学位論文

On the finiteness of support τ-tilting modules
(台τ傾加群の有限性について)

2022年3月

本間　孝拓

Acknowledgments

First of all I want to express my gratitude to Prof. Katsunori Sanada, my supervisor in Tokyo University of Science for him guidance, expertise, and all the opportunities he has given to me. I would also like to thank Prof. Tomohiro Itagaki for giving helpful and many useful suggestions. I also wish to thank Prof. Kengo Miyamoto and Dr. Qi Wang for many discussions during the production of this thesis. I would like to express my sincere gratitude to Prof. Takuma Aihara for his academic and other help since I was an undergraduate student. I also wish to thank my colleagues at Tokyo University of Science for scientific and personal exchange, and for providing happy distractions from work. Finally, special thanks to my parents for their unconditional trust.

Takahiro HONMA

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Chapter 1

Introduction

This thesis is based on [AH] and [AHMW].

Tilting theory was introduced by Brenner–Butler [BB] to compare the structure of two module categories, and was developed by Keller and Rickard to study the equivalence of derived categories [K, Ric1]. In tilting theory, *tilting objects* play a central role because tilting objects control the derived equivalence and the t-structure. Hence, several authors have studied *tilting mutation* to obtain many tilting objects [HU, O, Ric2, RS]. Tilting mutation is an operation to replace some direct summands of a given tilting objects to get a new one, but unfortunately some assumptions are required. Assumptions were taken away by extending tilting objects to *silting objects*, that is, *silting mutation* is always possible [AI].

Silting objects were introduced for the study of t-structures [HKM, KV]. In fact, Koenig and Yang [KY] constructed a bijection among (1) silting objects, (2) simple-minded collections, (3) co-t-structures and (4) t-structures with length heart .

One of the most crucial purposes in my research is to clarify the whole picture of silting objects. To realize the goal, we first discuss when a triangulated category is *silting-discrete*. Silting-discreteness is a finiteness condition, namely, a silting-discrete triangulated category admits only finitely many silting objects in any interval of silting objects [Ai1]. Actually, the set of silting objects has a poset structure [AI]. In the case, we can fully grasp the whole picture of silting objects, and such a triangulated category has so nice structure [AMY, PMZ]. To check if a given triangulated category is silting-discrete, an extremely powerful tool was introduced, and it is applied to the perfect derived category of a finite dimensional algebra over a field; we know several algebras with silting-discrete perfect derived categories, say silting-discrete algebras [AAC, AK, Ai1, AM, AD, EJR]. Typical examples of silting-discrete algebras are piecewise hereditary algebras of Dynkin type and representation-finite symmetric algebras.

On the other hand, we focus on two-term silting objects to grasp the structure of silting objects. Two-term silting objects correspond bijectively to *support τ-tilting modules* by taking 0-th cohomology and they play an important role in representation theory of module categories. In particular, we can obtain all support τ-tilting modules when the set of isomorphism classes of basic support τ-tilting modules is a finite set, that is, an algebra is *τ-tilting finite*. So our aim is to understand the structure of silting objects, we will classify silting-discrete triangulated categories and τ-tilting finite algebras.

The thesis is organized as follows: In Chapter 2, we discuss two subjects on the finiteness of silting objects. First, we discuss the most special case of silting-discrete triangulated categories that is the number of silting objects is zero. For example, the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ over a finite dimensional algebra Λ has no non-zero silting object if and only if Λ has infinite global dimension. When Λ is non-semisimple selfinjective, its stable module category $\underline{\mathsf{mod}}\,\Lambda$ admits no silting object (see [AI]). Inspired by these two cases, we ask if the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ has no non-zero silting object. We give the following theorem as an answer to this question.

Theorem 1 (Theorem 2.1 and Corollary 2.3)**.** $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ *admits no non-zero silting object if* $\mathrm{inj.dim}\,\Lambda_\Lambda < \infty$*. In particular, the stable category of the Cohen–Macaulay category over an Iwanaga–Gorenstein algebra has no non-zero silting object.*

Next we construct a new silting-discrete algebra from a given one. We denote by $\mathsf{silt}\,\Lambda$ the set of isomorphism classes of basic silting objects of the perfect derived category for Λ. The following is the second main theorem.

Theorem 2 (Theorem 2.4)**.** *Let R be a finite dimensional local K-algebra and put $\Gamma := R \otimes_K \Lambda$. If Λ is silting-discrete, then we have a poset isomorphism* $\mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Gamma$*. In particular, Γ is also silting-discrete.*

As an example of Theorem 2, the $n \times n$ (upper) triangular matrix algebra $T_n(R)$ over a local algebra R is actually isomorphic to $R \otimes_K K\overrightarrow{A_n}$. So, we get a corollary of this theorem (Proposition 3.6(1)).

In the context of *triangular matrix algebras* $T_n(\Lambda)$ over an algebra Λ (not necessarily local), it seems to be difficult to understand when $T_n(\Lambda)$ is silting-discrete. In Chapter 3, let us turn our attention to two-term silting objects, that is, support τ-tilting modules. As an analogue of Auslander–Reiten's results in [AR], we have the following theorem.

Theorem 3 (Theorem 3.1)**.** *Assume that Λ is representation-finite. Then we have:*

(1) *If the Auslander algebra of Λ is τ-tilting finite, then so is $T_2(\Lambda)$.*

(2) *If Λ is simply-connected, then the converse of (1) holds.*

We give sufficient conditions for algebras to be τ-tilting infinite. In particular, the converse of Theorem 3(1) is not necessarily true (see Example 3.4) . Moreover, we classify algebras Λ with $T_n(\Lambda)$ τ-tilting finite.

Theorem 4 (Theorem 3.8 and Theorem 3.10). *Let Λ be a finite dimensional nonlocal algebra over an algebraically closed field whose Gabriel quiver has no loop and $n \geq 3$. Then the following are equivalent:*

(1) $T_n(\Lambda)$ *is* τ*-tilting finite;*

(2) *One of the following cases holds:*

  (a) $n = 4$ *and* Λ *is the path algebra of type* A_2*;*

  (b) $n = 3$ *and* Λ *is a Nakayama algebra with precisely 2 simple modules;*

  (c) $n = 3$ *and* Λ *is a Nakayama algebra with radical square zero.*

This theorem tells us the fact that for a simply-connected algebra Λ and $n \geq 3$, $T_n(\Lambda)$ is τ-tilting finite if and only if it is representation-finite (Corollary 3.11).

In Chapter 4, we give two new classes of τ-tilting finite algebras. One is the class of *weakly symmetric algebras of tubular type with non-singular Cartan matrix* [BS1, BHS]. The other is the class of *non-standard selfinjective algebras which are socle equivalent to selfinjective algebras of tubular type* [BS2, BHS]. Here is the main theorem of Chapter 4. See Figures 4.1 and 4.2 for the notation of A_i's and Λ_i's.

Theorem 5 (Theorem 4.1, Corollary 4.2 and Theorem 4.3). (1) *Any weakly symmetric algebra* A_i *of tubular type with non-singular Cartan matrix is* τ*-tilting finite. In particular, we have the number of support* τ*-tilting modules:*

$A_1(\lambda)$	$A_2(\lambda)$	A_3	A_4	A_5	A_6	A_7	A_8
24	6	192	132	8	8	108	100

A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
108	116	100	32	28	32	30	30

(2) *Non-standard selfinjective algebras* $\Lambda_1, \ldots, \Lambda_9$ *(without* Λ_{10}*) which is socle equivalent to a selfinjective algebra of tubular type are* τ*-tilting finite. In particular, we have the number of support* τ*-tilting modules:*

Λ_1	Λ_2	$\Lambda_3(\lambda)$	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
8	8	6	32	28	32	30	30	192	≥ 500

(3) *Every algebra as in (1) and (2) (without* Λ_{10}*) is tilting-discrete.*

In Chapter 5, we discuss when τ-tilting finiteness implies representation-finiteness. This is a natural question because a representation-finite algebra is τ-tilting finite but the converse does not necessarily hold. A typical example is the hereditary case; that is, τ-tilting finite hereditary algebras are representation-finite. For more examples, it was proved that τ-tilting finite cycle-finite algebras are representation-finite [MS]. Recently, the gentle case was verified; τ-tilting finite gentle algebras are representation-finite [P]. Now, we give new classes of algebras which satisfy this property.

Theorem 6 (Corollary 5.2, 5.3, Theorem 5.8 and 5.11)**.** *The following algebras are representation-finite if they are τ-tilting finite:*

(1) *quasitilted algebras;*

(2) *algebras satisfying the separation condition;*

(3) *the trivial extensions of tree quiver algebras with radical square zero;*

(4) *locally hereditary algebras.*

Throughout this paper, algebras are always assumed to be finite dimensional over an algebraically closed field K. Modules are finite dimensional and right modules. For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ ($\mathsf{proj}\,\Lambda$) the category of (projective) modules over Λ. The perfect derived category of Λ is denoted by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$.

Chapter 2

The finiteness of silting objects

In this chapter, we consider the existence and the finiteness of silting objects. First, we consider a triangulated category which has no non-zero silting object in Section 2.1. And in Section 2.2, we construct a new silting-discrete algebra from a given silting-discrete algebra.

Throughout this chapter, $\mathcal{T}$ denotes a Krull–Schmidt triangulated category which is K-linear and Hom-finite. For example, it is the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ or the perfect derived category $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ over an algebra Λ.

Let us recall the definition of silting objects of $\mathcal{T}$. We say that an object T is *silting* if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i > 0$ and $\mathcal{T} = \mathsf{thick}\, T$. Here, $\mathsf{thick}\, T$ stands for the smallest thick subcategory of $\mathcal{T}$ containing T. It is known that the set $\mathsf{silt}\,\mathcal{T}$ of isomorphism classes of basic silting objects of $\mathcal{T}$ has a partial order $\geq$ and actions $\mu^{\pm}$ of *silting mutation*; see [AI] for details.

2.1. The existence of silting objects

In this section, we explore when a triangulated category has no silting object. A typical example of silting objects is the stalk complex Λ (and its shifts) in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. If we can find even one silting object, silting mutation produces infinitely many ones [AI]. However, we know triangulated categories with no silting object [AI, Example 2.5].

Let Λ be an algebra. We denote by $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ the singularity category of Λ; that is, it is the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. Here is the main result of this section.

Theorem 2.1. $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ *has no non-zero silting object if* $\mathrm{inj.dim}\,\Lambda_\Lambda < \infty$.

To prove this theorem, silting reduction [AI, IY] plays a crucial role.

In the rest, fix a presilting object T of $\mathcal{T}$ and define a subset $\mathsf{silt}_T\,\mathcal{T}$ of $\mathsf{silt}\,\mathcal{T}$ by

$$\mathsf{silt}_T\,\mathcal{T} := \{P \in \mathsf{silt}\,\mathcal{T} \mid T \text{ is a direct summand of } P\}.$$

Moreover, one puts $\mathcal{S} := \mathsf{thick}\, T$. The Verdier quotient of $\mathcal{T}$ by $\mathcal{S}$ is denoted by $\mathcal{T}/\mathcal{S}$.

