学位論文

Pseudo-Riemannian ruled surfaces with null curves of constant mean and scalar curvature

(平均曲率及びスカラー曲率一定の零的曲線に沿う擬リーマン線織面)

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小林 穂乃香

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Honoka Kobayashi

Contents

Chapter 1

Introduction

The thesis is based on [17] and [16].

In the thesis, we study pseudo-Riemannian surfaces of constant mean and scalar curvature and non-diagonalizable shape operator in pseudo-sphere or pseudo-hyperbolic space.

The notion of finite typeness of isometric immersions into a Euclidean space was introduced by B.-Y. Chen in the late 1970's (see [6], [7], [5], [9] etc.). Later, the finite typeness of isometric immersions into a Euclidean space or, in more general, a pseudo-Euclidean space has been studied by many geometers (see [6], [7], [5] etc.). B.-Y. Chen and P. Piccinni [8] extended the notion of the finite typeness to C^∞-maps and studied the finite typeness of the Gauss map (which is not an immersion) of isometric immersions. Recently, B. Bektaş, E.Ö. Canfes, U. Dursun and R. Yeğin have studied the finite typeness of the pseudo-spherical (resp. the pseudo-hyperbolic) Gauss maps of isometric immersions into the pseudo-sphere (resp. the pseudo-hyperbolic space) (see [4], [3], [2], [22]).

Let M be an n-dimensional pseudo-Riemannian manifold of index t, and let $\mathbb{E}^m_\nu$ be an m-dimensional pseudo-Euclidean space of index ν. The smooth map $\phi : M \to \mathbb{E}^m_\nu$ is said to be *of finite type* if ϕ has the spectral decomposition: $\phi = \phi_1 + \cdots + \phi_k$ with $\phi_i : M \to \mathbb{E}^m_\nu$'s non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i$, where Δ is the Laplacian operator of M and λ_i are constants. Here, if λ_i's are mutually distinct then ϕ is said to be *of k-type*. Denote by $\mathbb{S}^{m-1}_\nu$ the $(m-1)$-dimensional pseudo-sphere of constant curvature 1 and index ν, and by $\mathbb{H}^{m-1}_\nu$ the $(m-1)$-dimensional pseudo-hyperbolic space of constant curvature -1 and index ν. Let M be an n-dimensional oriented pseudo-Riemannian manifold of index t and $\mathbf{x} : M \hookrightarrow \mathbb{S}^{m-1}_\nu \subset \mathbb{E}^m_\nu$ an isometric immersion. Let $G(n+1,m)_t$ be the Grassmannian manifold consisting of $(n+1)$-dimensional oriented non-degenerate subspaces of index t of $\mathbb{E}^m_\nu$. Define the map $\tilde{\nu} : M \to G(n+1,m)_t$ by $\tilde{\nu}(p) = \mathbf{x}(p) \wedge \mathbf{x}_*(e^p_1) \wedge \mathbf{x}_*(e^p_2) \wedge \cdots \wedge \mathbf{x}_*(e^p_n)$ for $p \in M$, where $(e^p_1, \ldots, e^p_n)$ is an orthonormal frame of T_pM compatible with the orientation of M. This map $\tilde{\nu}$ is called *the pseudo-spherical Gauss map of* $\mathbf{x}$. Similarly, for an isometric immersion $\mathbf{x} : M \hookrightarrow \mathbb{H}^{m-1}_\nu \subset \mathbb{E}^m_{\nu+1}$, *the pseudo-hyperbolic Gauss map* $\tilde{\nu} : M \to G(n+1,m)_{t+1}$ is defined.

D. S. Kim and Y. H. Kim [14] classified such Lorentzian surfaces in the 3-dimensional de Sitter space $\mathbb{S}^3_1$ and anti-de Sitter space $\mathbb{H}^3_1$ as follows.

Theorem 1.1 ([14], see Theorem 3.1)**.** *Let M^2_1 be a Lorentzian surface in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$. If the mean and Gaussian curvatures are constant and the shape operator is not diagonalizable at a point, then M^2_1 is an open part of a complex circle or a B-scroll.*

B. Bektaş, E.Ö. Canfes and U. Dursun [2] determined the type number of the pseudo-spherical Gauss map of B-scroll in $\mathbb{S}^3_1$.

In the following, we describe the contents of this thesis.

We recall basic terminology and facts related to pseudo-Riemannian submanifolds and null curves in Chapter 2, as preliminaries.

In Chapter 3, we determine the type numbers of the pseudo-hyperbolic Gauss maps of a complex circle and a B-scroll in the 3-dimensional anti-de Sitter space. The following theorem is one of the main results in the thesis.

Theorem 1.2 (see Theorem 3.3)**.** *Let M be an oriented Lorentzian surface in $\mathbb{H}^3_1$ of constant mean and Gaussian curvatures and non-diagonalizable shape operator. The following facts hold.*

- (i) *M is an open part of a complex circle of radius -1 if and only if the pseudo-hyperbolic Gauss map is of 1-type.*
- (ii) *If M is the complex circle of radius κ ($\mathrm{Re}(\kappa) = -1$, $\kappa \neq -1$), then the pseudo-hyperbolic Gauss map is of infinite type.*
- (iii) *M is an open part of a non-flat B-scroll if and only if the pseudo-hyperbolic Gauss map is of null 2-type.*
- (iv) *If M is a flat B-scroll, then the pseudo-hyperbolic Gauss map is of infinite type.*

Also, we investigate the behavior of type numbers of the pseudo-hyperbolic Gauss map along the parallel family of such oriented Lorentzian surfaces in the 3-dimensional anti-de Sitter space. Furthermore, we investigate the type number of the pseudo-hyperbolic Gauss map of generalized umbilical hypersurfaces which are natural generalizations of B-scrolls in $\mathbb{S}^{n+1}_1$ and $\mathbb{H}^{n+1}_1$ given by [1].

In Chapter 4, we find ruled surfaces with the above null curves which have non-diagonalizable shape operators, constant mean curvatures and constant scalar curvatures. As the preparations, we construct Frenet frame fields along a null curve when it is not a bi-null Cartan curve in a pseudo-sphere with index 2 and a pseudo-hyperbolic space with index 2, respectively. The theory of the Frenet frame field along a null curve γ in a pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle)$ has been developed by [10], [11], [13] and [12], where γ is a null curve satisfying $\langle\dot{\gamma}, \dot{\gamma}\rangle = 0$ and $\dot{\gamma} \neq 0$. If M is a Lorentzian manifold, the Frenet frame field can be constructed uniquely for a null curve γ satisfying $\gamma^{(2)} \neq 0$. Such a frame field is called the *Cartan frame field* and a null curve γ with the Cartan frame field is called the *Cartan curve*. In the case where M is $\mathbb{E}^3_1$, a *B-scroll* is one of the most known ruled surfaces over a Cartan curve γ. It is defined as the image of the immersion $\mathbf{x} : I \times \mathbb{R} \to \mathbb{E}^3_1$ parameterized by

$$\mathbf{x}(s,t) = \gamma(s) + tB(s),$$

where B is a null vector field along γ such that $\langle \dot{\gamma}, B \rangle = -1$. Note that a B-scroll is a non-degenerate hypersurface in $\mathbb{E}^3_1$ and it has some properties as follows: the mean curvature is non-zero constant, Gaussian curvature is constant and the shape operator is non-diagonalizable everywhere. In the case where M is $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$, a B-scroll is defined by the same way as in $\mathbb{E}^3_1$. M. Magid [18] and L. J. Alías, A. Ferrández and P. Lucas [1] constructed some examples of hypersurfaces which are generalizations of B-scrolls in $\mathbb{E}^{n+1}_1$, $\mathbb{S}^{n+1}_1$ or $\mathbb{H}^{n+1}_1$, and they called such a hypersurface the *generalized umbilical hypersurface*. We remark that $\gamma^{(2)}$ is a spacelike vector field if γ is in a Lorentzian manifold.

If M is a pseudo-Riemannian manifold with index 2, Frenet frame fields cannot be constructed uniquely for a null curve γ since there are both cases where $\gamma^{(2)}$ is non-null and null (see [10], [11] and [13] for instance). In the case where γ is a bi-null curve, that is, $\langle \gamma^{(2)}, \gamma^{(2)} \rangle = 0$ and $\gamma^{(2)} \neq 0$, M. Sakaki [20] constructed a Frenet frame field more directly. It is called a *bi-null Cartan frame field* and a bi-null curve γ with a bi-null Cartan frame field is called a *bi-null Cartan curve*. M. Sakaki, A. Uçum and K. İlarslan [21] studied ruled surfaces over a bi-null Cartan curve in $\mathbb{E}^5_2$. In this paper, we consider the case where M is $\mathbb{S}^{n+1}_2$ or $\mathbb{H}^{n+1}_2$ and a null curve γ in M satisfies $\langle \gamma^{(2)}, \gamma^{(2)} \rangle \neq 0$, that is, γ is not a bi-null curve. In Chapter 4, we will show some examples of ruled hypersurfaces over a null curve γ in $\mathbb{S}^{n+1}_2$ and $\mathbb{H}^{n+1}_2$. In this paper, we consider the case where $\gamma^{(2)}$ is non-null everywhere. Then, there are both cases where $\gamma^{(2)}$ is spacelike and timelike. We prove that these hypersurfaces satisfy properties as the generalized umbilical hypersurfaces in Chapter 3: the mean curvature is non-zero constant, the scalar curvature is constant and the minimal polynomial of the shape operator is $(x + a)^2$ for some constant a. We explain those details in Chapter 4.

Finally, we consider the higher codimensional case of B-scroll. D. S. Kim, Y. H. Kim and D. W. Yoon [15] extended a B-scroll in $\mathbb{E}^3_1$ to in $\mathbb{E}^m_1$ and named it the *generalized B-scroll*. In Chapter 5, we construct ruled surfaces similar to the generalized B-scroll in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$.

Let γ be a null curve in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ and $\gamma^{(2)}$ is non-null. We put $A = \dot{\gamma}$. The following is one of the main results of this paper.

Theorem 1.3 (see Theorem 5.2). *Let (A, B, C, Z_1, Z_2) be a Frenet frame field along γ in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ such that B is a null vector field, $\langle A, B \rangle = -1$ and $\langle B, C \rangle = 0$. We define the immersion from $I \times \mathbb{R}$ into $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ by $\mathbf{x}(s,t) = \gamma(s) + tB(s)$ and denote an image of $\mathbf{x}$ by M. Then, M is a non-degenerate Lorentzian ruled surface along γ satisfying the following.*

(i) *In the case where Z_1 is non-null, we put $\varepsilon_C = \langle C, C \rangle$ and $\varepsilon_1 = \langle Z_1, Z_1 \rangle$. For some constants k_2 and k_3, the mean curvature and the minimal polynomial* *of the shape operator derived from the normalized mean curvature vector are* $\varepsilon_C k_2^2 + \varepsilon_1 k_3^2$ *and* $P(x) = (x - (\varepsilon_C k_2^2 + \varepsilon_1 k_3^2))^2$, *respectively.*

(ii) *In the case where* Z_1 *is null, for some constant* k_2, *the mean curvature and the minimal polynomial of the shape operator derived from the normalized mean curvature vector are* k_2^2 *and* $P(x) = (x - k_2^2)^2$, *respectively.*

Moreover, a non-degenerate Lorentzian ruled surface along γ *equipped with Frenet frame field is one of the above two cases.*

Chapter 2

Preliminaries

2.1 Basic notions and facts

Fix $0 \leq \nu \leq m$ and let $\langle\, ,\, \rangle$ be the symmetric non-degenerate $(0,2)$ tensor on $\mathbb{R}^m$ defined by

$$\langle v, w \rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{m} v_j w_j \tag{2.1}$$

for $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_m)$. By the standard isomorphism between $\mathbb{R}^m$ to $T_p\mathbb{R}^m$, we extended $\langle\, ,\, \rangle$ to a symmetric non-degenerate $(0,2)$ tensor field on $\mathbb{R}^m$. The pair $(\mathbb{R}^m, \langle\, ,\, \rangle)$ is called a *pseudo-Euclidean space with index* ν. We denote it by $\mathbb{E}^m_\nu$ and call $\langle\, ,\, \rangle$ a *pseudo-Euclidean metric with index* ν. In general, a pair (M, g) of an m-dimensional smooth manifold M and a symmetric non-degenerate $(0,2)$ tensor field g on M of constant index ν is called a *pseudo-Riemannian manifold*. According to the definition of g, M is a Riemannian manifold if index $\nu = 0$. In particular, M is called a *Lorentzian manifold* if index $\nu = 1$. Let $(x^1, \ldots, x^m)$ be a local coordinate system on M, and we put $\partial_i = \partial/\partial x^i$. The components of g can be written locally by $g_{ij} = \langle \partial_i, \partial_j \rangle$ for $1 \leq i, j \leq m$. Then, g is locally expressed as

$$g = \sum_{i,j=1}^{m} g_{ij} dx^i \otimes dx^j.$$

For $x \in \mathbb{E}^m_\nu$ and $c > 0$, we put

$$\mathbb{S}^{m-1}_\nu(c) = \{\, x = (x_1, \ldots, x_m) \in \mathbb{E}^m_\nu \mid \langle x, x \rangle = 1/c \,\}$$

and

$$\mathbb{H}^{m-1}_{\nu-1}(-c) = \{\, x = (x_1, \ldots, x_m) \in \mathbb{E}^m_\nu \mid \langle x, x \rangle = -1/c \,\},$$

where $\langle\, ,\, \rangle$ is defined by (2.1). Then $\mathbb{S}^{m-1}_\nu(c)$ (resp. $\mathbb{H}^{m-1}_{\nu-1}(-c)$) is an $(m-1)$-dimensional pseudo-Riemannian submanifold in $\mathbb{E}^m_\nu$ of constant curvature c and index

ν (resp. constant curvature $-c$ and index $\nu - 1$), and called a *pseudo-sphere* (resp. a *pseudo-hyperbolic space*). In particular, $\mathbb{S}^{m-1}_1(c)$ and $\mathbb{H}^{m-1}_1(-c)$ are called a *de Sitter space* and an *anti-de Sitter space*, respectively. For simplicity, we denote $\mathbb{S}^{m-1}_\nu(1)$ and $\mathbb{H}^{m-1}_{\nu-1}(-1)$ by $\mathbb{S}^{m-1}_\nu$ and $\mathbb{H}^{m-1}_{\nu-1}$, respectively.

Let M_t be an n-dimensional pseudo-Riemannian submanifold of index t in $\mathbb{E}^m_\nu$. Let $\tilde{\nabla}$ be the Levi-Civita connections of $\mathbb{E}^m_\nu$ and ∇ the induced connection on M_t. Also, let $\nabla^\perp$ be the normal connection of M_t in $\mathbb{E}^m_\nu$. For simplicity, we denote all metrics by the common symbol $\langle\, , \,\rangle$. We take a local orthonormal frame field $(e_1, \ldots, e_n)$ of the tangent bundle TM_t of M_t, and a local orthonormal frame field $(e_{n+1}, \ldots, e_m)$ of the normal bundle $T^\perp M_t$ of M_t in $\mathbb{E}^m_\nu$. Denote the signatures of $e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_m$ by $\varepsilon_A := \langle e_A, e_A\rangle = \pm 1$ where $A = 1, \ldots, m$. Let $\{\tilde{\omega}_{AB}\}_{A,B=1,\ldots,m}$ be the connection form of $\tilde{\nabla}$ with respect to $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_m)$. Similarly, let $\{\omega_{ij}\}_{i,j=1,\ldots,n}$ be the connection form of ∇ with respect to $(e_1, \ldots, e_n)$ and $\{\omega^\perp_{rs}\}_{r,s=n+1,\ldots,m}$ the connection form of $\nabla^\perp$ with respect to $(e_{n+1}, \ldots, e_m)$, that is, $\tilde{\nabla}$, ∇ and $\nabla^\perp$ are expressed by

$$\tilde{\nabla}_X(e_A) = \sum_{B=1}^{m} \varepsilon_B \tilde{\omega}_{AB}(X) e_B,$$
$$\nabla_X(e_i) = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(X) e_j$$

and

$$\nabla^\perp_X(e_r) = \sum_{s=n+1}^{m} \varepsilon_s \omega^\perp_{rs}(X) e_s$$

for $X \in TM_t$, respectively.

Throughout this section, we define the following range of indices: $1 \le A, B \le m$, $1 \le i, j, k, l \le n$ and $n + 1 \le r, s \le m$. Also, let h be the second fundamental form of M_t, and we describe the coefficients of h as h^r_{ij}, that is, $h(e_i, e_j) = \sum_{r=n+1}^{m} h^r_{ij} e_r$. For the simplicity, we denote A_{e_r} by A_r, where A_{e_r} is the shape operator of M_t in the direction e_r. Noting $\omega_{AB} + \omega_{BA} = 0$, the Gauss formula is given by

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h^r_{ki} e_r, \tag{2.2}$$

and the Weingarten formula is given by

$$\tilde{\nabla}_{e_k} e_r = -A_r(e_k) + \sum_{s=n+1}^{m} \varepsilon_s \omega^\perp_{rs}(e_k) e_s. \tag{2.3}$$

The mean curvature vector H and the scalar curvature S of M_t in $\mathbb{E}^m_\nu$ are defined by

$$H = \frac{1}{n} \sum_{r=n+1}^{m} \varepsilon_r \mathrm{tr} A_r e_r \tag{2.4}$$

and

$$S = n^2\langle H, H\rangle - \|h\|^2, \tag{2.5}$$

respectively, where $\|h\|^2 = \sum_{i,j=1}^{n}\sum_{r=n+1}^{m} \varepsilon_i\varepsilon_j\varepsilon_r h_{ij}^r h_{ji}^r$. Denote by $\hat{\nabla}h$ the covariant derivative of h with respect to ∇ and $\nabla^{\perp}$. Let

$$(\hat{\nabla}_{e_k}h)(e_i, e_j) = \sum_{r=n+1}^{m} \varepsilon_r h_{ij;k}^r e_r. \tag{2.6}$$

Then we have

$$h_{ij;k}^r = h_{jk;i}^r, \tag{2.7}$$

$$h_{jk;i}^r = e_i(h_{jk}^r) - \sum_{l=1}^{n} \varepsilon_l(h_{lk}^r\omega_{jl}(e_i) + h_{lj}^r\omega_{kl}(e_i)) + \sum_{s=n+1}^{m} \varepsilon_s h_{jk}^s \omega_{sr}^{\perp}(e_i), \tag{2.8}$$

$$R^{\perp}(e_j, e_k; e_r, e_s) = \langle [A_r, A_s](e_j), e_k\rangle = \sum_{i=1}^{n} \varepsilon_i(h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s), \tag{2.9}$$

where $R^{\perp}$ is the normal curvature tensor of M_t.

Let $\mathbf{x} : M_t \hookrightarrow \mathbb{S}_{\nu}^{m-1}(c)$ or $\mathbb{H}_{\nu-1}^{m-1}(-c) \subset \mathbb{E}_{\nu}^{m}$ be an isometric immersion. Denote by h and H the second fundamental form and the mean curvature vector of M_t in $\mathbb{E}_{\nu}^{m}$, respectively. Let $\hat{h}$ and $\hat{H}$ be the second fundamental form and of the mean curvature vector of M_t in $\mathbb{S}_{\nu}^{m-1}(c)$ or $\mathbb{H}_{\nu-1}^{m-1}(-c)$, respectively. Then, h and H are written by $\hat{h}$ and $\hat{H}$ as

$$H = \hat{H} - \varepsilon c\mathbf{x}, \tag{2.10}$$

$$h = \hat{h}(X, Y) - \varepsilon c\langle X, Y\rangle\mathbf{x}. \tag{2.11}$$

Hence, (2.5) is rewritten as

$$S = \varepsilon cn(n-1) + n^2\langle \hat{H}, \hat{H}\rangle - \|\hat{h}\|^2, \tag{2.12}$$

where $\varepsilon = +1$ if in $\mathbb{S}_{\nu}^{m-1}(c)$ and $\varepsilon = -1$ if in $\mathbb{H}_{\nu-1}^{m-1}(-c)$.

