and the MH model fits the data well, the S model does not hold due to the failure of the QS model.

The partial symmetry model (Saigusa et al., 2016) is defined as

$$p_{ij} = p_{ji} \text{ for at least one } (i, j) \quad (i = 1, \dots, R;\ j = 1, \dots, R;\ i \neq j).$$

The partial symmetry model indicates that the cell probability that an observation falls in the (i, j)th cell $(i < j)$ is equal to the probability that the observation falls in the (j, i)th cell for at least one (i, j). Note that the S model implies the partial symmetry model.

The local symmetry model (Saigusa et al., 2019b) is defined as

$$p_{ij} = p_{ji} \text{ for only one } (i, j) \quad (i = 1, \dots, R;\ j = 1, \dots, R;\ i \neq j).$$

The local symmetry model indicates that the cell probability that an observation falls in the (i,j)th cell $(i<j)$ is equal to the probability that the observation falls in the (j,i)th cell for only one (i,j).

The cumulative partial symmetry (CPS) model (Saigusa et al., 2019a) is defined as

$$C_{ij}=C_{ji} \text{ for at least one } (i,j) \quad (i=1,\ldots,R;\ j=1,\ldots,R;\ i\neq j).$$

Note that the S model implies the CPS model.

The cumulative local symmetry (CLS) model (Saigusa et al., 2020b) is defined as

$$C_{ij}=C_{ji} \text{ for only one } (i,j) \quad (i=1,\ldots,R;\ j=1,\ldots,R;\ i\neq j).$$

The partial marginal homogeneity (PMH) model (Saigusa et al., 2020a) is defined as

$$p_{i\cdot}=p_{\cdot i} \text{ for at least one } i \quad (i=1,\ldots,R).$$

This model indicates the homogeneity for one or more pairs of marginal probabilities. Note that the MH model implies the PMH model.

The cumulative partial marginal homogeneity (CPMH) model (Nakagawa et al., 2020) is defined as

$$G_{1(i)}=G_{2(i)} \text{ for at least one } i \quad (i=1,\ldots,R-1).$$

It should be noted that the structure of the CPMH model differs from those of the MH and PMH models. It is easy to see that the MH model implies the CPMH model.

The point-symmetry (PS) model is defined as

$$p_{ij}=p_{i^*j^*} \quad (i=1,\ldots,R;\ j=1,\ldots,R),$$

where $i^*=R+1-i$ (Tomizawa, 1985; Wall and Lienert, 1976). This model states that the probability that an observation falls in the (i,j)th cell is equal to the probability that it falls in a point symmetric (i^*,j^*)th cell with respect to the center point (or cell).

The marginal point-symmetry (MPS) model is defined as

$$p_{i\cdot}=p_{i^*\cdot} \text{ and } p_{\cdot j}=p_{\cdot j^*} \quad (i=1,\ldots,R;\ j=1,\ldots,R).$$

This indicates that the row (column) marginal distribution is point symmetric with respect to the midpoint of the row (column) categories.

Tomizawa (1985) defined the quasi point-symmetry (QPS) model as

$$p_{ij}=\mu\alpha_i\beta_j\psi_{ij} \quad (i=1,\ldots,R;\ j=1,\ldots,R),$$

where $\psi_{ij} = \psi_{i^*j^*}$. The PS model is a special case of the QPS model, which is obtained by putting $\{\alpha_i = \alpha_{i^*}\}$ and $\{\beta_j = \beta_{j^*}\}$. Using the odds ratios, the QPS model can also be expressed as

$$\theta_{i<j;s<t} = \theta_{j^*<i^*;t^*<s^*} \quad (i < j;\ s < t).$$

Note that the QPS model is unique in that the odds ratio is point symmetric.

Tomizawa (1985) gave the decomposition of the PS model as:

Theorem 1.2. *The PS model holds if and only if both QPS and MPS models hold.*

Consider an ordinal square contingency table. When the symmetric structure does not hold, McCullagh (1978) defined the conditional symmetry (CS) model as

$$p_{ij} = \gamma p_{ji} \quad (1 \leq i < j \leq R).$$

The CS model indicates that the probability that an observation falls in (i, j)th cell for $i < j$ is γ times higher than the probability that the observation falls in (j, i)th cell. Namely, the ratios of symmetric cell probabilities (i.e., $\{p_{ij}/p_{ji}\}$) depend on one parameter γ. The CS model with $\gamma = 1$ is the S model.

Goodman (1979a) defined the diagonals-parameter symmetry (DPS) model as

$$p_{ij} = \delta_{j-i} p_{ji} \quad (1 \leq i < j \leq R).$$

The DPS model indicates that the ratios of the symmetric cell probabilities depend on only the distance $j - i$ from the main diagonal. When $\{\delta_{j-i} = 1\}$, this model becomes the S model, but when $\{\delta_{j-i} = \gamma\}$, it becomes the CS model.

Agresti (1983a) defined the linear diagonals-parameter symmetry (LDPS) model as

$$p_{ij} = \eta^{j-i} p_{ji} \quad (1 \leq i < j \leq R).$$

The LDPS model indicates that the log ratios of symmetric probabilities can be expressed as a linear function of the distance from the main diagonal. The LDPS model has only one more parameter than the S model, and $\eta = 1$ is a special case equal to the S model. The LDPS model is a special case of the DPS model in which $\{\delta_{j-i} = \eta^{j-i}\}$.

Tomizawa (1990c) proposed the polynomial diagonals-parameter symmetry (PDPS) model which includes the S, CS, and LDPS models in special cases. Additionally, Tomizawa (1987, 1990a,b, 1991) gave several decompositions related to the PDPS model. Recently, Kateri and Papaioannou (1997), Kateri and Agresti (2007), and Tahata (2020) proposed various symmetry models based on the f-divergence (Csiszár and Shields, 2004). The f-divergence includes the Kullback-Leibler divergence and the power divergence (Read and Cressie, 1988).

(b) R^k tables

Consider R^k contingency tables ($k \geq 2$) with the same classifications. The complete symmetry (S^k) model is defined as

$$p_{\boldsymbol{i}} = p_{\boldsymbol{j}}$$

for any permutation $\boldsymbol{j} = (j_1, \ldots, j_k)$ of $\boldsymbol{i} = (i_1, \ldots, i_k)$. See Bhapkar and Darroch (1990) and Lovison (2000). The S^k model can be expressed in the log-linear form as

$$\log p_{\boldsymbol{i}} = \lambda_{(\boldsymbol{i})},$$

where $\lambda_{(\boldsymbol{i})} = \lambda_{(\boldsymbol{j})}$ for any permutation $\boldsymbol{j} = (j_1, \ldots, j_k)$ of $\boldsymbol{i} = (i_1, \ldots, i_k)$.

Bhapkar and Darroch (1990) defined the hth-order ($h = 1, \ldots, k-1$) quasi-symmetry (QS_h^k) model, which expressed as

$$\log p_{\boldsymbol{i}} = \lambda + \sum_{t=1}^{k} \lambda_{t(i_t)} + \sum\sum_{1 \leq t_1 < t_2 \leq k} \lambda_{t_1 t_2 (i_{t_1}, i_{t_2})} + \cdots + \sum \cdots \sum_{1 \leq t_1 < \cdots < t_h \leq k} \lambda_{t_1 \ldots t_h (i_{t_1}, \ldots, i_{t_h})} + \lambda_{(\boldsymbol{i})},$$

where $\lambda_{(\boldsymbol{i})} = \lambda_{(\boldsymbol{j})}$ for any permutation $\boldsymbol{j} = (j_1, \ldots, j_k)$ of $\boldsymbol{i} = (i_1, \ldots, i_k)$.

For a fixed h ($h = 1, \ldots, k-1$), the hth-order marginal symmetry (M_h^k) model is defined as

$$p_{\boldsymbol{i}}^{\boldsymbol{s}} = p_{\boldsymbol{j}}^{\boldsymbol{s}} = p_{\boldsymbol{i}}^{\boldsymbol{t}},$$

for any permutation $\boldsymbol{j} = (j_1, \ldots, j_h)$ of $\boldsymbol{i} = (i_1, \ldots, i_h)$ and for any $\boldsymbol{s} = (s_1, \ldots, s_h)$ and $\boldsymbol{t} = (t_1, \ldots, t_h)$ with $1 \leq t_1 \leq \cdots \leq t_h \leq k$ and $i_t = 1, \ldots, R$ ($t = 1, \ldots, k$) (Bhapkar and Darroch, 1990; Tomizawa and Tahata, 2007). For the case of $h = 1$, the M_1^k model is expressed as

$$p_i^{[1]} = \cdots = p_i^{[k]} \quad (i = 1, \ldots, R),$$

where $p_i^{[t]} = \Pr(X_t = i)$. This model indicates the homogeneity structure of the 1st-order marginal distribution.

Bhapkar and Darroch (1990) extended Theorem 1.1 to multi-way contingency tables as follows:

Theorem 1.3. *For an R^k table and fixed h ($h = 1, \ldots, k-1$), the S^k model holds if and only if both the QS_h^k and M_h^k models hold.*

The point-symmetry (PS^k) model is defined as

$$p_{\boldsymbol{i}} = p_{\boldsymbol{i}^*} \text{ for any } \boldsymbol{i}$$

where $\boldsymbol{i}^* = (i_1^*, \ldots, i_k^*)$ for $i_t^* = R + 1 - i_t$ ($t = 1, \ldots, k$) (Wall and Lienert, 1976).

For fixed h ($h = 1, \ldots, k-1$), Tahata and Tomizawa (2008) defined the hth-order quasi point-symmetry (QP_h^k) model as

$$\begin{aligned}\log p_{\boldsymbol{i}} = u &+ \sum_{t=1}^{k} u_{t(i_t)} + \sum\sum_{1\le t_1<t_2\le k} u_{t_1t_2(i_{t_1},i_{t_2})} \\ &+ \cdots + \sum_{1\le t_1<\cdots<t_{k-1}\le k}\cdots\sum u_{t_1\ldots t_{k-1}(i_{t_1},\ldots,i_{t_{k-1}})} + u_{12\ldots k(\boldsymbol{i})} \text{ for any } \boldsymbol{i},\end{aligned}$$

where $u_{t_1\ldots t_l(i_{t_1},\ldots,i_{t_l})} = u_{t_1\ldots t_l(i^*_{t_1},\ldots,i^*_{t_l})}$ ($l = h+1, \ldots, k;\ 1 \le t_1 < \cdots < t_l \le k$).

For fixed h ($h = 1, \ldots, k-1$), Tahata and Tomizawa (2008) also defined the hth-order marginal point-symmetry (MP_h^k) model as

$$p^{\boldsymbol{s}}_{\boldsymbol{i}} = p^{\boldsymbol{s}}_{\boldsymbol{i}^*} \text{ for any } \boldsymbol{s} = (s_1, \ldots, s_h),$$

where $\boldsymbol{i} = (i_1, \ldots, i_h)$ and $\boldsymbol{i}^* = (i^*_1, \ldots, i^*_h)$.

Tahata and Tomizawa (2008) extended Theorem 1.2 to multi-way contingency tables as follows:

Theorem 1.4. *For an R^k table and fixed h ($h = 1, \ldots, k-1$), the PS^k model holds if and only if both the QP_h^k and MP_h^k models hold.*

1.2.3 Measures of Departure from Models

In data analysis, when a model does not hold, it is interesting to measure the distribution of $\{p_{ij}\}$ degree of departure from the model. On the one hand, the goodness-of-fit test can be used to determine whether the distribution of $\{p_{ij}\}$ is adapted to the model. On the other hand, the test statistic cannot be used to compare the distance from the model across contingency tables since the test statistic depends on the dimension R (e.g., the number of categories) and sample size. Consequently, various measures have been proposed for the degree of departure from the model.

For square contingency tables with nominal categories, Tomizawa (1994) considered two kinds of measures (ϕ_S and ψ_S) to represent the departure from symmetry. Assuming that $\{p_{ij}+p_{ji} > 0\}$, $i \ne j$, the measures ϕ_S and ψ_S are expressed as

$$\phi_S = \frac{1}{\log 2}\sum_{i=1}^{R}\sum_{\substack{j=1\\j\ne i}}^{R} p^*_{ij}\log\left(\frac{p^*_{ij}}{p^s_{ij}}\right),$$

$$\psi_S = \sum_{i=1}^{R}\sum_{\substack{j=1\\j\ne i}}^{R} \frac{(p^*_{ij}-p^s_{ij})^2}{p^s_{ij}},$$

where

$$p_{ij}^* = \frac{p_{ij}}{\delta}, \quad p_{ij}^s = \frac{p_{ij}^* + p_{ji}^*}{2}, \quad \delta = \sum_{i=1}^{R} \sum_{\substack{j=1 \\ j \neq i}}^{R} p_{ij}.$$

Note that ϕ_S is $1/\log 2$ times Kullback-Leibler information, while ψ_S is Pearson's chi-squared type discrepancy. Note that (i) the measures lie between 0 and 1, (ii) the measures equal 0 if and only if the S model holds, and (iii) the measures equal 1 if and only if the degree of departure from symmetry is the largest in a sense (i.e., complete asymmetry) that $p_{ij} = 0$ $(p_{ji} > 0)$ or $p_{ji} = 0$ $(p_{ij} > 0)$ for $i = 1, \ldots, R;\ j = 1, \ldots, R;\ i \neq j$.

Tomizawa et al. (1998) defined a generalization of these measures as

$$\Phi_S^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)}(\{p_{ij}^*\}; \{p_{ij}^s\}) \text{ for } \lambda > -1$$

where

$$I^{(\lambda)}(\{p_{ij}^*\}; \{p_{ij}^s\}) = \frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{R} \sum_{\substack{j=1 \\ j \neq i}}^{R} p_{ij}^* \left\{ \left(\frac{p_{ij}^*}{p_{ij}^s} \right)^\lambda - 1 \right\}.$$

The value at $\lambda = 0$ is the continuous limit as $\lambda \to 0$, where λ is a real number selected by the user. Note that $\Phi_S^{(0)}$ and $\Phi_S^{(1)}$ are the same as ϕ_S and ψ_S, respectively. Indeed, $I^{(\lambda)}(\{p_{ij}^*\}; \{p_{ij}^s\})$ is the power divergence between the two conditional distributions $\{p_{ij}^*\}$ and $\{p_{ij}^s\}$.

Let $p_{ij}^c = p_{ij}/(p_{ij} + p_{ji})$ for $i = 1, \ldots, R;\ j = 1, \ldots, R;\ i \neq j$. Note that p_{ij}^c indicates the conditional probability that an observation falls in the (i, j)th cell, for the condition that the observation will fall in the (i, j)th or (j, i)th cell of the $R \times R$ table, and $p_{ij}^c = 1/2$ for all i and j if and only if the S model holds. Then, $\Phi_S^{(\lambda)}$ can be expressed as

$$\Phi_S^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} \sum\sum_{i<j} (p_{ij}^* + p_{ji}^*) I_{ij}^{(\lambda)} \left(\{p_{ij}^c, p_{ji}^c\}; \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right),$$

where

$$I_{ij}^{(\lambda)} \left(\{p_{ij}^c, p_{ji}^c\}; \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right) = \frac{1}{\lambda(\lambda+1)} \left\{ p_{ij}^c \left[\left(\frac{p_{ij}^c}{1/2} \right)^\lambda - 1 \right] + p_{ji}^c \left[\left(\frac{p_{ji}^c}{1/2} \right)^\lambda - 1 \right] \right\}.$$

Moreover, $\Phi_S^{(\lambda)}$ can be expressed as

$$\Phi_S^{(\lambda)} = \sum\sum_{i<j} (p_{ij}^* + p_{ji}^*) \phi_{ij}^{(\lambda)},$$

where

$$\phi_{ij}^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H_{ij}^{(\lambda)}(\{p_{ij}^c, p_{ji}^c\}),$$

$$H_{ij}^{(\lambda)}(\{p_{ij}^c, p_{ji}^c\}) = \frac{1}{\lambda} \left\{ 1 - (p_{ij}^c)^{\lambda+1} - (p_{ji}^c)^{\lambda+1} \right\}.$$

Additionally, note that $H_{ij}^{(\lambda)}(\{p_{ij}^c, p_{ji}^c\})$ is Patil and Taillie (1982)'s diversity index of degree λ for the conditional distribution $\{p_{ij}^c, p_{ji}^c\}$. In special cases, this index includes the Shannon entropy (when $\lambda = 0$) and the Gini concentration or the Simpson index (when $\lambda = 1$). The measure is expressed as the weighted *arithmetic* mean of $\{\phi_{ij}^{(\lambda)}\}$.

Saigusa et al. (2016) proposed a useful measure to determine the degree of departure from the partial symmetry model. Assuming that $\{p_{ij} + p_{ji} > 0\}$ for $i \neq j$, the measure is expressed as the weighted *geometric* mean of $\{\phi_{ij}^{(\lambda)}\}$, which is given as

$$\Phi_{S(G)}^{(\lambda)} = \prod_{i<j}\prod \left\{\phi_{ij}^{(\lambda)}\right\}^{(p_{ij}^* + p_{ji}^*)} \quad \text{for } \lambda > -1,$$

and the value at $\lambda = 0$ is the continuous limit as $\lambda \to 0$. Note that (i) the measure lies between 0 and 1, (ii) the measure equals 0 if and only if the partial symmetry model holds, and (iii) the measure equals 1 if and only if the degree of departure from partial symmetry is the largest.