Then, silting reduction [IY, Theorem 3.7] says:

Theorem 2.2. [IY, Theorem 3.7] *The canonical functor* $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ *induces a bijection* $\mathsf{silt}_T \mathcal{T} \to \mathsf{silt}\, \mathcal{T}/\mathcal{S}$ *if any object* X *of* $\mathcal{T}$ *satisfies* $\mathrm{Hom}_{\mathcal{T}}(T, X[\ell]) = 0 = \mathrm{Hom}_{\mathcal{T}}(X, T[\ell])$ *for* $\ell \gg 0$.

For example, this is the case where $\mathcal{T}$ has a silting object [AI, Proposition 2.4].

Now, we are ready to show our main theorem of this section.

Proof of Theorem 2.1. We will apply silting reduction to $\mathcal{T} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \Lambda)$ and $T = \Lambda$; in this setting, $\mathcal{S} = \mathsf{thick}\, \Lambda = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, \Lambda)$ and $\mathcal{T}/\mathcal{S} = \mathsf{D}_{\mathsf{sg}}(\Lambda)$. To do that, we check that the conditions $\mathrm{Hom}_{\mathcal{T}}(\Lambda, X[\ell]) = 0 = \mathrm{Hom}_{\mathcal{T}}(X, \Lambda[\ell])$ are satisfied for any object X and $\ell \gg 0$. The first equality holds evidently. Let us show that the second equality holds true. Since Λ has finite right selfinjective dimension, it can be regarded as a complex in $\mathsf{K}^{\mathsf{b}}(\mathsf{inj}\, \Lambda)$, which is obtained by applying the Nakayama functor $\nu := - \otimes^{\mathbf{L}}_{\Lambda} D\Lambda$ to some complex P in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, \Lambda)$. Then we get isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(X, \Lambda[\ell]) \simeq \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{mod}\, \Lambda)}(X, \nu P[\ell]) \simeq D\, \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{mod}\, \Lambda)}(P[\ell], X).$$

As the complex X is bounded, the last above is zero for sufficiently large ℓ. Thus, silting reduction brings us a bijection $\mathsf{silt}_{\Lambda}\, \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \Lambda) \to \mathsf{silt}\, \mathsf{D}_{\mathsf{sg}}(\Lambda)$. It follows from [AI, Example 2.5(1)] that the LHS of the bijection is $\{\Lambda\}$ if Λ has finite global dimension, and is otherwise empty. Hence, we conclude that $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ admits no non-zero silting object. □

An algebra Λ is said to be *Iwanaga–Gorenstein* if it has finite right and left selfinjective dimension. In that case, the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ is triangle equivalent to the stable category $\underline{\mathsf{CM}}\, \Lambda$ of the full subcategory of $\mathsf{mod}\, \Lambda$ consisting of Cohen–Macaulay modules M; i.e. $\mathrm{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for $i > 0$. So, we immediately obtain the following corollary.

Corollary 2.3. $\underline{\mathsf{CM}}\, \Lambda$ *has no non-zero silting object if* Λ *is Iwanaga–Gorenstein.*

2.2. A new example of silting-discrete algebras

In this section, we give a new construction of silting-discrete algebras.

A triangulated category $\mathcal{T}$ is said to be *silting-discrete* if it admits a silting object T, and for any $n > 0$ there are only finitely many (basic) silting objects U satisfying $T \geq U \geq T[n]$. We obtain from [Ai1, Corollary 3.9] that if $\mathcal{T}$ is silting-discrete, then the Hasse quiver of the poset $\mathsf{silt}\, \mathcal{T}$ is connected; namely, it is *silting-connected*.

When $\mathcal{T} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, \Lambda)$ for an algebra Λ, we write $\mathsf{silt}\, \mathcal{T}$ by $\mathsf{silt}\, \Lambda$ and say that Λ is *silting-discrete* if $\mathcal{T}$ is silting-discrete. The following theorem gives us a new way of constructing the silting-discrete algebra.

Theorem 2.4. *Let* R *be a local algebra and* Λ *a silting-discrete algebra. Put* $\Gamma := R \otimes_K \Lambda$. *Then we have a poset isomorphism* $\mathsf{silt}\, \Lambda \to \mathsf{silt}\, \Gamma$. *In particular,* Γ *is also silting-discrete.*

Proof. Let us consider the triangle functor $-\otimes_K R : \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda) \to \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)$. Then we have an isomorphism $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)}(X\otimes_K R, Y\otimes_K R) \simeq \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(X,Y)\otimes_K R$; see [Zim] for example. This leads to the fact that $-\otimes_K R$ keeps the indecomposability of objects; in fact, if E is a local algebra, then so is $E\otimes_K R$ since K is algebraically closed. We also observe that $-\otimes_K R$ induces an injection $\mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Gamma$ preserving the partial order.

We show that $-\otimes_K R$ preserves approximations. Let $\mathcal{Y}$ be a full subcategory of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ and $f : X \to Y$ be a left $\mathcal{Y}$-approximation of X in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$. For an object Z of $\mathcal{Y}$, the above isomorphism makes a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)}(Y\otimes_K R, Z\otimes_K R) & \xrightarrow{-\circ(f\otimes R)} & \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)}(X\otimes_K R, Z\otimes_K R) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(Y,Z)\otimes_K R & \xrightarrow[(-\circ f)\otimes R]{} & \mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(X,Z)\otimes_K R \end{array}$$

Since $-\circ f$ is surjective and $-\otimes R$ is (right) exact, we see that the two horizontal arrows are surjections, whence $f\otimes R$ is a left $\mathcal{Y}\otimes_K R$ approximation of $X\otimes_K R$.

Thus, it turns out that any arrow in $\mathsf{silt}\,\Lambda$ is also an arrow in $\mathsf{silt}\,\Gamma$ under the injection $-\otimes_K R : \mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Gamma$. Conversely, we obtain that all paths from/to Γ in $\mathsf{silt}\,\Gamma$ come from those from/to Λ in $\mathsf{silt}\,\Lambda$, because $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)$ have the same rank of the Grothendieck group.

Assume that there is a silting object U of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)$ with $\Gamma \geq U$ which is out of the image of the functor $-\otimes_K R$. By [AI, Proposition 2.36], we have a path $\Gamma =: U_0 \to U_1 \to \cdots \to U_\ell \to \cdots$ in $\mathsf{silt}\,\Gamma$ with $U_i \geq U$ for any i, which admits an infinite length, contrary to the assumption of Λ being silting-discrete. Therefore, all silting objects of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Gamma)$ smaller than Γ come from those of Λ. Then, we derive from [AM, Theorem 2.4] that Γ is silting-discrete. Moreover, one obtains that the map $-\otimes_K R : \mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Gamma$ is a bijection.

Finally, we see that the following are equivalent for any $T, U \in \mathsf{silt}\,\Lambda$:

(i) $T \geq U$;

(ii) there exists a path of finite length from T to U;

(iii) there is a path of finite length from $T\otimes_K R$ to $U\otimes_K R$;

(iv) $T\otimes_K R \geq U\otimes_K R$.

This implies that the map $-\otimes_K R$ is a poset isomorphism. □

We apply Theorem 2.4 to trivial extension algebras. The *trivial extension* of an algebra Λ by a (Λ,Λ)-bimodule M is defined to be $\Lambda\oplus M$ as a (Λ,Λ)-bimodule in which the composition of elements (a,m) and (b,n) is given by $(a,m)\cdot(b,n) := (ab, an+mb)$. Since the trivial extension of Λ by itself is isomorphic to $K[x]/(x^2)\otimes_K\Lambda$, we immediately obtain the following corollary from Theorem 2.4.

Corollary 2.5. *The trivial extension of* Λ *by itself is silting-discrete if* Λ *is so.*

Remark 2.6. The trivial extension of Λ by its K-dual is often called the *trivial extension* of Λ. Applying it frequently destroys the silting-discreteness of algebras. For instance, the algebra given by the quiver $\bullet \rightleftarrows \bullet$ with radical square zero is silting-discrete, but its trivial extension is neither silting-discrete nor even silting-connected [AGI].

We give a slight generalization of [EJR, Theorem 15].

Corollary 2.7. *Let* p *be the characteristic of* K *and suppose* $p \neq 0$. *Then every* p*-group is contained in the defect group of a nonlocal silting-discrete block of a group algebra.*

Proof. Let P be a p-group and G a finite group. Let Λ be a block of the group algebra KG with defect group D. As is well-known, KP is a local algebra and $KP \otimes_K \Lambda$ is a block of the group algebra $K[P \times G]$ whose defect group is $P \times D$. Thus, we apply Theorem 2.4 to get the desired block; for example, if D is cyclic, dihedral, semidihedral or quaternion, then Λ is silting-discrete, whence so is $KP \otimes_K \Lambda$. □

Chapter 3

τ-tilting finite triangular matrix algebras

In this chapter, we discuss the τ-tilting finiteness of triangular matrix algebras. First, we modify the representation-finiteness of second triangular matrix algebras in [AR] to the τ-tilting finiteness in Section 3.1. Next, we classify τ-tilting finite triangular matrix algebras $T_n(\Lambda)$ $(n \geq 3)$ in Section 3.2.

We start with recalling important facts on τ-tilting finite algebras. We call a module M over Λ a *support τ-tilting* module provided it is the 0th cohomology of a silting object T in $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ with $T^i = 0$ unless $i = 0, -1$ (see [AIR] for more details). Our interest in this paper is when an algebra Λ has only finitely many support τ-tilting modules; so-called, Λ is *τ-tilting finite*. Evidently, if Λ is silting-discrete, then it is τ-tilting finite. We also know that any factor algebra of a τ-tilting finite algebra is also τ-tilting finite [DIRRT, Theorem 5.12(d)]. A module M is said to be *brick* if $\mathrm{End}_\Lambda(M)$ is isomorphic to K. It was shown that Λ is τ-tilting finite iff there are only finitely many bricks of Λ [DIJ, Theorem 4.2].

3.1. Second triangular matrix algebras

The first aim of this section is to develop the Auslander–Reiten's results in [AR] to the τ-tilting finiteness.

A main algebra we study here is the $n \times n$ upper triangular matrix algebra $T_n(\Lambda)$, which is isomorphic to $\Lambda \otimes_K K\overrightarrow{A_n}$. Here, $\overrightarrow{A_n}$ denotes the linearly oriented A_n-quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$. As is well-known, we can identify the category $\mathsf{mod}\, T_2(\Lambda)$ with the category of homomorphisms in $\mathsf{mod}\,\Lambda$; that is, the objects are triples (M, N, f) of Λ-modules M, N and a Λ-homomorphism $f : M \to N$. A morphism $(M_1, N_1, f_1) \to (M_2, N_2, f_2)$ is a pair (α, β) of Λ-homomorphisms $\alpha : M_1 \to M_2$ and $\beta : N_1 \to N_2$ satisfying $f_2 \circ \alpha = \beta \circ f_1$.