The gradient vector field ∇f of $f \in C^{\infty}(M_t)$ is defined by $\nabla f = \sum_{i=1}^{n} \varepsilon_i e_i(f)e_i$, and Laplacian operator Δ of M_t with respect to the induced metric is given by $\Delta = \sum_{i=1}^{n} \varepsilon_i(\nabla_{e_i}e_i - e_ie_i)$.

Let λ be a real constant number, and let $\mathcal{H}$ be the mean curvature of M_t^n in $\bar{M}_{\nu}^{n+1}$, that is, $H = \varepsilon_{n+1}\mathcal{H}e_{n+1}$.

Proposition 2.1.1 ([1])**.** *Let M_t^n be a hypersurface in $\bar{M}_{\nu}^{n+1}$. Then $\Delta H = \lambda H$ if and only if one of the following statemnts holds:*

(i) *M_t^n is minimal in $\bar{M}_{\nu}^{n+1}$.*

(ii) *M_t^n has nonzero constant mean curvature $\mathcal{H}$ and* $\mathrm{tr}(S^2) = (1/n)\mathrm{tr}(S)^2$.

Moreover, the constant λ is always given by $\lambda = n\langle H, H\rangle = n(c + \varepsilon_{n+1}\mathcal{H}^2)$.

2.2 Frenet frame field along a null curves

Definition 2.2.1. A tangent vector v of pseudo-Riemannian manifold M is said to be *spacelike* if $\langle v, v\rangle > 0$ or $v = 0$, *timelike* if $\langle v, v\rangle < 0$, and *null* if $\langle v, v\rangle = 0$ and $v \neq 0$. Especially, a curve $\gamma(s)$ in M is said to be a null curve if $\dot{\gamma}(s)$ is null for all s.

Definition 2.2.2. A basis $\{E_1, \ldots, E_n\}$ of $\mathbb{E}^n_\nu$ is called an *orthonormal basis* if $E_1, \ldots, E_n$ satisfy

$$\langle E_i, E_j\rangle = \begin{cases} -\delta_{ij} & (i, j = 1, \ldots, \nu) \\ \delta_{ij} & (i, j = \nu + 1, \ldots, n) \\ 0 & (1 \leq i < j \leq n). \end{cases}$$

Let $B = \{X_1, Y_1, \ldots, X_p, Y_p, E_1, \ldots, E_{n-2p}\}$ be a basis of $\mathbb{E}^n_\nu$, where $1 \leq p \leq n/2$. Let k, l and i, j be $1 \leq k, l \leq p$ and $1 \leq i, j \leq n - 2p$, respectively. Then, B is called an *pseudo-orthonormal basis* if $X_1, Y_1, \ldots, X_p, Y_p, E_1, \ldots, E_{n-2p}$ satisfy

$$\langle X_k, X_k\rangle = \langle Y_k, Y_k\rangle = 0,\ \langle X_k, Y_l\rangle = -\delta_{kl},$$
$$\langle X_k, E_i\rangle = \langle Y_k, E_i\rangle = 0,\ \langle E_i, E_j\rangle = \varepsilon_i\delta_{ij},$$

where $\varepsilon_i = -1$ if $1 \leq i \leq q$ and $\varepsilon_i = 1$ if $q + 1 \leq i \leq n - 2p$ for $p + q = \nu$.

In the case where $(M, \langle\ ,\ \rangle, \nabla)$ is an n-dimensional Riemannian manifold, a *Frenet curve* and its *order* are defined for C^∞-curve γ in M as follows. Let γ be parameterized by s, and we put $V_1 := \dot{\gamma}$. Then, γ is called a *Frenet curve of order* d if there is an orthonormal frame field $(V_1, \ldots, V_d)$ and differential positive functions $k_1, \ldots, k_{d-1}$ such that

$$\nabla_{\dot{\gamma}} V_j(s) = -k_{j-1}(s)V_{j-1}(s) + k_{j+1}(s)V_{j+1}(s), \tag{2.13}$$

where $1 \leq d \leq n$, $1 \leq j \leq d$ and $V_0 \equiv V_{d+1} \equiv 0$, and $(V_1, \ldots, V_d)$ is said to be a *Frenet frame field of* γ. We put $\gamma^{(2)} = \nabla_{\dot{\gamma}}\dot{\gamma}$ and $\gamma^{(i)} = \nabla_{\dot{\gamma}}\gamma^{(i-1)}$ for $1 \leq i \leq d$. Then, we find $\{\dot{\gamma}, \ldots, \gamma^{(i)}\}$ is a linearly independent family by substituting (2.13) for $\nabla_{\dot{\gamma}}\gamma^{(i)}$ inductively. In the case where $(M, \langle\ ,\ \rangle, \nabla)$ is a pseudo-Riemannian manifold, an order of a curve in M is defined as follows.

Definition 2.2.3. Let γ be a curve in pseudo-Riemannian manifold M^n_ν and $1 \leq i \leq n$. Then, d is called an *order of* γ if d is the largest number of i such that $\{\dot{\gamma}, \ldots, \gamma^{(i)}\}$ is a linearly independent family.

Definition 2.2.4. Let γ be a null curve of order 3 in $\mathbb{S}^3_1(\subset \mathbb{E}^4_1)$ or $\mathbb{H}^3_1(\subset \mathbb{E}^4_2)$, and let (A, B, C) be a pseudo-orthonormal tangent frame field of $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ along γ. Then, (A, B, C) is called the *Cartan frame field along* γ if it satisfies

$$\begin{cases} \dot{\gamma}(s) = A(s), \\ \dot{A}(s) = k_1(s)C(s), \\ \dot{B}(s) = k_2C(s) + \varepsilon\gamma, \\ \dot{C}(s) = k_2A(s) + k_1(s)B(s) \end{cases} \tag{2.14}$$

for some positive-valued functions k_1 and k_2, where $\varepsilon = +1$ if γ is in $\mathbb{S}^3_1$ or $\varepsilon = -1$ if γ is in $\mathbb{H}^3_1$. Let M be the Lorentzian ruled surface in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ defined as the image of the immersion $\mathbf{x}(s,t) = \gamma(s) + tB(s)$. If k_2 is non-zero constant, then M is called the *B-scroll over* γ.

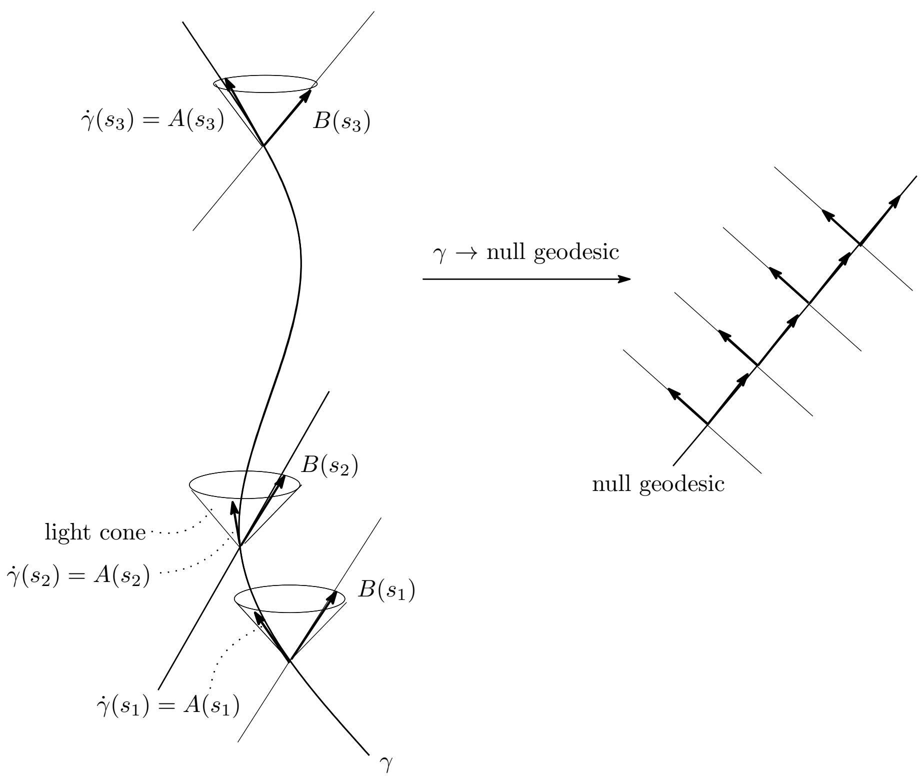

Figure 2.1: B-scroll over γ

Fact 2.2.1 ([2], [17])**.** The unit normal vector field N of the B-scroll in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ is given by $N = k_2 tB(s) + C(s)$. The shape operator A_N with respect to $(\partial\mathbf{x}/\partial s, \partial\mathbf{x}/\partial t)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 \\ -k_1(s) & -k_2 \end{pmatrix}.$$

Thus, the mean curvature H is a non-zero constant $-k_2$, Gaussian curvature is a constant and the minimal polynomial of its shape operator is $P(x) = (x + k_2)^2$.

Remark 2.2.1. Let M be a B-scroll in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ and H the mean curvature vector field of M in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$. Then M satisfies $\Delta H = \lambda H$ for a real constant λ.

Chapter 3

Classification of Lorentzian hypersurfaces in a de Sitter space and anti-de Sitter space

In this chapter, we determine the type number of the pseudo-hyperbolic Gauss map of Lorentzian hypersurfaces of constant mean and scalar curvature and non-diagonalizable shape operator in pseudo-sphere or pseudo-hyperbolic space. First, we consider the type number of the pseudo-hyperbolic Gauss map of such Lorentzian hypersurfaces in the 3-dimensional de Sitter space $\mathbb{S}^3_1$ and anti-de Sitter space $\mathbb{H}^3_1$.

3.1 Known results and main theorem

D.S. Kim and Y.H. Kim[14] classified the Lorentzian surfaces of constant mean and Gaussian curvatures and non-diagonalizable shape operator in the 3-dimensional de Sitter $\mathbb{S}^3_1$ and anti-de Sitter $\mathbb{H}^3_1$ space as follows.

Theorem 3.1 ([14]). *Let M^2_1 be a Lorentzian surface in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$. If the mean and Gaussian curvatures are constant and the shape operator is not diagonalizable at a point, then M^2_1 is an open part of a complex circle or a B-scroll.*

B. Bektaş, E.Ö. Canfes and U. Dursun [2] determined the type number of the pseudo-spherical Gauss map of an oriented Lorentzian surface in $\mathbb{S}^3_1$ of non-zero constant mean curvature and non-diagonalizable shape operator at a point.

Theorem 3.2 ([2]). *An oriented Lorentzian surface in $\mathbb{S}^3_1$ of constant mean curvature and non-diagonalizable shape operator is of null 2-type pseudo-spherical Gauss map if and only if it is an open part of a non-flat B-scroll over a null curve.*

In this chapter, we determined the type numbers of the pseudo-hyperbolic Gauss maps of such Lorentzian surfaces in $\mathbb{H}^3_1$ of constant mean and Gaussian curvatures and non- diagonalizable shape operator at a point. The following theorem is one of the main results.

Theorem 3.3. *Let M be an oriented Lorentzian surface in $\mathbb{H}^3_1$ of constant mean and Gaussian curvatures and non-diagonalizable shape operator. The following facts hold.*

(i) *M is an open part of a complex circle of radius -1 if and only if the pseudo-hyperbolic Gauss map is of 1-type.*

(ii) *If M is the complex circle of radius κ ($\mathrm{Re}(\kappa) = -1$, $\kappa \neq -1$), then the pseudo-hyperbolic Gauss map is of infinite type.*

(iii) *M is an open part of a non-flat B-scroll if and only if the pseudo-hyperbolic Gauss map is of null 2-type.*

(iv) *If M is a flat B-scroll, then the pseudo-hyperbolic Gauss map is of infinite type.*

3.2 Finite typeness of pseudo-hyperbolic Gauss map

Definition 3.2.1. Let $\phi : M_t \to \mathbb{H}^{m-1}_{\nu-1} \subset \mathbb{E}^m_\nu$ (resp. $\phi : M_t \to \mathbb{S}^{m-1}_\nu \subset \mathbb{E}^m_\nu$) be a smooth map. Then ϕ is said to be *of finite type* in $\mathbb{H}^{m-1}_{\nu-1}$ (resp. in $\mathbb{S}^{m-1}_\nu$) if ϕ has the following spectral decomposition:

$$\phi = \phi_1 + \phi_2 + \cdots + \phi_k,$$

where $\phi_i : M_t \to \mathbb{E}^m_\nu$'s are non-constant map such that $\Delta\phi_i = \lambda_i\phi_i$ with $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If ϕ has this spectral decomposition and λ_i's are mutually distinct constant, then the map ϕ is said to be *of k-type*, and when one of λ_i's is equal to zero, the map ϕ is said to be *of null k-type.*

For a map of finite type, the following fact holds.

Lemma 3.2.1. *Let $\phi : M_t \to \mathbb{H}^{m-1}_{\nu-1}$ or $\mathbb{S}^{m-1}_\nu$ be a smooth map. If $\Delta^2\phi = 0$, then $\Delta\phi = 0$ or ϕ is of infinite type.*

Proof. Assume that ϕ is of finite type and ϕ has the following spectral decomposition: $\phi = \phi_1 + \phi_2 + \cdots + \phi_k$ with $\Delta\phi_i = \lambda_i\phi_i$ for $\lambda_i \in \mathbb{R}$ and $i = 1, 2, \ldots, k$. Then we have

$$0 = \Delta^2\phi = \lambda_1^2\phi_1 + \cdots + \lambda_k^2\phi_k.$$

Therefore we have $k = 1$ and $\lambda_1 = 0$, that is, $\Delta\phi = 0$. □

Let $G(n+1, m)$ be the Grassmannian manifold consisting of $(n+1)$-dimensional oriented non-degenerate subspaces of $\mathbb{E}^m_\nu$, and let $G(n+1, m)_{t+1}$ be the submanifold of $G(n+1, m)$ consisting of $(n+1)$-dimensional oriented non-degenerate subspaces of index $t+1$ of $\mathbb{E}^m_\nu$. Let $(\widetilde{e}_1, \ldots, \widetilde{e}_m)$ and $(\widehat{e}_1, \ldots, \widehat{e}_m)$ be two orthonormal frames of

$\mathbb{E}^m_\nu$. Let $\widetilde{e}_{i_1} \wedge \cdots \wedge \widetilde{e}_{i_{n+1}}$ and $\widehat{e}_{j_1} \wedge \cdots \wedge \widehat{e}_{j_{n+1}}$ be two vectors in $\bigwedge^{n+1} \mathbb{E}^m_\nu$. Define an indefinite inner product $\langle\langle , \rangle\rangle$ on $\bigwedge^{n+1} \mathbb{E}^m_\nu$ by

$$\langle\langle \widetilde{e}_{i_1} \wedge \cdots \wedge \widetilde{e}_{i_{n+1}} , \widehat{e}_{j_1} \wedge \cdots \wedge \widehat{e}_{j_{n+1}} \rangle\rangle = \det\left(\langle \widetilde{e}_{i_l}, \widehat{e}_{j_k} \rangle\right), \quad (l, k = 1, \ldots, n+1).$$

Therefore, we may identify $\bigwedge^{n+1} \mathbb{E}^m_\nu$ with the pseudo-Euclidean space $\mathbb{E}^N_q$ for some positive integer q, where $N = \binom{m}{n+1}$. The Grassmannian manifold $G(n+1, m)_{t+1}$ can be imbedded into a pseudo-Euclidean space $\bigwedge^{n+1} \mathbb{E}^m_\nu \simeq \mathbb{E}^N_q$ by assigning $\Pi \in G(n+1, m)_{t+1}$ to $\widetilde{e}_1 \wedge \cdots \wedge \widetilde{e}_{n+1}$, where $(\widetilde{e}_1, \ldots, \widetilde{e}_{n+1})$ is an orthonormal basis of Π compatible with the orientation of Π.

Let $\mathbf{x} : M_t \hookrightarrow \mathbb{H}^{m-1}_{\nu-1} \subset \mathbb{E}^m_\nu$ be an isometric immersion. For the immersion $\mathbf{x}$, we define a map $\tilde{\nu} : M_t \to G(n+1, m)_{t+1}$ by

$$\tilde{\nu}(p) = \mathbf{x}(p) \wedge \mathbf{x}_*(e^p_1) \wedge \mathbf{x}_*(e^p_2) \wedge \cdots \wedge \mathbf{x}_*(e^p_n) \quad (p \in M_t),$$

where $(e^p_1, \ldots, e^p_n)$ is an orthonormal frame of T_pM_t compatible with the orientation of M_t. The map $\tilde{\nu}$ is called the *pseudo-hyperbolic Gauss map of* $\mathbf{x}$. In the sequel, we rewrite $\mathbf{x}_*(e_i)$ by e_i.

Let $(e_1, \ldots, e_n)$ be an local orthonormal frame field of TM_t compatible with the orientation of M_t and $(e_{n+1}, \ldots, e_m)$ an local orthonormal frame field of $T^\perp M_t$ defined an open set U of M_t, respectively.
The first derivative of the pseudo-hyperbolic Gauss map $\tilde{\nu}$ is given by

$$e_i\tilde{\nu} = \sum_{k=1}^{n} \sum_{r=n+1}^{m-1} \varepsilon_r h^r_{ik} \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n. \tag{3.1}$$

Yeğin and Dursun proved the following fact.

Lemma 3.2.2 ([22]). *Let M_t be an n-dimensional oriented pseudo-Riemannian submanifold of index t of a pseudo-hyperbolic $\mathbb{H}^{m-1}_\nu \subset \mathbb{E}^m_{\nu+1}$. Then the Laplacian of the pseudo-hyperbolic Gauss map $\tilde{\nu} : M_t \to G(n+1, m) \subset \mathbb{E}^N_q$, $N = \binom{m}{n+1}$ for some q is given by*

$$\begin{aligned} \Delta\tilde{\nu} = {} & \|\hat{h}\|^2\tilde{\nu} + n\hat{H} \wedge e_1 \wedge \cdots \wedge e_n - n\sum_{k=1}^{n} \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n \\ & + \sum_{\substack{i,k=1 \\ j\neq k}}^{n} \sum_{\substack{r,s=n+1 \\ r<s}}^{m-1} \varepsilon_r\varepsilon_s R^r_{sjk} \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge e_n \end{aligned} \tag{3.2}$$

where $R^r_{sjk} = R^D(e_j, e_k; e_r, e_s)$.

In case of $n = m - 2$, Yeğin and Dursun have the following fact.

Lemma 3.2.3 ([22])**.** *For an oriented pseudo-Riemannian hypersurface M_t with index t of $\mathbb{H}^{n+1}_{\nu-1} \subset \mathbb{E}^{n+2}_{\nu}$ we have*

$$\Delta(e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n) = -n\hat{\mathcal{H}}\tilde{\nu} - ne_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n \tag{3.3}$$

where $\hat{\mathcal{H}}$ is the mean curvature M_t in $\mathbb{H}^{n+1}_{\nu-1}$, that is, $\hat{H} = \varepsilon_{n+1}\hat{\mathcal{H}}e_{n+1}$.