Saigusa et al. (2019b) reported a measure to determine the degree of departure from the local symmetry model. Assume that $\{p_{ij} + p_{ji} \neq 0\}$ for $i < j$, and $p_{kl} \neq p_{lk}$ for any $k < l$ except $(k, l) = (a, b)$ with only one (a, b), $a < b$. Then the measure is expressed as the weighted *harmonic* mean of $\{\phi_{ij}^{(\lambda)}\}$, which is given as

$$\Phi_{S(H)}^{(\lambda)} = \frac{\prod_{i<j}\prod \phi_{ij}^{(\lambda)}}{\sum_{i<j}\sum \left\{ (p_{ij}^* + p_{ji}^*) \prod_{\substack{s<t \\ (s,t)\neq(i,j)}}\prod \phi_{st}^{(\lambda)} \right\}} \quad \text{for } \lambda > -1,$$

and the value at $\lambda = 0$ is the continuous limit as $\lambda \to 0$. Note that (i) the measure lies between 0 and 1, (ii) the measure equals 0 if and only if the local symmetry model holds, and (iii) the measure equals 1 if and only if the degree of departure from local symmetry is the largest. If all $\{\phi_{ij}^{(\lambda)}\}$ are not equal to 0, then the measure can be written as

$$\Phi_{S(H)}^{(\lambda)} = \frac{1}{\sum_{i<j}\sum \frac{p_{ij}^* + p_{ji}^*}{\phi_{ij}^{(\lambda)}}}.$$

For square contingency tables with ordered categories, Tomizawa et al. (2001) proposed a measure that represents the degree of departure from symmetry. Assuming $\{C_{ij}^* + C_{ji}^* \neq 0\}$ for $i \neq j$,

$$E_S^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)}(\{C_{ij}^*\}; \{C_{ij}^s\}) \text{ for } \lambda > -1$$

where

$$I^{(\lambda)}(\{C^*_{ij}\};\{C^s_{ij}\}) = \frac{1}{\lambda(\lambda+1)}\sum_{i=1}^{R}\sum_{\substack{j=1\\j\neq i}}^{R} C^*_{ij}\left\{\left(\frac{C^*_{ij}}{C^s_{ij}}\right)^{\lambda} - 1\right\},$$

and

$$C^*_{ij} = \frac{C_{ij}}{\Delta}, \quad C^s_{ij} = \frac{C^*_{ij} + C^*_{ji}}{2}, \quad \Delta = \sum_{i=1}^{R}\sum_{\substack{j=1\\j\neq i}}^{R} C_{ij},$$

and the value at $\lambda = 0$ is the continuous limit as $\lambda \to 0$. Note that (i) the measure lies between 0 and 1, (ii) the measure equals 0 if and only if the S model holds, and (iii) the measure equals 1 if and only if the degree of departure from symmetry is the largest in the sense that $C_{ij} = 0$ $(C_{ji} > 0)$ or $C_{ji} = 0$ $(C_{ij} > 0)$ for all $i < j$. Additionally, (iv) the value of $\Phi^{(\lambda)}_S$ is invariant to row and column reordering, but the value of $E^{(\lambda)}_S$ depends on the order of the rows and columns.

Tahata et al. (2009) proposed a measure that can distinguish two kinds of complete asymmetries. Assuming that $\{p_{ij} + p_{ji} \neq 0\}$, the measure φ is expressed as

$$\varphi = \frac{4}{\pi}\sum\sum_{i<j}(p^*_{ij} + p^*_{ji})\left(\theta_{ij} - \frac{\pi}{4}\right),$$

where

$$\theta_{ij} = \arccos\left(\frac{p_{ij}}{\sqrt{p^2_{ij} + p^2_{ji}}}\right).$$

Note that (i) the measure lies between -1 and 1, (ii) the measure equals -1 if and only if $p_{ji} = 0$ $(p_{ij} > 0)$ for all $i < j$, and (iii) the measure equals 1 if and only if $p_{ij} = 0$ $(p_{ji} > 0)$ for all $i < j$. Additionally, the measure equals 0 if the average of $\theta_{ij} - \pi/4$ for $i < j$ equals 0 when that an observation falls in one of the off-diagonal cells of the table. Therefore, we refer to this structure as the average symmetry when the measure equals 0. Note that if the symmetry model holds, then the average symmetry holds. However, the converse does not hold.

1.3 Review of Previous Studies

This section reviews previous studies related to Chapters 2 and 3.

1.3.1 Measures for MPS

Consider an $R \times C$ contingency table. Tomizawa (1985) defined the MPS model as

$$p_{i\cdot} = p_{i^*\cdot} \text{ and } p_{\cdot j} = p_{\cdot j^{**}} \quad (i = 1, \ldots, R;\ j = 1, \ldots, C),$$

where $i^* = R + 1 - i$ and $j^{**} = C + 1 - j$. Let

$$\delta_1 = \sum_{i=1}^{[\frac{R}{2}]} (p_{i\cdot} + p_{i^*\cdot}), \quad \delta_2 = \sum_{j=1}^{[\frac{C}{2}]} (p_{\cdot j} + p_{\cdot j^{**}}).$$

Additionally, let

$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1}, \quad q^c_{i\cdot} = \frac{q_{i\cdot}}{q_{i\cdot} + q_{i^*\cdot}}, \quad q^c_{i^*\cdot} = \frac{q_{i^*\cdot}}{q_{i\cdot} + q_{i^*\cdot}}, \left(i = 1, \dots, \left[\frac{R}{2}\right]\right),$$

and

$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2}, \quad q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2}, \quad q^c_{\cdot j} = \frac{q_{\cdot j}}{q_{\cdot j} + q_{\cdot j^{**}}}, \quad q^c_{\cdot j^{**}} = \frac{q_{\cdot j^{**}}}{q_{\cdot j} + q_{\cdot j^{**}}}, \left(j = 1, \dots, \left[\frac{C}{2}\right]\right).$$

Assuming $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$, Yamamoto et al. (2011) defined a measure of the degree of departure from MPS for a two-way table as

$$\psi^{(\lambda)}_{MPS} = \frac{\delta_1 \psi^{(\lambda)}_1 + \delta_2 \psi^{(\lambda)}_2}{\delta_1 + \delta_2} \text{ for } \lambda > -1,$$

where

$$\psi^{(\lambda)}_1 = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} \sum_{i=1}^{[\frac{R}{2}]} (q_{i\cdot} + q_{i^*\cdot})\, H^{(\lambda)}_{1i} \left(\{q^c_{i\cdot}, q^c_{i^*\cdot}\}\right)$$

with

$$H^{(\lambda)}_{1i} \left(\{q^c_{i\cdot}, q^c_{i^*\cdot}\}\right) = \frac{1}{\lambda} \left\{1 - (q^c_{i\cdot})^{\lambda+1} - (q^c_{i^*\cdot})^{\lambda+1}\right\},$$

and

$$\psi^{(\lambda)}_2 = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} \sum_{j=1}^{[\frac{C}{2}]} (q_{\cdot j} + q_{\cdot j^{**}})\, H^{(\lambda)}_{2j} \left(\{q^c_{\cdot j}, q^c_{\cdot j^{**}}\}\right)$$

with

$$H^{(\lambda)}_{2j} \left(\{q^c_{\cdot j}, q^c_{\cdot j^{**}}\}\right) = \frac{1}{\lambda} \left\{1 - \left(q^c_{\cdot j}\right)^{\lambda+1} - \left(q^c_{\cdot j^{**}}\right)^{\lambda+1}\right\}$$

The submeasures $\psi^{(\lambda)}_1$ and $\psi^{(\lambda)}_2$ must lie between 0 and 1. Therefore, $\psi^{(\lambda)}_{MPS}$ must lie between 0 and 1. Note that (i) $\psi^{(\lambda)}_{MPS}$ equals 0 (i.e., $\psi^{(\lambda)}_1 = \psi^{(\lambda)}_2 = 0$) if and only if the MPS model holds, and (ii) $\psi^{(\lambda)}_{MPS}$ equals 1 (i.e., $\psi^{(\lambda)}_1 = \psi^{(\lambda)}_2 = 1$) if and only if the degree of departure from MPS is the largest in the sense that $q^c_{i\cdot} = 0$ (then $q^c_{i^*\cdot} = 1$) or $q^c_{i^*\cdot} = 0$ (then $q^c_{i\cdot} = 1$) for $i = 1, \dots, \left[\frac{R}{2}\right]$, and $q^c_{\cdot j} = 0$ (then $q^c_{\cdot j^{**}} = 1$) or $q^c_{\cdot j^{**}} = 0$ (then $q^c_{\cdot j} = 1$) for $j = 1, \dots, \left[\frac{C}{2}\right]$.

There are four characteristic structures with the largest departure from MPS. When a contingency table is divided into four regions with respect to the center points of the row and column, the probability structure concentrates in either the upper right, upper left, lower right, or lower left regions of the contingency table.

Although these probability structures clearly differ, Yamamoto et al. (2011)'s measure cannot distinguish between them because all the values are 1. Therefore, Iki and Tomizawa (2018) defined a measure that distinguishes between the probability structures' lower right and upper left, as

$$\varphi_{MPS} = \frac{\delta_1\varphi_1 + \delta_2\varphi_2}{\delta_1 + \delta_2},$$

where

$$\varphi_1 = \frac{4}{\pi}\sum_{i=1}^{\left[\frac{R}{2}\right]}(q_{i\cdot} + q_{i^*\cdot})\left(\theta_{1(i)} - \frac{\pi}{4}\right),$$

with

$$\theta_{1(i)} = \arccos\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right),$$

and

$$\varphi_2 = \frac{4}{\pi}\sum_{j=1}^{\left[\frac{C}{2}\right]}(q_{\cdot j} + q_{\cdot j^{**}})\left(\theta_{2'(j)} - \frac{\pi}{4}\right),$$

with

$$\theta_{2'(j)} = \arccos\left(\frac{p_{\cdot j}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right).$$

The ranges of $\{\theta_{1(i)}\}$ and $\{\theta_{2(j)}\}$ are $0 \leq \theta_{1(i)} \leq \frac{\pi}{2}$ and $0 \leq \theta_{2(j)} \leq \frac{\pi}{2}$. Thus, the submeasures φ_1 and φ_2 lie between -1 and 1. Therefore, the measure φ_{MPS} also lies between -1 and 1. Note that (i) when the MPS model holds, $\varphi_{MPS} = 0$, but the converse does not hold, (ii) $\varphi_{MPS} = 1$ (i.e., $\varphi_1 = \varphi_2 = 1$) if and only if $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) and $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) for $i = 1, \ldots, \left[\frac{R}{2}\right]$ and $j = 1, \ldots, \left[\frac{C}{2}\right]$, and (iii) $\varphi_{MPS} = -1$ (i.e., $\varphi_1 = \varphi_2 = -1$) if and only if $p_{i^*\cdot} = 0$ (then $p_{i\cdot} > 0$) and $p_{\cdot j^{**}} = 0$ (then $p_{\cdot j} > 0$) for $i = 1, \ldots, \left[\frac{R}{2}\right]$ and $j = 1, \ldots, \left[\frac{C}{2}\right]$. Iki and Tomizawa (2018) called the $\varphi_1 = 1$ ($\varphi_2 = 1$) row (column) and $\varphi_1 = -1$ ($\varphi_2 = -1$) row (column) the upper complete asymmetry with respect to the midpoint and the lower complete asymmetry, respectively. Hence, only two of the four probability structures can be distinguished. We propose a measure that can distinguish the other two.

Consider an $R_1 \times R_2 \times \cdots \times R_k$ contingency table. According to Tahata and Tomizawa (2008), the 1st-order MP_1^k model written as

$$p_i^{[j]} = p_{i^*}^{[j]} \quad (i = 1, \ldots, R_j;\ j = 1, \ldots, k),$$

where $i^* = R_j + 1 - i$. Yamamoto et al. (2011) and Iki and Tomizawa (2018) showed that submeasures for a first-order marginal distribution can be expressed in terms of solely the marginal distribution. Therefore, the measure proposed by

Iki and Tomizawa (2018) and the proposed measure for a two-way table can be extended for a multi-way table by expressing it as a weighted arithmetic mean over the number of categories in the submeasures. See Chapter 2 for details and 1.4.1 for an outline.

1.3.2 Measures for Various MH

In the study of measures, determining whether a contingency table contains nominal or ordinal categories is crucial because the categories influence how the table is handled. For a contingency table with nominal categories, the value of the measure must be invariant upon replacing the categories. By contrast, for a table with ordinal categories, the value of the measure must change upon replacing the categories. For this reason, two models are defined. One uses cell or marginal probabilities, while the other uses cumulative probabilities. As indicated in 1.2.2 (a), six models related to the S model have been proposed. The S model is defined as "all," the partial symmetry model as "least one," and the local symmetry model as "only one." Models redefine each of these models in terms of cumulative probability. For each model, a measure is proposed. For nominal category contingency tables, Tomizawa et al. (1998), Saigusa et al. (2016), and Saigusa et al. (2019b) proposed measures for the S model, the partial symmetry model, and the local symmetry model are proposed, respectively. For ordinal category contingency tables, Tomizawa et al. (2001), Saigusa et al. (2019a), and Saigusa et al. (2020b) developed measures of the S model, the CPS model, and the CLS model, respectively. See Table 1.3.

Consider an $R \times R$ contingency table with nominal categories. The MH model is defined as

$$p_{i\cdot} = p_{\cdot i} \text{ for all } i \quad (i = 1, \ldots, R).$$

The PMH model (Saigusa et al., 2020a) is defined as

$$p_{i\cdot} = p_{\cdot i} \text{ for at least one } i \quad (i = 1, \ldots, R).$$

Then let

$$\pi_i = \frac{p_{i\cdot} + p_{\cdot i}}{2}, \quad p_{1(i)} = \frac{p_{i\cdot}}{p_{i\cdot} + p_{\cdot i}}, \quad p_{2(i)} = \frac{p_{\cdot i}}{p_{i\cdot} + p_{\cdot i}}.$$

Assuming that $\{p_{i\cdot} + p_{\cdot i} \neq 0\}$, Tomizawa and Makii (2001) defined a measure of the degree of departure from MH as

$$\psi^{(\lambda)}_{MH(A)} = \sum_{i=1}^{R} \pi_i \psi^{(\lambda)}_i \text{ for } \lambda > -1,$$

where

$$\psi_i^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} I_i^{(\lambda)},$$
$$I_i^{(\lambda)} = \frac{1}{\lambda}\left\{1 - (p_{1(i)})^{\lambda+1} - (p_{2(i)})^{\lambda+1}\right\}.$$

The measure $\psi_{MH(A)}^{(\lambda)}$ lies between 0 and 1. Note that (i) the MH model holds if and only if $\psi_{MH(A)}^{(\lambda)} = 0$ (i.e., all $\psi_i^{(\lambda)}$ are equal to 0) and (ii) $\psi_{MH(A)}^{(\lambda)} = 1$ if and only if the degree of departure from MH is the largest in the sense that $p_{i\cdot} = 0$ (then $p_{\cdot i} > 0$) or $p_{\cdot i} = 0$ (then $p_{i\cdot} > 0$) for all $i = 1, \ldots, R$. The measure is expressed as the weighted *arithmetic* mean of $\{\psi_i^{(\lambda)}\}$.

Saigusa et al. (2020a) defined a measure of the degree of departure from PMH as

$$\psi_{MH(G)}^{(\lambda)} = \prod_{i=1}^{R} \left(\psi_i^{(\lambda)}\right)^{\pi_i} \text{ for } \lambda > -1.$$

The measure $\psi_{MH(G)}^{(\lambda)}$ lies between 0 and 1. Note that (i) the PMH model holds if and only if $\psi_{MH(G)}^{(\lambda)} = 0$ (i.e., at least one $\psi_i^{(\lambda)}$ is equal to 0) and (ii) $\psi_{MH(G)}^{(\lambda)} = 1$ if and only if the degree of departure from PMH is the largest in the sense that $p_{i\cdot} = 0$ (then $p_{\cdot i} > 0$) or $p_{\cdot i} = 0$ (then $p_{i\cdot} > 0$) for all $i = 1, \ldots, R$. The measure is expressed as the weighted *geometric* mean of $\{\psi_i^{(\lambda)}\}$.

The MH model is also expressed as

$$G_{1(i)} = G_{2(i)} \text{ for all } i \quad (i = 1, \ldots, R-1).$$

The CPMH model (Nakagawa et al., 2020) is defined as

$$G_{1(i)} = G_{2(i)} \text{ for at least one } i \quad (i = 1, \ldots, R-1).$$

Let for $s = 1, 2$,

$$G_{s(i)}^{*} = \frac{G_{s(i)}}{\Delta}, \quad G_{s(i)}^{c} = \frac{G_{s(i)}}{G_{1(i)} + G_{2(i)}}, \quad \Delta = \sum_{i=1}^{R-1}(G_{1(i)} + G_{2(i)}).$$

Assuming that $\{G_{1(i)} + G_{2(i)} \neq 0\}$, Tomizawa et al. (2003) defined a measure of the degree of departure from MH as

$$\tau_{MH(A)}^{(\lambda)} = \sum_{i=1}^{R-1}(G_{1(i)}^{*} + G_{2(i)}^{*})\omega_i^{(\lambda)} \text{ for } \lambda > -1,$$

where

$$\omega_i^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H_i^{(\lambda)},$$
$$H_i^{(\lambda)} = \frac{1}{\lambda}\left\{1 - (G_{1(i)}^{c})^{\lambda+1} - (G_{2(i)}^{c})^{\lambda+1}\right\}.$$

The measure $\tau_{MH(A)}^{(\lambda)}$ lies between 0 and 1. Note that (i) the MH model holds if and only if $\tau_{MH(A)}^{(\lambda)} = 0$ (i.e., all $\omega_i^{(\lambda)}$ are equal to 0) and (ii) $\tau_{MH(A)}^{(\lambda)} = 1$ if and only if the degree of departure from MH is the largest in the sense that $G_{1(i)} = 0$ (then $G_{2(i)} > 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} > 0$) for all $i = 1, \ldots, R-1$. The measure is expressed as the weighted *arithmetic* mean of $\{\omega_i^{(\lambda)}\}$.

Nakagawa et al. (2020) defined a measure of the degree of departure from PMH as

$$\tau_{MH(G)}^{(\lambda)} = \prod_{i=1}^{R-1} \left(\omega_i^{(\lambda)}\right)^{(G_{1(i)}^{*}+G_{2(i)}^{*})} \quad \text{for } \lambda > -1.$$

The measure $\tau_{MH(G)}^{(\lambda)}$ lies between 0 and 1. Note that (i) the CPMH model holds if and only if $\tau_{MH(G)}^{(\lambda)} = 0$ (i.e., at least one $\omega_i^{(\lambda)}$ is equal to 0) and (ii) $\tau_{MH(G)}^{(\lambda)} = 1$ if and only if the degree of departure from CPMH is the largest in the sense that $G_{1(i)} = 0$ (then $G_{2(i)} > 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} > 0$) for all $i = 1, \ldots, R-1$. The measure is expressed as the weighted *geometric* mean of $\{\psi_i^{(\lambda)}\}$.