For an additive category $\mathcal{C}$, we denote by $\mathsf{mod}\,\mathcal{C}$ the full subcategory of the functor category of $\mathcal{C}$ consisting of finitely generated functors.

Inspired by [AR, Theorem 1.1], we have the second main result of this paper.

Theorem 3.1. *Assume that Λ is representation-finite. Then the following hold:*

(1) *If the Auslander algebra of Λ is τ-tilting finite, then so is $T_2(\Lambda)$.*

(2) *If Λ is simply-connected, then $T_2(\Lambda)$ is τ-tilting finite if and only if it is representation-finite. In particular, the converse of (1) holds.*

Proof. Let us first recall an argument in [AR, Theorem 1.1]. It was shown that the functor $\Phi : \mathsf{mod}\, T_2(\Lambda) \to \mathsf{mod}(\mathsf{mod}\,\Lambda)$ sending (M, N, f) to $\operatorname{Coker}\operatorname{Hom}_\Lambda(-, f)$ is full and dense. Denote by $\mathcal{D}$ the full subcategory of $\mathsf{mod}\, T_2(\Lambda)$ consisting of modules without indecomposable summands of the forms (M, M, id) and $(M, 0, 0)$, where M is an indecomposable module over Λ. Then the restriction of Φ is full and dense (not faithful!), and a morphism σ in $\mathcal{D}$ with $\Phi(\sigma)$ isomorphic is an isomorphism.

We show the assertion (1) holds true. As above, any brick over $T_2(\Lambda)$ lying in $\mathcal{D}$ is sent to some brick in $\mathsf{mod}(\mathsf{mod}\,\Lambda)$ by the functor Φ and the correspondence is objectively injective. Therefore, $T_2(\Lambda)$ inherits the finiteness of bricks from the Auslander algebra of Λ, whence the assertion follows from [DIJ, Theorem 4.2].

To prove the assertion (2), we assume that Λ is simply-connected and $T_2(\Lambda)$ is τ-tilting finite. Then, $T_2(\Lambda)$ does not contain a finite convex subcategory which is concealed of extended Dynkin type. The simple-connectedness of Λ (i.e. $\widetilde{\Lambda} = \Lambda$ in the sense of [LS1]) implies that $T_2(\Lambda)$ is representation-finite by [LS1, Theorem 4]. Moreover, we deduce from [AR, Theorem 1.1] that the Auslander algebra of Λ is also representation-finite, and so it is τ-tilting finite. □

Let Λ be an algebra whose Gabriel quiver is Q. The *separated quiver* Q^{sp} of Λ is defined as follows: The set of vertices consists of the vertices $i_1, \cdots, i_n$ of Q and their copies $i'_1, \cdots, i'_n$; we say that i and i' are the same character. We draw an arrow $i \to k$ if i is a vertex of Q, $k = j'$ for some vertex j of Q and if there is an arrow $i \to j$ in Q (see [ARS]). Observing the separated quiver, we can infer the representation-finiteness and τ-tilting finiteness of a radical-square-zero algebra [ARS, A1]; we use their results freely.

We know from [AR, Proposition 3.1] that if the separated quiver of an algebra Λ has a connected component which is not of type A_n, then $T_2(\Lambda)$ is representation-infinite. Here is a modification to τ-tilting finiteness.

Theorem 3.2. *Let Λ be an algebra given by a quiver without loop. If the separated quiver of Λ has a connected component which is not of type A_n, then $T_2(\Lambda)$ is τ-tilting infinite.*

Proof. Let $\mathcal{C}$ be a connected component of the separated quiver of Λ.

Suppose that $\mathcal{C}$ is of type $\widetilde{A_n}$. If all vertices of $\mathcal{C}$ have distinct characters from each other, then Λ is τ-tilting infinite by Adachi's theorem [A1, Theorem 3.1]. If $\mathcal{C}$ admits the same character i and i', then the vertex i of the (Gabriel) quiver Q of Λ is a source of precisely two arrows and a sink of exactly two arrows, which leads to the fact that the separated quiver of $T_2(\Lambda)$ contains a subquiver of type $\widetilde{D_5}$ with distinct characters. Hence, it turns out that $T_2(\Lambda)/\operatorname{rad} T_2(\Lambda)$, and so $T_2(\Lambda)$, are not τ-tilting finite.

Thus, we can assume that $\mathcal{C}$ is neither of type A_n nor of type $\widetilde{A_n}$. This means that $\mathcal{C}$ has a vertex v of degree at least 3; it does not matter if v is the original or the copy of a vertex of Q. We may suppose that Q possesses no multiple arrow. As Q admits no loop, it is seen that the 4 points around v (including also v) are distinct characters in $\mathcal{C}$. Therefore, we obtain that the separated quiver of $T_2(\Lambda)$ contains the diagram of type $\widetilde{E_6}$ whose vertices are distinct characters, whence $T_2(\Lambda)$ is not τ-tilting finite. $\square$

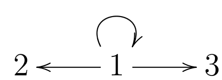
The converse of Theorem 3.2 does not necessarily hold.

Example 3.3 (See also Theorem 3.1(2))**.** Let $\Lambda := K\overrightarrow{A_n}$. Observe that $T_2(\Lambda)$ is the commutative ladder of degree n; see [AHMW, EH, LS1]. Then the following are equivalent: (i) $n \leq 4$; (ii) $T_2(\Lambda)$ is representation-finite; (iii) it is τ-tilting finite.

Combining this observation and Theorem 3.1(1), we recover [IX, Corollary 4.8]; that is, the following are equivalent: (i) $n \leq 4$; (ii) the Auslander algebra of Λ is representation-finite; (iii) it is τ-tilting finite.

We give an example which says that the converse of Theorem 3.1(1) does not necessarily hold and that the assumption of Λ having no loop as in Theorem 3.2 is required.

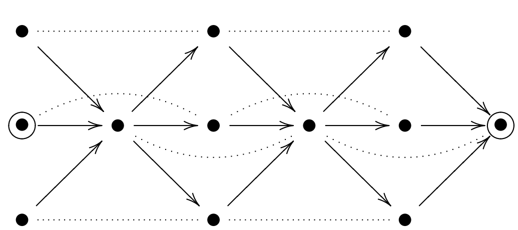
Example 3.4. Let Λ be the radical-square-zero algebra presented by the quiver:



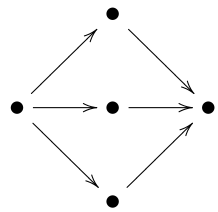
(i) The separated quiver of Λ consists of three connected components; one Dynkin quiver of type D_4 and two isolated points. So, Λ is representation-finite.

(ii) Let us show that $T_2(\Lambda)$ is τ-tilting finite. We consider the algebra A presented by the quiver $2 \longleftarrow 1 \longrightarrow 3$. Since $T_2(A)$ is derived equivalent to the path algebra of Dynkin type E_6 [La], it is seen that $T_2(A)$ is silting-discrete. By Theorem 2.4, we obtain that $T_2(A) \otimes_K K[x]/(x^2)$ is silting-discrete; in particular, it is τ-tilting finite. As there is an algebra epimorphism $T_2(A) \otimes_K K[x]/(x^2) \to T_2(\Lambda)$, we deduce that the target $T_2(\Lambda)$ is τ-tilting finite.

(iii) However, the Auslander algebra Γ of Λ is not τ-tilting finite. This is deduced by observing the Auslander–Reiten quiver of Λ (it gives a quiver presentation of Γ):



Here, the vertex ⊙ coincides. Factoring by an ideal, we find the factor algebra Γ_1 of Γ presented by the quiver

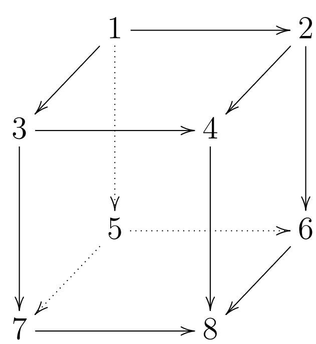


with a zero relation; the sum of the three paths of length 2 is zero. Truncating Γ_1 by idempotents, we get the Kronecker algebra, which implies that Γ_1, and so Γ, are τ-tilting infinite.

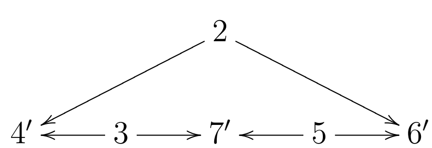
3.2. Higher triangulated matrix algebras

In this section, we focus on the τ-tilting finiteness of ${T_2}^3(\Lambda)$ and $T_n(\Lambda)$ $(n > 2)$.

In the paper [AR], it was also discussed that the third triangular matrix algebra ${T_2}^3(\Lambda)$ over an algebra Λ is not representation-finite [AR, Theorem 3.4]. To see this, we consider the triangular matrix algebra ${T_2}^3(\Lambda/\operatorname{rad}\Lambda)$. It is because this is a factor algebra of ${T_2}^3(\Lambda)$, since $T_2(\Lambda)/I \simeq T_2(\Lambda/\operatorname{rad}\Lambda)$. Here, I stands for the ideal $\begin{pmatrix}\operatorname{rad}\Lambda & \operatorname{rad}\Lambda \\ 0 & \operatorname{rad}\Lambda\end{pmatrix}$. As ${T_2}^3(\Lambda/\operatorname{rad}\Lambda)$ is the direct product of some copies of ${T_2}^3(K)$, the next step is to observe ${T_2}^3(K)$. We see that ${T_2}^3(K)$ is presented by the quiver



whose separated quiver contains the connected component:



This implies that ${T_2}^3(K)/\operatorname{rad}^2 {T_2}^3(K)$ is representation-infinite. Consequently, it turns out that ${T_2}^3(\Lambda)$ is of infinite representation type.

We can apply this argument to obtain the following result.

Proposition 3.5. (1) *The triangular matrix algebra* ${T_2}^3(\Lambda)$ *is* τ*-tilting infinite.*

(2) *For nonlocal algebras* Λ, Γ *and* Σ*,* $\Lambda \otimes_K \Gamma \otimes_K \Sigma$ *is* τ*-tilting infinite.*

Proof. (1) Combine the argument above and Adachi's theorem [A1, Theorem 3.1].

(2) By assumption, there is an algebra epimorphism from $\Lambda \otimes_K \Gamma \otimes_K \Sigma$ to $K\overrightarrow{A_2} \otimes_K K\overrightarrow{A_2} \otimes_K K\overrightarrow{A_2} \simeq {T_2}^3(K)$. Then apply (1). □

We expect that there is an upper bound of n such that the $n \times n$ triangular matrix algebra over a nonsemisimple algebra is τ-tilting finite; cf. [LS1, Theorem 6.1].