Lemma 3.2.4. *If there exists a polynomial $P(t) = (t - \lambda_1)(t - \lambda_2)$ with mutually distinct roots $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $P(\Delta)\tilde{\nu} = 0$, then $\tilde{\nu}$ is of at most 2-type or infinite type.*

Proof. Assume that $\tilde{\nu}$ is of finite type and it has the following spectral decomposition; $\tilde{\nu} = \tilde{\nu}_{\hat{\lambda}_1} + \cdots + \tilde{\nu}_{\hat{\lambda}_k}$ with $\Delta\tilde{\nu}_{\hat{\lambda}_i} = \hat{\lambda}_i\tilde{\nu}_{\hat{\lambda}_i}$ $(1 \leq i \leq k)$. Then we have

$$\begin{aligned} 0 = P(\Delta)\tilde{\nu} &= P(\Delta)(\tilde{\nu}_{\hat{\lambda}_1} + \cdots + \tilde{\nu}_{\hat{\lambda}_k}) \\ &= \sum_{i=1}^{k}(\hat{\lambda}_i - \lambda_1)(\hat{\lambda}_i - \lambda_2)\tilde{\nu}_{\hat{\lambda}_i}. \end{aligned}$$

Thus, we have $(\hat{\lambda}_i - \lambda_1)(\hat{\lambda}_i - \lambda_2) = 0$ $(1 \leq i \leq k)$. Hence, we have $\hat{\lambda}_i = \lambda_1$ or λ_2 for all i, that is, $\tilde{\nu}$ is of at most 2-type. Hence the statement of this lemma follows. $\square$

3.3 The proof of Theorem 3.3

Let $\mathbb{C}^{n+1}$ be the $(n+1)$-dimensional complex vector space which is identified with $\mathbb{R}^{2n+2}$. Define a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ of the $\mathbb{C}^{n+1}(= \mathbb{R}^{2n+2})$ by

$$\langle z, w\rangle = \mathrm{Re}\left(\sum_{i=1}^{n+1} z_i w_i\right), \tag{3.4}$$

where $z = (z_1, \ldots, z_{n+1})$ and $w = (w_1, \ldots, w_{n+1}) \in \mathbb{C}^{n+1}$. Note that $\langle z, z\rangle = \sum_{i=1}^{n+1} x_i^2 - \sum_{i=1}^{n+1} y_i^2$ when $z = (x_1 + \sqrt{-1}y_1, \ldots, x_{n+1} + \sqrt{-1}y_{n+1})$ $(x_i, y_i \in \mathbb{R},\ i = 1, \ldots, n+1)$. Let $\tilde{g}$ be a pseudo-Euclidean metric of index $n+1$ on the $(2n+2)$-dimensional affine space $\mathbb{R}^{2n+2}(= \mathbb{C}^{n+1})$ induced from $\langle\ ,\ \rangle$.

Definition 3.3.1. Fix a non-zero complex number κ. We put

$$S^n_{\mathbb{C}}(\kappa) = \{(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} = \mathbb{E}^{2n+2}_{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = \kappa\},$$

that is, $\sum_{i=1}^{n+1}(x_i^2 - y_i^2) = \mathrm{Re}(\kappa)$ and $2\sum_{i=1}^{n+1} x_i y_i = \mathrm{Im}(\kappa)$ for $z_i = x_i + \sqrt{-1}y_i$ $(x_i, y_i \in \mathbb{R})$. This submanifold $S^n_{\mathbb{C}}(\kappa)$ is called a *complex sphere of radius κ*. In particular, when $n = 1$, it is called a *complex circle of radius κ*.

Remark 3.3.1.
$$S^n_{\mathbb{C}}(\kappa) \subset \begin{cases} \mathbb{S}^{2n+1}_{n+1}(\mathrm{Re}(\kappa)) & (\text{if} \quad \mathrm{Re}(\kappa) > 0) \\ \\ \mathbb{H}^{2n+1}_{n}(\mathrm{Re}(\kappa)) & (\text{if} \quad \mathrm{Re}(\kappa) < 0) \end{cases}$$

The complex circle $S^1_{\mathbb{C}}(\kappa) \subset \mathbb{H}^3_1$ is parameterized as $\mathbf{x}(z) = \sqrt{\kappa}(\cos z, \sin z)$ $(z \in \mathbb{C})$.

Proof of (i) and (ii) of Theorem 3.3. Let M be the complex circle of radius κ in $\mathbb{H}^3_1$. Then, M is parameterized as

$$\mathbf{x}(z) = \sqrt{\kappa}(\cos z, \sin z) \quad (z \in \mathbb{C}),$$

where $\sqrt{\kappa}$ is one (with smaller argument) of squared roots of κ. Note that M is included by $\mathbb{H}^3_1$ because of $\mathrm{Re}(\kappa) = -1$. For the convenience, we put $\sqrt{\kappa} = d_1 + \sqrt{-1}d_2$ and $z = x + \sqrt{-1}y$. By simple calculations, we have

$$\langle \mathbf{x}_x, \mathbf{x}_x \rangle = -1, \quad \langle \mathbf{x}_x, \mathbf{x}_y \rangle = -2d_1d_2, \quad \langle \mathbf{x}_y, \mathbf{x}_y \rangle = 1.$$

Also, we can show that the unit normal vector field N of M in $\mathbb{H}^3_1$ is given by $N = (d_2 + \sqrt{-1}d_1)(\cos z, \sin z)$. With respect to the frame field $(\mathbf{x}_x, \mathbf{x}_y)$, the shape operator A_N in the direction N is expressed as

$$A_N = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where $\alpha = -2d_1d_2/(d_1^2 + d_2^2)$ and $\beta = 1/(d_1^2 + d_2^2)$. Put $e_1 := \mathbf{x}_x$, $\tilde{e}_2 := \mathbf{x}_y - \langle \mathbf{x}_y, e_1 \rangle e_1$ and $e_2 := \tilde{e}_2/|\tilde{e}_2|$. Then (e_1, e_2) forms an orthonormal tangent frame field on M. Note that $\langle e_1, e_1 \rangle = -1$, $\langle e_2, e_2 \rangle = 1$ and $|\tilde{e}_2|^2 = 1 + 4d_1^2d_2^2$. With respect to (e_1, e_2), the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} 0 & |\tilde{e}_2|\beta \\ -|\tilde{e}_2|\beta & 2\alpha \end{pmatrix}.$$

Thus, by remarking $\alpha = -2d_1d_2\beta$ and $d_1^2 - d_2^2 = -1$, we obtain $\hat{\mathcal{H}} = \alpha$ and $\|\hat{h}\|^2 = 2(\alpha^2 - \beta^2)$, where $\hat{\mathcal{H}}$ and $\hat{h}$ denote the mean curvature and the second fundamental form of M in $\mathbb{H}^3_1$. Hence, M is flat by (2.12). By (3.2) and (3.3), we obtain

$$\Delta\tilde{\nu} = 2(\alpha^2 - \beta^2)\tilde{\nu} + 2\alpha N \wedge e_1 \wedge e_2, \tag{3.5}$$
$$\Delta^2\tilde{\nu} = 4((\alpha^2 - \beta^2)^2 - \alpha^2)\tilde{\nu} + 4\alpha(\alpha^2 - \beta^2 - 1)N \wedge e_1 \wedge e_2. \tag{3.6}$$

Hence,

$$\Delta^2\tilde{\nu} - 2(\alpha^2 - \beta^2 - 1)\Delta\tilde{\nu} + 4\beta^2\tilde{\nu} = 0. \tag{3.7}$$

Therefore $\tilde{\nu}$ is either of finite type with type number $k \leq 2$ or of infinite type by Lemma 3.2.4.

If $\kappa = -1$, then we have $d_1 = 0$ and hence $\alpha = 0$. Therefore it follows from (3.5) that $\Delta\tilde{\nu} = -2\beta^2\tilde{\nu}$, that is, $\tilde{\nu}$ is of 1-type. If $\kappa \neq -1$, then we have $\alpha \neq 0$. Hence it follows from (3.5) that $\tilde{\nu}$ is of not 1-type. Therefore $\tilde{\nu}$ is of 2-type or of infinite type. Suppose that it is of 2-type, and that $\tilde{\nu}$ has a decomposition $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$ with $\Delta\tilde{\nu}_1 = \lambda_1\tilde{\nu}_1$ and $\Delta\tilde{\nu}_2 = \lambda_2\tilde{\nu}_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ are mutually distinct. From (3.5) and (3.6), $\tilde{\nu}_1$ and $\tilde{\nu}_2$ can be expressed as

$$\tilde{\nu}_1 = a\tilde{\nu} + bN \wedge e_1 \wedge e_2, \tag{3.8}$$

$$\tilde{\nu}_2 = (1-a)\tilde{\nu} - bN \wedge e_1 \wedge e_2 \tag{3.9}$$

for some constants a and b. By substituting (3.8) and (3.9) into (3.5) and (3.6), and comparing coefficient of $\tilde{\nu}$ and $N \wedge e_1 \wedge e_2$, we have

$$\begin{cases} (\lambda_1 - \lambda_2)a = 2(\alpha^2 - \beta^2) - \lambda_2, \\ (\lambda_1 - \lambda_2)b = 2\alpha, \\ (\lambda_1^2 - \lambda_2^2)a = 4(\alpha^2 - \beta^2)^2 - 4\alpha^2 - \lambda_2^2, \\ (\lambda_1^2 - \lambda_2^2)b = 4\alpha(\alpha^2 - \beta^2 - 1). \end{cases} \tag{3.10}$$

Thus, we obtain

$$\lambda_2^2 - 2(\alpha^2 - \beta^2 - 1)\lambda_2 + 4\beta^2 = 0. \tag{3.11}$$

The discriminant of (3.11) is

$$\begin{aligned} (\alpha^2 - \beta^2 - 1)^2 - 4\beta^2 &= (\alpha^2 - \beta^2)^2 - 1 \\ &= -\frac{1}{(d_1^2 + d_2^2)^4}(16d_1^8 + 16d_1^6 + 32d_1^4 + 16d_1^2) \\ &< 0. \end{aligned} \tag{3.12}$$

Therefore, there is no $\lambda_2 \in \mathbb{R}$ satisfying (3.11). Thus a contradiction arises. Hence, $\tilde{\nu}$ is of infinite type. Conversely, assume that the pseudo-hyperbolic Gauss map is of 1-type. From (3.2), we obtain that $\|\hat{h}\|^2$ is constant and $\hat{\mathcal{H}} = 0$. Hence, the Gaussian curvature is constant from (2.12). By Fact 3.1, (iii) and (iv) of Theorem 3.3 (which will be shown in the next section), M is an open part of a complex circle of radius -1. $\square$

Proof of (iii) and (iv) of Theorem 3.3. Let (A, B, C) be the Cartan frame field along a null curve γ in $\mathbb{H}^3_1$ given by Definition 2.2.4. Then, the immersion $\mathbf{x}(s,t) = \gamma(s) + tB(s)$ parametrizes the B-scroll over a null curve γ. We have

$$\mathbf{x}_s(s,t) = A(s) + t(k_2C(s) - \gamma(s)) \quad \text{and} \quad \mathbf{x}_t(s,t) = B(s),$$

and hence

$$\langle \mathbf{x}_s, \mathbf{x}_s \rangle = t^2(k_2^2 - 1), \quad \langle \mathbf{x}_s, \mathbf{x}_t \rangle = -1 \quad \text{and} \quad \langle \mathbf{x}_t, \mathbf{x}_t \rangle = 0.$$

The unit normal vector field of $\mathbf{x}$ is given by $N(s,t) = k_2 t B(s) + C(s)$. With respect to the frame $(\mathbf{x_s}, \mathbf{x_t})$, the shape operator A_N in the direction N is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 \\ -k_1(s) & -k_2 \end{pmatrix}. \tag{3.13}$$

When $|k_2| > 1$ and $t \neq 0$, an orthonormal frame field (e_1, e_2) on M is given by

$$e_1 = \frac{\mathbf{x_s}}{|\mathbf{x_s}|}, \quad e_2 = e_1 + |\mathbf{x_s}|\mathbf{x_t}.$$

By simple calculations, we have

$$\begin{aligned}
A_N(e_1) &= \frac{1}{|\mathbf{x_s}|} A_N(\mathbf{x_s}) \\
&= \frac{1}{|\mathbf{x_s}|}(-k_2 \mathbf{x_s} - k_1(s)\mathbf{x_t}) \\
&= -\frac{1}{|\mathbf{x_s}|}\left(k_2|\mathbf{x_s}|e_1 + k_1(s)\frac{1}{|\mathbf{x_s}|}(e_2 - e_1)\right) \\
&= \left(-k_2 + \frac{k_1(s)}{t^2|k_2^2 - 1|}\right) e_1 - \frac{k_1(s)}{t^2|k_2^2 - 1|} e_2, \\
A_N(e_2) &= A_N(e_1) + \frac{\langle \mathbf{x_s}, \mathbf{x_s}\rangle}{|\mathbf{x_s}|} A_N(\mathbf{x_t}) \\
&= \frac{k_1(s)}{t^2|k_2^2 - 1|} e_1 + \left(-k_2 - \frac{k_1(s)}{t^2|k_2^2 - 1|}\right) e_2.
\end{aligned}$$

Thus, with respect to an orthonormal frame (e_1, e_2), the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 + \frac{k_1(s)}{t^2|k_2^2-1|} & \frac{k_1(s)}{t^2|k_2^2-1|} \\ -\frac{k_1(s)}{t^2|k_2^2-1|} & -k_2 - \frac{k_1(s)}{t^2|k_2^2-1|} \end{pmatrix}$$

and hence we have $\hat{\mathcal{H}} = -k_2$ and $\|\hat{h}\|^2 = 2k_2^2$. When $|k_2| < 1$ and $t \neq 0$, we put $e_1 = \mathbf{x_s}/|\mathbf{x_s}|$ and $e_2 = e_1 - |\mathbf{x_s}|\mathbf{x_t}$. Similarly, we have $\hat{\mathcal{H}} = -k_2$ and $\|\hat{h}\|^2 = 2k_2^2$. Hence, M is a non-flat B-scroll by (2.12) when $k_2^2 \neq 1$ and $t \neq 0$. We put $e_3 := N$ and

$$\tilde{\nu}_1 := \frac{1}{k_2^2 - 1}(-\tilde{\nu} + k_2 e_3 \wedge e_1 \wedge e_2), \tag{3.14}$$

$$\tilde{\nu}_2 := \frac{1}{k_2^2 - 1}(k_2^2 \tilde{\nu} - k_2 e_3 \wedge e_1 \wedge e_2). \tag{3.15}$$

It is clear that $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$. Using (3.2) and (3.3), we obtain that $\Delta\tilde{\nu}_1 = 0$ and $\Delta\tilde{\nu}_2 = 2(k_2^2 - 1)\tilde{\nu}_2$. On the other hand, by using (3.1) and (3.3), we have

$$e_1(\tilde{\nu}) = \varepsilon_1\left(-k_2 + \frac{k_1(s)}{t^2(k_2^2 - 1)}\right)\mathbf{x} \wedge e_3 \wedge e_2 - \varepsilon_1 \frac{k_1(s)}{t^2(k_2^2 - 1)} \mathbf{x} \wedge e_1 \wedge e_3,$$

$$e_1(e_3 \wedge e_1 \wedge e_2) = \varepsilon_1 e_3 \wedge \mathbf{x} \wedge e_2,$$

and hence

$$e_1(\tilde{\nu}_1) = \frac{\varepsilon_1 k_1(s)}{t^2(k_2^2-1)^2}(\mathbf{x} \wedge e_3 \wedge e_2 - \mathbf{x} \wedge e_1 \wedge e_3) \neq 0.$$

Therefore $\tilde{\nu}$ is of null 2-type.

When $k_2^2 = 1$ or $t = 0$, an orthonormal frame field (e_1, e_2) on M is given by

$$e_1 = \frac{1}{\sqrt{2}}(\mathbf{x_s} + \mathbf{x_t}), \quad e_2 = \frac{1}{\sqrt{2}}(\mathbf{x_s} - \mathbf{x_t}).$$

By simple calculations, we have

$$\begin{aligned} A_N(e_1) &= \left(-k_2 - \frac{k_1(s)}{2}\right) e_1 + \frac{k_1(s)}{2} e_2, \\ A_N(e_2) &= -\frac{k_1(s)}{2} e_1 + \left(-k_2 + \frac{k_1(s)}{2}\right) e_2. \end{aligned}$$

Thus, with respect to an orthonormal frame (e_1, e_2), the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 - \frac{k_1(s)}{2} & -\frac{k_1(s)}{2} \\ \frac{k_1(s)}{2} & -k_2 + \frac{k_1(s)}{2} \end{pmatrix},$$

and hence we have $\hat{\mathcal{H}} = -k_2$ and $\|\hat{h}\|^2 = 2k_2^2$. Hence M is a flat B-scroll by (2.12). Using (3.2) and (3.3), we obtain $\Delta\tilde{\nu} = 2\tilde{\nu} - 2k_2 e_3 \wedge e_1 \wedge e_2 \neq 0$ and $\Delta^2\tilde{\nu} = 0$. Therefore $\tilde{\nu}$ is of infinite type by Lemma 3.2.1.

Conversely, assume that the pseudo-hyperbolic Gauss map $\tilde{\nu}$ is of null 2-type. Then, from (3.1), (3.2) and (3.3), we obtain

$$\Delta\tilde{\nu} = \|\hat{h}\|^2\tilde{\nu} + 2\hat{\mathcal{H}} e_3 \wedge e_1 \wedge e_2, \tag{3.16}$$

$$\begin{aligned} \Delta^2\tilde{\nu} &= (\|\hat{h}\|^2 - 2)\Delta\tilde{\nu} + (\Delta(\|\hat{h}\|^2) - 4\hat{\mathcal{H}}^2 + 2\|\hat{h}\|^2)\tilde{\nu} \\ &\quad - 2\sum_{j=1}^{2} \varepsilon_j e_j(\|\hat{h}\|^2) h_{j1}^3 \mathbf{x} \wedge e_3 \wedge e_2 - 2\sum_{j=1}^{2} \varepsilon_j e_j(\|\hat{h}\|^2) h_{j2}^3 \mathbf{x} \wedge e_1 \wedge e_3. \end{aligned} \tag{3.17}$$

Since $\tilde{\nu}$ is of null 2-type, we can put $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$ with $\Delta\tilde{\nu}_1 = 0$ and $\Delta\tilde{\nu}_2 = \lambda_2\tilde{\nu}_2$ ($\lambda_2 \neq 0$), where $\tilde{\nu}_1$ is non-constant. Then we have $\Delta^2\tilde{\nu} = \lambda_2\Delta\tilde{\nu}$. This together with (3.17) implies $e_j(\|\hat{h}\|^2) = 0$ (i.e. $\|\hat{h}\|^2$ is constant) and $\lambda_2 = \|\hat{h}\|^2 - 2$. Hence the Gaussian curvature is constant, that is, $\Delta(\|\hat{h}\|^2) = 0$. This together with (3.17) implies that $\|\hat{h}\|^2 = 2\hat{\mathcal{H}}$. Hence, from (2.12), we have

$$S = -2 + 4\hat{\mathcal{H}}^2 - \|h\|^2 = -2 + \|\hat{h}\|^2 = \lambda_2 \neq 0.$$

By Fact 3.1, (i) and (ii) of Theorem 3.3, M is an open part of a non-flat B-scroll. $\square$

3.4 Parallel surfaces of B-scroll and complex circle

In this section, we consider the parallel surface of a complex circle and a B-scroll in $\mathbb{H}^3_1$. We study the behavior of the type numbers of the pseudo-hyperbolic Gauss map along the parallel family of those.