To date, no model related to MH has been proposed that corresponds to the local symmetry and CLS models. See Table 1.4. Therefore, we propose models and measures that correspond to these models and their measures. See Chapter 3 for details and 1.4.2 for an outline.

1.4 Outline of the Chapters

1.4.1 Chapter 2

Chapter 2 proposes a measure of departure from another marginal average point-symmetry model. Then this measure is extended to multi-way contingency tables.

Yamamoto et al. (2011) showed that the measure of the marginal point symmetry is 1in four cases. Iki and Tomizawa (2018) proposed a measure to distinguish two of these cases. In Chapter 2, we propose a measure to differentiate the other two types in two-way contingency tables. We also extend the proposed measure to multi-way tables.

For an $R \times C$ contingency table, the MPS model is defined by

$$p_{i\cdot} = p_{i^*\cdot} \text{ and } p_{\cdot j} = p_{\cdot j^{**}} \quad (i = 1, \ldots, R;\ j = 1, \ldots, C).$$

Assuming that $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$, we defined a measure to represent the degree of departure from marginal point-symmetry as

$$\gamma_{MPS} = \frac{\delta_1\gamma_1 + \delta_2\gamma_2}{\delta_1 + \delta_2},$$

where

$$\delta_1 = \sum_{i=1}^{\left[\frac{R}{2}\right]} (p_{i\cdot} + p_{i^*\cdot}), \quad \delta_2 = \sum_{j=1}^{\left[\frac{C}{2}\right]} (p_{\cdot j} + p_{\cdot j^{**}})$$

and

$$\gamma_1 = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R}{2}\right]} (q_{i\cdot} + q_{i^*\cdot}) \left(\theta_{1(i)} - \frac{\pi}{4}\right)$$

with

$$\theta_{1(i)} = \arccos\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right)$$

and

$$\gamma_2 = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{C}{2}\right]} (q_{\cdot j} + q_{\cdot j^{**}}) \left(\theta_{2(j)} - \frac{\pi}{4}\right)$$

with

$$\theta_{2(j)} = \arccos\left(\frac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right),$$

The submeasure γ_1 has characteristics such that (1) $\gamma_1 = 1$ if and only if $p_{i\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$, and (2) $\gamma_1 = -1$ if and only if $p_{i^*\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$. Similarly, the submeasure γ_2 has characteristics such that (1) $\gamma_2 = 1$ if and only if $p_{\cdot j^{**}} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$, and (2) $\gamma_2 = -1$ if and only if $p_{\cdot j} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$. The measure γ_{MPS} has the following characteristics: (1) $\gamma_{MPS} = 1$ if and only if $\gamma_1 = \gamma_2 = 1$, and (2) $\gamma_{MPS} = -1$ if and only if $\gamma_1 = \gamma_2 = -1$.

Note that if the MPS model holds, $\gamma_1 = 0$ and $\gamma_2 = 0$, but the converse does not hold. Similarly, if the MPS model holds, then $\gamma_{MPS} = 0$, but the converse does not hold.

For an $R_1 \times R_2 \times \cdots \times R_k$ contingency table, let the 1st-order marginal probability of the jth dimension be

$$p_i^{[j]} = \Pr(X_j = i) \quad (i = 1, \ldots, R_j, \ j = 1, \ldots, k).$$

The MP_1^k model is defined as

$$p_i^{[j]} = p_{i^*}^{[j]} \quad (i = 1, \ldots, R_j; \ j = 1, \ldots, k).$$

Assuming that $\{p_i^{[j]} + p_{i^*}^{[j]} \neq 0\}$, we define a measure to represent the degree of departure from the 1st-order marginal point symmetry as

$$\Gamma_{MPS} = \frac{\sum_{j=1}^{k} (y_j \delta_j \Gamma_j + (1 - y_j)\delta_j \Gamma_{j^*})}{\sum_{l=1}^{k} \delta_l},$$

where

$$\Gamma_j = \frac{4}{\pi}\sum_{i=1}^{\left[\frac{R_j}{2}\right]}\left(q_i^{[j]} + q_{i^*}^{[j]}\right)\left(\theta_{j(i)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i)} = \arccos\left(\frac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$\Gamma_{j^*} = \frac{4}{\pi}\sum_{i=1}^{\left[\frac{R_j}{2}\right]}\left(q_i^{[j]} + q_{i^*}^{[j]}\right)\left(\theta_{j(i^*)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i^*)} = \arccos\left(\frac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$q_i^{[j]} = \frac{p_i^{[j]}}{\delta_j}, \quad q_{i^*}^{[j]} = \frac{p_{i^*}^{[j]}}{\delta_j},$$

$$\delta_j = \sum_{i=1}^{\left[\frac{R_j}{2}\right]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right) \quad \left(i = 1, \dots, \left[\frac{R_j}{2}\right], \quad j = 1, \dots, k\right),$$

where $\boldsymbol{y} = (y_1, \dots, y_j, \dots, y_k)$ and y_j is equal to 0 or 1 $(j = 1, \dots, k)$. For example, when we consider γ_{MPS} with $k = 2$, $\boldsymbol{y} = (1, 0)$. Similarly, when we consider the measure of Iki and Tomizawa (2018), $\boldsymbol{y} = (1, 1)$.

The submeasure Γ_j is satisfied when (1) $\Gamma_j = 1$ if and only if $p_i^{(j)} = 0$ for $i = 0, \dots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_j = -1$ if and only if $p_{i^*}^{(j)} = 0$ for $i = 0, \dots, \left[\frac{R_j}{2}\right]$. Similarly, the submeasure Γ_{j^*} is satisfied when (1) $\Gamma_{j^*} = 1$ if and only if $p_{i^*}^{(j)} = 0$ for $i = 0, \dots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_{j^*} = -1$ if and only if $p_i^{(j)} = 0$ for $i = 1, \dots, \left[\frac{R_j}{2}\right]$. The measure Γ_{MPS} is satisfied when (1) $\Gamma_{MPS} = 1$ if and only if all significant Γ_j or Γ_{j^*} equal 1 $(j = 1, \dots, k)$, and (2) $\Gamma_{MPS} = -1$ if and only if all significant Γ_j or Γ_{j^*} equal -1. (3) When y_j corresponds to the j digit of a binary number (e.g., $\boldsymbol{y} = (1, 0, 0)$ correspond to 100), Γ_{MPS} of the corresponding complement of $\boldsymbol{y}$ is obtained by changing the sign of Γ_{MPS} of $\boldsymbol{y}$.

1.4.2 Chapter 3

In Chapter 3, we propose (i) the local marginal homogeneity model and a measure of the degree of departure from this model, and (ii) the cumulative local marginal homogeneity model and its measure.

For square contingency tables with nominal categories, we define the local marginal homogeneity (LMH) model as

$$p_{i\cdot} = p_{\cdot i} \quad \text{for only one } i \quad (i = 1, \ldots, R).$$

Assuming that $p_{i\cdot} + p_{\cdot i} \neq 0$ $(i = 1, \ldots, R)$ and $p_{i\cdot} \neq p_{\cdot i}$ for any i except for only one a, we propose the following measure:

$$\psi^{(\lambda)}_{MH(H)} = \frac{\displaystyle\prod_{s=1}^{R} \psi^{(\lambda)}_{s}}{\displaystyle\sum_{i=1}^{R} \left(\pi_i \prod_{\substack{s=1 \\ s \neq i}}^{R} \psi^{(\lambda)}_{s} \right)} \quad (\lambda > -1),$$

where $\pi_i = (p_{i\cdot} + p_{\cdot i})/2$, $p_{1(i)} = p_{i\cdot}/(p_{i\cdot} + p_{\cdot i})$, $p_{2(i)} = p_{\cdot i}/(p_{i\cdot} + p_{\cdot i})$,

$$\psi^{(\lambda)}_{i} = 1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} I^{(\lambda)}_{i},$$

$$I^{(\lambda)}_{i} = \frac{1}{\lambda} \left\{ 1 - \left(p_{1(i)}\right)^{\lambda+1} - \left(p_{2(i)}\right)^{\lambda+1} \right\}.$$

For $\lambda = 0$, $\psi^{(0)}_{MH(H)} = \lim_{\lambda \to 0} \psi^{(\lambda)}_{MH(H)}$. Note that λ is a real value chosen by the user.

For any $\lambda > -1$, $\psi^{(\lambda)}_{MH(H)}$ has the following characteristics:

1. Measure $\psi^{(\lambda)}_{MH(H)}$ must lie between 0 and 1;
2. $\psi^{(\lambda)}_{MH(H)} = 0$ if and only if the LMH model holds;
3. $\psi^{(\lambda)}_{MH(H)} = 1$ if and only if the degree of departure from LMH is the largest in the sense that $p_{i\cdot} = 0$ (then $p_{\cdot i} > 0$) or $p_{\cdot i} = 0$ (then $p_{i\cdot} > 0$) for all $i = 1, \ldots, R$.

When the LMH model does not hold, it is easy to see that

$$\psi^{(\lambda)}_{MH(H)} = \left(\sum_{i=1}^{R} \frac{\pi_i}{\psi^{(\lambda)}_{i}} \right)^{-1}.$$

Namely, the measure is expressed as the weighted *harmonic* mean of $\{\psi^{(\lambda)}_{i}\}$.

For square contingency tables with ordered categories, we define the cumulative local marginal homogeneity (CLMH) model as

$$G_{1(i)} = G_{2(i)} \text{ for only one } i \ (i = 1, \ldots, R-1).$$

Assuming that $G_{1(i)} + G_{2(i)} \neq 0$ $(i = 1, \ldots, R-1)$ and $G_{1(i)} \neq G_{2(i)}$ for any i expect for only one a, we propose the following measure:

$$\tau_{MH(H)}^{(\lambda)} = \frac{\displaystyle\prod_{s=1}^{R-1} \omega_s^{(\lambda)}}{\displaystyle\sum_{i=1}^{R-1} \left\{ \left(G_{1(i)}^* + G_{2(i)}^*\right) \prod_{\substack{s=1 \\ s \neq i}}^{R-1} \omega_s^{(\lambda)} \right\}} \quad (\lambda > -1),$$

where $G_{s(i)}^* = G_{s(i)}/\Delta$ $(\Delta = \sum_{i=1}^{R-1}(G_{1(i)} + G_{2(i)}))$, $G_{s(i)}^c = G_{s(i)}/(G_{1(i)} + G_{2(i)})$ $(s = 1, 2)$,

$$\omega_i^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H_i^{(\lambda)},$$
$$H_i^{(\lambda)} = \frac{1}{\lambda}\left\{1 - \left(G_{1(i)}^c\right)^{\lambda+1} - \left(G_{2(i)}^c\right)^{\lambda+1}\right\}.$$

For $\lambda = 0$, we define $\tau_{MH(H)}^{(0)} = \lim_{\lambda \to 0} \tau_{MH(H)}^{(\lambda)}$. This measure holds the following properties, which are the same as the measure of the CLMH model. For any $\lambda > -1$,

1. Measure $\tau_{MH(H)}^{(\lambda)}$ must lie between 0 and 1;
2. $\tau_{MH(H)}^{(\lambda)} = 0$ if and only if the probability table has the structure of CLMH;
3. $\tau_{MH(H)}^{(\lambda)} = 1$ if and only if the probability table has the structure of complete marginal inhomogeneity in the sense that $G_{1(i)} = 0$ (then $G_{2(i)} \neq 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} \neq 0$) for all $i = 1, \ldots, R-1$.

It should be noted that the measure $\tau_{MH(H)}^{(\lambda)}$ is expressed as the weighted *harmonic* mean of $\{\omega_i^{(\lambda)}\}$.

Table 1.1: Voting changes in the 1964 and 1966 British Elections; from Upton (1977).

	1964				
1966	Conservative	Labor	Liberal	Abstention	Total
Conservative	63	3	8	3	77
Labor	6	72	8	1	87
Liberal	2	3	25	0	30
Abstention	5	4	5	5	19
Total	76	82	46	9	213

Table 1.2: Stationary transitions in a panel study of potential voters in Erie County, Ohio, 1940 (Bishop et al., 1975).

	Time t			
Time t-1	Republican	Undecided	Democrat	Total
Republican	646	32	7	685
Undecided	83	391	69	543
Democrat	18	28	506	552
Total	747	451	582	1780

Table 1.3: Six measures involved in the S model.

	For nominal categories	For ordered categories
Symmetry model	Tomizawa et al. (1998)	Tomizawa et al. (2001)
Partial Symmetry model	Saigusa et al. (2016)	Saigusa et al. (2019a)
Local Symmetry model	Saigusa et al. (2019b)	Saigusa et al. (2020b)

Table 1.4: Six measures involved in the MH model.

	For nominal categories	For ordered categories
MH model	Tomizawa and Makii (2001)	Tomizawa et al. (2003)
PMH model	Saigusa et al. (2020a)	Nakagawa et al. (2020)
LMH model	Chapter 3	Chapter 3

Chapter 2

Measure of Departure from Marginal Point-Symmetry for Multi-Way Contingency Tables

2.1 Introduction

Firstly, consider $R \times C$ rectangular contingency tables with ordered categories to call two-way tables. Let p_{ij} denote the probability that an observation will fall in the (i, j)th cell of the table $(i = 1, \ldots, R;\ j = 1, \ldots, C)$. Tomizawa (1985) proposed the point-symmetry model for $R \times C$ contingency tables as follows:

$$p_{ij} = p_{i^* j^{**}} \ (i = 1, \ldots, R;\ j = 1, \ldots, C), \tag{2.1.1}$$

where $i^* = R + 1 - i$ and $j^{**} = C + 1 - j$. See Tomizawa (1985), Tomizawa and Tahata (2007), and Tahata and Tomizawa (2014) for the details. Tomizawa (1985) also proposed the marginal point-symmetry model defined by

$$\begin{aligned} p_{i\cdot} &= p_{i^*\cdot} \ (i = 1, \ldots, R), \\ p_{\cdot j} &= p_{\cdot j^{**}} \ (j = 1, \ldots, C), \end{aligned} \tag{2.1.2}$$

where

$$p_{i\cdot} = \sum_{j=1}^{C} p_{ij} \ \text{ and } \ p_{\cdot j} = \sum_{i=1}^{R} p_{ij}.$$

The model (2.1.2) indicates that the row marginal distribution is point-symmetric with respect to the midpoint and the column marginal distribution is also point-symmetric with respect to the midpoint. Let $[x]$ denote the maximum integer which is not larger than a real number x. For example, when $R = 4$, $\left[\frac{R}{2}\right] = 2$, and when $C = 7$, $\left[\frac{C}{2}\right] = 3$. The marginal point-symmetry model is also expressed as essentially

$$p_{i\cdot} = p_{i^*\cdot} \quad \left(i = 1, \ldots, \left[\frac{R}{2}\right]\right),$$
$$p_{\cdot j} = p_{\cdot j^{**}} \quad \left(j = 1, \ldots, \left[\frac{C}{2}\right]\right).$$

Secondly, suppose we have R^k contingency tables ($k \geq 2$) with ordered categories, to call multi-way tables. Let X_l ($l = 1, \ldots, k$) be random variables. Let $p_{\boldsymbol{i}}$ denote the probability that an observation will fall in the $\boldsymbol{i} = (i_1, \ldots, i_k)$th cell of the table ($i_n = 1, \ldots, R$; $n = 1, \ldots, k$). Wall and Lienert (1976) proposed the point-symmetry model defined by

$$p_{\boldsymbol{i}} = p_{\boldsymbol{i}^*} \text{ for any } \boldsymbol{i} = (i_1, \ldots, i_k), \tag{2.1.3}$$

where $\boldsymbol{i}^* = (i_1^*, \ldots, i_k^*)$ and $i_t^* = R + 1 - i_t$.

The hth-order ($1 \leq h < k$) marginal probability is defined by $p_{\boldsymbol{i}}^{\boldsymbol{s}}$ that is $p_{\boldsymbol{i}}^{\boldsymbol{s}} = \Pr(X_{s_1} = i_1, \ldots, X_{s_h} = i_h)$, where $\boldsymbol{s} = (s_1, \ldots, s_h)$, $1 \leq s_1 < \cdots < s_h \leq k$ and $i_n = 1, \ldots, R$ ($n = 1, \ldots, h$). For a fixed $h = 1, \ldots, k-1$, Tahata and Tomizawa (2008) proposed the marginal point-symmetry model defined by

$$p_{\boldsymbol{i}}^{\boldsymbol{s}} = p_{\boldsymbol{i}^*}^{\boldsymbol{s}} \text{ for any } \boldsymbol{s} = (s_1, \ldots, s_h). \tag{2.1.4}$$

When the model does not hold, we are interested in measuring the degree of departure from the model. Tomizawa et al. (2007) proposed the measure from the point-symmetry model (2.1.1). For the measure from the marginal point-symmetry model (2.1.2), Yamamoto et al. (2011) proposed the power-divergence type measure of $\psi^{(\lambda)}$. When the measure $\psi^{(\lambda)} = 1$, there are four types of complete asymmetry for $i = 1, \ldots, [R/2]$; $j = 1, \ldots, [C/2]$, (i) $p_{i\cdot} = 0$ and $p_{\cdot j} = 0$, (ii) $p_{i^*\cdot} = 0$ and $p_{\cdot j} = 0$, (iii) $p_{i\cdot} = 0$ and $p_{\cdot j^{**}} = 0$, and (iv) $p_{i^*\cdot} = 0$ and $p_{\cdot j^{**}} = 0$. However, we cannot distinguish four types of complete asymmetry the type (i) to (iv). In some cases, it is important to know which type of asymmetry we have. In a clinical trial, when row and column variables denote conditions before treatment and after treatment, respectively, the type (i) denotes that treatment has no effect, but the type (ii) denotes that treatment has a remarkable effect.

Iki and Tomizawa (2018) proposed a measure using marginal average point-symmetry that is expanded marginal point-symmetry. That measure lets us distinguish the type (i) and type (iv) complete asymmetry. However, that measure cannot judge the type (ii) and type (iii) complete asymmetry.

This chapter proposes a measure expanded to 1st-ordered marginal point-symmetry for multi-way tables. In Section 2, we propose an improved measure of Iki and Tomizawa (2018) and give a large-sample confidence interval. In Section 3, we extend the measure to multi-way tables.