Proposition 3.6. (1) *Let* $n > 0$*. If* Λ *is local, then* $T_n(\Lambda)$ *is silting-discrete. Hence, it is* τ*-tilting finite.*

(2) *Assume that* Λ *is nonlocal. If* $T_n(\Lambda)$ *is* τ*-tilting finite, then we have* $n \leq 4$*.*

Proof. (1) The algebra $T_n(\Lambda)$ is isomorphic to $\Lambda \otimes_K K\overrightarrow{A_n}$, and then apply Theorem 2.4.

(2) Let $n \geq 5$. As Λ is nonlocal, we obtain algebra epimorphisms $T_n(\Lambda) \to T_5(K\overrightarrow{A_2}) \simeq T_2(K\overrightarrow{A_5})$. The target is not τ-tilting finite by Example 3.3, whence neither is $T_n(\Lambda)$. □

In the rest of this section, we explore when $T_n(\Lambda)$ is τ-tilting finite for $n \geq 3$.

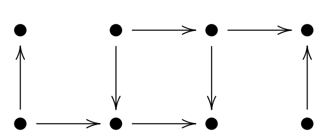
First, we treat radical-square-zero Nakayama algebras, which play a role in our goal.

Lemma 3.7. *Let* Λ *be a nonlocal Nakayama algebra with radical square zero. If* $\Lambda \neq K\overrightarrow{A_2}$*, then* $T_4(\Lambda)$ *is* τ*-tilting infinite.*

Proof. Assume that Λ is linear Nakayama with at least 3 simple modules. Since $T_4(\Lambda)$ is strongly simply-connected and representation-infinite by [LS1, Theorem 6.2], we obtain from [W, Theorem 2.6] that it is τ-tilting infinite.

If Λ is cyclic Nakayama with at least 3 simple modules, then there is an algebra epimorphism $\Lambda \to \Gamma$, which induces $T_4(\Lambda) \to T_4(\Gamma)$. Here, $\Gamma := K\overrightarrow{A_3}/\operatorname{rad}^2 K\overrightarrow{A_3}$. As above, this implies that $T_4(\Lambda)$ is τ-tilting infinite.

We show that $T_4(\Lambda)$ is not τ-tilting finite if Λ is a radical-square-zero cyclic Nakayama algebra with precisely 2 simple modules. Then one sees from the Happel–Vossieck List [HV] that it has a tame concealed factor algebra of type $\widetilde{E_7}$ as follows:



Hence, it turns out that $T_4(\Lambda)$ is not τ-tilting finite. □

We solve the problem for the case that given algebras have at least 3 simple modules.

Theorem 3.8. *Let Λ be an algebra given by a quiver Q which has no loops and at least 3 vertices. Let $n \geq 3$. Then the following are equivalent:*

(1) *$T_n(\Lambda)$ is τ-tilting finite;*

(2) *It is representation-finite;*

(3) *$n = 3$ and Λ is a Nakayama algebra with radical square zero.*

Proof. It is trivial that (2) implies (1). It follows from [LS1, Theorem 6.1] that the implications (2)$\Leftrightarrow$(3) hold true.

We show that (1) implies (3). One may suppose that Q admits no multiple arrow. Assume that Q has $\bullet \longleftarrow \bullet \longrightarrow \bullet$ or $\bullet \longrightarrow \bullet \longleftarrow \bullet$ as a subquiver. Then, we see that there is an algebra epimorphism $T_n(\Lambda) \to T_3(A)$, where A is the path algebra of $\bullet \longleftarrow \bullet \longrightarrow \bullet$ or $\bullet \longrightarrow \bullet \longleftarrow \bullet$. By the Happel–Vossieck List [HV], we observe that a tame concealed algebra of type $\widetilde{E_7}$ appears as a factor algebra of $T_3(A)$, which is τ-tilting infinite, and hence, so is $T_n(\Lambda)$. Thus, we find out that Λ is a Nakayama algebra. As a similar argument above, we deduce the fact that Q does not admit $\bullet \longrightarrow \bullet \longrightarrow \bullet$ without zero relation, which implies that Λ has radical square zero. Finally, apply Proposition 3.6(2) and Lemma 3.7 to get $n = 3$. $\square$

Let us turn to the case where a given algebra has precisely 2 simple modules. We prepare a lemma to reduce the length.

Lemma 3.9. *Let Λ be a cyclic Nakayama algebra with precisely 2 simple modules. Then we have a poset isomorphism $\mathsf{s}\tau\text{-}\mathsf{tilt}\, T_n(\Lambda) \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\, T_n(\Lambda/\operatorname{rad}^2\Lambda)$.*

Proof. By assumption, Λ is given by the quiver $1 \underset{y}{\overset{x}{\rightleftarrows}} 2$. Then it is seen that $z := xy + yx$ belongs to the center and the radical of Λ, whence zI is in those of $T_n(\Lambda)$. Here, I is the identity matrix. We observe that the factor algebra of $T_n(\Lambda)$ by the ideal generated by zI is isomorphic to $T_n(\Lambda/\operatorname{rad}^2\Lambda)$, which completes the proof by [EJR, Theorem 11]. $\square$

Now, we totally realize our goal.

Theorem 3.10. *Let Λ be an algebra whose quiver has precisely 2 vertices and no loops. Let $n \geq 3$. Then the following are equivalent:*

(1) *$T_n(\Lambda)$ is τ-tilting finite;*

(2) *$n = 3$ and Λ is a Nakayama algebra, or $n = 4$ and $\Lambda = K\overrightarrow{A_2}$.*

Proof. If $T_n(\Lambda)$ is τ-tilting finite, then we observe that $n \leq 4$ by Proposition 3.6(2), and Λ is also τ-tilting finite. This implies that Λ has no multiple arrow, so it is a Nakayama algebra.

Let $n = 4$ and $\Lambda \neq K\overrightarrow{A_2}$; so Λ is cyclic Nakayama. By Lemma 3.9, we can suppose that Λ has radical square zero, but we then obtain from Lemma 3.7 that $T_2(\Lambda)$ is not τ-tilting finite, contrary. Thus, if $n = 4$, then we have $\Lambda = K\overrightarrow{A_2}$.

Let us show that the implication (2)⇒(1) holds true. By Example 3.3, we have only to check the case where $n = 3$ and Λ is cyclic Nakayama. From Lemma 3.9, one obtains $\mathsf{s\tau\text{-}tilt}\, T_3(\Lambda) \simeq \mathsf{s\tau\text{-}tilt}\, T_3(\Lambda/\operatorname{rad}^2\Lambda)$, which is a finite set because $T_3(\Lambda/\operatorname{rad}^2\Lambda)$ is representation-finite by [LS1, Theorem 6.1]. Thus, we have done. □

As a corollary of Theorems 3.8 and 3.10, we get the following.

Corollary 3.11. *Let Λ be a simply-connected algebra and $n \geq 3$. Then $T_n(\Lambda)$ is τ-tilting finite if and only if it is representation-finite.*

Finally, we give a complete list of positive integers n and r such that $T_n(\Lambda)$ is silting-discrete for $\Lambda := K\overrightarrow{A_r}/\operatorname{rad}^2 K\overrightarrow{A_r}$.

Theorem 3.12. *Let Λ be a radical-square-zero linear Nakayama algebra with r simple modules. Then $T_n(\Lambda)$ is silting-discrete if and only if one of the following cases occurs: (i) $n = 1$; (ii) $r = 1$; (iii) $n = 2$ and $1 < r \leq 4$; (iv) $1 < n \leq 4$ and $r = 2$.*

Proof. It is well-known that Λ is derived equivalent to $K\overrightarrow{A_r}$, and so $T_n(\Lambda)$ is derived equivalent to $T_n(K\overrightarrow{A_r})$, which is τ-tilting infinite if $n \geq 3$ and $r \geq 3$ by Theorem 3.8. In the case, it is not silting-discrete.

We already know that $T_n(K\overrightarrow{A_2}) \simeq T_2(K\overrightarrow{A_n})$ is not silting-discrete for $n \geq 5$; see Example 3.3. For $n = 1, 2, 3$ and 4, we have the ADE-chain A_2, D_4, E_6 and E_8, respectively. This means that $T_n(K\overrightarrow{A_2})$ is derived equivalent to the path algebra of each type [La], which is silting-discrete. This completes the proof. □

Chapter 4

The τ-tilting finiteness of nondomestic polynomial growth selfinjective algebras

In this chapter, we discuss τ-tilting finiteness of weakly symmetric algebras of tubular type with non-singular Cartan matrix. First, we explore the τ-tilting finiteness of these algebras in Section 4.1. Moreover, we determine the number of support τ-tilting modules in Section 4.2.

4.1. The τ-tilting finiteness and the tilting-discreteness

Weakly symmetric algebras A_i of tubular type with non-singular Cartan matrix were completely classified up to Morita equivalence by [BS1] as the next page (Figure 4.1).

The main theorem of this section is the following.

Theorem 4.1. *Any weakly symmetric algebra of tubular type with non-singular Cartan matrix is τ-tilting finite. In particular, we have the number of support τ-tilting modules:*

$A_1(\lambda)$	$A_2(\lambda)$	A_3	A_4	A_5	A_6	A_7	A_8
24	6	192	132	8	8	108	100

A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
108	116	100	32	28	32	30	30

Proof. We will proove the τ-tilting finiteness. Note that A_i is symmetric for all i but $i = 3$ [BS1, Theorem 2]. Observe that the Cartan matrix of A_i has positive definite. We then apply [EJR, Theorem 13] to deduce the conclusion that A_i is τ-tilting finite for all i but $i = 3$. The algebra A_3 is just the preprojective algebra of Dynkin type $\mathbb{D}_4$, and so it is τ-tilting finite by [Mi, Theorem 2.21]. In Section 4.2, we will calculate the numbers of support τ-tilting modules of A_i's where A_i is in Figure 4.1. □