Definition 3.4.1. Let $\bar{M}$ be a pseudo-Riemannian manifold and M a pseudo-Riemannian hypersurface of $\bar{M}$ with unit normal vector field N. Let α_{N_p} denote the geodesic in $\bar{M}$ with $\dot{\alpha}_{N_p}(0) = N_p$ at $p \in M$. Then we know that, for $u \in \mathbb{R}$ sufficiently close to 0, the map $\mathbf{x}^u : M \to \bar{M}$ defined by $\mathbf{x}^u(p) := \exp_{\mathbf{x}(p)} uN_p = \alpha_{N_p}(u)$ is an immersion, where $\exp_{\mathbf{x}(p)}$ denotes the exponential map of $\bar{M}$ at $\mathbf{x}(p)$. We denote the image of the immersion $\mathbf{x}$ by M_u. Then, M_u is called the *parallel surface of M at distance u.*

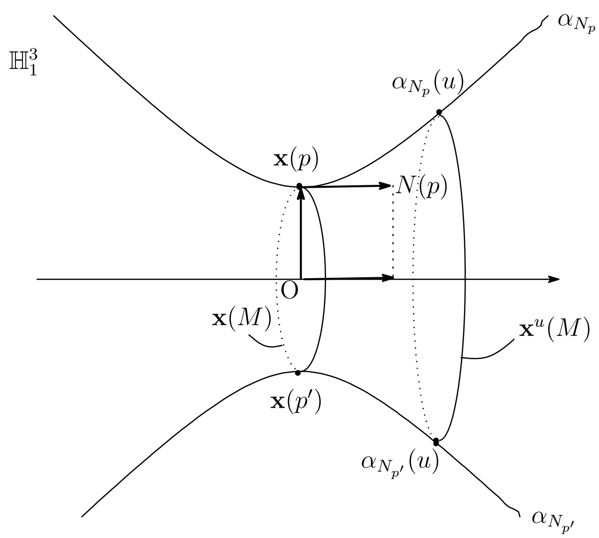

Figure 3.1: parallel surface

Example 3.4.1. Let $\mathbf{x} : M \hookrightarrow \mathbb{S}^{n+1}_1 (\subset \mathbb{E}^{n+2}_1)$ be a Lorentzian hypersurface and N its unit normal vector field. Then $\mathbf{x}^u$ is given by $\mathbf{x}^u(p) = (\cos u)\mathbf{x}(p) + (\sin u)N_p$ $(p \in M)$.

Example 3.4.2. Let $\mathbf{x} : M \hookrightarrow \mathbb{H}^{n+1}_1 (\subset \mathbb{E}^{n+2}_2)$ be a Lorentzian hypersurface and N its unit normal vector field. Then $\mathbf{x}^u$ is given by $\mathbf{x}^u(p) = (\cosh u)\mathbf{x}(p) + (\sinh u)N_p$ $(p \in M)$.

We prove the following two theorem for the pseudo-hyperbolic Gauss map of the parallel family of a complex circle and a B-scroll.

Theorem 3.4. *Let M be a complex circle in $\mathbb{H}^3_1$, and let u be any real number. The parallel surface M^u of M at distance u is a complex circle and the radius κ^u of the complex circle M^u moves over the whole of $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) = -1\}$ when u moves* *over $\mathbb{R}$. Hence the only parallel surface of M has the pseudo-hyperbolic Gauss map of 1-type and other parallel surfaces of M have the pseudo-hyperbolic Gauss map of infinite type.*

Proof. Let M be a complex circle in $\mathbb{H}^3_1$ and κ the radius of M. Let $\sqrt{\kappa} = d_1 + \sqrt{-1}d_2$ for $d_1, d_2 \in \mathbb{R}$. Since $M \subset \mathbb{H}^3_1$, we have $\mathrm{Re}(\kappa) = d_1^2 - d_2^2 = -1$. We remember that a unit normal vector field N of M is given by $N(z) = (d_2 + \sqrt{-1}d_1)(\cos z, \sin z)$. Hence the parallel surface M^u of M at distance u is parameterized by

$$\begin{aligned}\mathbf{x}^u(z) &= \cosh u \cdot (d_1 + \sqrt{-1}d_2)(\cos z, \sin z) + \sinh u \cdot (d_2 + \sqrt{-1}d_1)(\cos z, \sin z)\\ &= \kappa^u(\cos z, \sin z),\end{aligned}$$

where κ^u is the complex number satisfying

$$\sqrt{\kappa^u} = (d_1 \cosh u + d_2 \sinh u) + \sqrt{-1}(d_2 \cosh u + d_1 \sinh u).$$

Thus, M^u is the complex circle of radius κ^u. It is easy to show that κ^u moves over the whole of $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) = -1\}$ when u moves over $(-\infty, \infty)$. There the statement of Theorem 3.4 follows from (i) and (ii) of Theorem 3.3. $\square$

Theorem 3.5. *Let M be a B-scroll in $\mathbb{H}^3_1$ and $u \in \mathbb{R}$ sufficiently close to 0. If M is flat (resp. non-flat), then the parallel surface M_u also is a flat (resp. non-flat) B-scroll. Hence the type numbers of the pseudo-hyperbolic Gauss maps of the parallel surfaces of a B-scroll are equal to that of the original B-scroll.*

Proof. We consider a B-scroll M in $\mathbb{H}^3_1$ parameterized as $\mathbf{x}(s,t) = \gamma(s) + tB(s)$. Since $N(s,t) = k_2 tB(s) + C(s)$, the parallel surface M^u of M is parameterized as

$$\mathbf{x}^u(s,t) = r(u)tB(s) + (\sinh u)C(s) + (\cosh u)\gamma(s),$$

where we set $r(u) = \cosh u + k_2 \sinh u$. By simple calculations, we have

$$\begin{aligned}\frac{\partial \mathbf{x}^u}{\partial s} &= r(u)A(s) + k_1(s)(\sinh u)B(s) + r(u)t(-\gamma(s) + k_2 C(s)),\\ \frac{\partial \mathbf{x}^u}{\partial t} &= r(u)B(s),\end{aligned}$$

and hence

$$\begin{aligned}&\left\langle \frac{\partial \mathbf{x}^u}{\partial s}, \frac{\partial \mathbf{x}^u}{\partial s} \right\rangle = r(u)(-2k_2 \sinh u + r(u)t^2(k_2^2 - 1)),\\ &\left\langle \frac{\partial \mathbf{x}^u}{\partial s}, \frac{\partial \mathbf{x}^u}{\partial t} \right\rangle = -r(u)^2,\\ &\left\langle \frac{\partial \mathbf{x}^u}{\partial t}, \frac{\partial \mathbf{x}^u}{\partial t} \right\rangle = 0.\end{aligned} \tag{3.18}$$

If $r(u) = 0$, then $\mathbf{x}^u$ is not immersion. Hence, we need to assume that

$$\begin{array}{ccc} \operatorname{arctanh}\frac{-1}{k_2} < u < \infty & \text{if} & k_2 > 1, \\ -\infty < u < \operatorname{arctanh}\frac{-1}{k_2} & \text{if} & k_2 < -1, \\ -\infty < u < \infty & \text{if} & |k_2| \leq 1. \end{array} \tag{3.19}$$

The unit normal vector field N^u of M^u is given by

$$N^u(s,t) = \frac{k_2 + r(u)\sinh u}{\cosh u} tB(s) + (\cosh u)C(s) + (\sinh u)\gamma(s).$$

Hence the shape operator A_{N^u} in the direction N^u is expressed with respect to the usual frame $(\partial \mathbf{x}^u/\partial s, \partial \mathbf{x}^u/\partial t)$ as

$$A_{N^u} = \begin{pmatrix} -\alpha & 0 \\ -\frac{k_1(s)}{r(u)^2} & -\alpha \end{pmatrix},$$

where $\alpha = (k_2 + r(u)\sinh u)/(r(u)\cosh u)$. When $\partial \mathbf{x}^u/\partial s$ is non-null, we put

$$e_1 := \frac{1}{\left|\frac{\partial \mathbf{x}^u}{\partial s}\right|}\frac{\partial \mathbf{x}^u}{\partial s}, \quad \tilde{e}_2 := \frac{1}{\left|\frac{\partial \mathbf{x}^u}{\partial s}\right|}\left(-r(u)^2\frac{\partial \mathbf{x}^u}{\partial s} - \left\langle \frac{\partial \mathbf{x}^u}{\partial s}, \frac{\partial \mathbf{x}^u}{\partial s}\right\rangle \frac{\partial \mathbf{x}^u}{\partial t}\right).$$

We have $|\tilde{e}_2| = r(u)^2$. We put $e_2 := (1/r(u)^2)\tilde{e}_2$. Then, with respect to an orthonormal frame (e_1, e_2), the shape operator A_{N^u} is expressed as

$$A_{N^u} = \begin{pmatrix} -\alpha + \beta & -\beta \\ \beta & -\alpha - \beta \end{pmatrix},$$

where $\beta = k_1(s)/(r(u)(-2k_2 \sinh u + r(u)t^2(k_2^2 - 1)))$. Thus, the mean curvature $\hat{\mathcal{H}}_u = -\alpha$ and $\|\hat{h}_u\|^2 = 2\alpha^2$.

When $\partial \mathbf{x}^u/\partial s$ is null, an orthonormal frame field (e_1, e_2) on M_u is given by

$$e_1 = \frac{1}{\sqrt{2}r(u)}\left(\frac{\partial \mathbf{x}^u}{\partial s} + \frac{\partial \mathbf{x}^u}{\partial t}\right), \quad e_2 = \frac{1}{\sqrt{2}r(u)}\left(\frac{\partial \mathbf{x}^u}{\partial s} - \frac{\partial \mathbf{x}^u}{\partial t}\right).$$

With respect to (e_1, e_2), the shape operator A_{N_u} is expressed as

$$A_{N^u} = \begin{pmatrix} -\alpha - \beta' & -\beta' \\ \beta' & -\alpha + \beta' \end{pmatrix},$$

where $\beta' = k_1(s)/2r(u)^2$. Thus, the mean curvature $\hat{\mathcal{H}}_u = -\alpha$ and $\|\hat{h}_u\|^2 = 2\alpha^2$. In both cases, it follows that M_u has constant Gaussian curvature. Hence, M_u is a B-scroll or a complex circle by Fact 3.1. By Theorem 3.4, M_u is a B-scroll. By (2.12), M_u is flat if $\alpha^2 = 1$ and M_u is non-flat if $\alpha^2 \neq 1$.

Assume that M_u is flat for some $u \neq 0$. Then we have $\alpha^2 = 1$ and hence $r(u)(-\sinh u \pm \cosh u) = k_2$. From this relation, we have $k_2 = \pm 1$. Hence we obtain the second-half of statement of this theorem. $\square$

Remark 3.4.1. We put

$$
\begin{aligned}
&u_+ := \begin{cases} \infty & (k_2 \geq -1) \\ \operatorname{arctanh}(\frac{-1}{k_2}) & (k_2 < -1) \end{cases} \qquad \text{and} \\
&u_- := \begin{cases} -\infty & (k_2 \leq 1) \\ \operatorname{arctanh}(\frac{-1}{k_2}) & (k_2 > 1). \end{cases}
\end{aligned} \tag{3.20}
$$

If $u_+ < \infty$ (resp. $u_- > -\infty$), then M_{u_+} (resp. M_{u_-}) is a focal submanifold of M by (3.18).

Remark 3.4.2. In $\mathbb{S}^3_1$, we can derive the following fact similar to Theorem 3.5 and Remark 3.4.1. We put

$$
\begin{aligned}
&u_+ := \begin{cases} \arctan \frac{-1}{k_2} & (k_2 < 0) \\ \frac{\pi}{2} & (k_2 > 0) \end{cases} \qquad \text{and} \\
&u_- := \begin{cases} -\frac{\pi}{2} & (k_2 < 0) \\ \arctan \frac{-1}{k_2} & (k_2 > 0). \end{cases}
\end{aligned} \tag{3.21}
$$

M_{u_+} and M_{u_-} are a focal submanifold of M.

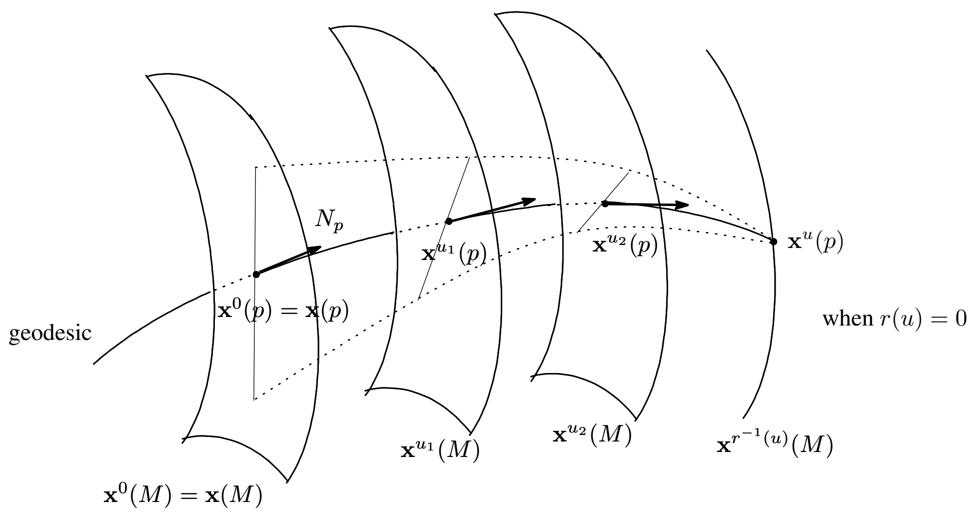

Figure 3.2: focal submanifold of B-scroll

3.5 Generalized umbilical hypersurfaces in $\mathbb{S}^{n+1}_1$ and $\mathbb{H}^{n+1}_1$

In this section, we determined the type number of the pseudo-hyperbolic Gauss map of two hypersurfaces which are natural generalizations of B-scrolls in $\mathbb{H}^{n+1}_1$ and $\mathbb{S}^{n+1}_1$

given by [1].

Definition 3.5.1 ([18])**.** Let M_1^n be an n-dimensional Lorentzian hypersurface in $(n+1)$-dimensional Lorentzian manifold $\tilde{M}_1^{n+1}$, and we assume that the shape operator of M_1^n is not diagonalizable. If the shape operator has the only non-zero real eigenvalue, then M_1^n is called the *generalized umbilical hypersurface*.

Remark 3.5.1. Let M_1^n be a generalized umbilical hypersurface in $\mathbb{S}_1^{n+1}$ or $\mathbb{H}_1^{n+1}$ and H the mean curvature vector field of M in $\mathbb{S}_1^{n+1}$ or $\mathbb{H}_1^{n+1}$. Then M_1^n satisfies $\Delta H = \lambda H$ for a real constant λ (see [1]).

Let $(V, \langle\ ,\ \rangle)$ be a Lorentzian space, that is, a semi-Euclidean space with index 1. If A is a self-adjoint symmetric linear operator on V, A can be put into one of the following four forms [19];

$$\text{(I)}\ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \qquad \text{(II)}\ \begin{pmatrix} a_0 & 0 & & & \\ 1 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{pmatrix},$$

$$\text{(III)}\ \begin{pmatrix} a_0 & 0 & 0 & & & \\ 0 & a_0 & 1 & & & \\ -1 & 0 & a_0 & & & \\ & & & a_1 & & \\ & & & & \ddots & \\ & & & & & a_{n-3} \end{pmatrix}, \quad \text{(IV)}\ \begin{pmatrix} a_0 & b_0 & & & \\ -b_0 & a_0 & & & \\ & & a_1 & & \\ & & & \ddots & \\ & & & & a_{n-2} \end{pmatrix},$$

where $b_0 \neq 0$. The eigenvalues of (IV) are complex, while those of (I), (II) and (III) are real. A is represented with respect to an orthonormal basis in the case (I) and (IV) and with a pseudo-orthonormal basis in the case (II) and (III), respectively. Recalling (3.13), the shape operator of the B-scroll in $\mathbb{S}_1^3$ or $\mathbb{H}_1^3$ is (II).

L. J. Alías, A. Ferrández and P. Lucas [1] gave some examples of generalized umbilical hypersurfaces whose shape operators are (II) and (III) satisfying $\Delta H = \lambda H$ for a real constant λ. In this paper, we consider two hypersurfaces given in [1] whose shape operators are (II). First, we describe the construction of a frame field along a null curve γ in a Lorentzian manifold, and its existence and uniqueness.

Next proposition is about the frame field along a null curve in a Lorentzian manifold which is constructed by the same way as the Frenet frame field. E. Cartan proved that the existence and uniqueness of the Frenet type frame field along a null curve of $\mathbb{E}_1^3$, which is called a Cartan frame field. Later, K. L. Duggal and A. Bejancu [10] extended it to a null curve in a general Lorentzian manifold.

Moreover, D. H. Jin [13] simplified their equations for the Frenet frame field by taking a special parameter. The following proposition is already proved by their way. We remark that the proof is more direct and plainer and we proved it without assuming that a parameter of a null curve is specific.

Proposition 3.5.1. *Let γ be a null curve of order $d \geq 3$ in an n-dimensional Lorentzian manifold $(M, \langle\ ,\ \rangle, \nabla)$. Then, there is a frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ along γ satisfying*

$$\begin{aligned}
&\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,\\
&\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,\\
&\langle A, Z_i\rangle = \langle B, Z_i\rangle = \langle C, Z_i\rangle = 0,\\
&\langle Z_i, Z_j\rangle = \delta_{ij}
\end{aligned} \tag{3.22}$$

and

$$\begin{cases}
\dot\gamma = A,\\
\nabla_{\dot\gamma} A = k_1 C,\\
\nabla_{\dot\gamma} C = k_2 A + k_1 B,\\
\nabla_{\dot\gamma} B = k_2 C + k_3 Z_1,\\
\nabla_{\dot\gamma} Z_1 = k_3 A + k_4 Z_2,\\
\nabla_{\dot\gamma} Z_2 = -k_4 Z_1 + k_5 Z_3,\\
\vdots\\
\nabla_{\dot\gamma} Z_{d-4} = -k_{d-2} Z_{d-5} + k_{d-1} Z_{d-3},\\
\nabla_{\dot\gamma} Z_{d-3} = -k_{d-1} Z_{d-4},
\end{cases} \tag{3.23}$$

where k_i $(i = 1, \ldots, d-1)$ are positive-valued functions.

Definition 3.5.2. This frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ is called the *Cartan frame field along γ.*

We remark that our definition of the Cartan frame field along $\gamma(s)$ is slightly different from the definition in the paper of A. Ferrández, A. Giménez and P. Lucas [12] since their definition is applied only to the case where a null curve γ is parametrized by a pseudo-arc parameter, that is, γ satisfies $\langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 1$.

A null curve in a Lorentzian manifold M equipped with the Cartan frame field is called the *Cartan curve*. The next proposition ensures the uniqueness and existence of the Cartan curve for any non-zero functions. It is proved in [12], [10] and [11] and a simple proof is also found in [17].