2.2 Two-way tables

2.2.1 Measure

Consider the $R \times C$ contingency tables. Let

$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1} \quad \left(i = 1, \ldots, \left[\frac{R}{2}\right]\right),$$
$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2} \ \text{ and } \ q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2} \quad \left(j = 1, \ldots, \left[\frac{C}{2}\right]\right).$$

Assume that $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$. We propose the measure to represent the degree of departure from marginal point-symmetry defined by

$$\gamma_{MPS} = \frac{\delta_1\gamma_1 + \delta_2\gamma_2}{\delta_1 + \delta_2}, \tag{2.2.1}$$

where

$$\delta_1 = \sum_{i=1}^{\left[\frac{R}{2}\right]}(p_{i\cdot} + p_{i^*\cdot}), \quad \delta_2 = \sum_{j=1}^{\left[\frac{C}{2}\right]}(p_{\cdot j} + p_{\cdot j^{**}})$$

and

$$\gamma_1 = \frac{4}{\pi}\sum_{i=1}^{\left[\frac{R}{2}\right]}(q_{i\cdot} + q_{i^*\cdot})\left(\theta_{1(i)} - \frac{\pi}{4}\right) \tag{2.2.2}$$

with

$$\theta_{1(i)} = \arccos\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right)$$

and

$$\gamma_2 = \frac{4}{\pi}\sum_{i=1}^{\left[\frac{C}{2}\right]}(q_{\cdot j} + q_{\cdot j^{**}})\left(\theta_{2(j)} - \frac{\pi}{4}\right) \tag{2.2.3}$$

with

$$\theta_{2(j)} = \arccos\left(\frac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right),$$

We indicate that the sub-measure γ_1 in (2.2.2) represents the degree of departure from point-symmetry of row marginal distribution, and the sub-measure γ_2 in (2.2.3) represents the degree of departure from point-symmetry of column marginal distribution. The measure γ_{MPS} in (2.2.1), which is the weighted sum of the sub-measure γ_1 and γ_2, represents the degree of departure from marginal point-symmetry.

The ranges of $\{\theta_{1(i)}\}$ and $\{\theta_{2(j)}\}$ are $0 \leq \theta_{1(i)} \leq \frac{\pi}{2}$ and $0 \leq \theta_{2(j)} \leq \frac{\pi}{2}$. Thus, the submeasure γ_1 and γ_2 lie between -1 and 1. Therefore, the measure γ_{MPS} lies between -1 and 1.

The submeasure γ_1 has characteristics that (1) $\gamma_1 = 1$ if and only if $p_{i\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$, and (2) $\gamma_1 = -1$ if and only if $p_{i^*\cdot} = 0$ for $i = 1, \ldots, \left[\frac{R}{2}\right]$. Similarly, the submeasure γ_2 has characteristics that (1) $\gamma_2 = 1$ if and only if $p_{\cdot j^{**}} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$, and (2) $\gamma_2 = -1$ if and only if $p_{\cdot j} = 0$ for $j = 1, \ldots, \left[\frac{C}{2}\right]$. The measure γ_{MPS} has characteristics that (1) $\gamma_{MPS} = 1$ if and only if $\gamma_1 = \gamma_2 = 1$, and (2) $\gamma_{MPS} = -1$ if and only if $\gamma_1 = \gamma_2 = -1$.

Note that if the marginal point-symmetry model (2.1.2) holds, we have $\gamma_1 = 0$ and $\gamma_2 = 0$; but the converse dose not hold. Similarly, if the marginal point-symmetry model holds, then we have $\gamma_{MPS} = 0$; but the converse dose not hold. We also note that if the submeasure $\gamma_1 = 0$ and $\gamma_2 = 0$, then measure $\gamma_{MPS} = 0$; but the converse dose not hold.

For example, consider the artificial probabilities in Tables 2.1a, 2.1b and 2.1c. For Table 2.1a, since there is the structure of the type (iii) complete asymmetry that is $p_{i\cdot} = 0$ (i.e., $\gamma_1 = 1$) and $p_{\cdot j^{**}} = 0$ (i.e., $\gamma_2 = 1$), we see that the measure $\gamma_{MPS} = 1$. Also for Table 2.1b, since there is the structure of the type (ii) complete asymmetry that is $p_{i^*\cdot} = 0$ (i.e., $\gamma_1 = -1$) and $p_{\cdot j} = 0$ (i.e., $\gamma_2 = -1$), we see that the measure $\gamma_{MPS} = -1$. For Table 2.1c, since there is the structure of the type (i) complete asymmetry that is $p_{i\cdot} = 0$ (i.e., $\gamma_1 = 1$) and $p_{\cdot j} = 0$ (i.e., $\gamma_2 = -1$), we see that the measure $\gamma_{MPS} = 0$.

2.2.2 Approximate confidence interval

Let n_{ij} denote the observed frequency in the (i, j)th cell of the table ($i = 1, \ldots, R$; $j = 1, \ldots, C$). Assuming that a multinomial distribution applies to the $R \times C$ table, we shall consider an approximate standard error and large-sample confidence interval for the measure γ_{MPS} and the submeasure γ_1 and γ_2 using the delta method, description of which are given by, for example, Bishop et al. (1975). The sample version of γ_{MPS}, i.e., $\widehat{\gamma}_{MPS}$, is given by γ_{MPS} with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$, where $\widehat{p}_{ij} = n_{ij}/N$ and $N = \sum\sum n_{ij}$. Using the delta method, $\sqrt{N}(\widehat{\gamma}_{MPS} - \gamma_{MPS})$ has asymptotically (as $N \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\gamma_{MPS}] = \sum_{i=1}^{R}\sum_{j=1}^{C} p_{ij}\left(\frac{\partial \gamma_{MPS}}{\partial p_{ij}}\right)^2,$$

where

$$\begin{aligned}\frac{\partial \gamma_{MPS}}{\partial p_{ij}} =&(\delta_1+\delta_2)^{-2}\left\{(\delta_1+\delta_2)\left(\delta_1\frac{\partial \gamma_1}{\partial p_{ij}} + \delta_2\frac{\partial \gamma_2}{\partial p_{ij}}\right)\right\} \\ &+ (\delta_1+\delta_2)^{-2}\left\{(\gamma_1-\gamma_2)\left(\delta_2\frac{\partial \delta_1}{\partial p_{ij}} - \delta_1\frac{\partial \delta_2}{\partial p_{ij}}\right)\right\},\end{aligned}$$

with

$$\frac{\partial \delta_1}{\partial p_{ij}} = \begin{cases} 1 & (i = 1, \ldots, \left[\frac{R}{2}\right], \left[\frac{R+1}{2}\right] + 1, \ldots, R,\ j = 1, \ldots, C), \\ 0 & \text{otherwise}, \end{cases}$$

$$\frac{\partial \delta_2}{\partial p_{ij}} = \begin{cases} 1 & (i = 1, \ldots, R,\ j = 1, \ldots, \left[\frac{C}{2}\right], \left[\frac{C+1}{2}\right] + 1, \ldots, C), \\ 0 & \text{otherwise}, \end{cases}$$

$$\frac{\partial \gamma_1}{\partial p_{ij}} = \begin{cases} \frac{4}{\pi\delta_1}\left\{\arccos\left(\frac{p_{i\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right) - \frac{p_{i^*\cdot}(p_{i\cdot} + p_{i^*\cdot})}{p_{i^*\cdot}^2 + p_{i\cdot}^2}\right\} - \frac{\gamma_1 + 1}{\delta_1} & \\ \qquad (i = 1, \ldots, \left[\frac{R}{2}\right],\ j = 1, \ldots, C), & \\ \frac{4}{\pi\delta_1}\left\{\arccos\left(\frac{p_{i^*\cdot}}{\sqrt{p_{i\cdot}^2 + p_{i^*\cdot}^2}}\right) + \frac{p_{i^*\cdot}(p_{i\cdot} + p_{i^*\cdot})}{p_{i^*\cdot}^2 + p_{i\cdot}^2}\right\} - \frac{\gamma_1 + 1}{\delta_1} & \\ \qquad (i = \left[\frac{R+1}{2}\right] + 1, \ldots, R,\ j = 1, \ldots, C), & \\ 0 \qquad \text{otherwise}, & \end{cases}$$

and

$$\frac{\partial \gamma_2}{\partial p_{ij}} = \begin{cases} \frac{4}{\pi\delta_2}\left\{\arccos\left(\frac{p_{\cdot j^{**}}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right) + \frac{p_{\cdot j^{**}}(p_{\cdot j} + p_{\cdot j^{**}})}{p_{\cdot j^{**}}^2 + p_{\cdot j}^2}\right\} - \frac{\gamma_2 + 1}{\delta_2} & \\ \qquad (i = 1, \ldots, R,\ j = 1, \ldots, \left[\frac{C}{2}\right]), & \\ \frac{4}{\pi\delta_2}\left\{\arccos\left(\frac{p_{\cdot j}}{\sqrt{p_{\cdot j}^2 + p_{\cdot j^{**}}^2}}\right) - \frac{p_{\cdot j^{**}}(p_{\cdot j} + p_{\cdot j^{**}})}{p_{\cdot j^{**}}^2 + p_{\cdot j}^2}\right\} - \frac{\gamma_2 + 1}{\delta_2} & \\ \qquad (i = 1, \ldots, R,\ j = \left[\frac{C+1}{2}\right] + 1, \ldots, C), & \\ 0 \qquad \text{otherwise}. & \end{cases}$$

Let $\widehat{\sigma}^2[\gamma_{MPS}]$ denote $\sigma^2[\gamma_{MPS}]$ with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$. Then, $\widehat{\sigma}[\gamma_{MPS}]/\sqrt{N}$ is an estimator of approximate standard error of $\widehat{\gamma}_{MPS}$, and

$$\left(\widehat{\gamma}_{MPS} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_{MPS}]}{N}},\quad \widehat{\gamma}_{MPS} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_{MPS}]}{N}}\right)$$

is an approximate $(1-\alpha)$ confidence interval for γ_{MPS}. Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution.

As for $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$, $\sqrt{N}(\widehat{\gamma}_k - \gamma_k)$ asymptotically has (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\gamma_k] = \sum_{i=1}^{R}\sum_{j=1}^{C} p_{ij}\left(\frac{\partial \gamma_k}{\partial p_{ij}}\right)^2,$$

and

$$\left(\widehat{\gamma}_k - Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_k]}{N}},\quad \widehat{\gamma}_k + Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\gamma_k]}{N}}\right)$$

is an approximate $(1-\alpha)$ confidence interval for γ_k $(k = 1, 2)$.

2.3 Multi-way tables

Consider the $R_1 \times \cdots \times R_k$ contingency tables $(k \geq 2)$. Let X_l $(l = 1, \ldots, k)$ be lth random variables. Let $p_{\boldsymbol{i}}$ denote the probability that an observation will fall in the $\boldsymbol{i} = (i_1, \ldots, i_k)$th cell of the table $(i_t = 1, \ldots, R_t;\ t = 1, \ldots, k)$.

2.3.1 Measure

Let

$$\delta_j = \sum_{i=1}^{\left[\frac{R_j}{2}\right]} \left(p_i^{[j]} + p_{i^*}^{[j]}\right),$$

where 1st-order marginal probability of the jth dimension is

$$p_i^{[j]} = \Pr(X_j = i) \quad (i = 1, \ldots, R_j,\ j = 1, \ldots, k).$$

Assume that $\{p_i^{[j]} + p_{i^*}^{[j]} \neq 0\}$. We propose a measure to represent the degree of departure from the marginal point-symmetry defined by

$$\Gamma_{MPS} = \frac{\displaystyle\sum_{j=1}^{k} (y_j \delta_j \Gamma_j + (1 - y_j)\delta_j \Gamma_{j^*})}{\displaystyle\sum_{l=1}^{k} \delta_l},$$

where

$$\Gamma_j = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R_j}{2}\right]} \left(q_i^{[j]} + q_{i^*}^{[j]}\right) \left(\theta_{j(i)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i)} = \arccos\left(\frac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$\Gamma_{j^*} = \frac{4}{\pi} \sum_{i=1}^{\left[\frac{R_j}{2}\right]} \left(q_i^{[j]} + q_{i^*}^{[j]}\right) \left(\theta_{j(i^*)} - \frac{\pi}{4}\right)$$

with

$$\theta_{j(i^*)} = \arccos\left(\frac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right)$$

and

$$q_i^{[j]} = \frac{p_i^{[j]}}{\delta_j}, \quad q_{i^*}^{[j]} = \frac{p_{i^*}^{[j]}}{\delta_j} \quad \left(i = 1, \ldots, \left[\frac{R_j}{2}\right], \quad j = 1, \ldots, k\right),$$

where $\boldsymbol{y} = (y_1, \ldots, y_j, \ldots, y_k)$ and y_j is equal to 0 or 1 $(j = 1, \ldots, k)$. For example, when we consider γ_{MPS} with $k = 2$, we see $\boldsymbol{y} = (1, 0)$. Similarly, when we consider the measure of Iki and Tomizawa (2018), we see $\boldsymbol{y} = (1, 1)$.

We point out that the sub-measure Γ_j and Γ_{j^*} represent the degree of departure from point-symmetry of jth marginal distribution. The measure Γ_{MPS}, being the weighted sum of all of significant Γ_j or Γ_{j^*} $(j = 1, \ldots, k)$, represents the degree of departure from marginal point-symmetry.

The ranges of $\{\theta_{j(i)}\}$ and $\{\theta_{j(i^*)}\}$ are $0 \leq \theta_{j(i)} \leq \frac{\pi}{2}$ and $0 \leq \theta_{j(i^*)} \leq \frac{\pi}{2}$. Thus, the submeasure Γ_j and Γ_{j^*} lie between -1 and 1. Therefore, the measure Γ_{MPS} lies between -1 and 1. The submeasure Γ_j satisfies that (1) $\Gamma_j = 1$ if and only if $p_i^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_j = -1$ if and only if $p_{i^*}^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$. Similarly, the submeasure Γ_{j^*} satisfies that (1) $\Gamma_{j^*} = 1$ if and only if $p_{i^*}^{[j]} = 0$ for $i = 0, \ldots, \left[\frac{R_j}{2}\right]$ and (2) $\Gamma_{j^*} = -1$ if and only if $p_i^{[j]} = 0$ for $i = 1, \ldots, \left[\frac{R_j}{2}\right]$. The measure Γ_{MPS} satisfied that (1) $\Gamma_{MPS} = 1$ if and only if all of significant Γ_j or Γ_{j^*} is equal to 1 $(j = 1, \ldots, k)$, and (2) $\Gamma_{MPS} = -1$ if and only if all of significant Γ_j or Γ_{j^*} is equal to -1. (3) When y_j corresponds to j digit of binary number, e.g. $\boldsymbol{y} = (1, 0, 0)$ correspond to 100, Γ_{MPS} of correspondent ones complement of $\boldsymbol{y}$ is obtained by changing the sign of Γ_{MPS} of $\boldsymbol{y}$.

2.3.2 Approximate confidence interval

We give an approximate standard error and large-sample confidence interval for the measure Γ_{MPS} using the delta method. Let $n_{i_1 \cdots i_k}$ denote the observed frequency in the $(i_1, \ldots, i_k)$th cell of the table $(i_t = 1, \ldots, R_t;\ t = 1, \ldots, k)$. Let

$$N = \sum_{i_1=1}^{R_1} \cdots \sum_{i_k=1}^{R_k} n_{i_1 \cdots n_k}.$$

We estimate Γ_{MPS} by $\widehat{\Gamma}_{MPS}$ is given by Γ_{MPS} with $\{p_{i_1 \cdots i_k}\}$ replaced by $\{\widehat{p}_{i_1 \cdots i_k}\}$, where $\widehat{p}_{i_1 \cdots i_k} = n_{i_1 \cdots i_k}/N$. Using the delta method, as $N \to \infty$, $\sqrt{N}(\widehat{\Gamma}_{MPS} - \Gamma_{MPS})$ asymptotically has a normal distribution with mean zero and variance

$$\sigma^2[\Gamma_{MPS}] = \sum_{i_1=1}^{R_1} \cdots \sum_{i_k=1}^{R_k} p_{i_1 \cdots i_k} \left(\frac{\partial \Gamma_{MPS}}{\partial p_{i_1 \cdots i_k}}\right)^2,$$

where

$$\frac{\partial \Gamma_{MPS}}{\partial p_i^{[j]}} = \left(\sum_{l=1}^{k} \delta_l\right)^{-2} \left[\left\{\sum_{j=1}^{k}\left(y_j \delta_j \frac{\partial \Gamma_j}{\partial p_i^{[j]}} + (1-y_j)\delta_j \frac{\partial \Gamma_{j^*}}{\partial p_i^{[j]}}\right)\right\}\left(\sum_{l=1}^{k} \delta_l\right)\right]$$
$$+ \left(\sum_{l=1}^{k} \delta_l\right)^{-2} \left[\sum_{j=1}^{k}(y_j\Gamma_j + (1-y_j)\Gamma_{j^*})\left(\sum_{\substack{l=1 \\ l\neq j}}^{k} \delta_l \frac{\partial \delta_j}{\partial p_i^{[j]}} - \delta_j \sum_{\substack{l=1 \\ l\neq j}}^{k} \frac{\partial \delta_l}{\partial p_i^{[j]}}\right)\right]$$

with

$$\frac{\partial \delta_j}{\partial p_i^{[j]}} = \begin{cases} 1 & \left(i = 1, \ldots, \left[\frac{R_j}{2}\right], \left[\frac{R_j+1}{2}\right] + 1, \ldots, R_j\right), \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial \Gamma_j}{\partial p_i^{[j]}} =$$
$$\begin{cases} \frac{4}{\pi \delta_j}\left\{\arccos\left(\frac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right) - \frac{p_{i^*}^{[j]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2}\right\} - \frac{\Gamma_j + 1}{\delta_j} \\ \qquad \left(i = 1, \ldots, \left[\frac{R_j}{2}\right]\right), \\ \frac{4}{\pi \delta_j}\left\{\arccos\left(\frac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right) + \frac{p_{i^*}^{[j]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2}\right\} - \frac{\Gamma_j + 1}{\delta_j} \\ \qquad \left(i = \left[\frac{R_j+1}{2}\right] + 1, \ldots, R\right), \\ 0 \qquad \text{otherwise,} \end{cases}$$

and

$$\frac{\partial \Gamma_{j^*}}{\partial p_i^{[j]}} =$$
$$\begin{cases} \frac{4}{\pi \delta_j}\left\{\arccos\left(\frac{p_{i^*}^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right) + \frac{p_{i^*}^{[j]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2}\right\} - \frac{\Gamma_{j^*} + 1}{\delta_j} \\ \qquad \left(i = 1, \ldots, \left[\frac{R_j}{2}\right]\right), \\ \frac{4}{\pi \delta_j}\left\{\arccos\left(\frac{p_i^{[j]}}{\sqrt{\left(p_i^{[j]}\right)^2 + \left(p_{i^*}^{[j]}\right)^2}}\right) - \frac{p_{i^*}^{[j]}\left(p_i^{[j]} + p_{i^*}^{[j]}\right)}{\left(p_{i^*}^{[j]}\right)^2 + \left(p_i^{[j]}\right)^2}\right\} - \frac{\Gamma_{j^*} + 1}{\delta_j} \\ \qquad \left(i = \left[\frac{R_j+1}{2}\right] + 1, \ldots, R\right), \\ 0 \qquad \text{otherwise.} \end{cases}$$

Let $\widehat{\sigma}^2[\Gamma_{MPS}]$ denote $\sigma^2[\Gamma_{MPS}]$ with $\{p_{i_1\cdots i_k}\}$ replaced by $\{\widehat{p}_{i_1\cdots i_k}\}$. Then, $\widehat{\sigma}[\Gamma_{MPS}]/\sqrt{N}$ is an estimator of the approximate standard error of $\widehat{\Gamma}_{MPS}$, and

$$\left(\widehat{\Gamma}_{MPS} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\Gamma_{MPS}]}{N}},\ \ \widehat{\Gamma}_{MPS} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{\sigma}^2[\Gamma_{MPS}]}{N}}\right)$$

is an approximate $1-\alpha$ confidence interval for Γ_{MPS}. Here, $Z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution.