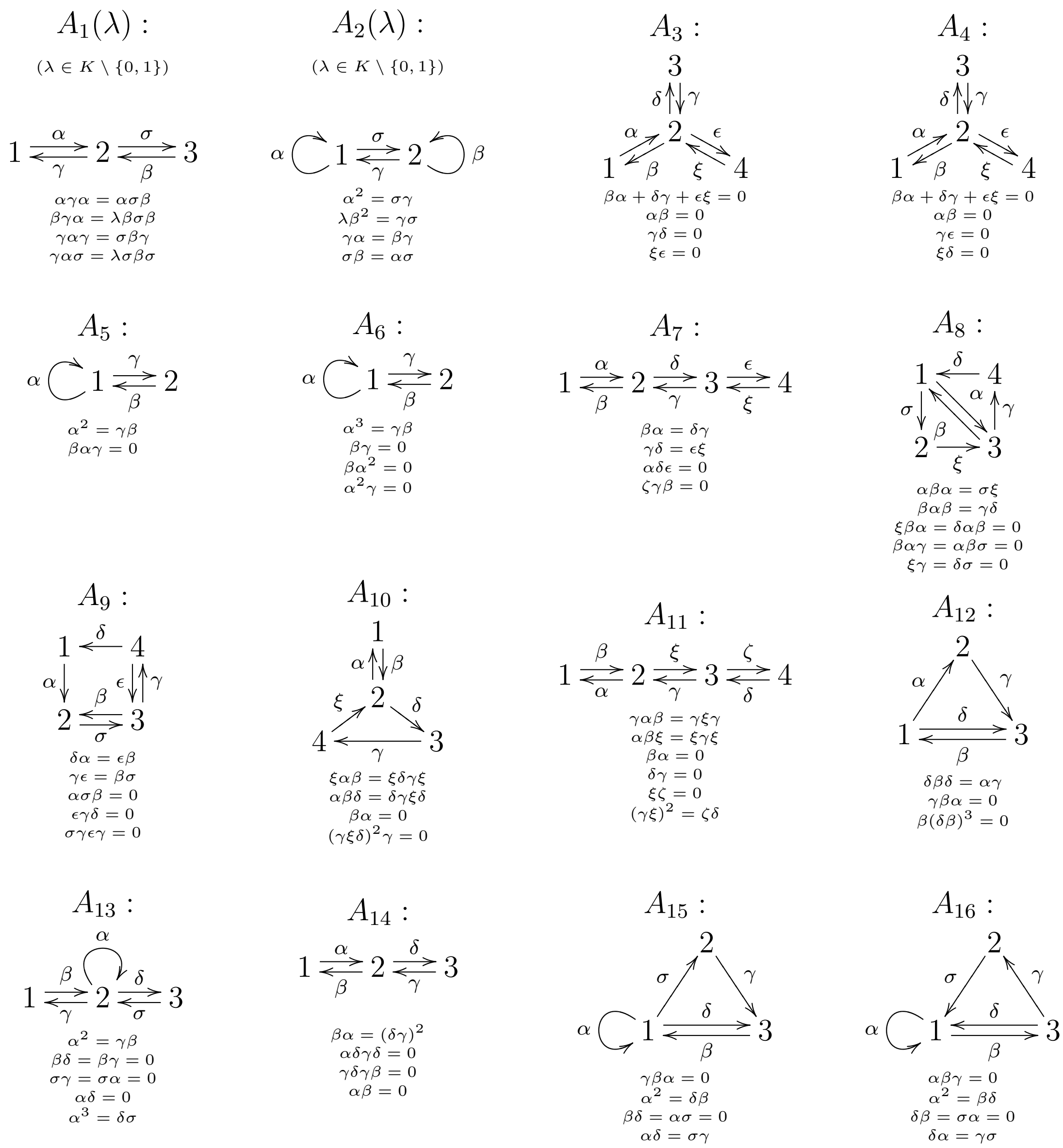


Figure 4.1: List of weakly symmetric algebras of tubular type

A selfinjective algebra is said to be *tilting-discrete* if for any $n > 0$, there are only finitely many tilting objects of length n. Here is a corollary of Theorem 4.1.

Corollary 4.2. *Any weakly symmetric algebra of tubular type with non-singular Cartan matrix is tilting-discrete.*

Proof. A weakly symmetric algebra of tubular type with non-singular Cartan matrix is derived equivalent to one of A_i's [BHS], which is τ-tilting finite by Theorem 4.1. It follows from [AM, Corollary 2.11] that the algebra is tilting-discrete. □

Thanks to Białkowski–Skowroński [BS2], we also have a complete list of Morita equivalence classes of selfinjective algebras which are socle equivalent to selfinjective algebras of tubular type. We focus on such algebras which are not of tubular type. The following classes of algebras coincide [S1]:

(i) selfinjective algebras which are socle equivalent to selfinjective algebras of tubular type but not of tubular type;

(ii) non-standard selfinjective algebras which are socle equivalent to selfinjective algebras of tubular type;

(iii) non-standard non-domestic selfinjective algebras of polynomial growth;

(iv) algebras Λ_i presented by the quivers and relations as in Figure 4.2.

Then we have a similar result as Theorem 4.1 for the algebras in (iv).

Theorem 4.3. *The algebras $\Lambda_1, \dots, \Lambda_9$ (without Λ_{10}) are τ-tilting finite. In particular, we have the number of support τ-tilting modules:*

Λ_1	Λ_2	$\Lambda_3(\lambda)$	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
8	8	6	32	28	32	30	30	192	≥ 500

Moreover, the algebras $\Lambda_1, \dots, \Lambda_9$ are tilting-discrete.

The proof of this theorem is by direct calculation, and we will give the numbers of support τ-tilting modules of Λ_i in Section 4.2.

4.2. The numbers of support τ-tilting modules over weakly symmetric algebras of tubular type

In this section, we give the numbers of support τ-tilting modules of A_i and Λ_i as in Section 4.1. (see the Theorem 4.1 and Theorem 4.3 for the tables of the numbers.)

The following theorem by Eisele–Janssens–Raedschelders plays an important role.

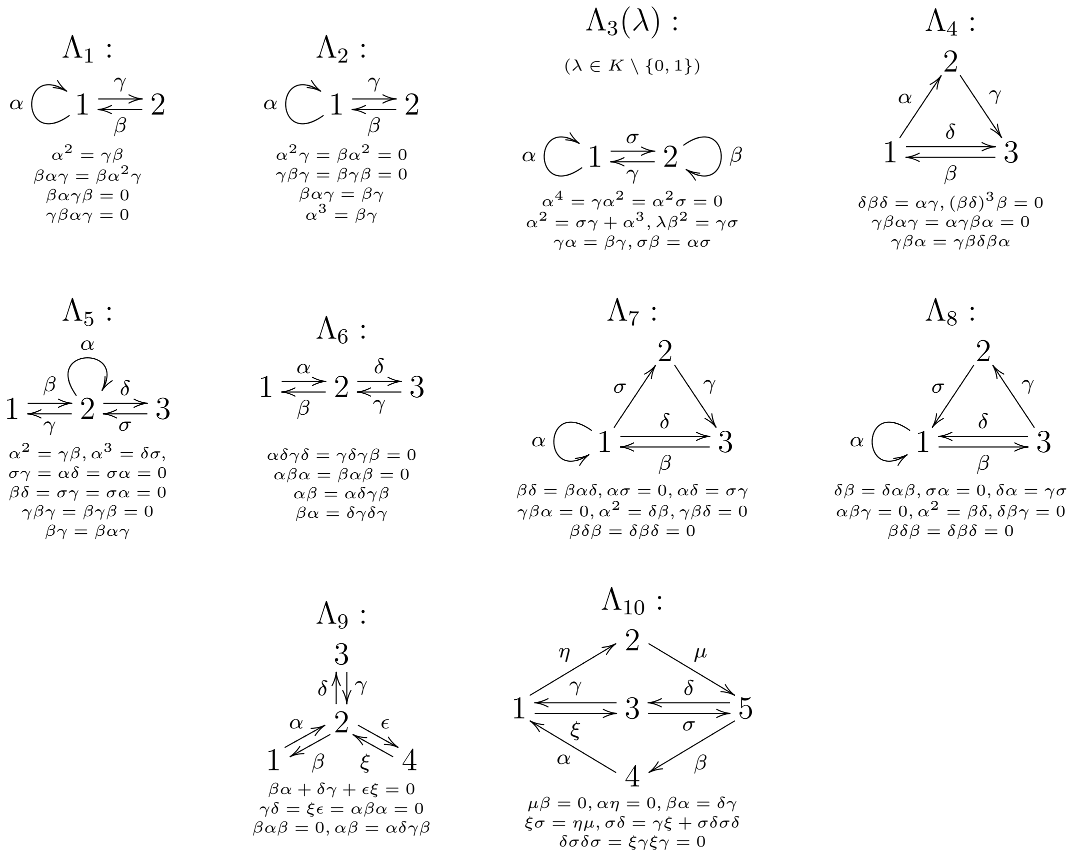


Figure 4.2: List of non-standard selfinjective algebras which are socle equivalent to selfinjective algebras of tubular type

Theorem 4.4. [EJR, Theorem 11] *Let I be a two-sided ideal of Λ which is contained in the center and the radical of Λ. Then we have an isomorphism of posets* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$.

For our algebra Λ, the strategy is the following.

(i) Find central elements which are in the radical.

(ii) Construct an ideal I generated by the elements as in (i).

(iii) Consider the factor algebra Λ/I. By Theorem 4.4, we have an isomorphism of posets $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$. Then, one counts the number or draws the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda/I$. If possible, we may find a nice algebra whose factor algebra is isomorphic to Λ/I and which admits a well-known Hasse quiver of support τ-tilting modules.

4.2.1. The number of $\mathsf{s\tau\text{-}tilt}\, A_i$

First, let us discuss for A_i's. In any case, we can easily check that the following elements belong to the center.

$i=1$: $\alpha\gamma+\gamma\alpha$ and $\beta\sigma+\sigma\beta$;	$i=2$: $\alpha+\beta$;
$i=3$: –;	$i=4$: $\beta\alpha-\gamma\delta-\xi\varepsilon$ and $\alpha\delta\gamma\beta$;
$i=5$: $\alpha\beta+\beta\alpha$;	$i=6$: α^2 and $\beta\alpha\gamma$;
$i=7$: $\alpha\beta+\beta\alpha+\gamma\delta+\xi\varepsilon$;	$i=8$: $\alpha\beta+\beta\alpha$;
$i=9$: $\beta\sigma+\varepsilon\gamma+\sigma\beta$;	$i=10$: $\alpha\beta+\gamma\xi\delta+\xi\delta\gamma$;
$i=11$: $\alpha\beta+\gamma\xi$;	$i=12$: $\alpha\gamma\beta+\beta\alpha\gamma$ and $\gamma\beta\delta\beta\alpha$;
$i=13$: $\alpha^2, \sigma\delta$ and $\beta\alpha\gamma$;	$i=14$: $\alpha\delta\gamma\beta, \delta\gamma\beta\alpha, \gamma\beta\alpha\delta$ and $\beta\alpha+\gamma\delta\gamma\delta$;
$i=15$: $\alpha^2, \beta\alpha\delta$ and $\gamma\beta\sigma$;	$i=16$: $\alpha^2, \delta\alpha\beta$ and $\sigma\alpha\beta$.

Let I_i be the ideal of A_i generated by the elements above and the socle, and $\overline{A_i} := A_i/I_i$.

In the following, we feel free to utilize Theorem 4.4 and refer to [Mi] for support τ-tilting modules over preprojective algebras of Dynkin type.

$i=1$

It is seen that $\overline{A_1}$ is isomorphic to the factor algebra of the preprojective algebra of Dynkin type $\mathbb{A}_3$ by the intersection of the center and the radical. This implies that A_1 has 24 support τ-tilting modules.