Proof. First we show that $\nabla_{\dot\gamma} A$ is non-null. Suppose that $\nabla_{\dot\gamma} A$ is null. Since γ is of order $d(\geq 3)$, A and $\nabla_{\dot\gamma} A$ are linearly independent. Denote by W_1 the 2-dimensional subspace spanned by A and $\nabla_{\dot\gamma} A$. Also we have

$$\langle A, \nabla_{\dot\gamma} A\rangle = \frac{1}{2}\dot\gamma\langle A, A\rangle = 0.$$

Let $(e_1 = A, e_2 = \nabla_{\dot\gamma} A, e_3, \ldots, e_n)$ be a frame field along $\gamma(s)$. Then we have

$$(\langle e_i, e_j\rangle) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & \\ & * & * \end{pmatrix}.$$

This contradicts the fact that $\langle , \rangle$ is Lorentzian. Therefore $\nabla_{\dot\gamma}A$ is non-null.
We put $k_1 = \sqrt{|\langle \nabla_{\dot\gamma}A, \nabla_{\dot\gamma}A\rangle|}$, $C = (1/k_1)\nabla_{\dot\gamma}A$ and $\varepsilon_C := \langle C, C\rangle$. We have

$$\nabla_{\dot\gamma}C = \left(\frac{1}{k_1}\right)^{\cdot} \nabla_{\dot\gamma}A + \frac{1}{k_1}\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) \quad \in \mathrm{Span}\{\dot\gamma, \nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma)\}. \tag{3.24}$$

Also we have $\langle \nabla_{\dot\gamma}C, C\rangle = 0$ and $\langle A, C\rangle = 0$, hence

$$\langle \nabla_{\dot\gamma}C, A\rangle = -\langle C, \nabla_{\dot\gamma}A\rangle = -k_1\langle C, C\rangle \quad (\neq 0). \tag{3.25}$$

Let $(\hat e_1 = C, \hat e_2 = A, \hat e_3, \dots, \hat e_n)$ be a frame field along γ such that $(\hat e_2, \dots, \hat e_n)$ is a frame field of $\mathrm{Span}\{C\}^\perp$. Then we have

$$(\langle \hat e_i, \hat e_j\rangle) = \begin{pmatrix} \varepsilon_C & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ 0 & & & & \\ \vdots & & & * & \\ 0 & & & & \end{pmatrix}.$$

Therefore we have $\varepsilon_C = 1$ and $\langle \nabla_{\dot\gamma}C, A\rangle = -k_1$ because $\langle , \rangle$ is Lorentzian. Since γ is of order $d(\geq 3)$, we have $\dim(\mathrm{Span}\{A, \nabla_{\dot\gamma}C\}) = 2$. From these facts, it follows that $\mathrm{Span}\{A, \nabla_{\dot\gamma}C\}$ is a 2-dimensional Lorentzian space. There is a unique null vector $B \in \mathrm{Span}\{A, \nabla_{\dot\gamma}C\}$ such that $\langle A, B\rangle = -1$. It can be expressed as $\nabla_{\dot\gamma}C = aA + bB$ for some functions a and b since $\nabla_{\dot\gamma}C \in \mathrm{Span}\{A, B\}$. We have $b = k_1$ by (3.25). We set $k_2 := a$. Put $W_2 = \mathrm{Span}\{A, B, C\}^\perp$, which is an $(n-3)$-dimensional Euclidean space. We put

$$\nabla_{\dot\gamma}B = \hat a A + \hat b B + cC + Z \quad (Z \in W_2).$$

We have

$$\begin{aligned} -\hat a &= \langle \nabla_{\dot\gamma}B, B\rangle = 0, \\ -\hat b &= \langle \nabla_{\dot\gamma}B, A\rangle = \langle B, \nabla_{\dot\gamma}A\rangle = -k_1\langle B, C\rangle = 0, \\ \hat c &= \langle \nabla_{\dot\gamma}B, C\rangle = -\langle B, \nabla\dot\gamma C\rangle = k_2. \end{aligned}$$

If $d = 3$, then we have $Z = 0$. In the sequel, we consider the case of $d \geq 4$. We put $k_3 = |Z|$ and $Z_1 = Z/|Z|$. Then $\nabla_{\dot\gamma}B$ can be expressed as $\nabla_{\dot\gamma}B = k_2C + k_3Z_1$. We put

$$\nabla_{\dot\gamma}Z_1 = \check a A + \check b B + \check c C + \check z_1 Z_1 + \hat Z,$$

where $\hat Z \in \mathrm{Span}\{A, B, C, Z_1\}^\perp$. Then we have

$$\begin{aligned} -\check b &= \langle \nabla_{\dot\gamma}Z_1, A\rangle = -\langle Z_1, k_1C\rangle = 0, \\ -\check a &= \langle \nabla_{\dot\gamma}Z_1, B\rangle = -\langle Z_1, k_2C + k_3Z_1\rangle = -k_3, \\ \check c &= \langle \nabla_{\dot\gamma}Z_1, C\rangle = \langle Z_1, k_2A + k_1B\rangle = 0, \\ \check z_1 &= \langle \nabla_{\dot\gamma}Z_1, Z_1\rangle = 0. \end{aligned}$$

If $d = 4$, then we have $\hat{Z} = 0$. In the sequel, we consider the case of $d \geq 5$. Then we have $\hat{Z} \neq 0$. We put $k_4 = |\hat{Z}|$ and $Z_2 = \hat{Z}/|\hat{Z}|$. Then $\nabla_{\dot{\gamma}} Z_1$ can expressed as $\nabla_{\dot{\gamma}} Z_1 = k_3 A + k_4 Z_2$. We put

$$\nabla_{\dot{\gamma}} Z_2 = \tilde{a} A + \tilde{b} B + \tilde{c} C + \tilde{z}_1 Z_1 + \tilde{z}_2 Z_2 + \tilde{Z},$$

where $\tilde{Z} \in \mathrm{Span}\{A, B, C, Z_1, Z_2\}^\perp$. Then we have $\tilde{a} = \tilde{b} = \tilde{c} = \tilde{z}_2 = 0$ and $\tilde{z}_1 = -k_4$. If $d = 5$, then we have $\tilde{Z} = 0$. In the sequel, we consider the case of $d \geq 6$. Then we have $\tilde{Z} \neq 0$. We put $k_5 = |\tilde{Z}|$ and $Z_3 = \tilde{Z}/|\tilde{Z}|$. Then $\nabla_{\dot{\gamma}} Z_2$ is expressed as $\nabla_{\dot{\gamma}} Z_2 = -k_4 Z_1 + k_5 Z_3$. By repeating the same discussion, we can derive the relations in Lemma 3.5.1. $\square$

Proposition 3.5.2. *Let $k_1, \ldots, k_{d-1}$ be differentiable non-zero functions on $(s_0 - \varepsilon, s_0 + \varepsilon)$ for some small $\varepsilon > 0$. Let p_0 be a point in $\mathbb{E}^n_1$ and $(A^0, B^0, C^0, Z^0_1, \ldots, Z^0_{d-3})$ pseudo-orthonormal vectors in $\mathbb{E}^n_1$ at p_0. Then, there is a unique Cartan curve γ of order d in $\mathbb{E}^n_1$ such that $\gamma(s_0) = p_0$, and its Cartan frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ satisfies*

$$A(s_0) = A^0,\ B(s_0) = B^0,\ C(s_0) = C^0,\ Z_1(s_0) = Z^0_1, \ldots, Z_{d-3}(s_0) = Z^0_{d-3}.$$

Proof. We put $V := (A+B)/\sqrt{2}$ and $W := (A-B)/\sqrt{2}$. Let F be a matrix consisting column vector fields $V, W, C, Z_1, \ldots, Z_{d-3}$, that is $F = (V\, W\, C\, Z_1 \cdots Z_{d-3})$. We put

$$X = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}}(k_1 + k_2) \\ 0 & 0 & -\frac{1}{\sqrt{2}}(k_1 - k_2) \\ \frac{1}{\sqrt{2}}(k_1 + k_2) & \frac{1}{\sqrt{2}}(k_1 - k_2) & 0 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & -k_4 & & & & \\ k_4 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & -k_{d-2} & 0 \\ & & & k_{d-2} & 0 & -k_{d-1} \\ & & & 0 & k_{d-1} & 0 \end{pmatrix},$$

and we define a coefficient matrix K by

$$K = \left(\begin{array}{ccc|cccc} & & & \frac{1}{\sqrt{2}}k_3 & 0 & \cdots & 0 \\ & X & & \frac{1}{\sqrt{2}}k_3 & 0 & \cdots & 0 \\ & & & 0 & 0 & \cdots & 0 \\ \hline \frac{1}{\sqrt{2}}k_3 & -\frac{1}{\sqrt{2}}k_3 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & & & Y & \\ 0 & 0 & 0 & & & & \end{array}\right).$$

Then, by (3.23), we have

$$\dot{F} = FK. \tag{3.26}$$

If an initial value is given, then the solution is unique. We prove that the solution satisfies (3.22). We put $E := \mathrm{diag}(-1, 1, \ldots, 1)$. Let Φ be a solution of (3.26) with E as the initial value. It is easy to check that the solution is expressed as $F = F^0 E\Phi$ for any initial value F^0 and EK is skew-symmetric. Hence

$$\begin{aligned}\frac{d}{ds}(EF^{-1}E)(s) &= -EK(s)F^{-1}(s)E \\ &= {}^t(EK(s))F^{-1}(s)E \\ &= {}^tK(s)(EF^{-1}E)(s).\end{aligned}$$

By

$$\begin{aligned}{}^t\Phi(s_0) &= {}^tE = E, \\ (E\Phi^{-1}E)(s_0) &= E,\end{aligned}$$

we obtain ${}^tF = EF^{-1}E$. Hence, the columns of F form an orthonormal basis for $\mathbb{R}^n_1$ and V is timelike. Thus F satisfies (3.22). When we put $\gamma(s) = \int_{s_0}^s A(t)dt$, γ is null curve satisfying (3.22) and (3.23).

Finally, we prove the uniqueness of γ with Cartan frame field F. Let γ and $\tilde{\gamma}$ be Cartan curves. Assume that k_i of γ in (3.23) coincides with that of $\tilde{\gamma}$ for all i. Let F and $\tilde{F}$ be pseudo-orthogonal matrices defined by $F = (V\,W\,C\,Z_1 \cdots Z_{d-3})$ for γ and $\tilde{F} = (\tilde{V}\,\tilde{W}\,\tilde{C}\,\tilde{Z}_1 \cdots \tilde{Z}_{d-3})$ for $\tilde{\gamma}$, respectively. We put $F^0 := F(s_0)$ and $\tilde{F}^0 := \tilde{F}(s_0)$. When we put $L := F^0(\tilde{F}^0)^{-1}$, by the uniqueness of the solution of ODE (3.26) with the same initial value, we have $F = L\tilde{F}$. In particular, $A = L\tilde{A}$ because $V = L\tilde{V}$ and $W = L\tilde{W}$. Put $b := \gamma(s_0) - L\tilde{\gamma}(s_0)$. Then we have

$$\begin{aligned}\gamma(s) &= \gamma(s_0) + \int_{s_0}^s A(t)dt \\ &= L\tilde{\gamma}(s_0) + b + \int_{s_0}^s L\tilde{A}(t)dt \\ &= L\tilde{\gamma}(s) + b.\end{aligned}$$

Since L is a pseudo-orthogonal matrix, γ and $\tilde{\gamma}$ are congruent. □

If M in Proposition 3.5.1 is $\mathbb{H}^n_1$ or $\mathbb{S}^n_1$, we can rewrite (3.23) by replacing ∇ with the standard connection $\tilde{\nabla}$ of the ambient space $\mathbb{E}^{n+1}_2$ or $\mathbb{E}^{n+1}_1$, and one can easily see that the following corollary follows.

Corollary 3.5.1. *Let γ be a null curve γ of order $d \geq 3$ in $\mathbb{H}^n_1$ (resp. $\mathbb{S}^n_1$) and $(A, B, C, Z_1, \ldots, Z_{d-3})$ the Cartan frame field along γ. Then, with respect to the* *standard connection* $\tilde{\nabla}$ *of* $\mathbb{E}^{n+1}_2$ *(resp.* $\mathbb{E}^{n+1}_1$*), the following relations hold;*

$$\begin{cases} \tilde{\nabla}_{\dot\gamma} A = k_1 C, \\ \tilde{\nabla}_{\dot\gamma} C = k_2 A + k_1 B, \\ \tilde{\nabla}_{\dot\gamma} B = k_2 C + k_3 Z_1 + \varepsilon\gamma, \\ \tilde{\nabla}_{\dot\gamma} Z_1 = k_3 A + k_4 Z_2, \\ \tilde{\nabla}_{\dot\gamma} Z_i = -k_{i+2} Z_{i-1} + k_{i+3} Z_{i+1} \quad (2 \le i \le d-4), \\ \tilde{\nabla}_{\dot\gamma} Z_{d-3} = -k_{d-1} Z_{d-4}, \end{cases}$$

where $\varepsilon = -1$ *(resp.* $\varepsilon = +1$*).*

Next, we determine the type number of the pseudo-hyperbolic Gauss map of hypersurfaces given by [1] whose shape operator is (II). For simplicity, we put $\mathbf{x}(s,t,z) = \mathbf{x}(s,t,z_1,z_2,\dots,z_{n-2})$, $|z|^2 = z_1^2 + z_2^2 + \cdots + z_{n-2}^2$ and $\mathbf{x}_s = \partial\mathbf{x}/\partial s$ (resp. $\mathbf{x}_t$ and $\mathbf{x}_{z_j}$).

Proposition 3.5.3. *Let* $(A, B, C, Z_1, \dots, Z_{n-2})$ *be the Cartan frame field of a null curve* $\gamma \subset \mathbb{H}^{n+1}_1$*. Assume that* $k_1(s) \neq 0$*,* $k_2^2 = 1$ *and* k_i *are non-zero constants for* $i = 3, \dots, n-2$*. The immersion* $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{H}^{n+1}_1 \subset \mathbb{E}^{n+2}_2$ *given by*

$$\mathbf{x}(s,t,z) = \left(1 + \frac{|z|^2}{2}\right)\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2|z|^2}{2} C(s) \tag{3.27}$$

parametrizes an oriented Lorentzian hypersurface of $\mathbb{H}^{n+1}_1$*, where* $z = (z_1, \dots, z_{n-2}) \in \mathbb{R}^{n-1}$*. Denote by* M^n_1 *the image of* $\mathbf{x}$*. Then, the pseudo-hyperbolic Gauss map of* M^n_1 *is infinite type.*

Proof. By a straightforward computation, we have

$$\begin{aligned} \mathbf{x}_s &= A(s) - \frac{k_1(s)k_2|z|^2}{2} B(s) + k_2 t C(s) + k_3 t Z_1(s) + \sum_{j=1}^{n-2} z_j \dot{Z}_j(s) - t\gamma(s), \\ \mathbf{x}_t &= B(s), \\ \mathbf{x}_{z_i} &= z_i\gamma(s) + Z_i(s) - k_2 z_i C(s). \end{aligned}$$

The unit normal vector field N of the Lorentzian hypersurface M^n_1 in $\mathbb{H}^{n+1}_1$ is given by

$$N(s,t,z) = k_2 t B(s) + k_2 \sum_{j=1}^{n-2} z_j Z_j(s) + \left(1 - \frac{|z|^2}{2}\right) C(s) + \frac{k_2|z|^2}{2}\gamma(s),$$

where $z = (z_1, \dots, z_{n-2})$. With respect to $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_{z_1}, \dots, \mathbf{x}_{z_{n-2}})$, the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ -k_1(s) & -k_2 & & \text{\Large 0} & \\ & & -k_2 & & \\ & \text{\Large 0} & & \ddots & \\ & & & & -k_2 \end{pmatrix}. \tag{3.28}$$

By Corollary 3.5.1, we obtain

$$
\begin{array}{ll}
\langle \dot Z_1, \dot Z_1\rangle = k_4^2, & \\
\langle \dot Z_i, \dot Z_i\rangle = k_{i+2}^2 + k_{i+3}^2 & (2 \le i \le n-3), \\
\langle \dot Z_{n-3}, \dot Z_{n-3}\rangle = k_n^2, & \\
\langle \dot Z_i, \dot Z_j\rangle = 0 & (i \ne j \quad \text{or} \quad i \ne j \pm 2), \\
\langle \dot Z_i, \dot Z_{i+2}\rangle = -k_{i+3}k_{i+4} & (1 \le i \le n-4)
\end{array}
\tag{3.29}
$$

and

$$
\begin{aligned}
\sum_{i,j=1}^{n-2} z_i z_j \langle \dot Z_i, \dot Z_j\rangle &= k_4^2 z_1^2 + \sum_{i=2}^{n-3}(k_{i+2}^2 + k_{i+3}^2) z_i^2 + k_n^2 z_{n-3}^2 \\
&\quad + 2(-k_4k_5z_1z_3 - \cdots - k_{n-1}k_n z_{n-4}z_{n-2}) \\
&= k_4^2 z_1^2 + k_n^2 z_{n-3}^2 + \sum_{i=1}^{n-3}(k_{i+2}z_i - k_{i+3}z_{i+1})^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle \mathbf{x}_s, \mathbf{x}_s\rangle &= -k_1k_2|z|^2\langle A, B\rangle + k_2^2t^2\langle C, C\rangle + k_3^2t^2\langle Z_1, Z_1\rangle + \sum_{i,j=1}^{n-2} z_iz_j\langle \dot Z_i, \dot Z_j\rangle \\
&\quad + t^2\langle \gamma, \gamma\rangle + 2k_3t\langle Z_1, \sum_{j=1}^{n-2} z_j \dot Z_j\rangle - k_1k_2|z|^2\langle B, z_1\dot Z_1\rangle \\
&= k_1k_2|z|^2(1 + k_3z_1) + k_3^2t^2 + k_4^2z_1^2 + \sum_{i=1}^{n-3}(k_{i+2}z_i - k_{i+3}z_{i+1})^2 \\
&\quad + k_n^2 z_{n-3}^2 - 2k_3k_4z_2t, \\
\langle \mathbf{x}_s, \mathbf{x}_{z_i}\rangle &= k_3t\delta_{1i} + \sum_{j=1}^{n-2} z_j\langle Z_i, \dot Z_j\rangle \\
&= \begin{cases} k_3t - k_4z_2 & (i = 1) \\ k_{i+2}z_{i-1} - k_{i+3}z_{i+1} & (2 \le i \le n-3) \\ k_nz_{n-3} & (i = n-2) \end{cases}, \\
\langle \mathbf{x}_s, \mathbf{x}_t\rangle &= -(1 + |z|^2 + k_3z_1), \qquad \langle \mathbf{x}_t, \mathbf{x}_t\rangle = \langle \mathbf{x}_t, \mathbf{x}_{z_i}\rangle = 0, \qquad \langle \mathbf{x}_{z_i}, \mathbf{x}_{z_j}\rangle = \delta_{ij}.
\end{aligned}
\tag{3.30}
$$

We put $e_j = \mathbf{x}_{z_j}$ $(1 \le j \le n-2)$ and

$$
\tilde e_{n-1} := \mathbf{x}_s - \sum_{j=1}^{n-2}\langle \mathbf{x}_s, e_j\rangle e_j. \tag{3.31}
$$

If $\tilde e_{n-1}$ is non-null, we put

$$
e_{n-1} := \frac{1}{|\tilde e_{n-1}|}\tilde e_{n-1}, \quad \tilde e_n := \frac{1}{|\tilde e_{n-1}|}(\langle \mathbf{x}_s, \mathbf{x}_t\rangle e_{n-1} - \langle \tilde e_{n-1}, \tilde e_{n-1}\rangle \mathbf{x}_t) \tag{3.32}
$$

and $e_n := \tilde{e}_n/\langle \mathbf{x}_s, \mathbf{x}_t\rangle$. With respect to the orthonormal frame field $(e_1, \ldots, e_n)$, the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 & & & & \\ & \ddots & & \text{\Large 0} & \\ & & -k_2 & & \\ & \text{\Large 0} & & -k_2-\alpha & -\alpha \\ & & & \alpha & -k_2+\alpha \end{pmatrix},$$

where $\alpha = \varepsilon_{n-1}k_1|\tilde{e}_n|/|\tilde{e}_{n-1}|^2$. If $\tilde{e}_{n-1}$ is null, we put

$$\begin{aligned} e_{n-1} &:= \frac{1}{\sqrt{2}\langle \tilde{e}_{n-1}, \mathbf{x}_t\rangle^{\frac{1}{2}}}(\tilde{e}_{n-1} + \mathbf{x}_t), \\ e_n &:= \frac{1}{\sqrt{2}\langle \tilde{e}_{n-1}, \mathbf{x}_t\rangle^{\frac{1}{2}}}(\tilde{e}_{n-1} - \mathbf{x}_t). \end{aligned} \tag{3.33}$$

With the pseudo-orthonormal frame field $(e_1, \ldots, e_n)$, the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 & \text{\Large 0} & & & \\ & \ddots & & \text{\Large 0} & \\ & & -k_2 & & \\ & \text{\Large 0} & & -k_2-\alpha & -\alpha \\ & & & \alpha & -k_2+\alpha \end{pmatrix},$$

where $\alpha = -k_1(s)/2$. In both cases, $\|\hat{h}\|^2 = n$ and the mean curvature $\hat{\mathcal{H}} = -k_2$. By (2.12), M_1^n is scalar flat. Hence, $\Delta\tilde{\nu} = n\tilde{\nu} - 2k_2 N \wedge e_1 \wedge e_2 \neq 0$ and $\Delta^2\tilde{\nu} = 0$ by (3.2) and (3.3). Therefore, $\tilde{\nu}$ is of infinite type by Lemma 3.2.1. $\square$

Proposition 3.5.4. *Let $(A, B, C, Z_1, \ldots, Z_{n-2})$ be the Cartan frame field of a null curve $\gamma \subset \mathbb{H}_1^{n+1}$. Assume that $k_1(s) \neq 0$ and that $k_2^2 \neq 1$ and k_i are non-zero constants for $i = 3, 4, \ldots, n-2$. Define the immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{H}_1^{n+1} \subset \mathbb{E}_2^{n+2}$ by*

$$\mathbf{x}(s,t,z) = \frac{k_2^2 - f(z)}{-1+k_2^2}\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2(1-f(z))}{-1+k_2^2}C(s), \tag{3.34}$$

where $f(z) = \sqrt{1-(-1+k_2^2)|z|^2}$. Then, it parametrizes a Lorentzian hypersurface M_1^n of $\mathbb{H}_1^{n+1}$ in a neighborhood of the origin. Then, the pseudo-hyperbolic Gauss map of M_1^n is null 2-type.