2.4 Examples

2.4.1 Example 1 (Two-way table)

Consider the data in Tables 2.2a and 2.2b taken directly from Agresti (2002). These data describe the results of a randomized, double-blind clinical trial comparing an active hypnotic drug with a placebo in patients with insomnia. The outcome variable is a patient's reported time to fall asleep going to bed, measured using four categories (<20 minutes, 20-30 minutes, 30-60 minutes, and >60 minutes).

We see from Table 2.3a that for the data in Table 2.2a, the estimated value of the sub-measure γ_1 is 0.545, and the confidence interval for γ_1 does not include zero. Also, Table 2.3a shows that the estimated value of the sub-measure γ_2 is 0.584, and the confidence interval for γ_2 does not include zero. Since the importance of sub-measure γ_1 and γ_2 are almost the same, the measure γ_{MPS} is estimated to lie between γ_1 and γ_2, and the confidence interval for γ_{MPS} does not include zero.

As for Table 2.3b for the data in Table 2.2b, the estimated value of the sub-measure γ_1 is 0.512 and the confidence interval for γ_1 does not include zero. From Table 2.3b, the estimated value of the sub-measure γ_2 is 0.000, and the confidence interval for γ_2 contains zero. Iki and Tomizawa (2018) considered the structure of the column is the average column point-symmetry.

In addition, when we compare the data in Tables 2.2a and 2.2b using the estimated sub-measure γ_1, the degree of departure toward $p_{i\cdot} = 0$ (then $p_{i^*\cdot} > 0$) for $i = 1, 2$ is almost same for the data in Tables 2.2a and 2.2b. However, when we compare using the estimated submeasure γ_2, for patient's reported time after treatment, the degree of departure toward $p_{\cdot j} = 0$ (then $p_{\cdot j^{**}} > 0$) for $j = 1, 2$ is greater in Active treatment than in Placebo treatment. Therefore, patient's reported time after treatment in Active treatment would tend to be shorter than that in Placebo treatment.

2.4.2 Example 2 (Three-way table)

Consider the data in Tables 2.4a and 2.4b taken from the 2016 General Social Survey (Smith et al., 2018) conducted by the National Opinion Research Center at the University of Chicago. These describe the cross classifications of subject's opinions regarding government spending on Education, Environment, and Assistance to the poor in 1984 and 2016. The common response categories are 'too little', 'about right', and 'too much'.

When $\boldsymbol{y} = (1, 1, 1)$, the measure takes 1 when Education, Environment, and Assistance to the poor are all too much and takes -1 when all are too little. When $\boldsymbol{y} = (1, 1, 0)$, it takes 1 when Education and Environment are too much, and Assistance to the poor is too little and takes -1 when Education and Environment are too little, and Assistance to the poor is too much. When $\boldsymbol{y} = (1, 0, 1)$, it takes 1 when Education and Assistance to the poor are too much, and Environment is too little, and the measure takes -1 when Education and Assistance to the poor are too little, and Environment is too much. When $\boldsymbol{y} = (1, 0, 0)$, it takes 1 when Education is too much, and Environment and Assistance to the poor are too little, and it takes -1 when Education is too little, and Environment and Assistance to the poor are too much. By changing the value of $\boldsymbol{y}$, we can see where the frequencies are concentrated in the three-way contingency table.

No apparent difference in the measure values for any $\boldsymbol{y}$ in Table 2.5 indicates that the trend in answers has not changed between 1984 and 2016. The measures of $\boldsymbol{y} = (1, 1, 1)$ are -0.820 in Table 2.5a and -0.857 in Table 2.5b, respectively, which indicates that many people believe that government spending is not sufficient on the environment, education, and assistance to the poor.

As mentioned in Section 3.1, Property (3), comparing $\boldsymbol{y} = (1, 1, 1)$ and $\boldsymbol{y} = (0, 0, 0)$ in Table 2.5a, the measures estimate only change sign. This is a natural result if we note that when $\boldsymbol{y} = (0, 0, 0)$, the measure is 1 when Education, Environment, and Assistance to the poor are too little.

2.4.3 Example 3 (Three-way table)

Consider the data in Tables 2.6a and 2.6b obtained from Japan Meteorological Agency. These are obtained from the daily atmospheric temperatures at Sapporo, Tokyo, and Naha in Japan in 2010 and 2016, using three levels, 'below normal', 'normal', and 'above normal'. $\boldsymbol{y} = (1, 1, 0)$ and $\boldsymbol{y} = (1, 0, 1)$ are greatly different between 2010 and 2016. Comparing $\boldsymbol{y} = (1, 1, 0)$ and $\boldsymbol{y} = (1, 1, 1)$, we can see that in 2010, the average temperature in Naha was slightly below normal on many days. On the other hand, in 2016, there were considerably more days with above-normal temperatures. Next, comparing $\boldsymbol{y} = (1, 0, 1)$ and $\boldsymbol{y} = (1, 1, 1)$, we can see that there were more days in 2010 than in 2016 when the temperature in Tokyo was slightly above normal. Finally, comparing the measures for $\boldsymbol{y} = (1, 1, 1)$ for 2010 and 2016 shows that 2016 has somewhat higher values, indicating that the overall temperature is higher in 2016 for these three cities.

2.5 Concluding remarks

We proposed a new measure to distinguish two kinds of complete asymmetry for the midpoint. Since the measure Γ_{MPS} always ranges between -1 and 1 independent of the dimension k and the sample size N, Γ_{MPS} is useful for comparing the degrees of departure from marginal point-symmetry in several tables. Our measure is the extension of the measure given by Iki and Tomizawa (2018). Since sub-measures Γ_j and Γ_{j^*} depend only on the marginal frequency of jth dimension, one can easily calculate our measure even though k increases.

Table 2.1: Artificial probabilities

(a) $\gamma_{MPS} = 1$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0	0
(2)	0	0	0	0	0
(3)	0.3	0.2	0	0	0.5
(4)	0.2	0.3	0	0	0.5
Total	0.5	0.5	0	0	1

(b) $\gamma_{MPS} = -1$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0.3	0.2	0.5
(2)	0	0	0.2	0.3	0.5
(3)	0	0	0	0	0
(4)	0	0	0	0	0
Total	0	0	0.5	0.5	1

(c) $\gamma_{MPS} = 0$

	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0	0
(2)	0	0	0	0	0
(3)	0	0	0.3	0.2	0.5
(4)	0	0	0.2	0.3	0.5
Total	0	0	0.5	0.5	1

Table 2.2: Insomniac patients reported time (in minutes) to fall asleep after going to bed from Agresti (2002).

(a) Active treatment

	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	1	0	12
20-30	11	5	2	2	20
30-60	13	23	3	1	40
> 60	9	17	13	8	47
Total	40	49	19	11	119

(b) Placebo treatment

	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	2	1	14
20-30	14	5	1	0	20
30-60	6	9	18	2	35
> 60	4	11	14	22	51
Total	31	29	35	25	120

Table 2.3: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 2.2a and 2.2b.

(a) For Table 2.2a

Measure	Estimated measure	Standard error	Confidence interval
γ_{MPS}	0.564	0.056	$(+0.454, +0.675)$
γ_1	0.545	0.087	$(+0.375, +0.714)$
γ_2	0.584	0.082	$(+0.424, +0.745)$

(b) For Table 2.2b

Measure	Estimated measure	Standard error	Confidence interval
γ_{MPS}	0.256	0.053	$(+0.152, +0.361)$
γ_1	0.512	0.089	$(+0.337, +0.688)$
γ_2	0.000	0.115	$(-0.226, +0.226)$

Table 2.4: Opinions regarding government on "Education", "Environment", and "Assistance to the poor" in 1984 and 2016 from the 2016 General Social Survey (Smith et al., 2018).

(a) Opinions about government spending in 1984

		Assistance to the poor		
Education	Environment	too little	about right	too much
too little	too little	152	34	14
	about right	45	20	8
	too much	19	2	2
about right	too little	34	19	4
	about right	18	26	7
	too much	5	3	2
too much	too little	4	4	5
	about right	9	1	6
	too much	2	2	1

(b) Opinions about government spending in 2016

		Assistance to the poor		
Education	Environment	too little	about right	too much
too little	too little	612	110	30
	about right	134	55	11
	too much	51	11	11
about right	too little	85	30	6
	about right	46	43	9
	too much	9	11	5
too much	too little	12	8	3
	about right	16	16	8
	too much	13	8	13

Table 2.5: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 2.4a and 2.4b.

(a) For Table 2.4a

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	-0.820	0.075	$(-0.968, -0.673)$
(1,1,0)	-0.277	0.025	$(-0.326, -0.229)$
(1,0,1)	-0.301	0.027	$(-0.353, -0.249)$
(1,0,0)	0.242	0.040	$(+0.165, +0.319)$
For the complement of (1,1,1)			
(0,0,0)	0.820	0.075	$(+0.673, +0.968)$

(b) For Table 2.4b

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	-0.857	0.044	$(-0.943, -0.772)$
(1,1,0)	-0.274	0.012	$(-0.298, -0.250)$
(1,0,1)	-0.338	0.015	$(-0.368, -0.309)$
(1,0,0)	0.245	0.022	$(+0.201, +0.289)$

Table 2.6: The daily atmospheric temperatures at Sapporo, Tokyo, and Naha in Japan in 2010 and 2016 (Japan Meteorological Agency, 2022).

(a) The daily atmospheric temperatures in 2010

		Naha		
Sapporo	Tokyo	below normal	normal	above normal
below normal	below normal	19	4	5
	normal	5	2	3
	above normal	35	12	45
normal	below normal	4	1	3
	normal	1	0	1
	above normal	11	3	11
above normal	below normal	49	4	16
	normal	8	0	6
	above normal	41	11	62

(b) The daily atmospheric temperatures in 2016

		Naha		
Sapporo	Tokyo	below normal	normal	above normal
below normal	below normal	6	6	29
	normal	2	0	12
	above normal	8	4	63
normal	below normal	4	1	7
	normal	1	1	3
	above normal	3	0	15
above normal	below normal	35	5	31
	normal	6	0	24
	above normal	21	7	71

Table 2.7: The estimated measures, approximate standard errors, and approximate 95% confidence interval for measures are applied to Tables 2.6a and 2.6b.

(a) For Table 2.6a

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	0.213	0.039	(+0.137, +0.290)
(1,1,0)	0.268	0.036	(+0.198, +0.337)
(1,0,1)	−0.097	0.030	(−0.156, −0.039)
(1,0,0)	−0.043	0.036	(−0.113, +0.027)

(b) For Table 2.6b

$\boldsymbol{y}$	Estimated measure	Standard error	Confidence interval
(1,1,1)	0.378	0.047	(+0.286, +0.470)
(1,1,0)	−0.027	0.026	(−0.078, +0.024)
(1,0,1)	0.205	0.030	(+0.146, +0.264)
(1,0,0)	−0.200	0.040	(−0.278, −0.122)

Chapter 3

Measures of Departure from Local Marginal Homogeneity for Square Contingency Tables

3.1 Introduction

Consider $R \times R$ contingency tables with the same row and column classifications. In such contingency tables, the test of independence is meaningless because the observations are concentrated on the main diagonal cell. Therefore, we would like to perform an analysis with respect to the symmetry of the contingency table. Let p_{ij} denote the probability that an observation will fall in the (i, j)th cell of the table $(i = 1, \ldots, R;\ j = 1, \ldots, R)$. For nominal contingency tables, several symmetry models with respect to the main diagonal are considered. The symmetry (S) model (Bishop et al., 1975; Bowker, 1948) is defined as

$$p_{ij} = p_{ji} \text{ for all } (i, j; \quad i \neq j).$$

The partial symmetry (PS) model (Saigusa et al., 2016) is defined as

$$p_{ij} = p_{ji} \text{ for at least one } (i, j; \quad i \neq j).$$

The local symmetry (LS) model (Saigusa et al., 2019b) is defined as

$$p_{ij} = p_{ji} \text{ for only one } (i, j; \quad i \neq j).$$

The LS model indicates that the cell probability that an observation will fall in the ith row category and the jth $(> i)$ column category is equal to the probability that the observation falls in the jth row category and the ith column category for only one (i, j). Because of the strong constraints of the S model, various models using the marginal probabilities have been proposed to loosen the constraints. The marginal homogeneity (MH) model (Stuart, 1955) is defined as

$$p_{i\cdot} = p_{\cdot i} \text{ for all } i = 1, \ldots, R,$$

where $p_{i\cdot} = \sum_{t=1}^{R} p_{it}$, and $p_{\cdot i} = \sum_{s=1}^{R} p_{si}$. The partial marginal homogeneity (PMH) model (Saigusa et al., 2020a) is defined as

$$p_{i\cdot} = p_{\cdot i} \text{ for at least one } i = 1, \ldots, R.$$

In addition to these, other symmetry (e.g., quasi symmetry (Caussinus, 1965)) models or asymmetry (e.g., conditional symmetry (McCullagh, 1978), diagonal-parameter symmetry (Goodman, 1979a), linear diagonals-parameter symmetry (Agresti, 1983a)) models are proposed.

For square contingency tables with ordered categories, some symmetry models are also proposed, including cumulative probabilities from the upper-right and lower-left corners of the table. Denote the row and column variables by X and Y. The cumulative probability is defined as

$$C_{ij} = \begin{cases} P(X \leq i, Y \geq j) = \displaystyle\sum_{s=1}^{i} \sum_{t=j}^{R} p_{st} & \text{when } i < j, \\ P(X \geq i, Y \leq j) = \displaystyle\sum_{s=i}^{R} \sum_{t=1}^{j} p_{st} & \text{when } i > j. \end{cases}$$

Then the S model can also be expressed as

$$C_{ij} = C_{ji} \text{ for all } (i, j; \quad i \neq j).$$

The cumulative partial symmetry (CPS) model (Saigusa et al., 2019a) is defined as

$$C_{ij} = C_{ji} \text{ for at least one } (i, j; \quad i \neq j).$$

The cumulative local symmetry (CLS) model (Saigusa et al., 2020b) is defined as

$$C_{ij} = C_{ji} \text{ for only one } (i, j; \quad i \neq j).$$

The CLS model describes the probability that an observation will fall in the ith row category or below and the jth ($> i$) column category or above (upper-right corner) is equivalent to the probability that the observation falls in the jth row category or above and the ith column category or below (lower-left corner) for only one (i, j). Some marginal homogeneity models are also proposed which have cumulative probabilities. The cumulative probability is defined as

$$G_{1(i)} = P(X \leq i, Y \geq i+1) = \sum_{s=1}^{i} \sum_{t=i+1}^{R} p_{st},$$

$$G_{2(i)} = P(X \geq i+1, Y \leq i) = \sum_{s=i+1}^{R} \sum_{t=1}^{i} p_{st}.$$

Then the MH model is also expressed as

$$G_{1(i)} = G_{2(i)} \text{ for all } i = 1, \ldots, R-1.$$

The cumulative partial marginal homogeneity (CPMH) model (Nakagawa et al., 2020) is defined as

$$G_{1(i)} = G_{2(i)} \text{ for at least one } i = 1, \ldots, R-1.$$

Some statistics for testing the goodness of fit of the MH model are given by, for example, Stuart (1955), Bhapkar (1966), Fleiss and Everitt (1971), Bishop et al. (1975) and Agresti (1983b). Consider now several square tables. When there is no structure of MH in each of these tables, we are interested in measuring and comparing the degrees of departure from MH in the tables. The test statistic can be used for testing the goodness-of-fit of the MH model, but it is not suitable to use the test static for the comparison of the degrees of departure from the MH model in several square tables. See Tomizawa et al. (2003) details.