$i=2$

Observe that $\overline{A_2}$ is the Nakayama algebra presented by the quiver $\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet$ with relations $xy=0=yx$, whence there are 6 support τ-tilting modules of A_2.

$i=3$

A_3 is the preprojective algebra of type $\mathbb{D}_4$, which has 192 support τ-tilting modules.

$i=5,6$

It is obvious that $\overline{A_5}$ and $\overline{A_6}$ are isomorphic, which are furthermore isomorphic to $R(2AB)$ in Table 2 of [EJR]. Hence, A_5 and A_6 have 8 support τ-tilting modules.

$i=7$

By Theorem 4.4, we have an isomorphism of posets $\mathsf{s\tau\text{-}tilt}\, A_7 \simeq \mathsf{s\tau\text{-}tilt}\, \overline{A_7}$. Moreover, one observes that $\overline{A_7}$ is isomorphic to the factor algebra of the preprojective algebra Γ of type $\mathbb{A}_4$ by the central elements in the radical, and the socle. However, the socle of

Γ is not contained in the center, and so we can not apply Theorem 4.4 to obtain the Hasse quiver of support τ-tilting modules.

Now, let us apply Adachi's method [A2]. We fix the numbering of the vertices of $\mathbb{A}_4$ by $1 — 2 — 3 — 4$ and let $\overline{\Gamma}$ be the factor algebra of Γ by the central elements in the radical. We can still apply Theorem 4.4 to get an isomorphism $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Gamma \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$. Let P be the indecomposable projective module of $\overline{\Gamma}$ corresponding to the vertex 1 and define a subset $\mathcal{N}$ of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$ by

$$\mathcal{N} := \{N \in \mathsf{s}\tau\text{-}\mathsf{tilt}\left(\overline{\Gamma}/\operatorname{soc} P\right) \mid P/\operatorname{soc} P \in \mathsf{add}\, N \text{ and } \operatorname{Hom}_{\overline{\Gamma}}(N, P) = 0\}.$$

Here, $\operatorname{soc} P$ stands for the socle of P. We see that $\mathcal{N}$ has 6 elements; see [Mi] for example. It follows from [A2, Theorem 3.3(1)] that the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\overline{\Gamma}$ can be constructed by $\mathsf{s}\tau\text{-}\mathsf{tilt}\left(\overline{\Gamma}/\operatorname{soc} P\right)$ and the copy of $\mathcal{N}$. A similar argument works for the indecomposable projective module P' of $\overline{\Gamma}$ at the vertex 4 instead of P. As $\overline{A_7}$ is isomorphic to the factor algebra of $\overline{\Gamma}$ by the socle of P and P', it turns out that $\overline{A_7}$ has precisely 12 support τ-tilting modules fewer than $\overline{\Gamma}$, so than Γ. Consequently, we obtain that A_7 has 108 supoort τ-tilting modules.

$i = 8, 9, 11$

We can use 'String Applet' (`https://www.math.uni.-bielefeld.de/~jgeuenich/string-applet/`); apply it to $\overline{A_i}$.

Remark 4.5. The applet can be also run for A_7.

$i = 4, 10$

We count the number of τ-tilting modules over the factor algebra by each idempotent. Let $\{e_1, \cdots, e_n\}$ be a complete set of primitive orthogonal idempotents of an algebra Λ and I be a subset of $\{1, \cdots, n\}$ (possibly, $I = \emptyset$). We denote by t_I the number of τ-tilting modules of $\Lambda/(e)$, where $e = \sum_{i \in I} e_i$. Here, $t_\emptyset$ means the number of τ-tilting modules of Λ. Note that the number of support τ-tilting modules over Λ is equal to $\sum_I t_I$.

We demonstrate the way of counting for $i = 4$; it similarly works for $i = 10$. Putting $\Lambda := \overline{A_4}$, e_i denotes the primitive idempotent corresponding to the vertex i.

(i) We observe that $\Lambda/(e_1)$ is the factor algebra of the Brauer tree algebra of the Brauer tree $\circ — \circ — \circ — \circ$ by some socles, and so one easily obtains $t_{\{1\}} = 9$.

(ii) When I has the vertex 2, $\Lambda/(e)$ is semisimple, so $t_I = 1$; there are 8 cases.

(iii) In the cases that $I = \{3\}$ and $\{4\}$, $\Lambda/(e)$ is the preprojective algebra of type $\mathbb{A}_3$, so $t_I = 13$; see [Mi] for example.

(iv) For $I = \{1,3\}, \{1,4\}, \{3,4\}$, see the case of $i = 2$; $t_I = 3$.

(v) We easily get $t_{\{1,3,4\}} = 1$.

There remains to count the number of τ-tilting modules of Λ. To do that, we use the GAP-package QPA; Λ is representation-finite, and so all indecomposable τ-rigid modules can be got on QPA. Then, we obtain $t_{\emptyset} = 79$. Consequently, one sees that there are 132 support τ-tilting modules of Λ, so of A_4.

We only put the table for A_{10}.

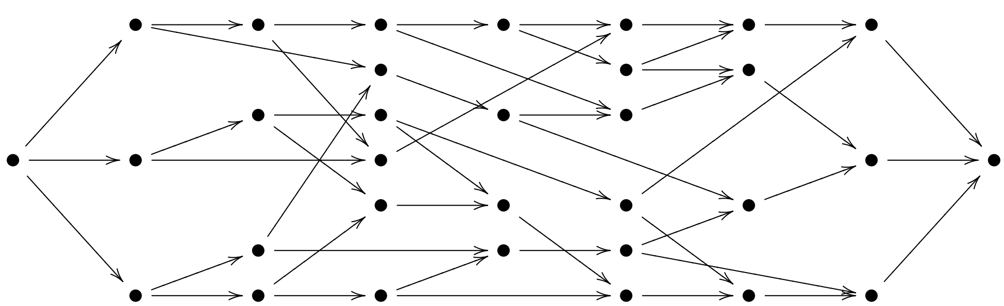
I	4 points 3 points	{1,2} {1,3} {1,4} {2}	{2,3} {2,4}	$\{3,4\}$	$\{1\}$	{3} {4}	$\emptyset$	total
t_I	1 (5 cases)	2	1	3	10	8	72	116

Remark 4.6. It is not difficult to draw the Hasse quivers directly, but they are too large.

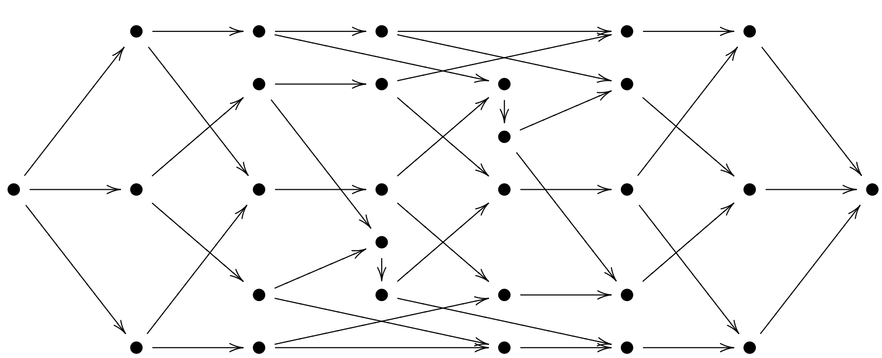
$i = 12, 13, 14, 15$

We directly construct the Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_i$ as follows.

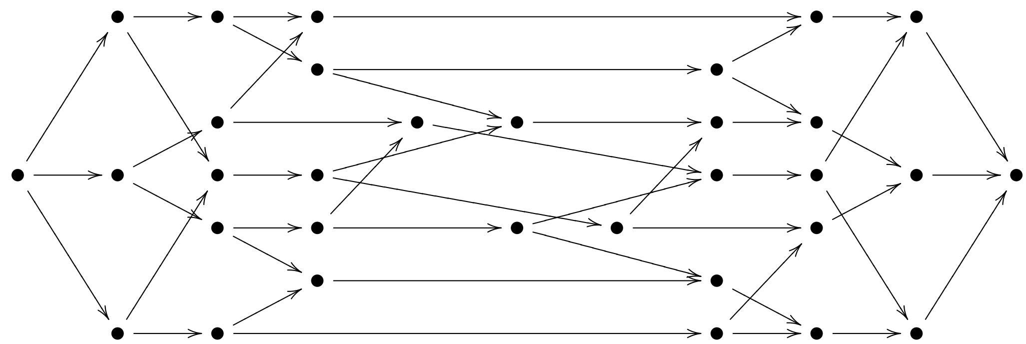
- The Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{12}$:



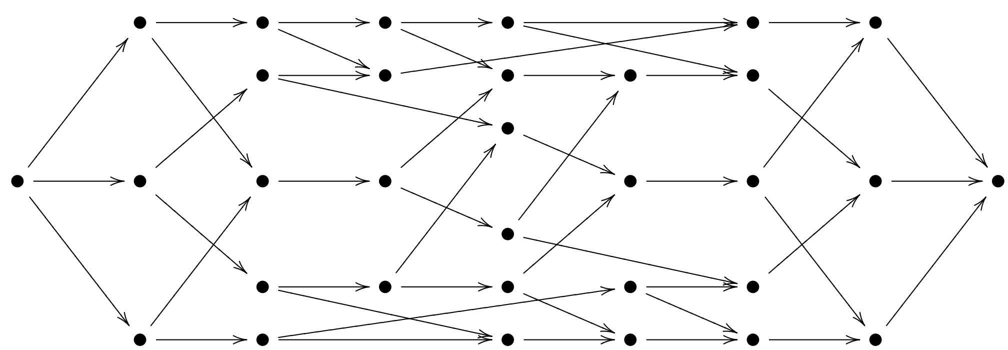
- The Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{13}$:



- The Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{14}$:



- The Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{15}$:



Remark 4.7. Note that A_{16} is the opposite algebra of A_{15}. So, it follows from [AIR, Theorem 2.14] that there is a bijection between $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{15}$ and $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A_{16}$. We also remark that $\overline{A_{15}}$ is not representation-finite.

4.2.2. The number of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, \Lambda_i$

Next, we discuss for Λ_i's. One gets central elements.

$i=1$: $\alpha^2+\beta\gamma$ and $\beta\alpha\gamma$;	$i=2$: $\alpha^2, \beta\gamma$ and $\gamma\beta$;
$i=3$: $\alpha+\beta, \sigma\gamma$ and $\gamma\sigma$;	$i=4$: $\gamma\beta\alpha$ and $\beta\alpha\gamma+\alpha\gamma\beta$;
$i=5$: $\beta\gamma, \gamma\beta, \delta\sigma$ and $\sigma\delta$;	$i=6$: $\alpha\beta$ and $\beta\alpha+\gamma\delta\gamma\delta$;
$i=7$: $\beta\delta, \delta\beta$ and $\gamma\beta\sigma$;	$i=8$: –;
$i=9$: $\alpha\beta, \gamma\beta\alpha\delta, \delta\gamma\beta\alpha$ and $\xi\delta\gamma\epsilon$;	$i=10$: $\gamma\xi-\sigma\delta$.