Proof. The unit normal vector field N of the Lorentzian hypersurface M_1^n in $\mathbb{H}_1^{n+1}$ is given by

$$N(s,t,z) = k_2 t B(s) + k_2 \sum_{j=1}^{n-2} z_j Z_j(s) + \frac{-1+k_2^2 f(z)}{-1+k_2^2}C(s) + \frac{k_2(1-f(z))}{-1+k_2^2}\gamma(s).$$

With respect to $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_{z_1}, \ldots, \mathbf{x}_{z_{n-2}})$, the shape operator A_N derived from N is the same form as (3.28).

We prove that $\mathbf{x}_{z_i}$ is spacelike for any $i = 1, 2, \ldots, n-2$. By a straightforward computation, we have

$$\langle \mathbf{x}_{z_i}, \mathbf{x}_{z_i} \rangle = \frac{z_i^2}{f(z)^2}(-1 + k_2^2) + 1. \tag{3.35}$$

In the case where $-1 + k_2^2 > 0$, $\mathbf{x}_{z_i}$ is spacelike. In the case where $-1 + k_2^2 < 0$, we obtain

$$\begin{aligned} \langle \mathbf{x}_{z_i}, \mathbf{x}_{z_i} \rangle &= \frac{z_i^2}{f(z)^2}(-1 + k_2^2) + 1 \\ &> \frac{|z|^2}{f(z)^2}(-1 + k_2^2) + 1 \\ &= \frac{1}{1 - (-1 + k_2^2)|z|^2} \\ &> 0 \end{aligned} \tag{3.36}$$

since $f(z) > 0$. Thus, $\mathbf{x}_{z_i}$ is spacelike in both cases. We put $e_1 := \mathbf{x}_{z_1}$, $\tilde{e}_j := \mathbf{x}_j - \sum_{k=1}^{j-1} \langle \mathbf{x}_j, e_k \rangle e_k$ and $e_j := \tilde{e}_j / |\tilde{e}_j|$ for $2 \leq j \leq n-2$. Then, e_j is spacelike. We obtain $A_N(e_j) = -k_2 e_j$ by a straightforward computation. We put

$$\tilde{e}_{n-1} := \mathbf{x}_s - \sum_{j=1}^{n-2} \langle \mathbf{x}_s, e_j \rangle e_j.$$

If $\tilde{e}_{n-1}$ is non-null, we define e_{n-1}, $\tilde{e}_n$ and e_n by (3.31) and (3.32), respectively. With respect to the orthonormal frame field $(e_1, \ldots, e_n)$, the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 & & & & \\ & \ddots & & \text{\huge 0} & \\ & & -k_2 & & \\ & \text{\huge 0} & & -k_2 - \alpha & -\alpha \\ & & & \alpha & -k_2 + \alpha \end{pmatrix}, \tag{3.37}$$

where $\alpha = -k_1(s)|\tilde{e}_n| / |\tilde{e}_{n-1}|$. If $\tilde{e}_{n-1}$ is null, we define e_{n-1} and e_n by (3.33). With respect to the pseudo-orthonormal frame field $(e_1, \ldots, e_n)$, the shape operator A_N is expressed as

$$A_N = \begin{pmatrix} -k_2 & & & & \\ & \ddots & & \text{\huge 0} & \\ & & -k_2 & & \\ & \text{\huge 0} & & -k_2 - \alpha & -\alpha \\ & & & \alpha & -k_2 + \alpha \end{pmatrix}, \tag{3.38}$$

where $\alpha = -k_1(s)/2$. In both cases, $\|\hat{h}\|^2 = nk_2^2$, the mean curvature $\hat{\mathcal{H}} = -k_2$ and the scalar curvature $S = n(n-1)(k_2^2 - 1) \neq 0$. Hence,

$$\begin{aligned} \Delta\tilde{\nu} &= nk_2^2\tilde{\nu} - nk_2 e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n, \\ \Delta^2\tilde{\nu} &= n^2k_2^2(k_2^2-1)\tilde{\nu} - nk_2(k_2^2-1)e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n, \end{aligned} \tag{3.39}$$

where $e_{n+1} = N$. We put

$$\begin{aligned} \tilde{\nu}_1 &:= \frac{1}{k_2^2-1}(-\tilde{\nu} + k_2 e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n), \\ \tilde{\nu}_2 &:= \frac{1}{k_2^2-1}(k_2^2\tilde{\nu} - k_2 e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n). \end{aligned}$$

It is clear that $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2$. Using (3.2) and (3.3) we obtain that $\Delta\tilde{\nu}_1 = 0$ and $\Delta\tilde{\nu}_2 = n(k_2^2-1)\tilde{\nu}_2$. On the other hand, by using (3.1) and (3.3), we have

$$\begin{aligned} e_{n-1}(\tilde{\nu}) = \varepsilon_{n-1}&(-k_2 + \beta)\mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{n-2} \wedge e_{n+1} \wedge e_n \\ &- \varepsilon_{n-1}\beta\mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{n-2} \wedge e_{n-1} \wedge e_{n+1} \end{aligned} \tag{3.40}$$

and

$$e_{n-1}(e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n) = \varepsilon_i e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{n-2} \wedge \mathbf{x} \wedge e_n, \tag{3.41}$$

where $\beta = k_1(s)|\tilde{e}_n|/|\tilde{e}_{n-1}|$ if $\tilde{e}_{n-1}$ is non-null or $\beta = -k_1(s)/2$ if $\tilde{e}_{n-1}$ is null. Hence,

$$\begin{aligned} e_{n-1}(\tilde{\nu}_1) &= \frac{\varepsilon_{n-1}\beta}{k_2^2-1}(\mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{n-2} \wedge e_{n+1} \wedge e_n - \mathbf{x} \wedge e_1 \wedge \cdots \wedge e_{n-1} \wedge e_n) \\ &\neq 0 \end{aligned}$$

by (3.40) and (3.41). Therefore $\tilde{\nu}$ is of null 2-type. □

Remark 3.5.2. Submanifolds parametrized by $\mathbf{x}$ in (3.27) and (3.34) are Lorentzian hypersurfaces of $\mathbb{H}_1^{n+1}$ satisfying that each shape operator is non-diagonalizable everywhere and the mean and scalar curvature are constants. In particular, the mean curvature H is $-k_2(\neq 0)$ and the minimal polynomial $P(x)$ of the shape operator is $(x+k_2)^2$.

Remark 3.5.3. There is a relation between Example 3.5.3 and 3.5.4. One can easily check that the parametrization $\mathbf{x}$ defined by (3.34) uniformly converges in C^∞ to $\mathbf{x}$ defined by (3.27) on a fixed neighborhood of $z = 0$ as $k_2^2 \to 1$.

Chapter 4

Generalized umbilical hypersurfaces in $\mathbb{S}_2^{n+1}$ and $\mathbb{H}_2^{n+1}$

The purpose of this chapter is to find some pseudo-Riemannian hypersurfaces in $\mathbb{S}_2^{n+1}$ and $\mathbb{H}_2^{n+1}$ similar to generalized umbilical hypersurfaces in Chapter 3.5. First, we describe the construction of a frame field along a null curve γ in a pseudo-Riemannian manifold with index 2, and its existence and uniqueness. Second, we construct a non-degenerate hypersurface in $\mathbb{S}_2^{n+1}$ and $\mathbb{H}_2^{n+1}$ satisfying following conditions. The mean curvature H and scalar curvature are constant and $H \neq 0$, and the shape operator is non-diagonalizable.

Let V be an n-dimensional vector space with index 2 and A a self-adjoint symmetric linear operator on V. Put $G := (\langle v_i, v_j \rangle)_{ij}$ for a basis $(v_1, \ldots, v_n)$ of V, where $\langle\, ,\, \rangle$ is a scalar product on V. In the case of index 1, the form A is one of the only four forms in Section 3.5, however the forms A can take are various when the index is 2 (see [19]). Thus, we just introduce some of forms needed in this paper;

$$
\text{(I) } G = \begin{pmatrix} 0 & -1 & & & & \\ -1 & 0 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} \lambda & 0 & & & \\ 1 & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix}
$$

$$
\text{(II) } G = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & -1 & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda & 0 & & & & & \\ 1 & \lambda & & & & & \\ & & \lambda & 0 & & & \\ & & 1 & \lambda & & & \\ & & & & \lambda & & \\ & & & & & \ddots & \\ & & & & & & \lambda \end{pmatrix},
$$

where λ is a real number. The matrix G defined by $\langle\, ,\, \rangle$ with respect to the frame field $(A, B, C, Z_1, \dots, Z_{d-3})$ in Proposition 4.1.1 is of the form (I). In the following Proposition 4.1.1, we will consider a null curve γ satisfying that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is non-null. On the other hand, the Frenet type frame field proven by M. Sakaki, A. Uçum and K. İlarslan [21] is of the form (II). In that paper, they assume that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is null for a null curve γ. A null curve γ in $\mathbb{E}^n_2$ is called a *bi-null curve* if $\nabla_{\dot{\gamma}}\dot{\gamma}$ is null. As we described in Chapter 4.1, a frame field in Proposition 4.1.1 and the existence and uniqueness are already proved by K. L. Duggal, A. Bejancu and D. H. Jin (see [10] and [11]). We give another proof of those which is more direct and simple.

4.1 Cartan fame field along a null curves in a pseudo-Riemannian manifold with index 2

Let $(M, \langle\, ,\, \rangle, \nabla)$ be an n-dimensional pseudo-Riemannian manifold with index 2 and γ a null curve of order $d \geq 4$ in M. Assume that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is non-null. Put $A = \dot{\gamma}$ and $k_1 = \sqrt{|\langle \nabla_{\dot{\gamma}}A, \nabla_{\dot{\gamma}}A\rangle|}$. We defined a vector field C by $C = (1/k_1)\nabla_{\dot{\gamma}}A$. Note that $\langle C, C\rangle = \pm 1$. Let B be a null vector field along γ with $\langle A, B\rangle = -1$ and $\langle B, C\rangle = 0$. In this section, we assume the following.

Assumption 4.1. There exists a one parameter family of linearly independent non-null vectors $(\tilde{Z}_1, \dots, \tilde{Z}_{d-3})$ along γ such that

$$\gamma^{(j+3)} \in \operatorname{Span}\{A, B, C, \tilde{Z}_1, \dots, \tilde{Z}_j\} \quad \text{for all } 1 \leq j \leq d-3, \tag{4.1}$$

$$\operatorname{Span}\{\gamma^{(1)}, \dots, \gamma^{(d)}\} = \operatorname{Span}\{A, B, C, \tilde{Z}_1, \dots, \tilde{Z}_{d-3}\} \tag{4.2}$$

and

$$\operatorname{Span}\{A, B, C\} \perp \operatorname{Span}\{\tilde{Z}_1, \dots, \tilde{Z}_{d-3}\}. \tag{4.3}$$

Remark 4.1.1. We remark that a one parameter family of linearly independent vectors $(\tilde{Z}_1, \dots, \tilde{Z}_{d-3})$ along γ satisfying (4.1), (4.2) and (4.3) always exists. However, we cannot choose all of $\tilde{Z}_1, \dots, \tilde{Z}_{d-3}$ which are non-null always. Thus, the nontrivial point of Assumption 4.1 is what all $\tilde{Z}_j$ are non-null.

Proposition 4.1.1. *Let γ be a null curve of order $d \geq 4$ in an n-dimensional pseudo-Riemannian manifold $(M, \langle\, ,\, \rangle, \nabla)$ with index 2. Assume that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is non-null. Under Assumption 4.1, there exists uniquely a frame field $(A, B, C, Z_1, \dots, Z_{d-3})$ along γ satisfying the following conditions;*

$$\begin{aligned}
&\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,\\
&\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = \varepsilon_C,\\
&\langle A, Z_i\rangle = \langle B, Z_i\rangle = \langle C, Z_i\rangle = 0,\\
&\langle Z_i, Z_j\rangle = \varepsilon_i\delta_{ij}
\end{aligned} \tag{4.4}$$

and

$$
(4.5)\qquad \begin{cases} \dot\gamma = A, \\ \nabla_{\dot\gamma}A = k_1 C, \\ \nabla_{\dot\gamma}C = k_2 A + \varepsilon_C k_1 B, \\ \nabla_{\dot\gamma}B = \varepsilon_C k_2 C + k_3 Z_1, \\ \nabla_{\dot\gamma}Z_1 = \varepsilon_1 k_3 A + k_4 Z_2, \\ \varepsilon_2 \nabla_{\dot\gamma}Z_2 = -\varepsilon_1 k_4 Z_1 + \varepsilon_2 k_5 Z_3, \\ \qquad \vdots \\ \varepsilon_{d-4}\nabla_{\dot\gamma}Z_{d-4} = -\varepsilon_{d-5}k_{d-2}Z_{d-5} + \varepsilon_{d-4}k_{d-1}Z_{d-3}, \\ \varepsilon_{d-3}\nabla_{\dot\gamma}Z_{d-3} = -\varepsilon_{d-4}k_{d-1}Z_{d-4} \end{cases}
$$

for some positive-valued functions k_i $(i = 1, \dots, d-1)$, $\varepsilon_C = \pm 1$ and $\varepsilon_i = \pm 1$. If $\varepsilon_C = -1$ then $\varepsilon_i = 1$ for all $1 \le i \le d-3$ and if $\varepsilon_C = 1$ then there is a unique $1 \le j \le d-3$ such that $\varepsilon_j = -1$ and $\varepsilon_i = 1$ $(i \neq j)$.

Proof. We put $A := \dot\gamma$. In the case where $\nabla_{\dot\gamma}A$ is timelike, this proposition is proved by the same way as in the proof of Proposition 3.5.1 since the index of M is 2 (see [17]). Thus, we prove this proposition in the case where $\nabla_{\dot\gamma}A$ is spacelike. Recall that $k_1 = \sqrt{|\langle \nabla_{\dot\gamma}A, \nabla_{\dot\gamma}A\rangle|}$ and $C = (1/k_1)\nabla_{\dot\gamma}A$. In this case, $\varepsilon_C = \langle C, C\rangle = 1$ since $\nabla_{\dot\gamma}A$ is spacelike. Then, $\mathrm{Span}\{A, \nabla_{\dot\gamma}C\}$ is a non-degenerate 2-dimensional Lorentzian space since $\langle A, \nabla_{\dot\gamma}C\rangle = -k_1 \neq 0$ and A is null. Therefore, there is a unique null vector field B in $\mathrm{Span}\{A, \nabla_{\dot\gamma}C\}$ such that $\langle A, B\rangle = -1$ and $\langle B, C\rangle = 0$. Putting $k_2 = -\langle \nabla_{\dot\gamma}C, B\rangle$, $\nabla_{\dot\gamma}C$ can be written as $\nabla_{\dot\gamma}C = k_2 A + k_1 B$. By $\langle C, C\rangle = 1$, $\mathrm{Span}\{A, B, C\}$ is a non-degenerate 3-dimensional Lorentzian space, that is, $\mathrm{Span}\{A, B, C\}^\perp$ is an $(n-3)$-dimensional Lorentzian space since M is of index 2. By Assumption 4.1, there is a vector field $\tilde Z_1 \in \mathrm{Span}\{A, B, C\}^\perp$ along γ satisfying

$$
(4.6)\qquad \nabla_{\dot\gamma}B = aA + bB + cC + d\tilde Z_1
$$

with some functions a, b, c and d. Actually, we have $-b = \langle \nabla_{\dot\gamma}B, A\rangle = 0$, $-a = \langle \nabla_{\dot\gamma}B, B\rangle = 0$ and $c = \langle \nabla_{\dot\gamma}B, C\rangle = k_2$. Thus, (4.6) becomes $\nabla_{\dot\gamma}B = k_2 C + \tilde Z_1$. Put $k_3 = |\tilde Z_1|$ and $Z_1 = (d/k_3)\tilde Z_1$. Remark that k_3 is a positive function since $\tilde Z_1$ is non-null by Assumption 4.1. Then, (4.6) can be rewritten as

$$
\nabla_{\dot\gamma}B = k_2 C + k_3 Z_1.
$$

Also, $\nabla_{\dot\gamma}Z_1$ is expressed as

$$
\nabla_{\dot\gamma}Z_1 = a'A + b'B + c'C + d'Z_1 + e'\tilde Z_2
$$

with some functions a', b', c', d' and e'. By a straightforward computation, it can be rewritten as

$$
\nabla_{\dot\gamma}Z_1 = \varepsilon_1 k_3 A + e'\tilde Z_2,
$$

where $\varepsilon_1 = \langle Z_1, Z_1\rangle$ and $\tilde{Z}_2$ is a vector field along γ in $\mathrm{Span}\{A, B, C, Z_1\}^{\perp}$. Put $k_4 = |\tilde{Z}_2|$ and $Z_2 = (e'/k_4)\tilde{Z}_2$. Hence, we have

$$\nabla_{\dot\gamma} Z_1 = \varepsilon_1 k_3 A + k_4 Z_2$$

and put $\varepsilon_2 = \langle Z_2, Z_2\rangle$.