We mention that statistics cannot measure the degree of departure from the model for some contingency tables that do not fit the model. Therefore, measures have been proposed to measure the degree of departure from the model. In the analysis of two-way contingency tables, the degree of departure from independence is assessed by using measures of association between the row and column variables. Measures of the association include, for example, Yule' s coefficients of association and colligation (Yule, 1912, 1900), Cramér' s coefficient (Cramér, 1946), and Goodman and Kruskal' s coefficient (Goodman and Kruskal, 1954). For contingency tables with nominal categories, measures to represent the degree of departure from the S, PS, and LS models have been developed (Saigusa et al., 2016, 2019b; Tomizawa et al., 1998). These measures are given as forms of weighted arithmetic, geometric and harmonic means of a diversity index by Patil and Taillie (1982), consisting of cell probabilities. In the sense that the values of these measures do not depend on the order of the categories, these measures may not be suitable for ordered contingency tables. For square contingency tables with ordered categories, several measures of the structure of cumulative probability are proposed, which incorporate information about the order of the categories. The measures for the S, CPS, and CLS models are given as weighted arithmetic, geometric and harmonic means of the diversity index consisting of the cumulative probabilities C_{ij} (Saigusa et al., 2019a, 2020b; Tomizawa et al., 2001). Similarly, measures to represent the degree of departure from several MH models are proposed. For square contingency tables with nominal categories, the measures for the MH and PMH models are given as weighted arithmetic and geometric means of the diversity index consisting of the marginal probabilities Tomizawa and Makii (Altun and Aktaş, 2018; Saigusa et al., 2020a; Tomizawa and Makii, 2001). The values of these measures do not depend on the order of the categories. For square contingency tables with ordered categories, the measures for the MH and CPMH models are given as weighted arithmetic and geometric means of the diversity index consisting of the cumulative probabilities $G_{1(i)}$ and $G_{2(i)}$ (Nakagawa et al., 2020; Tomizawa et al., 2003).

On the other hand, Rand index (Rand, 1971) is proposed as a correspondence measure between different partitions. Hubert and Arabie (1985) introduce an extension of the Rand index and its application to rows and columns of contingency tables. The application to contingency tables is based on dividing the entire sample with respect to row and column categories to form a contingency table. Therefore, the symmetry-related measures and Rand index have different objectives. Also, the Rand index is calculated based on the number of samples in each contingency table cell, while the measures proposed in prior studies and this chapter are not.

This chapter aims to propose local marginal homogeneity models for the marginal probabilities and the cumulative probabilities. Moreover, we propose weighted harmonic mean measures for the proposed models. Section 3.2 proposes new measures for the local homogeneity of marginal probabilities $p_{i\cdot}$ and $p_{\cdot i}$ with nominal categories and cumulative probabilities $G_{1(i)}$ and $G_{2(i)}$ with ordered categories. Section 3.3 consists approximate confidence interval of the measures. Section 3.4 denotes the properties of the measures using artificial data sets. Section 3.6 shows examples that apply to the measures.

3.2 New models and measures

In section 3.2.1, we propose a new model which has the structure of the local marginal homogeneity for a square contingency table with nominal categories and its measure, which expresses the degree of departure from the model. In section 3.2.2, we also define another model with the cumulative local marginal homogeneity structure for a square contingency table with ordered categories and its measure.

3.2.1 For nominal category

For square contingency tables with nominal categories, we propose the local marginal homogeneity (LMH) model defined by

$$p_{i\cdot} = p_{\cdot i} \quad \text{for only one } i \quad (i = 1, \dots, R).$$

The LMH model describes that the probability that an observation will fall in the ith row category is equal to that of the observation falling in the ith column category for only one i.

Assume that $p_{i\cdot} + p_{\cdot i} \neq 0$ $(i = 1, \ldots, R)$ and $p_{i\cdot} \neq p_{\cdot i}$ for any i except for only one a. We propose the following measure:

$$\psi^{(\lambda)}_{MH(H)} = \frac{\displaystyle\prod_{s=1}^{R} \psi^{(\lambda)}_{s}}{\displaystyle\sum_{i=1}^{R} \left(\pi_i \prod_{\substack{s=1 \\ s\neq i}}^{R} \psi^{(\lambda)}_{s} \right)} \quad (\lambda > -1),$$

where $\pi_i = (p_{i\cdot} + p_{\cdot i})/2$, $p_{1(i)} = p_{i\cdot}/(p_{i\cdot} + p_{\cdot i})$, $p_{2(i)} = p_{\cdot i}/(p_{i\cdot} + p_{\cdot i})$,

$$\psi^{(\lambda)}_{i} = 1 - \frac{\lambda 2^{\lambda}}{2^{\lambda} - 1} I^{(\lambda)}_{i},$$

$$I^{(\lambda)}_{i} = \frac{1}{\lambda} \left\{ 1 - \left(p_{1(i)}\right)^{\lambda+1} - \left(p_{2(i)}\right)^{\lambda+1} \right\}.$$

For $\lambda = 0$, we define that $\psi^{(0)}_{MH(H)} = \lim_{\lambda \to 0} \psi^{(\lambda)}_{MH(H)}$. Note that λ is a real value chosen by users. The index $I^{(\lambda)}_{i}$ is a diversity index of degree-λ for $\{p_{1(i)}, p_{2(i)}\}$. We note that the diversity index includes the Shanon entropy (when $\lambda = 0$) and the Gini concentration (when $\lambda = 1$) in special cases. For more details of this diversity index, see Patil and Taillie (1982). We can rewrite submeasure $\psi^{(\lambda)}_{i}$ as follows:

$$\psi^{\lambda}_{i} = \frac{\lambda(\lambda - 1)}{2^{\lambda} - 1} D^{(\lambda)}_{i} \left(\{p_{k(i)}\}; \left\{ \frac{1}{2} \right\} \right),$$

$$D^{(\lambda)}_{i} \left(\{p_{k(i)}\}; \left\{ \frac{1}{2} \right\} \right) = \frac{1}{\lambda(\lambda + 1)} \left[p_{1(i)} \left\{ \left(\frac{p_{1(i)}}{1/2} \right)^{\lambda} - 1 \right\} + p_{2(i)} \left\{ \left(\frac{p_{2(i)}}{1/2} \right)^{\lambda} - 1 \right\} \right].$$

$D^{(\lambda)}_{i}$ is a powerdivergence between two distributions $\{p_{1(i)}, p_{2(i)}\}$ and $\{1/2, 1/2\}$. We note that the power divergence includes teh Kullback-Leibler (KL) information (when $\lambda = 0$) and the Peason chi-wquared type discrepancy (when $\lambda = 1$) in special cases. For more ditails of teh power divergence, see Cressie and Read (1984) and Read and Cressie (1988). For any $\lambda > -1$, the $\psi^{(\lambda)}_{MH(H)}$ has characteristics that

1. the measure $\psi^{(\lambda)}_{MH(H)}$ must lie between 0 and 1.

2. $\psi^{(\lambda)}_{MH(H)} = 0$ if and only if the LMH model holds.

3. $\psi^{(\lambda)}_{MH(H)} = 1$ if and only if the degree of departure from LMH is the maximum in the sense that $p_{i\cdot} = 0$ (then $p_{\cdot i} > 0$) or $p_{\cdot i} = 0$ (then $p_{i\cdot} > 0$) for all $i = 1, \ldots, R$.

When the LMH model does not hold, it is easy to see that

$$\psi^{(\lambda)}_{MH(H)} = \left(\sum_{i=1}^{R} \frac{\pi_i}{\psi^{(\lambda)}_i} \right)^{-1}.$$

Namely, the measure is expressed as the weighted harmonic mean of $\{\psi^{(\lambda)}_i\}$.

The measure $\psi^{(\lambda)}_{MH(H)}$ is appropriate for analyzing the data on a nominal scale because the value of $\psi^{(\lambda)}_{MH(H)}$ is invariant under the same arbitrary permutation of the row and column categories.

3.2.2 For ordered category

For square contingency tables with ordered categories, we propose the cumulative local marginal homogeneity (CLMH) model defined by

$$G_{1(i)} = G_{2(i)} \text{ for only one } i \ (i = 1, \ldots, R-1).$$

The CLMH model describes that the probability that an observation will fall in the ith row category or below and the $i+1$th column category or above is equal to the probability that the observation falls in the $i+1$th row category or above and the ith column category or below for only one i.

Assume that $G_{1(i)} + G_{2(i)} \neq 0$ $(i = 1, \ldots, R-1)$ and $G_{1(i)} \neq G_{2(i)}$ for any i expect for only one a. We propose the following measure:

$$\tau^{(\lambda)}_{MH(H)} = \frac{\displaystyle\prod_{s=1}^{R-1} \omega^{(\lambda)}_s}{\displaystyle\sum_{i=1}^{R-1} \left\{ \left(G^*_{1(i)} + G^*_{2(i)}\right) \prod_{\substack{s=1 \\ s\neq i}}^{R-1} \omega^{(\lambda)}_s \right\}} \quad (\lambda > -1),$$

where $G^*_{s(i)} = G_{s(i)}/\Delta$ $(\Delta = \sum_{i=1}^{R-1}(G_{1(i)} + G_{2(i)}))$, $G^c_{s(i)} = G_{s(i)}/(G_{1(i)} + G_{2(i)})$,

$$\omega^{(\lambda)}_i = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H^{(\lambda)}_i,$$
$$H^{(\lambda)}_i = \frac{1}{\lambda}\left\{1 - \left(G^c_{1(i)}\right)^{\lambda+1} - \left(G^c_{2(i)}\right)^{\lambda+1}\right\}.$$

For $\lambda = 0$, we define that $\tau^{(0)}_{MH(H)} = \lim_{\lambda\to 0} \tau^{(\lambda)}_{MH(H)}$. The measure holds the following properties same as the measure of the LMH model in Section 3.2.1. For any $\lambda > -1$,

1. the measure $\tau^{(\lambda)}_{MH(H)}$ must lie between 0 and 1.

2. $\tau^{(\lambda)}_{MH(H)} = 0$ if and only if the probability table has the structure of CLMH,

3. $\tau_{MH(H)}^{(\lambda)} = 1$ if and only if the probability table has the structure of complete marginal inhomogeneity in the sense that $G_{1(i)} = 0$ (then $G_{2(i)} \neq 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} \neq 0$) for all $i = 1, \ldots, R-1$.

It should be noted that the measure $\tau_{MH(H)}^{(\lambda)}$ is expressed as the weighted harmonic mean of $\{\omega_s^{(\lambda)}\}$.

3.3 Approximate confidence interval of the measures

In this section, we construct an approximate confidence interval for $\psi_{MH(H)}^{(\lambda)}$ and $\tau_{MH(H)}^{(\lambda)}$. As see in Section 3.2, the measures $\psi_{MH(H)}^{(\lambda)}$ and $\tau_{MH(H)}^{(\lambda)}$ are the function of p_{ij}. For the sake of general discussion, we firstly consider $\Phi^{(\lambda)}$ as a function of p_{ij} and construct an approximate confidence interval for it. Then, we obtain the approximate confidence intervals of the measures $\psi_{MH(H)}^{(\lambda)}$ and $\tau_{MH(H)}^{(\lambda)}$ by replacing $\Phi_{MH(H)}^{(\lambda)}$ with $\psi_{MH(H)}^{(\lambda)}$ and $\tau_{MH(H)}^{(\lambda)}$. Let n_{ij} denote the observed frequency in the (i,j)th cell of the table ($i = 1, \ldots, R;\ j = 1, \ldots, R$). Assuming that a multinomial distribution applies to the $R \times R$ table, we shall consider the approximate standard error and the large-sample confidence interval of the measure $\Phi^{(\lambda)}$ using the delta method, the description of which is given by, for example, Bishop et al. (1975) and Agresti (2002). The sample version of $\Phi^{(\lambda)}$, *i.e.* $\hat{\Phi}^{(\lambda)}$, is given by $\Phi^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/N$ and $N = \sum_{i=1}^{R}\sum_{j=1}^{R} n_{ij}$. Using the delta method, $\sqrt{N}\left(\hat{\Phi}^{(\lambda)} - \Phi^{(\lambda)}\right)$ asymptotically (as $N \to \infty$) has a normal distribution with mean zero and variance σ^2, where

$$\sigma^2 = \sum_{i=1}^{R}\sum_{j=1}^{R} p_{ij}\left(\frac{\partial \Phi^{(\lambda)}}{\partial p_{ij}}\right)^2 - \left(\sum_{i=1}^{R}\sum_{j=1}^{R} p_{ij}\frac{\partial \Phi^{(\lambda)}}{\partial p_{ij}}\right)^2 \quad (\lambda > -1).$$

Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}/\sqrt{N}$ is an estimated approximate standard error for $\hat{\Phi}^{(\lambda)}$, and $\hat{\Phi}^{(\lambda)} \pm z_{\alpha/2}\hat{\sigma}/\sqrt{N}$ is approximate $(1-\alpha)$ confidence limits for $\Phi^{(\lambda)}$, where $z_{\alpha/2}$ is the upper $\alpha/2$ point of the standard normal distribution.

The confidence interval of the measure $\psi_{MH(H)}^{(\lambda)}$ is given by $\partial\Phi^{(\lambda)}/\partial p_{ij}$ replaced by $\gamma_{ij}^{(\lambda)}$, where

$$\gamma_{ij}^{(\lambda)} = -\left(\psi_{MH(H)}^{(\lambda)}\right)^2 \left\{ \frac{1}{\left(\psi_i^{(\lambda)}\right)^2} A_{12}(i) + \frac{1}{\left(\psi_j^{(\lambda)}\right)^2} A_{21}(j) \right\} \quad (\lambda \neq 0),$$

with

$$A_{12}(i) = \frac{\psi_i^{(\lambda)}}{2} - \frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} p_{2(i)} \left\{ \left(p_{1(i)}\right)^\lambda - \left(p_{2(i)}\right)^\lambda \right\},$$
$$A_{21}(i) = \frac{\psi_i^{(\lambda)}}{2} + \frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} p_{1(i)} \left\{ \left(p_{1(i)}\right)^\lambda - \left(p_{2(i)}\right)^\lambda \right\},$$

and the confidence interval of the measure $\tau_{MH(H)}^{(\lambda)}$ is also given by $\partial\Phi^{(\lambda)}/\partial p_{ij}$ replaced by $\beta_{ij}^{(\lambda)}$, where

$$\beta_{ij}^{(\lambda)} = \begin{cases} \dfrac{\left(\tau_{MH(H)}^{(\lambda)}\right)^2}{\Delta} \displaystyle\sum_{k=i}^{j-1} B_{12}(k) + (j-i)\dfrac{\tau_{MH(H)}^{(\lambda)}}{\Delta} & (i < j), \\ \dfrac{\left(\tau_{MH(H)}^{(\lambda)}\right)^2}{\Delta} \displaystyle\sum_{k=t}^{i-1} B_{21}(k) + (i-j)\dfrac{\tau_{MH(H)}^{(\lambda)}}{\Delta} & (i > j), \end{cases}$$

with

$$B_{12}(k) = \frac{2^\lambda(\lambda+1)G_{2(k)}^c}{(2^\lambda - 1)(\omega_k^{(\lambda)})^2} \left\{ \left(G_{1(k)}^c\right)^\lambda - \left(G_{2(k)}^c\right)^\lambda \right\} - \frac{1}{\omega_k^{(\lambda)}},$$
$$B_{21}(k) = \frac{2^\lambda(\lambda+1)G_{1(k)}^c}{(2^\lambda - 1)(\omega_k^{(\lambda)})^2} \left\{ \left(G_{2(k)}^c\right)^\lambda - \left(G_{1(k)}^c\right)^\lambda \right\} - \frac{1}{\omega_k^{(\lambda)}},$$

and $\gamma_{ij}^{(0)} = \lim_{\lambda\to 0} \gamma_{ij}^{(\lambda)}$, $\beta_{ij}^{(0)} = \lim_{\lambda\to 0} \beta_{ij}^{(\lambda)}$.

3.4 Properties of measures

In this section, we check the properties of measures in this chapter and their relationship to the measures proposed in previous studies using artificial data. Firstly, we show that the proposed measures are the smallest in each of the nominal contingency tables and ordered contingency tables. Denote the measures for MH and PMH for nominal contingency tables $\psi_{MH(A)}$ and $\psi_{MH(G)}$, respectively (see Appendix 3.A). Since arithmetic mean is larger than geometric mean, it holds that

$$\psi_{MH(H)}^{(\lambda)} \le \psi_{MH(G)}^{(\lambda)} \le \psi_{MH(A)}^{(\lambda)} \tag{3.4.1}$$

and the equal signs can be used only when

$$\psi_1^{(\lambda)} = \psi_2^{(\lambda)} = \cdots = \psi_R^{(\lambda)}.$$

This means that, from the formula $\psi_i^{(\lambda)}$, the ratio of $p_{1(i)}$ and $p_{2(i)}$ is equal for all i.

Denote the measure for MH and CPMH for ordered contingency tables $\tau_{MH(A)}$ and $\tau_{MH(G)}$, respectively (see Appendix 3.A). In the same matter as in the discussion above, it holds that

$$\tau_{MH(H)}^{(\lambda)} \leq \tau_{MH(G)}^{(\lambda)} \leq \tau_{MH(A)}^{(\lambda)} \tag{3.4.2}$$

and equal signs can be used only when

$$\omega_1^{(\lambda)} = \omega_2^{(\lambda)} = \cdots = \omega_{R-1}^{(\lambda)}.$$

From the formula $\omega_i^{(\lambda)}$, the ratio of $G_{1(i)}^c$ and $G_{2(i)}^c$ is also equal for all i.

Now we check the above properties by using the artificial data, Table 3.1 and Table 3.2. As we can see from a glance at Table 3.2, the properties (3.4.1) and (3.4.2) are satisfied. Table 3.1a is a table with $p_{1\cdot} = p_{\cdot 1}$ and $G_{1(1)} = G_{2(1)}$. From Table 3.2a(a) and Table 3.2b(a), it can be confirmed that $\psi_{MH(H)}^{(\lambda)} = \tau_{MH(H)}^{(\lambda)} = 0$. In Table 3.1c and 3.1d, as we can see from the actual calculation, $G_{1(i)}^c/G_{2(i)}^c$ is equivalent to 1/2 or 2, $\omega_1^{(\lambda)} = \omega_2^{(\lambda)} = \omega_3^{(\lambda)}$ and $p_{1(i)}/p_{2(i)}$ are equal to 1/3 or 3, $\psi_1^{(\lambda)} = \psi_2^{(\lambda)} = \psi_3^{(\lambda)} = \psi_4^{(\lambda)}$, respectively. Therefore, it can be confirmed that $\psi_{MH(H)}^{(\lambda)} = \psi_{MH(G)}^{(\lambda)} = \psi_{MH(A)}^{(\lambda)}$ and $\tau_{MH(H)}^{(\lambda)} = \tau_{MH(G)}^{(\lambda)} = \tau_{MH(A)}^{(\lambda)}$ from Table 3.2a(d) and Table 3.2b(c). Tables 3.1b and 3.1c have numbers (1) and (4) interchanged. $\psi_{MH(H)}^{(\lambda)}$ is invariant from Table 3.2a(b) and (c), but $\tau_{MH(H)}^{(\lambda)}$ has changed from Table 3.2b(b) and (c). Therefore, it can be confirmed that $\tau_{MH(H)}^{(\lambda)}$ is the measure that takes the order into account. Table 3.1e and 3.1f are examples of contingency tables that have the structure of the greatest departure from CLMH and LMH, respectively. They do not necessarily have the same structure.