Let I_i be the ideal of Λ_i generated by the elements above and the socle. Putting $\overline{\Lambda_i} := \Lambda_i/I_i$, we observe isomorphisms as follows.

	$\overline{\Lambda_1}$	$\overline{\Lambda_2}$	$\overline{\Lambda_3}$	$\overline{\Lambda_4}$	$\overline{\Lambda_5}$	$\overline{\Lambda_6}$	$\overline{\Lambda_7}$	Λ_8	$\overline{\Lambda_9}$
$\simeq$	$\overline{A_5}$	$\overline{A_5}$	$\overline{A_2}$	$\overline{A_{12}}$	$\overline{A_{13}}$	$\overline{A_{14}}$	$\overline{A_{15}}$	Λ_7^{op}	$\overline{A_3}$

Here, Λ^{op} stands for the opposite algebra of an algebra Λ. Thus it turns out that Λ_i for every i except $i=10$ is τ-tilting finite by Theorem 4.1. Moreover, we have the number of support τ-tilting modules of Λ_i as in Theorem 4.3.

Chapter 5

Representation-finiteness vs. τ-tilting finiteness

The aim of this chapter is to provide several classes of algebras whose τ-tilting finiteness implies representation-finiteness.

5.1. When does the τ-tilting finiteness imply the representation-finiteness

Let C be a connected component of the Auslander–Reiten quiver of an algebra. We say that C is *preprojective* if it has no oriented cycle, and any module in C is of the form $\tau^{-n}P$ for some non-negative integer n and some indecomposable projective module P. Dually, define *preinjective* components.

We start with the following proposition, which was given in [Mo] as a remark; see also [A1].

Proposition 5.1. *A τ-tilting finite algebra with preprojective or preinjective component is representation-finite.*

We give several classes of algebras as in Proposition 5.1.

A *quasitilted* algebra is defined to be the endomorphism algebra of a tilting object T over a hereditary abelian K-category $\mathcal{H}$. When $\mathcal{H} = \mathsf{mod}\, K\Delta$ for some acyclic quiver Δ, the algebra is called *tilted* of type Δ. If in addition, T is preprojective, then the algebra is said to be *concealed*. We know from [CH] that every quasitilted algebra admits a preprojective component. This leads to the following corollary, which is a slight generalization of Zito's result [Zit, Theorem 3.1].

Corollary 5.2. *A τ-tilting finite quasitilted algebra is representation-finite.*

Let Λ be an algebra associated to an acyclic quiver Q and i a vertex of Q. We write the full subquiver of Q generated by the non-predecessors of i by $Q(i)$. An algebra Λ

is said to *satisfy the separation condition* if for any vertex i of Q, all distinct indecomposable summands of $\operatorname{rad} P_i$ have supports lying in different connected components of $Q(i)$. Here, P_i denotes the indecomposable projective module corresponding to i. In the case, Λ admits a preprojective component [ASS, IX, Theorem 4.5]. So, we get the following corollary.

Corollary 5.3. *A τ-tilting-fintie algebra satisfying the separation condition is representation-finite.*

Since every tree quiver algebra satisfies the separation condition [ASS, IX, Lemma 4.3], the following is also obtained.

Corollary 5.4. *A τ-tilting finite tree quiver algebra is representation-finite.*

We study the nice (isomorphism) class $\mathcal{C}$ of algebras which are representation-finite or have a tame concealed algebra as a factor. Such a class contains the classes of algebras with a preprojective component [SS, XIV, Theorem 3.1], cycle-finite algebras [MS] and loop-finite algebras [S2, Theorem 4.5]. Here is a generalization of Proposition 5.1.

Proposition 5.5. *A τ-tilting finite algebra in $\mathcal{C}$ is representation-finite.*

Proof. Combine Corollary 5.2 and [DIRRT, Theorem 5.12(d)]. □

A *commutative ladder* of degree n is an algebra presented by the quiver

$$\begin{array}{ccccccc} 1 & \longrightarrow & 2 & \longrightarrow & \cdots & \longrightarrow & n \\ \downarrow & & \downarrow & & & & \downarrow \\ 1' & \longrightarrow & 2' & \longrightarrow & \cdots & \longrightarrow & n' \end{array}$$

with all possible commutative relations, which is isomorphic to $K\overrightarrow{\mathbb{A}_2} \otimes K\overrightarrow{\mathbb{A}_n}$. Here, $\overrightarrow{\mathbb{A}_n}$ stands for the linearly oriented quiver of type $\mathbb{A}_n$. By [EH, Theorem 3], a commutative ladder of degree n is representation-finite if and only if $n \leq 4$. We derive a corollary from Proposition 5.5.

Corollary 5.6. *A τ-tilting finite commutative ladder is representation-finite.*

Proof. Let Λ be a commutative ladder of degree 5. As the Happel–Vossieck list [HV] (see also [Rin]), the factor algebra of Λ by the idempotents corresponding to the vertices 1 and $5'$ is a tame concealed algebra of type $\widetilde{\mathbb{E}_7}$. Observe that a commutative ladder of degree ≥ 5 has Λ as a factor. Thus the class of commutative ladders is contained in $\mathcal{C}$. □

We can also deduce Corollary 5.6 from Corollary 5.3; this is because a commutative ladder satisfies the separation condition, since all indecomposable projectives have indecomposable radicals.

Remark 5.7. Inspired by this work, the fourth named author of this paper showed that any τ-tilting finite strongly simply-connected algebra is representation-finite [W, Theorem 2.6], which generalizes Corollary 5.4 and 5.6.

Let us discuss algebras with radical square zero. To do that, we first recall the definition of separated quivers.

For a quiver Q, we construct a new quiver Q^s as follows:

- the vertices of Q^s are those of Q and their copies; we donote by i' the copy of a vertex i of Q.
- an arrow $a \to b$ of Q^s are drawn whenever a is a vertex i of Q, b is the copy of a vertex j of Q, and Q has an arrow $i \to j$.

We call the acyclic quiver Q^s the *separated quiver* of Q. As is well-known, a radical square zero algebra presented by a quiver Q is stable equivalent to the hereditary algebra KQ^s [ARS, X, Theorem 2.4]. A full subquiver of a separated quiver Q^s is said to be *single* if it has at most one of vertices i and i' for each vertex i of Q. Then an algebra given by a quiver Q with radical square zero is τ-tilting finite if and only if every single subquiver of Q^s is a disjoint union of Dynkin quivers [A1, Theorem 3.1].

Thanks to these results, we show the following result.

Theorem 5.8. *Let Λ be an algebra presented by a tree quiver with radical square zero.*

(1) *If Λ is τ-tilting finite, then it is representation-finite.*

(2) *If the trivial extension of Λ is τ-tilting finite, then it is representation-finite.*

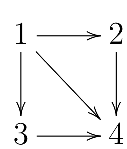
Proof. (1) This is due to Corollary 5.4, but we give another proof here, in which we use combinatorial discussion.

As the quiver of Λ is tree, we observe that every connected component R of the separated quiver has no same latter i and i'. Then we can apply [A1, Theorem 3.1] for R to deduce the fact that R is of Dynkin type, since Λ is τ-tilting finite. Hence, it follows from [ARS, X, Theorem 2.6] that Λ is representation-finite.

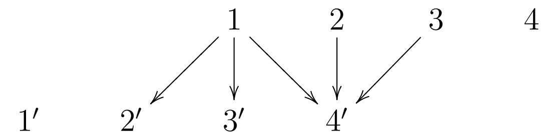
(2) If the trivial extension $T(\Lambda)$ of Λ is τ-tilting finite, then so is Λ by [DIRRT, Theorem 5.12(d)], and hence Λ is representation-finite by (1). We observe that Λ is simply-connected and has the quadratic form of positive definite, which implies that it is an iterated tilted algebra of Dynkin type [AS, Proposition 5.1]. (see also [H].) It follows from [AHR, Theorem 3.1] that $T(\Lambda)$ is representation-finite. □

Theorem 5.8 does not necessarily hold if Λ is given by a non-tree acyclic quiver.

Example 5.9. (1) Let Λ be an algebra presented by the quiver

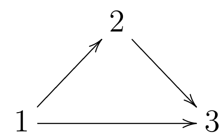


with radical square zero. Then the separated quiver is the following:

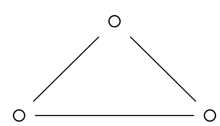


Observe that it contains the extended Dynkin diagram $\widetilde{\mathbb{D}}_5$ as an underlying graph, whence Λ is τ-tilting finite by [A1] but not representation-finite by [ARS].

(2) Let us consider the algebra presented by the quiver



with radical square zero. Then the trivial extension is the Brauer graph algebra given by the Brauer graph



This is τ-tilting finite by [AAC, Theorem 6.7] but not representation-finite.

Let Q be a quiver. The *double quiver* of Q, denoted by Q^d, is constructed from Q by adding the inverse arrow of every arrow in Q. Here is an easy observation.

Proposition 5.10. *Let Q be a tree quiver and I an admissible ideal of KQ^d. Put $\Lambda := KQ^d/I$. If Λ is τ-tilting finite, then Q is of Dynkin type.*

Proof. By assumtion, it follows from [DIRRT] that $\Lambda/\operatorname{rad}^2\Lambda$ is τ-tilting finite. We observe that the separated quiver of Q^d is the disjoint union of two quivers R_1 and R_2 which satisfy $i \in R_j \Leftrightarrow i' \notin R_j$ $(j = 1, 2)$ and whose underlying graphs coincide with that of Q. We apply [A1] to deduce the fact that R_1, R_2, and hence Q, are of Dynkin type. □

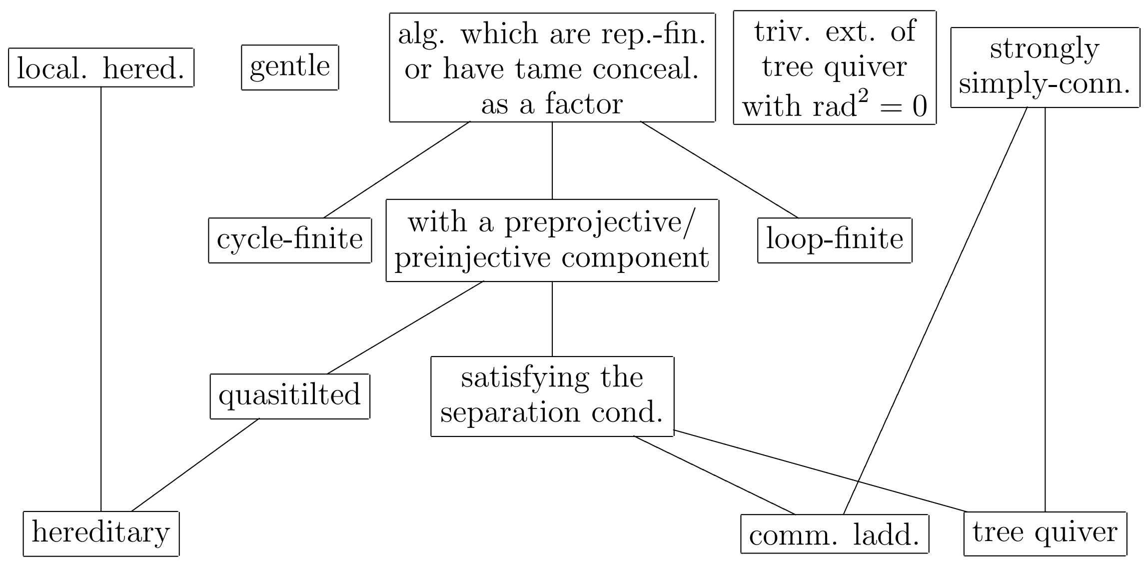
Let us discuss the locally hereditary case. An algebra is said to be *locally hereditary* provided every homomorphism between indecomposable projective modules is a monomorphism or zero; see [B, LS1, MV]. We know that such an algebra is presented by an acyclic quiver and the relations contain no monomials. We show the following theorem.