Hereafter, we prove inductively that a vector field Z_j along γ satisfies (4.4) and (4.5) for $j = 2, \ldots, d-4$, that is, Z_j satisfies the following equation;

$$\varepsilon_j \nabla_{\dot\gamma} Z_j = -\varepsilon_{j-1} k_{j+2} Z_{j-1} + \varepsilon_j k_{j+3} Z_{j+1} \tag{4.7}$$

for $j = 2, \ldots, d-4$, where $\varepsilon_j = \langle Z_j, Z_j\rangle = \pm 1$ and $\langle Z_i, Z_j\rangle = \varepsilon_j \delta_{ij}$.

In the case where $j = 2$, there is a vector field $\tilde{Z}_3$ in $\mathrm{Span}\{A, B, C, Z_1, Z_2\}^{\perp}$ such that

$$\nabla_{\dot\gamma} Z_2 \in \mathrm{Span}\{A, B, C, Z_1, Z_2\} \oplus \mathrm{Span}\{\tilde{Z}_3\}$$

since γ is order $d \geq 3$. If $\tilde{Z}_3$ is non-null, we put $k_5 = |\tilde{Z}_3|$, $Z_3 = (1/k_5)\tilde{Z}_3$ and $\varepsilon_3 = \langle Z_3, Z_3\rangle$. By a straight forward computation, we have

$$\varepsilon_2 \nabla_{\dot\gamma} Z_2 = -\varepsilon_1 k_4 Z_1 + \varepsilon_2 k_5 Z_3.$$

Next, we prove that (4.7) holds up for $j = l+1$ if (4.7) holds up for $j = l$, where $3 \leq l \leq d-4$. We put $W(l) = \mathrm{Span}\{A, B, C, Z_1, \ldots, Z_l\}$. Especially, $W(l)$ is a non-degenerate vector space. Note that Z_l is non-null for all l. Then, there is a vector field $\tilde{Z}_{l+2}$ along γ satisfying that $\{Z_1, \ldots, Z_{l+1}, \tilde{Z}_{l+2}\}$ is a linearly independent family and

$$\nabla_{\dot\gamma} Z_{l+1} \in W(l+1) \oplus \mathrm{Span}\{\tilde{Z}_{l+2}\},$$

since the order of γ is d. If $\tilde{Z}_{l+1}$ is non-null, put $k_{l+4} = |\tilde{Z}_{l+2}|$, $Z_{l+2} = (1/k_{l+4})\tilde{Z}_{l+2}$ and $\varepsilon_{l+2} = \langle Z_{l+2}, Z_{l+2}\rangle$. Therefore, we obtain

$$\varepsilon_{l+1} \nabla_{\dot\gamma} Z_{l+1} = -\varepsilon_j k_{l+3} Z_l + \varepsilon_{l+1} k_{l+4} Z_{l+2}$$

by a straight computation. In the case where $j = d-3$, $\nabla_{\dot\gamma} Z_{d-3}$ is in $W(d-3)$ since γ is order d. In the same way, we have

$$\varepsilon_{d-3} \nabla_{\dot\gamma} Z_{d-3} = -\varepsilon_{d-4} k_{d-1} Z_{d-4}.$$

Thus, the proof is completed. □

The frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ given by Proposition 4.1.1 is called the *Cartan frame field along* γ. A null curve in a pseudo-Riemannian manifold with index 2 equipped with the Cartan frame field is called the *Cartan curve*.

In the case where $\nabla_{\dot\gamma}\dot\gamma$ is null, a Frenet type frame field along γ is called the *bi-null Cartan frame field* in [21]. Moreover, a bi-null curve γ in $\mathbb{E}^n_2$ is called a *bi-null Cartan curve* if $\{\dot\gamma, \ldots, \gamma^{(n-1)}\}$ is linearly independent. Ruled surfaces along bi-null Cartan curves are studied also in [21].

The next proposition ensures the existence and uniqueness of a Cartan curve for any non-zero functions.

Proposition 4.1.2. *Fix $s_0 \in \mathbb{R}$. Let $k_1, \ldots, k_{d-1}$ be differentiable non-zero functions on $(s_0 - \varepsilon, s_0 + \varepsilon)$ for some small $\varepsilon > 0$. Let p_0 be a point in $\mathbb{E}^n_2$ and $(A^0, B^0, C^0, Z^0_1, \ldots, Z^0_{d-3})$ pseudo-orthonormal vectors in $\mathbb{E}^n_2$ at p_0. Then, there is a unique Cartan curve γ of order d in $\mathbb{E}^n_2$ such that $\gamma(s_0) = p_0$, and its Cartan frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ satisfies*

$$
\begin{aligned}
A(s_0) = A^0,\ B(s_0) = B^0,\ C(s_0) = C^0, \\
Z_1(s_0) = Z^0_1,\ \ldots,\ Z_{d-3}(s_0) = Z^0_{d-3}.
\end{aligned}
\tag{4.8}
$$

Proof. First, we prove the existence of γ with a frame field $(A, B, C, Z_1, \ldots, Z_{d-3})$ satisfying the condition only (4.5). Let $A, B, C, Z_1, \ldots, Z_{d-3}$ be vector fields satisfying

$$
\begin{cases}
\dot{A} = k_1 C, \\
\dot{C} = k_2 A + \varepsilon_C k_1 B, \\
\dot{B} = \varepsilon_C k_2 C + k_3 Z_1, \\
\dot{Z}_1 = \varepsilon_1 k_3 A + k_4 Z_2, \\
\varepsilon_2 \dot{Z}_2 = -\varepsilon_1 k_4 Z_1 + \varepsilon_2 k_5 Z_3, \\
\quad\vdots \\
\varepsilon_{d-4} \dot{Z}_{d-4} = -\varepsilon_{d-5} k_{d-2} Z_{d-5} + \varepsilon_{d-4} k_{d-1} Z_{d-3}, \\
\varepsilon_{d-3} \dot{Z}_{d-3} = -\varepsilon_{d-4} k_{d-1} Z_{d-4}
\end{cases}
\tag{4.9}
$$

with initial condition (4.8). Actually, the existence of such vector fields follows from the general theory of ODE. Define a null curve γ by $\gamma(s) := p_0 + \int_{s_0}^{s} A(t)dt$. Put $V := (A + B)/\sqrt{2}$ and $W := (A - B)/\sqrt{2}$. Let F be a matrix consisting of column vector fields $V, W, C, Z_1, \ldots, Z_{d-3}$, that is, $F = (V\,W\,C\,Z_1 \cdots Z_{d-3})$. We put

$$
X = \begin{pmatrix}
0 & 0 & \frac{1}{\sqrt{2}}(\varepsilon_C k_1 + k_2) \\
0 & 0 & \frac{1}{\sqrt{2}}(-\varepsilon_C k_1 + k_2) \\
\frac{1}{\sqrt{2}}(k_1 + \varepsilon_C k_2) & \frac{1}{\sqrt{2}}(k_1 - \varepsilon_C k_2) & 0
\end{pmatrix}
$$

and

$$
Y = \begin{pmatrix}
0 & -\varepsilon_1 \varepsilon_2 k_4 & & & & \\
k_4 & 0 & & & & \\
& & \ddots & & & \\
& & & 0 & -\varepsilon_{d-4}\varepsilon_{d-5} k_{d-2} & 0 \\
& & & k_{d-2} & 0 & -\varepsilon_{d-3}\varepsilon_{d-4} k_{d-1} \\
& & & 0 & k_{d-1} & 0
\end{pmatrix},
$$

and we define a coefficient matrix K by

$$K = \left(\begin{array}{ccc|cccc} & & & \frac{1}{\sqrt{2}}k_3 & 0 & \cdots & 0 \\ & X & & \frac{1}{\sqrt{2}}k_3 & 0 & \cdots & 0 \\ & & & 0 & 0 & \cdots & 0 \\ \hline \frac{1}{\sqrt{2}}k_3 & -\frac{1}{\sqrt{2}}k_3 & 0 & & & & \\ 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & & & Y & \\ 0 & 0 & 0 & & & & \end{array}\right).$$

Then, (4.9) is equivalent to

$$\dot{F} = FK. \tag{4.10}$$

We remark that the solution of (4.10) is unique for a given initial condition.

Next, we prove that $A, B, C, Z_1, \ldots, Z_{d-3}$ along γ satisfy the condition (4.4). Recalling the notation of Proposition 4.1.1, $-\varepsilon_j\varepsilon_{j+1} = -1$ for all $1 \leq j \leq d-3$ if $\varepsilon_C = -1$. If $\varepsilon_C = 1$, then $-\varepsilon_{i-1}\varepsilon_i = -\varepsilon_i\varepsilon_{i+1} = 1$ for a unique $1 \leq i \leq d-3$ such that $\varepsilon_i = -1$ and $-\varepsilon_j\varepsilon_{j+1} = 1$ for j and $j+1 \neq i$. We put $E := \mathrm{diag}(-1, 1, \ldots, -1, \ldots, 1)$. Let Φ be the solution of (4.10) with initial value E. Put $F^0 := F(s_0)$. Then, we have $F^0E\Phi(s_0) = F^0EE = F^0$. Thus, by the uniqueness of the solution of ODE (4.10) with the same initial value, $F = F^0E\Phi$. Note that EK is skew-symmetric. Hence,

$$\begin{aligned} \frac{d}{ds}(EF^{-1}E)(s) &= -EK(s)F^{-1}(s)E \\ &= {}^t(EK(s))F^{-1}(s)E \\ &= {}^tK(s)(EF^{-1}E)(s). \end{aligned}$$

By ${}^t\Phi(s_0) = {}^tE = E$ and $(E\Phi^{-1}E)(s_0) = E$, we obtain ${}^tF = EF^{-1}E$, that is, F is a pseudo-orthogonal matrix. Hence, F satisfies (4.4) because the columns of F form an orthonormal basis for $\mathbb{E}^n_2$ and V is timelike. Therefore, a proof of the existence of Cartan curve γ for functions $k_1. \ldots, k_{d-1}$ is completed.

Finally, we prove the uniqueness of γ with Cartan frame field F. Let γ and $\tilde{\gamma}$ be Cartan curves. Assume that k_i of γ in (4.5) coincides with that of $\tilde{\gamma}$ for all i. Let F and $\tilde{F}$ be pseudo-orthogonal matrices defined by $F = (V\,W\,C\,Z_1 \cdots Z_{d-3})$ for γ and $\tilde{F} = (\tilde{V}\,\tilde{W}\,\tilde{C}\,\tilde{Z}_1 \cdots \tilde{Z}_{d-3})$ for $\tilde{\gamma}$, respectively. We put $F^0 := F(s_0)$ and $\tilde{F}^0 := \tilde{F}(s_0)$. When we put $L := F^0(\tilde{F}^0)^{-1}$, by the uniqueness of the solution of ODE (4.10) with the same initial value, we have $F = L\tilde{F}$. In particular, $A = L\tilde{A}$ because $V = L\tilde{V}$ and $W = L\tilde{W}$. Put $b := \gamma(s_0) - L\tilde{\gamma}(s_0)$. Then we have

$$\begin{aligned} \gamma(s) &= \gamma(s_0) + \int_{s_0}^{s} A(t)dt \\ &= L\tilde{\gamma}(s_0) + b + \int_{s_0}^{s} L\tilde{A}(t)dt \\ &= L\tilde{\gamma}(s) + b. \end{aligned}$$

Since L is a pseudo-orthogonal matrix, γ and $\tilde{\gamma}$ are congruent. $\square$

4.2 Examples of the generalized umbilical hypersurface in $\mathbb{S}_2^{n+1}$ and $\mathbb{H}_2^{n+1}$

In this section, we constructed ruled surfaces satisfying the properties similar to generalized umbilical hypersurfaces in Chapter 3.5. Ruled hypersurfaces given in examples in this section satisfy the same conditions of Definition 3.5.1 and Remark 3.5.1.

We construct the following six examples of hypersurfaces M in $\mathbb{H}_2^{n+1}$ and $\mathbb{S}_2^{n+1}$ by using frame fields of Proposition 4.1.1. Let N be a unit normal vector field of M in $\mathbb{H}_2^{n+1}$ or $\mathbb{S}_2^{n+1}$ and A_N the shape operator of M derived from N. For simplicity, we put $\mathbf{x}(s,t,z) = \mathbf{x}(s,t,z_1,z_2,\ldots,z_{n-2})$, $|z|^2 = \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \cdots + \varepsilon_{n-2} z_{n-2}^2$ and $\mathbf{x}_s = \partial\mathbf{x}/\partial s$ (resp. $\mathbf{x}_t$ and $\mathbf{x}_{z_j}$). Moreover, assume that $k_1(s) \neq 0$ for all s and k_2 is a nonzero constant.

First, we consider the case where $\nabla_{\dot\gamma} A$ is timelike, in other words, $\varepsilon_j = +1$ for all $1 \leq j \leq n-2$.

Example 4.2.1. Let γ be a null curve in $\mathbb{H}_2^{n+1}$. The immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{H}_2^{n+1} \subset \mathbb{R}_3^{n+2}$ given by

$$\mathbf{x}(s,t,z) = \frac{k_2^2 - f(z)}{1+k_2^2}\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2(1+f(z))}{1+k_2^2}C(s)$$

parametrizes an oriented Lorentzian hypersurface M of $\mathbb{H}_2^{n+1}$, where

$$f(z) = \sqrt{1 + (1+k_2^2)|z|^2}.$$

By a straightforward computation, the unit normal vector field of M in $\mathbb{H}_2^{n+1}$ is given by

$$N(s,t,z) = -\frac{k_2(1+f(z))}{1+k_2^2}\gamma(s) + k_2 tB(s) + k_2\sum_{j=1}^{n-1} z_j Z_j(s) - \frac{-1+k_2^2 f(z)}{1+k_2^2}C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

Then, one can easily see that the mean curvature H is $-k_2$, the scalar curvature S is a non-zero constant and the minimal polynomial $P(x)$ of the shape operator is $(x+k_2)^2$.

Example 4.2.2. Let γ be a null curve $\mathbb{S}_2^{n+1}$. Assume that $k_1(s) \neq 0$ and $k_2^2 \neq 1$. As in Example 4.2.1, the immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{S}_2^{n+1} \subset \mathbb{R}_2^{n+2}$ given by

$$\mathbf{x}(s,t,z) = \frac{-k_2^2 + f(z)}{-1+k_2^2}\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) + \frac{k_2(1-f(z))}{-1+k_2^2}C(s)$$

parametrizes an oriented Lorentzian hypersurface of $\mathbb{S}_2^{n+1}$ in a neighborhood of the origin, where

$$f(z) = \sqrt{1 + (-1+k_2^2)|z|^2}$$

for z with $1 + (-1+k_2^2)|z|^2 > 0$. By a straightforward computation, the unit normal vector field of M in $\mathbb{S}_2^{n+1}$ is given by

$$N(s,t,z) = \frac{-k_2(1+f(z))}{-1+k_2^2}\gamma(s) + k_2 tB(s) + k_2\sum_{j=1}^{n-1} z_j Z_j(s) + \frac{1-k_2^2 f(z)}{-1+k_2^2}C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ -k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

Then, one can easily see that the mean curvature H is $-k_2$, the scalar curvature S is a non-zero constant and the minimal polynomial $P(x)$ of the shape operator is $(x+k_2)^2$.

Example 4.2.3. Let γ be a null curve in $\mathbb{S}_2^{n+1}$ with $k_2^2 = 1$. As in Example 4.2.1, the immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{S}_2^{n+1} \subset \mathbb{R}_2^{n+2}$ given by

$$\mathbf{x}(s,t,z) = (-1 + \frac{|z|^2}{2})\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2|z|^2}{2}C(s)$$

parametrizes an oriented Lorentaisn hypersurface of $\mathbb{S}_2^{n+1}$. By a straightforward computation, the unit normal vector field of M in $\mathbb{S}_2^{n+1}$ is given by

$$N(s,t,z) = -\frac{k_2|z|^2}{2}\gamma(s) + k_2 tB(s) + k_2\sum_{j=1}^{n-1} z_j Z_j(s) - (1 + \frac{|z|^2}{2})C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ -k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

Then, one can easily see that the mean curvature H is $-k_2$, the scalar curvature S is zero and the minimal polynomial $P(x)$ of the shape operator is $(x+k_2)^2$.

Remark 4.2.1. There is a relation between Example 4.2.2 and 4.2.3. In the case where $0 < k_2^2 < 1$, one can easily check that the parametrization $\mathbf{x}$ defined by Example 4.2.2 uniformly converges in C^∞ to $\mathbf{x}$ defined by Example 4.2.3 on a fixed neighborhood of $z = 0$ as $k_2^2 \to 1$. In the case where $k_2^2 > 1$, one can easily check a similar relation between Example 4.2.2 and 4.2.3 as in the case where $0 < k_2^2 < 1$ without any condition for z.

Next, we consider the case where $\nabla_{\dot\gamma} A$ is spacelike.

Example 4.2.4. Let γ be a null curve in $\mathbb{S}_2^{n+1}$. The immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{S}_2^{n+1} \subset \mathbb{R}_2^{n+2}$ given by

$$\mathbf{x}(s,t,z) = \frac{k_2^2 - f(z)}{1+k_2^2}\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2(1+f(z))}{1+k_2^2}C(s)$$

parametrizes an oriented hypersurface with index 2 of $\mathbb{S}_2^{n+1}$ in a neighborhood of the origin, where

$$f(z) = \sqrt{1-(1+k_2^2)|z|^2}$$

for z with $1-(1+k_2^2)|z|^2 > 0$. By a straightforward computation, the unit normal vector field of M in $\mathbb{S}_2^{n+1}$ is given by

$$N(s,t,z) = -\frac{k_2(1+f(z))}{1+k_2^2}\gamma(s) + k_2 tB(s) + k_2\sum_{j=1}^{n-1} z_j Z_j(s) - \frac{-1+k_2^2 f(z)}{1+k_2^2}C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ -k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

One can easily see that the mean curvature H is $-k_2$, the scalar curvature S is a non-zero constant and the minimal polynomial $P(x)$ of the shape operator is $(x+k_2)^2$.

Example 4.2.5. Let γ be a null curve in $\mathbb{H}_2^{n+1}$. Assume that $k_1(s) \neq 0$ and $k_2^2 \neq 1$. As in Example 4.2.4, the immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{H}_2^{n+1} \subset \mathbb{R}_3^{n+3}$ given by

$$\mathbf{x}(s,t,z) = \frac{k_2^2 - f(z)}{-1+k_2^2}\gamma(s) + tB(s) + \sum_{j=1}^{n-2} z_j Z_j(s) - \frac{k_2(1-f(z))}{-1+k_2^2}C(s)$$

parametrizes an oriented hypersurface with index 2 of $\mathbb{S}_2^{n+1}$ in a neighborhood of the origin, where

$$f(z) = \sqrt{1 - (-1 + k_2^2)|z|^2}$$

for z with $1 - (-1 + k_2^2)|z|^2 > 0$. By a straightforward computation, the unit normal vector field of M in $\mathbb{H}_2^{n+1}$ is given by

$$N(s,t,z) = \frac{k_2(1 - f(z))}{-1 + k_2^2}\gamma(s) + k_2 t B(s) + k_2 \sum_{j=1}^{n-1} z_j Z_j(s) + \frac{-1 + k_2^2 f(z)}{-1 + k_2^2} C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ -k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

Then, one can easily see that the mean curvature H is $-k_2$, the scalar curvature S is a non-zero constant and the minimal polynomial $P(x)$ of the shape operator is $(x + k_2)^2$.