3.5 Simulation

This section simulates the probability of coverage of confidence intervals for the LMH and CLMH model measures.

Simulations were performed on 4×4 randomly generated contingency tables. The tables with sample sizes of 200, 500, and 1000 were generated 1000 times according to the probability structure of the contingency tables. Confidence intervals for the LMH and CLMH measures were calculated with eight lambda values (-0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0) to determine the probability of the actual measures falling within the 95% confidence interval.

The confidence interval is sufficiently reliable since it exceeds 90% in most of the cells in Table 3.3. The probability of an actual measure falling in the confidence interval increases as the sample size increases, but this is not the case for some cells, e.g., sample size 1000 for $\lambda = 0.0$ in Table 3.3a. This may be because when the sample size is large, the simulation completes without problems, even when the scale takes extreme values.

3.6 Example

In this section, we show examples of the adaptation of each measure for nominal or ordered contingency tables.

This actual data is an example of a contingency table with nominal categories taken from Upton (1977), showing the change in voting party for the three parties (Conservative, Labour, and Liberal) and abstention in 1964, 1966, and 1970. Table 3.4a shows the results of estimating the measure $\psi^{(\lambda)}_{MH(H)}$ for the change in voting party from 1964 to 1966, and Table 3.4b estimates the measure for the difference in voting party from 1966 to 1970 to see the degree of departure from the LMH model. Table 3.4a shows that the change in 1964 and 1966 fit the LMH model well. Table 3.4b shows that the degree of departure from the LMH model is more significant for the changes in voting parties between 1966 and 1970 than between 1964 and 1966.

This real data is an example of a contingency table with ordered categories are taken from Tominaga (1979), which shows the cross-classifications of Japanese fathers' and their son's occupational status in 1955 and 1975, respectively. Although it may be nonsense to think of occupational classes in modern society, we will treat them as an ordered category according to the references. The status of each category number is (1)Professional and Managers, (2)Clerical and Sales, (3)Skilled manual, Semiskilled manual, and Unskilled manual, and (4)Farmers. Table 3.5a shows the results of estimating the measure $\tau^{(\lambda)}_{MH(H)}$ for the father's and son's occupational class as of 1955, and Table 3.5b estimates the measure for the father's and son's occupational class as of 1975 to see the degree of departure from the CLMH model. From Table 3.5, the values in the confidence interval of $\tau^{(\lambda)}_{MH(H)}$ are greater for Table 3.5b than for Table 3.5a. Therefore, the degree of departure from the CLMH model for father-son pairs is estimated to be larger in 1975 than in 1955.

3.7 Concluding remarks

For $R \times R$ square contingency tables, we proposed the LMH model for nominal categories and the CLMH model for ordered categories. Also, we proposed harmonic mean-type measures of departure from them. As shown in the example in Section 3.6, there are two types of categories which are nominal and ordered.

Suppose we apply an ordered measure to a nominal contingency table. In that case, we introduce extra information, and if we use a nominal measure to an ordered contingency table, information about the order will be lost. Therefore, to analyze the contingency table, it is necessary to consider whether the elements of the categories are ordered or not.

As described in Section 3.1, the measures of MH, PMH, and LMH models are constructed using arithmetic, geometric, and harmonic mean, respectively. We would like to express these three measures in a single formula.

Table 3.1: Artificial data

(a)

	(1)	(2)	(3)	(4)	Total
(1)	0.12	0.09	0.07	0.02	0.30
(2)	0.08	0.09	0.12	0.02	0.31
(3)	0.06	0.03	0.06	0.05	0.20
(4)	0.04	0.01	0.08	0.06	0.19
Total	0.30	0.22	0.33	0.15	1.00

(b)

	(1)	(2)	(3)	(4)	Total
(1)	0.16	0.12	0.05	0.03	0.36
(2)	0.02	0.10	0.03	0.02	0.17
(3)	0.04	0.01	0.14	0.02	0.21
(4)	0.04	0.10	0.00	0.12	0.26
Total	0.26	0.33	0.22	0.19	1.00

(c)

	(1)	(2)	(3)	(4)	Total
(1)	0.12	0.10	0.00	0.04	0.26
(2)	0.02	0.10	0.03	0.02	0.17
(3)	0.02	0.01	0.14	0.04	0.21
(4)	0.03	0.12	0.05	0.16	0.36
Total	0.19	0.33	0.22	0.26	1.00

(d)

	(1)	(2)	(3)	(4)	Total
(1)	0.02	0.09	0.12	0.04	0.27
(2)	0.02	0.03	0.03	0.02	0.10
(3)	0.02	0.01	0.08	0.04	0.15
(4)	0.03	0.17	0.22	0.06	0.48
Total	0.09	0.30	0.45	0.16	1.00

(e)

	(1)	(2)	(3)	(4)	Total
(1)	0.00	0.20	0.00	0.10	0.30
(2)	0.00	0.00	0.30	0.05	0.35
(3)	0.00	0.00	0.00	0.35	0.35
(4)	0.00	0.00	0.00	0.00	0.00
Total	0.00	0.20	0.30	0.50	1.00

(f)

	(1)	(2)	(3)	(4)	Total
(1)	0.00	0.20	0.00	0.45	0.65
(2)	0.00	0.00	0.00	0.00	0.00
(3)	0.00	0.05	0.00	0.30	0.35
(4)	0.00	0.00	0.00	0.00	0.00
Total	0.00	0.25	0.00	0.75	1.00

Table 3.2: Values of six measures for Table 1 which are related to various Marginal Homogeneity model.

(a) Measures of nominal categories

			Applied tables					
			(a)	(b)	(c)	(d)	(e)	(f)
$\hat{\psi}^{(\lambda)}_{MH(A)}$	λ	0.00	0.019	0.029	0.029	0.189	0.416	1.000
		0.50	0.024	0.036	0.036	0.230	0.420	1.000
		1.00	0.026	0.039	0.039	0.250	0.422	1.000
$\hat{\psi}^{(\lambda)}_{MH(G)}$	λ	0.00	0.000	0.011	0.011	0.189	0.076	1.000
		0.50	0.000	0.014	0.014	0.230	0.087	1.000
		1.00	0.000	0.016	0.016	0.250	0.092	1.000
$\hat{\psi}^{(\lambda)}_{MH(H)}$	λ	0.00	0.000	0.002	0.002	0.189	0.012	1.000
		0.50	0.000	0.002	0.002	0.230	0.015	1.000
		1.00	0.000	0.002	0.002	0.250	0.017	1.000

(b) Measures of ordered categories

			Applied tables					
			(a)	(b)	(c)	(d)	(e)	(f)
$\hat{\tau}^{(\lambda)}_{MH(A)}$	λ	0.00	0.022	0.060	0.082	0.180	1.000	0.877
		0.50	0.028	0.075	0.101	0.217	1.000	0.897
		1.00	0.031	0.082	0.111	0.234	1.000	0.905
$\hat{\tau}^{(\lambda)}_{MH(G)}$	λ	0.00	0.000	0.052	0.082	0.046	1.000	0.847
		0.50	0.000	0.065	0.101	0.056	1.000	0.878
		1.00	0.000	0.071	0.111	0.060	1.000	0.889
$\hat{\tau}^{(\lambda)}_{MH(H)}$	λ	0.00	0.000	0.044	0.082	0.004	1.000	0.811
		0.50	0.000	0.055	0.101	0.005	1.000	0.855
		1.00	0.000	0.061	0.111	0.006	1.000	0.871

Table 3.3: Simulation results for LMH and CLMH.

(a) Result of LMH

	sample size		
λ	200	500	1000
-0.5	0.941	0.955	0.949
0.0	0.939	0.929	0.897
0.5	0.874	0.890	0.918
1.0	0.949	0.941	0.965
1.5	0.940	0.956	0.910
2.0	0.962	0.940	0.951
2.5	0.939	0.851	0.923
3.0	0.934	0.948	0.943

(b) Result of CLMH

	Sample size		
λ	100	500	1000
-0.5	0.874	0.885	0.940
0.0	0.946	0.951	0.954
0.5	0.906	0.948	0.885
1.0	0.942	0.940	0.947
1.5	0.937	0.950	0.952
2.0	0.934	0.962	0.917
2.5	0.939	0.934	0.948
3.0	0.936	0.927	0.875

Table 3.4: The estimated measures, estimated approximate standard errors, and approximate 95% confidence interval for $\psi^{(\lambda)}_{MH(H)}$, applied to Voting changes among 1964, 1966 and 1970 British Elections; from Upton (1977).

(a) Result from voting changes between 1966 and 1964 British Election

λ	Estimated measure	Standard error	Confidence interval
−0.5	0.0000	0.0005	(−0.0009, 0.0010)
0.0	0.0001	0.0008	(−0.0015, 0.0016)
0.5	0.0001	0.0010	(−0.0019, 0.0021)
1.0	0.0001	0.0011	(−0.0021, 0.0023)
1.5	0.0001	0.0011	(−0.0021, 0.0023)
2.0	0.0001	0.0011	(−0.0021, 0.0023)
2.5	0.0001	0.0010	(−0.0019, 0.0021)
3.0	0.0001	0.0009	(−0.0018, 0.0020)

(b) Result from voting changes between 1966 and 1970 British Election

λ	Estimated measure	Standard error	Confidence interval
−0.5	0.0079	0.0033	(0.0014, 0.0144)
0.0	0.0133	0.0056	(0.0024, 0.0243)
0.5	0.0167	0.0070	(0.0030, 0.0304)
1.0	0.0184	0.0077	(0.0033, 0.0335)
1.5	0.0188	0.0079	(0.0034, 0.0343)
2.0	0.0184	0.0077	(0.0033, 0.0335)
2.5	0.0173	0.0072	(0.0031, 0.0315)
3.0	0.0158	0.0066	(0.0028, 0.0288)

Table 3.5: The estimated measures, estimated approximate standard errors, and approximate 95% confidence interval for $\tau^{(\lambda)}_{MH(H)}$, applied to cross-classifications of Japanese father's and his son's occupational status in 1955 and 1975 (Tominaga, 1979).

(a) Result in 1955

λ	Estimated measure	Standard error	Confidence interval
-0.5	0.0032	0.0094	(-0.0151, 0.0216)
0.0	0.0055	0.0158	(-0.0255, 0.0364)
0.5	0.0068	0.0198	(-0.0319, 0.0456)
1.0	0.0076	0.0218	(-0.0352, 0.0504)
1.5	0.0078	0.0224	(-0.0361, 0.0516)
2.0	0.0076	0.0218	(-0.0352, 0.0504)
2.5	0.0071	0.0205	(-0.0331, 0.0474)
3.0	0.0065	0.0188	(-0.0303, 0.0433)

(b) Result in 1975

λ	Estimated measure	Standard error	Confidence interval
-0.5	0.0713	0.0196	(0.0328, 0.1098)
0.0	0.1172	0.0314	(0.0556, 0.1788)
0.5	0.1443	0.0379	(0.0700, 0.2187)
1.0	0.1576	0.0410	(0.0773, 0.2379)
1.5	0.1611	0.0417	(0.0793, 0.2428)
2.0	0.1576	0.0410	(0.0773, 0.2379)
2.5	0.1495	0.0392	(0.0726, 0.2265)
3.0	0.1385	0.0369	(0.0662, 0.2109)

Appendix 3.A Measures proposed in previous studies

The measures for the MH and PMH models for nominal contingency tables and the MH and CPMH models for ordered contingency tables are shown. Assuming that $p_{i\cdot} + p_{\cdot i} \neq 0$, Tomizawa and Makii (2001) proposed a measure to represent the degree of departure from the MH model as follows:

$$\psi^{(\lambda)}_{MH(A)} = \sum_{i=1}^{R} \pi_i \psi^{(\lambda)}_i \quad \text{for } \lambda > -1$$

where

$$\pi_i = \frac{p_{i\cdot} + p_{\cdot i}}{2}, \quad p_{1(i)} = \frac{p_{i\cdot}}{p_{i\cdot} + p_{\cdot i}}, \quad p_{2(i)} = \frac{p_{\cdot i}}{p_{i\cdot} + p_{\cdot i}},$$

$$\psi^{(\lambda)}_i = \begin{cases} 1 - \dfrac{\lambda 2^{\lambda}}{2^{\lambda} - 1} I^{(\lambda)}_i & \text{for } \lambda \neq 0, \\ 1 - \dfrac{1}{\log 2} I^{(0)}_i & \text{for } \lambda = 0, \end{cases}$$

$$I^{(\lambda)}_i = \begin{cases} \dfrac{1}{\lambda}\left\{1 - \left(p_{1(i)}\right)^{\lambda+1} - \left(p_{2(i)}\right)^{\lambda+1}\right\} & \text{for } \lambda \neq 0, \\ -p_{1(i)} \log p_{1(i)} - p_{2(i)} \log p_{2(i)} & \text{for } \lambda \neq 0. \end{cases}$$

Saigusa et al. (2020a) proposed a measure for the PMH model defined by

$$\psi^{(\lambda)}_{MH(G)} = \prod_{i=1}^{R} \left(\psi^{(\lambda)}_i\right)^{\pi_i} \quad \text{for } \lambda > -1.$$

Assuming that $G_{1(i)} + G_{2(i)} \neq 0$, Tomizawa et al. (2003) proposed a measure to represent the degree of departure from the MH model as follows:

$$\tau^{(\lambda)}_{MH(A)} = \sum_{i=1}^{R-1} \left(G^{*}_{1(i)} + G^{*}_{2(i)}\right) \omega^{(\lambda)}_i \quad \text{for } \lambda > -1$$

where

$$G^{*}_{s(i)} = \frac{G_{s(i)}}{\Delta}, \quad \Delta = \sum_{i=1}^{R-1} \left(G_{1(i)} + G_{2(i)}\right), \quad G^{c}_{s(i)} = \frac{G_{s(i)}}{G_{1(i)} + G_{2(i)}} \ (s = 1 \text{ or } 2),$$

$$\omega^{(\lambda)}_i = \begin{cases} 1 - \dfrac{\lambda 2^{\lambda}}{2^{\lambda} - 1} H^{(\lambda)}_i & \text{for } \lambda \neq 0, \\ 1 - \dfrac{1}{\log 2} H^{(0)}_i & \text{for } \lambda = 0, \end{cases}$$

$$H^{(\lambda)}_i = \begin{cases} \dfrac{1}{\lambda}\left\{1 - \left(G^{c}_{1(i)}\right)^{\lambda+1} - \left(G^{c}_{2(i)}\right)^{\lambda+1}\right\} & \text{for } \lambda \neq 0, \\ -G^{c}_{1(i)} \log G^{c}_{1(i)} - G^{c}_{2(i)} \log G^{c}_{2(i)} & \text{for } \lambda \neq 0. \end{cases}$$

Nakagawa et al. (2020) proposed a measure for the CPMH model defined by

$$\tau^{(\lambda)}_{MH(G)} = \prod_{i=1}^{R-1} \left(\omega_i^{(\lambda)}\right)^{\left(G^*_{1(i)}+G^*_{2(i)}\right)} \quad \text{for } \lambda > -1.$$

It can be seen that the measure $\psi^{(\lambda)}_{MH(A)}$ and $\tau^{(\lambda)}_{MH(A)}$ are weighted arithmetic means of the submeasure $\psi_i^{(\lambda)}$ and $\omega_i^{(\lambda)}$, respectively. $\psi^{(\lambda)}_{MH(G)}$ and $\tau^{(\lambda)}_{MH(G)}$ are also weighted geometric means of the submeasure $\psi_i^{(\lambda)}$ and $\omega_i^{(\lambda)}$, respectively.