Example 4.2.6. Let γ be a null curve in $\mathbb{H}_2^{n+1}$ with $k_2^2 = 1$. As in Example 4.2.4, the immersion $\mathbf{x} : I \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{H}_2^{n+1} \subset \mathbb{R}_3^{n+3}$ given by

$$\mathbf{x}(s,t,z) = (1 + \frac{|z|^2}{2})\gamma(s) + tB(s) + \sum_{j=1}^{n-2} \varepsilon_j z_j Z_j(s) - \frac{k_2|z|^2}{2} C(s)$$

parametrizes an oriented hypersurface with index 2 in $\mathbb{S}_2^{n+1}$. By a straightforward computation, the unit normal vector field of M in $\mathbb{H}_2^{n+1}$ is given by

$$N(s,t,z) = \frac{k_2|z|^2}{2}\gamma(s) + k_2 t B(s) + k_2 \sum_{j=1}^{n-1} \varepsilon_j z_j Z_j(s)(1 - \frac{|z|^2}{2})C(s),$$

and the shape operator A_N with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_z)$ is expressed as

$$A_N = \begin{pmatrix} -k_2 & 0 & & & \\ k_1(s) & -k_2 & & & \\ & & -k_2 & & \\ & & & \ddots & \\ & & & & -k_2 \end{pmatrix}.$$

Then, one can easily see that the mean curvature H is $-k_2$, the scalar curvature S is zero and the minimal polynomial $P(x)$ of the shape operator is $(x + k_2)^2$.

Remark 4.2.2. There is a relation between Example 4.2.5 and 4.2.6. One can easily check that the parametrization $\mathbf{x}$ defined by Example 4.2.5 uniformly converges in C^∞ to $\mathbf{x}$ defined by Example 4.2.6 on a fixed neighborhood of $z = 0$ as $k_2^2 \to 1$.

Remark 4.2.3. By Proposition 2.1.1, those six examples given by this chapter satisfy $\Delta H = \lambda H$ for real consant λ.

By Remark 4.2.1, 4.2.2 and 4.2.3, we emphasize that examples in this section also satisfy properties of generalized umbilical hypersurfaces in Remark 3.5.3.

Chapter 5

Generalized B-scroll in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$

In the previous chapter, we studied Lorentzian hypersurfaces with index 2. We remark that those results are only about pseudo-Riemannian hypersurfaces M, that is, the case where the codimension of M in $\bar{M}$ $(= \mathbb{S}^m_2$ or $\mathbb{H}^m_2)$ is 1. In this chapter, we consider the case where the codimension of M is larger than 1. More precisely, we construct 2-dimensional Lorentzian ruled surfaces along null curve γ in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$.

5.1 Known results of a generalized B-scroll in $\mathbb{E}^m_1$

On the higher codimensional case, D. S. Kim, Y. H. Kim and D. W. Yoon [15] extended a B-scroll in $\mathbb{E}^3_1$ to in $\mathbb{E}^m_1$ and named it the *generalized B-scroll.* Moreover, they proved the following theorem about the minimal polynomial of the shape operator of a generalized B-scroll. Hence, at first, to consider 2-dimensional Lorentzian ruled surfaces along null curves in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$, we introduce their results.

Definition 5.1.1 ([15])**.** Let γ be a null curve in $\mathbb{E}^m_1$ and β a null vector field along γ. Let M be a ruled surface in $\mathbb{E}^m_1$ parameterized as an immersion $\mathbf{x}(s,t) = \gamma(s) + t\beta(s)$. Then, M is called a *null scroll* if β satisfies $\dot{\gamma}(s) \wedge \beta(s) \neq 0$ for all s.

Definition 5.1.2 ([15])**.** Let γ be a null curve in $\mathbb{E}^m_1$ and $(A, B, C, Z_1, \dots, Z_{m-3})$ a frame field along γ satisfying the conditions of (3.22). Let M be a null scroll such that it is a non-degenerate pseudo-Riemannian surface along γ parameterized as $\mathbf{x}(s,t) = \gamma(s) + t\beta(s)$. Then, M is called a *generalized B-scroll* if $\beta(s) = B(s)$ for all s.

Let H be a mean curvature vector field of a null scroll M. The shape operator of M derived from H has the minimal polynomial as follows.

Theorem 5.1 ([15])**.** *Let M be a null scroll in an m-dimensional Lorentzian space $\mathbb{E}^m_1$. If the shape operator A_H of M derived from the mean curvature vector field H has the minimal polynomial of the form $(x - a^2)^2$ for some constant a then M is a generalized B-scroll.*

Conversely, they proved that the minimal polynomial of A_H of a generalized B-scroll M is given by $(x - k_2^2)^2$, where k_2 is a constant appeared in the following Corollary 5.2.1 (the coefficient of A of $\dot{C}$). Similarly, in this chapter, we also study the minimal polynomial of the shape operator of a ruled surface in $\mathbb{S}_2^5$ and $\mathbb{H}_2^5$.

5.2 Exapmles of the generalized B-scroll in $\mathbb{S}_2^5$ revisioand $\mathbb{H}_2^5$

In the case where in M_2^5, Frenet frame field along a null curve γ in M_2^5 is one of the following two cases if $\nabla_{\dot{\gamma}}\dot{\gamma}$ is non-null. Recalling Proposotion 4.1.1, one can easily see that the following corollary follows.

Corollary 5.2.1. *Let γ be a null curve of order $d = 5$ in an 5-dimensional pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle, \nabla)$ with index 2. Assume that $\nabla_{\dot{\gamma}}\dot{\gamma}$ is non-null. Under Assumption 4.1, there exists uniquely a frame field (A, B, C, Z_1, Z_2) along γ satisfying the following conditions;*

$$\begin{aligned}
&\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,\\
&\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = \varepsilon_C,\\
&\langle A, Z_i\rangle = \langle B, Z_i\rangle = \langle C, Z_i\rangle = 0,\\
&\langle Z_i, Z_j\rangle = \varepsilon_i\delta_{ij}
\end{aligned} \tag{5.1}$$

and

$$\begin{cases}
\dot{A} = k_1C,\\
\dot{C} = k_2A + \varepsilon_C k_1 B,\\
\dot{B} = \varepsilon_C k_2 C + k_3 Z_1 + \varepsilon\gamma,\\
\dot{Z}_1 = \varepsilon_1 k_3 A + k_4 Z_2,\\
\dot{Z}_2 = \varepsilon_C k_4 Z_1
\end{cases} \tag{5.2}$$

for some positive-valued functions k_i $(i = 1, \ldots, d-1)$, ε_C, $\varepsilon_i = \pm 1$ and $i = 1, 2$. Note that $\varepsilon_1 = \varepsilon_2 = 1$ if $\varepsilon_C = -1$ and $\varepsilon_1 = -\varepsilon_2$ if $\varepsilon_C = 1$.

The next proposition holds only in the case that a null curve γ is in M_2^5, because a null vector field Z_2 satisfying $\langle Z_1, Z_2\rangle = -1$ is not uniquely if $m \geq 6$.

Proposition 5.2.1. *Let γ be a null curve in M_2^5, and we put $A = \dot{\gamma}$. If $\nabla_{\dot{\gamma}}A$ is non-null and Z_1 is null, there is a unique frame field (A, B, C, Z_1, Z_2) satisfying*

$$\begin{aligned}
&\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = -1,\\
&\langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,\\
&\langle Z_1, Z_1\rangle = \langle Z_2, Z_2\rangle = 0, \quad \langle Z_1, Z_2\rangle = -1,\\
&\langle A, Z_i\rangle = \langle B, Z_i\rangle = \langle C, Z_i\rangle = 0 \quad (i = 1, 2)
\end{aligned} \tag{5.3}$$

and

$$
(5.4) \qquad \begin{cases} \dot{A} = k_1 C, \\ \dot{C} = k_2 A + k_1 B, \\ \dot{B} = k_2 C + Z_1 + \varepsilon\gamma, \\ \dot{Z}_1 = h Z_1, \\ \dot{Z}_2 = -A - h Z_2, \end{cases}
$$

where k_2 and h are non-zero functions.

The vector field Z_2 in (5.4) could not be unique when $\dim M \geq 6$. Note that a null curve γ in Proposition 5.2.1 is order 4. We also remark that a frame field (A, B, C, Z_1, Z_2) in the previous two propositions is a pseudo-orthonormal frame field along null curve γ, but it is not a bi-null Cartan frame introduced in [21] because $\nabla_{\dot{\gamma}} A$ is null in the bi-null Cartan frame.

Next, we give examples of non-degenerate 2-dimensional Lorentzian ruled surfaces in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ satisfying conditions of the generalized B-scroll in the sense of [15]. Namely, the shape operator A_H is non-diagonalizable at any point, the mean curvature is non-zero constant and Gaussian curvature is constant. Here, we recall Theorem 5.1, that is, A_H derived from the mean curvature vector field H has the minimal polynomial of the form $(x - a^2)^2$ for some constant a.

Example 5.2.1. We consider the case where (A, B, C, Z_1, Z_2) satisfies (5.1) and (5.2). Let γ be a null curve in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$, and we put $A = \dot{\gamma}$. Assume that $\nabla_{\dot{\gamma}} A$ is non-null and k_2 and k_3 are non-zero constant. The immersion defined by $\mathbf{x}(s,t) = \gamma(s) + tB(s)$ parameterizes a non-degenerate 2-dimensional Lorentzian surface, since (A, B, C, Z_1, Z_2) satisfies (5.1) and (5.2). By a straightforward computation, its unit normal vector fields are

$$
\begin{cases} N_1(s,t) = k_2 t B(s) + C(s), \\ N_2(s,t) = \varepsilon_1 k_3 t B(s) + Z_1(s), \\ N_3(s,t) = Z_2(s), \end{cases}
$$

and the shape operator A_{N_r} derived from N_r are

$$
A_{N_1} = \begin{pmatrix} -k_2 & 0 \\ -\varepsilon_1 k_3 & -k_2 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} -\varepsilon_1 k_3 & 0 \\ 0 & -\varepsilon_1 k_3 \end{pmatrix}
$$

and

$$
A_{N_3} = \begin{pmatrix} 0 & 0 \\ -\varepsilon_C \varepsilon_1 t k_3 k_4 & 0 \end{pmatrix}.
$$

Thus, the scalar curvature is non-zero constant and the mean curvature vector field $\tilde{H}$ is written by

$$
\tilde{H} = -\varepsilon_C k_2 N_1 - k_3 N_2.
$$

We put $H = \tilde{H}/\|\tilde{H}\|$. Then, the shape operator A_H derived from H with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t)$ and the minimal polynomial $P(x)$ of A_H are given by

$$A_H = \begin{pmatrix} \varepsilon_C k_2^2 + \varepsilon_1 k_3^2 & 0 \\ k_1 k_2 & \varepsilon_C k_2^2 + \varepsilon_1 k_3^2 \end{pmatrix} \tag{5.5}$$

and

$$P(x) = (x - (\varepsilon_C k_2^2 + \varepsilon_1 k_3^2))^2, \tag{5.6}$$

where $\varepsilon_C k_2^2 + \varepsilon_1 k_3^2$ is constant. Note that a null curve γ is order 5 by the condition (5.2).

Example 5.2.2. We consider the case where (A, B, C, Z_1, Z_2) satisfies (5.3) and (5.4). Let γ be a null curve in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$, and we put $A = \dot{\gamma}$. Assume that $\nabla_{\dot{\gamma}} A$ is non-null and k_2 is non-zero constant. The immersion defined by $\mathbf{x}(s,t) = \gamma(s) + tB(s)$ parameterizes a non-degenerate 2-dimensional Lorentzian surface, since (A, B, C, Z_1, Z_2) satisfies (5.3) and (5.4). By a straightforward computation, its unit normal vector fields are

$$\begin{cases} N_1(s,t) = k_2 tB(s) + C(s), \\ N_2(s,t) = -tB(s) - \dfrac{1}{2} Z_1(s) + Z_2(s), \\ N_3(s,t) = -tB(s) + \dfrac{1}{2} Z_1 + Z_2(s), \end{cases}$$

and the shape operator A_{N_r} derived from N_r are

$$A_{N_1} = \begin{pmatrix} -k_2 & 0 \\ k_1 & -k_2 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$A_{N_3} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the scalar curvature is non-zero constant and the mean curvature vector field $\tilde{H}$ is written by

$$\tilde{H} = -k_2 N_1 - Z_1.$$

We put $H = \tilde{H}/\|\tilde{H}\|$. Then, the shape operator A_H derived from H with respect to the frame $(\mathbf{x}_s, \mathbf{x}_t)$ and the minimal polynomial $P(x)$ of A_H are given by

$$A_H = \begin{pmatrix} k_2^2 & 0 \\ k_1 k_2 & k_2^2 \end{pmatrix}$$

and

$$P(x) = (x - k_2^2)^2, \tag{5.7}$$

where k_2^2 is constant.

Remark 5.2.1. In the case where γ is in $\mathbb{S}^4_2$ or $\mathbb{H}^4_2$, we can construct a ruled surface satisfying conditions of the generalized B-scroll by the same way as Example 5.2.1. Since $\mathbb{S}^4_2$ and $\mathbb{H}^4_2$ are non-degenerate pseudo-Riemannian manifolds and we consider in the case where $\nabla_{\dot\gamma}\dot\gamma$ is non-null, there is no ruled surface in $\mathbb{S}^4_2$ or $\mathbb{H}^4_2$ as Example 5.2.2.

Remark 5.2.2. Let Z_i $(i = 1, 2)$ be a vector field along γ defined by Example 5.2.2. Because $\mathbf{x}(s,t) = \gamma(s) + tZ_i(s)$ is a degenerate surface, there is no non-degenerate null scroll equipped with a Cartan frame whose mean curvature vector is non-null except for Example 5.2.1 and 5.2.2 in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$. In the case where $\nabla_{\dot\gamma}\dot\gamma$ is null, there is no non-degenerate pseudo-Riemannian null scroll equipped with a bi-null Cartan frame in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$ (see [21]).

Remark 5.2.3. Each form of the minimal polynomial (5.6) and (5.7) satisfies the assumption of Theorem 5.1. Since a B-scroll in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ is equipped with a Cartan frame, null scrolls given by Example 5.2.1 and 5.2.2 are candidates of a generalized B-scroll in $\mathbb{S}^5_2$ and $\mathbb{H}^5_2$. Meanwhile, the eigenvalue of (5.5) is not equal to the coefficient of a vector field A in $\dot C$. Also, in Example 5.2.2, a null curve γ in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ is order 4.

Summarizing these examples and Remarks in this chapter, we have the following theorem.

Theorem 5.2. *Let (A, B, C, Z_1, Z_2) be a Frenet frame field along γ in $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ such that B is a null vector field, $\langle A, B\rangle = -1$ and $\langle B, C\rangle = 0$. We define the immersion from $I \times \mathbb{R}$ into $\mathbb{S}^5_2$ or $\mathbb{H}^5_2$ by $\mathbf{x}(s,t) = \gamma(s) + tB(s)$ and denote an image of $\mathbf{x}$ by M. Then, M is a non-degenerate Lorentzian ruled surface along γ satisfying the following.*

(i) *In the case where Z_1 is non-null, we put $\varepsilon_C = \langle C, C\rangle$ and $\varepsilon_1 = \langle Z_1, Z_1\rangle$. For some constants k_2 and k_3, the mean curvature and the minimal polynomial of the shape operator derived from the normalized mean curvature vector are $\varepsilon_C k_2^2 + \varepsilon_1 k_3^2$ and $P(x) = (x - (\varepsilon_C k_2^2 + \varepsilon_1 k_3^2))^2$, respectively.*

(ii) *In the case where Z_1 is null, for some constant k_2, the mean curvature and the minimal polynomial of the shape operator derived from the normalized mean curvature vector are k_2^2 and $P(x) = (x - k_2^2)^2$, respectively.*

Moreover, a non-degenerate Lorentzian ruled surface along γ equipped with Frenet frame field is one of the above two cases.

Bibliography

[1] Luis J. Alías, Angel Ferrández, and Pascual Lucas. Hypersurfaces in the non-flat Lorentzian space forms with a characteristic eigenvector field. *J. Geom.*, 52(1-2):10–24, 1995.

[2] Burcu Bektaş, Elif Özkara Canfes, and Uğur Dursun. Classification of surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map. *Math. Nachr.*, 290(16):2512–2523, 2017.

[3] Burcu Bektaş, Elif Özkara Canfes, and Uğur Dursun. Pseudo-spherical submanifolds with 1-type pseudo-spherical Gauss map. *Results Math.*, 71(3-4):867–887, 2017.

[4] Burcu Bektaş and Uğur Dursun. On spherical submanifolds with finite type spherical Gauss map. *Adv. Geom.*, 16(2):243–251, 2016.

[5] Bang-Yen Chen. *Total mean curvature and submanifolds of finite type*, volume 1 of *Series in Pure Mathematics*. World Scientific Publishing Co., Singapore, 1984.

[6] Bang-Yen Chen. Finite type submanifolds in pseudo-euclidean space and its applicatiions. *Kodai math. J*, 8:358–374, 1985.

[7] Bang-Yen Chen. A report on submanifolds of finite type. *Soochow J. Math.*, 22(2):117–337, 1996.

[8] Bang-Yen Chen and Paolo Piccinni. Submanifolds with finite type Gauss map. *Bull. Austral. Math. Soc.*, 35(2):161–186, 1987.

[9] Bang-Yen Chen and Joeri Van der Veken. Complete classification of parallel surfaces in 4-dimensional Lorentzian space forms. *Tohoku Math. J. (2)*, 61(1):1–40, 2009.

[10] Krishan L. Duggal and Aurel Bejancu. *Lightlike submanifolds of semi-Riemannian manifolds and applications*, volume 364 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[11] Krishan L. Duggal and Dae Ho Jin. *Null curves and hypersurfaces of semi-Riemannian manifolds*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[12] Angel Ferrández, Angel Giménez, and Pascual Lucas. Null helices in Lorentzian space forms. *Internat. J. Modern Phys. A*, 16(30):4845–4863, 2001.

[13] Dae Ho Jin. Natural Frenet equations of null curves. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.*, 12(3):211–221, 2005.

[14] Dong-Soo Kim and Young Ho Kim. *B*-scrolls with non-diagonalizable shape operators. *Rocky Mountain J. Math.*, 33(1):175–190, 2003.

[15] Dong-Soo Kim, Young Ho Kim, and Dae Won Yoon. Characterization of generalized *B*-scrolls and cylinders over finite type curves. *Indian J. Pure Appl. Math.*, 34(11):1523–1532, 2003.

[16] Honoka Kobayashi. Ruled surfaces with null curves and frenet frame in pseudo-sphere and pseudo-hyperbolic space. *to appear in Tsukuba Journal of Mathematics.*

[17] Honoka Kobayashi and Naoyuki Koike. Pseudo-hyperbolic Gauss maps of Lorentzian surfaces in anti–de Sitter space. *Math. Nachr.*, 293(5):923–944, 2020.

[18] Martin A. Magid. Lorentzian isoparametric hypersurfaces. *Pacific J. Math.*, 118(1):165–197, 1985.

[19] Barrett O'Neill. *Semi-Riemannian geometry With Applications to Relativity.* Pure and Applied Mathematics, 1983.

[20] Makoto Sakaki. Bi-null Cartan curves in semi-Euclidean spaces of index 2. *Beitr. Algebra Geom.*, 53(2):421–436, 2012.

[21] Makoto Sakaki, Ali Uçum, and Kazı m İlarslan. Ruled surfaces with bi-null curves in $\mathbb{R}^5_2$. *Rend. Circ. Mat. Palermo (2)*, 66(3):485–493, 2017.

[22] Rüya Yeğin and Uğur Dursun. On submanifolds of pseudo-hyperbolic space with 1-type pseudo-hyperbolic Gauss map. *Zh. Mat. Fiz. Anal. Geom.*, 12(4):315–337, 2016.