Appendix 3.B Differentiation of the proposed measures

3.B.1 Measure of LMH

Consider $p_{ij}(i = 1, \ldots, R,\ j = 1, \ldots, R)$. Differentiating $\psi^{(\lambda)}_{MH(H)}$ by p_{ij}, we obtain

$$\begin{aligned}
\frac{\partial}{\partial p_{ij}}(\psi^{(\lambda)}_{MH(H)}) &= \left[\sum_{i=1}^{R}\left(\pi_i \prod_{\substack{s=1\\ s\neq i}}^{R} \psi_s^{(\lambda)}\right)\right]^{-1} \cdot \frac{\partial}{\partial p_{ij}}\left\{\prod_{s=1}^{R} \psi_s^{(\lambda)}\right\} \\
&\quad + \prod_{s=1}^{R} \psi_s^{(\lambda)} \cdot \frac{\partial}{\partial p_{ij}}\left[\sum_{i=1}^{R}\left(\pi_i \prod_{\substack{s=1\\ s\neq i}}^{R-1} \psi_s^{(\lambda)}\right)\right]^{-1} \\
&= \left(\psi^{(\lambda)}_{MH(H)}\right)^2 \left\{\frac{\pi_i}{(\psi_i^{(\lambda)})^2} \cdot \frac{\partial \psi_i^{(\lambda)}}{\partial p_{ij}} + \frac{\pi_j}{(\psi_j^{(\lambda)})^2} \cdot \frac{\partial \psi_j^{(\lambda)}}{\partial p_{ij}}\right\} \\
&\quad - \left(\psi^{(\lambda)}_{MH(H)}\right)^2 \left\{\frac{1}{\psi_i^{(\lambda)}} \cdot \frac{\partial \pi_i}{\partial p_{ij}} + \frac{1}{\psi_j^{(\lambda)}} \cdot \frac{\partial \pi_j}{\partial p_{ij}}\right\}.
\end{aligned}$$

Considering the derivative of $\psi_i^{(\lambda)}$ and $\psi_j^{(\lambda)}$, we obtain

$$\begin{aligned}
\frac{\partial \psi_i}{\partial p_{ij}} &= \frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} \frac{p_{2(i)}}{\pi_i} \left\{\left(p_{1(i)}\right)^\lambda - \left(p_{2(i)}\right)^\lambda\right\}, \\
\frac{\partial \psi_j}{\partial p_{ij}} &= -\frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} \frac{p_{1(j)}}{\pi_j} \left\{\left(p_{1(j)}\right)^\lambda - \left(p_{2(j)}\right)^\lambda\right\}.
\end{aligned}$$

Because $\partial\pi_i/\partial p_{ij}$ and $\partial\pi_j/\partial p_{ij}$ is equal to 1/2, we get

$$
\begin{aligned}
\frac{\partial}{\partial p_{ij}}(\psi_{MH(H)}^{(\lambda)}) &= \left(\psi_{MH(H)}^{(\lambda)}\right)^2 \left\{ \frac{\pi_i}{(\psi_i^{(\lambda)})^2} \cdot \frac{\partial \psi_i^{(\lambda)}}{\partial p_{ij}} + \frac{\pi_j}{(\psi_j^{(\lambda)})^2} \cdot \frac{\partial \psi_j^{(\lambda)}}{\partial p_{ij}} \right\} \\
&\quad - \left(\psi_{MH(H)}^{(\lambda)}\right)^2 \left\{ \frac{1}{\psi_i^{(\lambda)}} \cdot \frac{\partial \pi_i}{\partial p_{ij}} + \frac{1}{\psi_j^{(\lambda)}} \cdot \frac{\partial \pi_j}{\partial p_{ij}} \right\} \\
&= -\left(\psi_{MH(H)}^{(\lambda)}\right)^2 \left[\frac{1}{2\psi_i^{(\lambda)}} - \frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} \frac{p_{2(i)}}{(\psi_i^{(\lambda)})^2} \left\{ \left(p_{1(i)}\right)^\lambda - \left(p_{2(i)}\right)^\lambda \right\} \right] \\
&\quad - \left(\psi_{MH(H)}^{(\lambda)}\right)^2 \left[\frac{1}{2\psi_j^{(\lambda)}} + \frac{2^{\lambda-1}(\lambda+1)}{2^\lambda - 1} \frac{p_{2(j)}}{(\psi_j^{(\lambda)})^2} \left\{ \left(p_{1(j)}\right)^\lambda - \left(p_{2(j)}\right)^\lambda \right\} \right].
\end{aligned}
$$

3.B.2 Measure of CLMH

Consider $p_{st}(s < t)(s = 1, \ldots, R,\ t = 1, \ldots, R)$. Differentiating $\tau_{MH(H)}^{(\lambda)}$ by p_{st}, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial p_{st}}(\tau_{MH(H)}^{(\lambda)}) &= \left[\sum_{i=1}^{R-1} \left((G_{1(i)}^* + G_{2(i)}^*) \prod_{\substack{s=1 \\ s \neq i}}^{R-1} \omega_s^{(\lambda)} \right) \right]^{-1} \cdot \frac{\partial}{\partial p_{st}} \left\{ \prod_{s=1}^{R-1} \omega_s^{(\lambda)} \right\} \\
&\quad + \prod_{s=1}^{R-1} \omega_s^{(\lambda)} \cdot \frac{\partial}{\partial p_{st}} \left[\sum_{i=1}^{R-1} \left((G_{1(i)}^* + G_{2(i)}^*) \prod_{\substack{s=1 \\ s \neq i}}^{R-1} \omega_s^{(\lambda)} \right) \right]^{-1} \\
&= \left(\tau_{MH(H)}^{(\lambda)}\right)^2 \left\{ \frac{G_{1(s)}^* + G_{2(s)}^*}{(\omega_s^{(\lambda)})^2} \cdot \frac{\partial \omega_s^{(\lambda)}}{\partial p_{st}} + \cdots + \frac{G_{1(t-1)}^* + G_{2(t-1)}^*}{(\omega_{t-1}^{(\lambda)})^2} \cdot \frac{\partial \omega_{t-1}^{(\lambda)}}{\partial p_{st}} \right\} \\
&\quad - \left(\tau_{MH(H)}^{(\lambda)}\right)^2 \left\{ \frac{1}{\omega_1^{(\lambda)}} \cdot \frac{\partial (G_{1(1)}^* + G_{2(1)}^*)}{\partial p_{st}} + \cdots + \frac{1}{\omega_R^{(\lambda)}} \cdot \frac{\partial (G_{1(r)}^* + G_{2(r)}^*)}{\partial p_{st}} \right\}.
\end{aligned}
$$

Considering the derivative of $\omega_s^{(\lambda)}$, we obtain

$$
\frac{\partial \omega_s^{(\lambda)}}{\partial p_{st}} = \frac{2^\lambda(\lambda+1)G_{2(s)}^c}{(2^\lambda - 1)(G_{1(s)} + G_{2(s)})}((G_{1(s)}^c)^\lambda - (G_{2(s)}^c)^\lambda).
$$

Consider with respect to the derivative of $G_{1(i)}^* + G_{2(i)}^*$. Assume that $G_{1(n)}^*$ contains p_{st} and $G_{1(m)}^*$ does not contain p_{st}, we have

$$
\begin{aligned}
\frac{\partial (G_{1(n)}^* + G_{2(n)}^*)}{\partial p_{st}} &= \frac{1}{\Delta}\{1 - (t-s)(G_{1(n)}^* + G_{2(n)}^*)\}, \\
\frac{\partial (G_{1(m)}^* + G_{2(m)}^*)}{\partial p_{st}} &= -(t-s)\left(\frac{1}{\Delta}\right)(G_{1(n)}^* + G_{2(n)}^*).
\end{aligned}
$$

Substituting these derivatives into the derivative of $\tau^{(\lambda)}_{MH(H)}$, we get

$$
\begin{aligned}
\frac{\partial}{\partial p_{st}}(\tau^{(\lambda)}_{MH(H)}) &= (\tau^{(\lambda)}_{MH(H)})^2 \left\{ \frac{G^*_{1(s)} + G^*_{2(s)}}{(\omega^{(\lambda)}_s)^2} \cdot \frac{\partial \omega^{(\lambda)}_s}{\partial p_{st}} + \cdots + \frac{G^*_{1(t-1)} + G^*_{2(t-1)}}{(\omega^{(\lambda)}_{t-1})^2} \cdot \frac{\partial \omega^{(\lambda)}_{t-1}}{\partial p_{st}} \right\} \\
&\quad - (\tau^{(\lambda)}_{MH(H)})^2 \left\{ \frac{1}{\omega^{(\lambda)}_1} \cdot \frac{\partial (G^*_{1(1)} + G^*_{2(1)})}{\partial p_{st}} + \cdots + \frac{1}{\omega^{(\lambda)}_R} \cdot \frac{\partial (G^*_{1(r)} + G^*_{2(r)})}{\partial p_{st}} \right\} \\
&= \frac{(\tau^{(\lambda)}_{MH(H)})^2}{\Delta} \sum_{k=s}^{t-1} \left(\frac{2^\lambda (\lambda+1) G^c_{2(k)}}{(2^\lambda - 1)(\omega^{(\lambda)}_k)^2} ((G^c_{1(k)})^\lambda - (G^c_{2ks})^\lambda) - \frac{1}{\omega^{(\lambda)}_k} \right) \\
&\quad + (t-s) \frac{\tau^{(\lambda)}_{MH(H)}}{\Delta}.
\end{aligned}
$$

Similarly consider $p_{st}(s > t)(s = 1, \ldots, R,\ t = 1, \ldots, R)$. Noting that the derivative of $\omega^{(\lambda)}_s$ is

$$
\frac{\partial \omega^{(\lambda)}_s}{\partial p_{st}} = \frac{2^\lambda (\lambda+1) G^c_{1(s)}}{(2^\lambda - 1)(G_{1(s)} + G_{2(s)})} ((G^c_{2(s)})^\lambda - (G^c_{1(s)})^\lambda),
$$

the derivative of $\tau^{(\lambda)}_{MH(H)}$ is

$$
\begin{aligned}
\frac{\partial}{\partial p_{st}}(\tau^{(\lambda)}_{MH(H)}) &= (\tau^{(\lambda)}_{MH(H)})^2 \left\{ \frac{G^*_{1(t)} + G^*_{2(t)}}{(\omega^{(\lambda)}_t)^2} \cdot \frac{\partial \omega^{(\lambda)}_t}{\partial p_{st}} + \cdots + \frac{G^*_{1(s-1)} + G^*_{2(s-1)}}{(\omega^{(\lambda)}_{s-1})^2} \cdot \frac{\partial \omega^{(\lambda)}_{s-1}}{\partial p_{st}} \right\} \\
&\quad - (\tau^{(\lambda)}_{MH(H)})^2 \left\{ \frac{1}{\omega^{(\lambda)}_1} \cdot \frac{\partial (G^*_{1(1)} + G^*_{2(1)})}{\partial p_{st}} + \cdots + \frac{1}{\omega^{(\lambda)}_R} \cdot \frac{\partial (G^*_{1(r)} + G^*_{2(r)})}{\partial p_{st}} \right\} \\
&= \frac{(\tau^{(\lambda)}_{MH(H)})^2}{\Delta} \sum_{k=t}^{s-1} \left(\frac{2^\lambda (\lambda+1) G^c_{1(k)}}{(2^\lambda - 1)(\omega^{(\lambda)}_k)^2} ((G^c_{2(k)})^\lambda - (G^c_{1(k)})^\lambda) - \frac{1}{\omega^{(\lambda)}_k} \right) \\
&\quad + (s-t) \frac{(\tau^{(\lambda)}_{MH(H)})}{\Delta}.
\end{aligned}
$$

Appendix 3.C Data tables

3.C.1 The data table of Table 3.2

Table 3.6: Voting changes among 1964, 1966 and 1970 British Elections; from Upton (1977).

(a) Voting changes between 1966 and 1964 British Election

	1964				
1966	Conservative	Labor	Liberal	Abstention	Total
Conservative	63	3	8	3	77
Labor	6	72	8	1	87
Liberal	2	3	25	0	30
Abstention	5	4	5	5	19
Total	76	82	46	9	213

(b) Voting changes between 1966 and 1970 British Election

	1970				
1966	Conservative	Labor	Liberal	Abstention	Total
Conservative	68	1	1	7	77
Labor	12	60	5	10	87
Liberal	12	3	13	2	30
Abstention	8	2	3	6	19
Total	100	66	22	25	213

3.C.2 The data table of Table 3.5

Table 3.7: Cross-classifications of father's and his son's occupational status in 1955 and 1975 (Tominaga, 1979)

(a) in 1955

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1) Professional and Managers	80	72	37	19	208
(2) Clerical and Sales	44	155	61	31	291
(3) Skilled manual and Semiskilled manual	26	73	218	45	362
(4) Unskilled manual Farmers	69	156	166	614	1005
Total	219	456	482	709	1866

(b) in 1975

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1) Professional and Managers	127	101	54	12	294
(2) Clerical and Sales	86	207	125	13	431
(3) Skilled manual and Semiskilled manual	78	124	310	24	536
(4) Unskilled manual Farmers	109	206	437	325	1077
Total	400	638	926	374	2338

Chapter 4

Discussions and Concluding Remarks

4.1 Discussions

4.1.1 Properties of measures in Chapter 2

We discuss the characteristics of each measure in Yamamoto et al. (2011), Iki and Tomizawa (2018), and Saito et al. (2022a) in Table 1.2. Yamamoto et al. (2011) sets the measure to 1 when the probability is concentrated in one of the four corner cells. However, from an analytical point of view, each of the four corner cells has a different meaning: R $\rightarrow$ R is the number of Republican supporters who have always supported the Republican party, D $\rightarrow$ D is the number of Democrat supporters who have always supported the Democrat party, R $\rightarrow$ D is the number of supporters who changed their support from Republic to Democrat, and D $\rightarrow$ R is the opposite. Iki and Tomizawa (2018) focuses on whether Republican or Democrat supporters are more likely to remain. Saito et al. (2022a), on the other hand, focuses on whether there is a greater flow from Republican to Democrat or vice versa.

Sub-measure Γ_j measures whether the probability of jth classification is concentrated in the first or the latter half. Therefore, the analyst can determine the value of y_j from the perspective of which part of the classification is more concentrated. Table 4.1 shows the probability structure of a $4 \times 4 \times 4$ contingency table with a $\Gamma_{MPS} = 1$. Using Table 2.6 as an example, $\boldsymbol{y} = (1, 1, 1)$ if we are interested in whether all three cities are warming, and $\boldsymbol{y} = (1, 1, 0)$ if Naha is cooling and the other two cities are warming. Note that $\Gamma_j^* = -\Gamma_j$, and we can get a rough estimate of Γ_3 by comparing $\boldsymbol{y} = (1, 1, 1)$ and $\boldsymbol{y} = (1, 1, 0)$. Therefore, it is possible to predict whether the probability of the third classification is concentrated in the first or the latter half. However, if you want to predict this, it is better to calculate the sub-measure Γ_j. If this is impossible, you need to predict Γ_j, paying attention to whether the number of categories is even or odd and whether the data is likely concentrated in the center of categories.

4.1.2 Select of λ

For the measure $\psi^{(\lambda)}_{MH(H)}$, the analyst may be interested in which value of λ should be preferred. However, in comparing tables, it seems best to compare the values of $\psi^{(\lambda)}_{MH(H)}$ for a range of values of λ. For example, consider the artificial data in Table 4.2. Table 4.3 shows that the confidence intervals slightly overlap for the λ marked with an asterisk (*). So, for these cases, it may be impossible to decide whether the degree of asymmetry is greater for Tables 4.2a or 4.2b. But generally, for the comparison between the two tables, it would be possible to conclude if $\psi^{(\lambda)}_{MH(H)}$ is always greater (or always less) for one table than for the other table. The same can be concluded for $\tau^{(\lambda)}_{MH(H)}$.

4.2 Concluding Remarks

In this paper, a new method for contingency table analysis is proposed. The measures of departure from the marginal point symmetry model for a two-way contingency table with ordered categories are verified and extended to a multi-way contingency table. Then a measure of the departure from two local marginal homogeneity models is devised.

Similar to Tahata et al. (2009), Chapter 2 uses the inverse trigonometric function to express the direction by allowing the measure to range from -1 to 1. By extending these functions, we propose a measure of the departure from the 1st-order marginal point symmetry model in a multi-way contingency table. We also confirm that this multi-way measure includes the proposed two-way measure. We only extended the measure to the first-order case, so it seems we could consider extending it to the MP^k_h model. However, this would not be easy.

Chapter 3 proposes a measure of departure from the local marginal homogeneity in the case of a contingency table with nominal categories and the case of a contingency table with ordinal categories. The difference between these measures is whether or not the category order of the contingency table is interchangeable. We also consider the relationship between the proposed measures and previous ones related to the marginal homogeneity model.

As a further development of this study, in Chapter 3, we will consider models or measures with similar properties to those proposed in this study using similar or completely new methods. There have been various proposals for marginal homogeneity models and their measures for multi-way contingency tables. Further investigations of these topics are future works.

In summary, this study develops new models and measures for marginal point symmetry and marginal homogeneity of contingency tables. These models and measures may contribute to analyzing marginalized elements of contingency tables.

Table 4.1: Probability structures for Γ_{MPS} to be 1

(a) $\boldsymbol{y} = (1,1,1)$

X_3	1				2				3				4			
X_1/X_2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	p_{333}	p_{343}	0	0	p_{334}	p_{344}
4	0	0	0	0	0	0	0	0	0	0	p_{433}	p_{443}	0	0	p_{434}	p_{444}

(b) $\boldsymbol{y} = (1,1,0)$

X_3	1				2				3				4			
X_1/X_2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	p_{331}	p_{341}	0	0	p_{332}	p_{342}	0	0	0	0	0	0	0	0
4	0	0	p_{431}	p_{441}	0	0	p_{432}	p_{442}	0	0	0	0	0	0	0	0

(c) $\boldsymbol{y} = (1,0,1)$

X_3	1				2				3				4			
X_1/X_2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	p_{313}	p_{323}	0	0	p_{314}	p_{324}	0	0
4	0	0	0	0	0	0	0	0	p_{413}	p_{423}	0	0	p_{414}	p_{424}	0	0

(d) $\boldsymbol{y} = (0,1,1)$

X_3	1				2				3				4			
X_1/X_2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	0	0	0	0	0	0	0	0	0	0	p_{133}	p_{143}	0	0	p_{134}	p_{144}
2	0	0	0	0	0	0	0	0	0	0	p_{233}	p_{243}	0	0	p_{234}	p_{244}
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.2: Artificial data

(a)

	(1)	(2)	(3)	Total
(1)	20	90	120	230
(2)	20	30	30	80
(3)	20	10	80	110
Total	60	130	230	420

(b)

	(1)	(2)	(3)	Total
(1)	30	50	100	180
(2)	20	30	10	60
(3)	50	10	90	150
Total	100	90	200	390

Table 4.3: The estimated measures, estimated approximate standard errors, and approximate 95% confidence intervals for $\psi^{(\lambda)}_{MH(H)}$, applied to Table 4.2.

(a) Result from 4.2a

λ	Estimated measure	Standard error	Confidence interval
-0.5^*	0.05119	0.01348	(0.02476, 0.07761)
0.0^*	0.08497	0.02197	(0.04191, 0.12804)
0.5	0.10529	0.02687	(0.05262, 0.15796)
1.0	0.11542	0.02923	(0.05813, 0.17271)
1.5	0.11807	0.02983	(0.05961, 0.17653)
2.0	0.11542	0.02923	(0.05813, 0.17271)
2.5	0.10922	0.02786	(0.05462, 0.16382)
3.0^*	0.10081	0.02601	(0.04983, 0.15179)

(b) Result from 4.2b

λ	Estimated measure	Standard error	Confidence interval
-0.5^*	0.01382	0.00570	(0.00265, 0.02499)
0.0^*	0.02325	0.00954	(0.00455, 0.04195)
0.5	0.02908	0.01190	(0.00577, 0.05240)
1.0	0.03206	0.01309	(0.00641, 0.05771)
1.5	0.03285	0.01340	(0.00659, 0.05912)
2.0	0.03206	0.01309	(0.00641, 0.05771)
2.5	0.03018	0.01234	(0.00599, 0.05437)
3.0^*	0.02763	0.01134	(0.00541, 0.04984)

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