Theorem 5.11. *A τ-tilting finite locally hereditary algebra is representation-finite.*

Proof. Let Λ be a τ-tilting finite locally hereditary algebra. As is easy to see, the local hereditariness yields that Λ has no monomial relation and the quiver Q is acyclic. The τ-tilting finiteness implies that Q does not contain a subquiver of extended Dynkin type, whence Λ admits all possible commutative relations. Then, we figure out that Λ is strongly simply-connected; see [Le] for example. The assertion follows from [W, Theorem 2.6]. $\square$

5.2. Inclusion relationships of classes

We close this chapter by giving an interesting observation. Denote by $\mathcal{A}$ the class of algebras in which τ-tilting finiteness implies representation-finiteness; we put a hierarchy of classes contained in $\mathcal{A}$:



Proposition 5.12. *The class $\mathcal{A}$ is closed under taking factors by ideals contained in the center and the radical.*

Proof. Let Λ be in $\mathcal{A}$ and put $\Gamma := \Lambda/I$, where I is an ideal of Λ contained in the center and the radical. By [EJR, Theorem 11] , these algebras have the same poset of support τ-tilting modules. Therefore, if Γ is τ-tilting finite, then so is Λ. By assumption, it turns out that Λ is representation-finite, so is Γ. $\square$

Bibliography

[A1] T. Adachi, Characterizing τ-tilting finite algebras with radical square zero. *Proc. Amer. Math. Soc.* **144** (2016), no. 11, 4673–4685.

[A2] T. Adachi, The classification of τ-tilting modules over Nakayama algebras. *J. Algebra* **452** (2016), 227–262.

[AAC] T. Adachi, T. Aihara and A. Chan, Classification of two-term tilting complexes over Brauer graph algebras. *Math. Z.* **290** (2018), no. 1–2, 1–36.

[AK] T. Adachi and R. Kase, Examples of tilting-discrete self-injective algebras which are not silting-discrete. Preprint (2020), arXiv: 2012.14119.

[AMY] T. Adachi, Y. Mizuno and D. Yang, Discreteness of silting objects and t-structures in triangulated categories. *Proc. Lond. Math. Soc. (3)* **118** (2019), no. 1, 1–42.

[AIR] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory. *Compos. Math.* **150**, no. 3, 415–452 (2014).

[Ai1] T. Aihara, Tilting-connected symmetric algebras. *Algebr. Represent. Theory* **16** (2013), no. 3, 873–894.

[Ai2] T. Aihara, On silting-discrete triangulated categories. *Proceedings of the 47th Symposium on Ring Theory and Representation Theory*, 7–13, *Symp. Ring Theory Represent. Theory Organ. Comm., Okayama*, 2015.

[AGI] T. Aihara, J. Grant and O. Iyama, Private communication.

[AH] T. Aihara and T. Honma, τ-tilting finite triangular matrix algebras. *J. Pure Appl. Algebra* **225** (2021), no. 12, 106785, 10 pp.

[AHMW] T. Aihara, T. Honma, K. Miyamoto and Q. Wang, Report on the finiteness of silting objects. *Proc. Edinb. Math. Soc. (2)* **64** (2021), no. 2, 217–233.

[AI] T. Aihara and O. Iyama, Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)* **85** (2012), no. 3, 633–668.

[AM] T. Aihara and Y. Mizuno, Classifying tilting complexes over preprojective algebras of Dynkin type. *Algebra Number Theory* **11** (2017), no. 6, 1287–1315.

[AHR] I. Assem, D. Happel and O. Roldán, Representation-finite trivial extension algebras. . *J. Pure Appl. Algebra* **33** (1984), no. 3, 235–242.

[ASS] I. Assem, D. Simson and A. Skowroński, Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory. London Mathematical Society Student Texts, **65**. *Cambridge University Press, Cambridge*, 2006.

[AS] I. Assem and A. Skowroński, Quadratic forms and iterated tilted algebras. *J. Algebra* **128** (1990), no. 1, 55–85.

[AD] J. August and A. Dugas, Silting and tilting for weakly symmetric algebras. Preprint (2021), arXiv: 2101.03097.

[AR] M. Auslander and I. Reiten, On the representation type of triangular matrix rings. *J. London Math. Soc. (2)* **12** (1975/76), no. 3, 371–382

[ARS] M. Auslander, I. Reiten and S. O. Smalø, Representation theory of Artin algebras. Cambridge Studies in Advanced Mathematics, **36**. *Cambridge University Press, Cambridge*, 1995.

[B] R. Bautista, Classification of certain algebras of finite representation type. *An. Int. Mat. Univ. Nac. Autónoma México* **22** (1982), 1–82.

[BHS] J. Białkowski, T. Holm and A. Skowroński, Derived equivalences for tame weakly symmetric algebras having only periodic modules. *J. Algebra* **269** (2003), no. 2, 652–668.

[BS1] J. Białkowski and A. Skowroński, On tame weakly symmetric algebras having only periodic modules. *Arch. Math. (Basel)* **81** (2003), no. 2, 142–154.

[BS2] J. Białkowski and A. Skowroński, Socle deformations of self-injective algebras of tubular type. *J. Math. Soc. Japan* **56** (2004), no. 3, 687-716.

[BB] S. Brenner and M. C. R. Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors. *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, pp. 103–169, Lecture Notes in Math., 832, *Springer, Berlin-New York*, 1980.

[BPP] N. Broomhead, D. Pauksztello and D. Ploog, Discrete derived categories II: the silting pairs CW complex and the stability manifold. *J. Lond. Math. Soc. (2)* **93** (2016), no. 2, 273–300.

[CH] F. U. Coelho and D. Happel, Quasitilted algebras admit a preprojective component. *Proc. Amer. Math. Soc.* **125** (1997), no. 5, 1283–1291.

[DIJ] L. Demonet, O. Iyama and G. Jasso, τ-tilting finite algebras, bricks, and g-vectors. *Int. Math. Res. Not. IMRN* 2019, no. 3, 852–892.

[DIRRT] L. Demonet, O. Iyama, N. Reading, I. Reiten and H. Thomas, Lattice theory of torsion classes. Preprint (2017), arXiv: 1711.01785.

[EJR] F. Eisele, G. Janssens and T. Raedschelders, A reduction theorem for τ-rigid modules. *Math. Z.* **290** (2018), no. 3–4, 1377–1413.

[EH] E. G. Escolar and Y. Hiraoka, Persistence modules on commutative ladders of finite type. *Discrete Comput. Geom.* **55** (2016), no. 1, 100–157.

[H] D. Happel, Tilting sets on cylinders. *Proc. London Math. Soc. (3)* **51** (1985), no. 1, 21–55.

[HU] D. Happel and L. Unger, On a partial order of tilting modules. *Algebr. Represent. Theory* **8** (2005), no. 2, 147–156.

[HV] D. Happel and D. Vossieck, Minimal algebras of infinite representation type with preprojective component. *Manuscripta Math.* **42** (1983), no. 2–3, 221–243.

[HKM] M. Hoshino, Y. Kato and J. Miyachi, On t-structures and torsion theories induced by compact objects. *J. Pure Appl. Algebra* **167** (2002), no. 1, 15–35.

[IX] O. Iyama and Z. Xiaojin, Tilting modules over Auslander–Gorenstein algebras. *Pacific J. Math.* **298** (2019), no. 2, 399–416.

[IY] O. Iyama and D. Yang, Silting reduction and Calabi-Yau reduction of triangulated categories. *Trans. Amer. Math. Soc.* **370** (2018), no. 11, 7861–7898.

[K] B. Keller, Deriving DG categories. *Ann. Sci. Ecole Norm. Sup. (4)* **27** (1994), no. 1, 63–102.

[KV] B. Keller and D. Vossieck, Aisles in derived categories. *Bull. Soc. Math. Belg. Ser. A* **40** (1988), no. 2, 239–253. Deuxieme Contact Franco-Belge en Algebre (Faulx–es-Tombes, 1987).

[KY] S. Koenig and D. Yang, Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. *Doc. Math.* **19** (2014), 403–438.

[La] S. Ladkani, On derived equivalences of lines, rectangles and triangles. *J. Lond. Math. Soc. (2)* **87** (2013), no. 1, 157–176.

[Le] Z. Leszczyński, The completely separating incidence algebras of tame representation type. *Colloq. Math.* **94** (2002), no.2, 243–262.

[LS1] Z. Leszczyński and D. Simson, On triangular matrix rings of finite representation type. *J. Lond. Math. Soc. (2)* **20** (1979), no. 3, 396–402.

[LS2] Z. Leszczyński and A. Skowroński, Tame tensor products of algebras. *Colloq. Math.* **98** (2003), no. 1, 125–145.

[MS] P. Malicki and A. Skowroński, Cycle-finite algebras with finitely many τ-rigid indecomposable modules. *Comm. Algebra* **44** (2016), no. 5, 2048–2057.

[MV] R. Martínez-Villa, Algebras stably equivalent to ℓ-hereditary. *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, 396–431, Lecture Notes in Math., **832**, *Springer, Berlin*, 1980.

[Mi] Y. Mizuno, Classifying τ-tilting modules over preprojective algebras of Dynkin type. *Math. Z.* **277** (2014), no. 3–4, 665–690.

[Mo] K. Mousavand, τ-tilting finiteness of biserial algebras. Preprint (2019), arXiv: 1904.11514.

[O] T. Okuyama, Some examples of derived equivalent blocks of finite groups. Preprint, 1998.

[PMZ] D. Pauksztell, M. Saorin and A. Zvonareva, Contractibility of the stability manifold of silting-discrete algebras. *Forum Math.* **30** (2018), no. 5, 1255–1263.

[P] P.-G. Plamondon, τ-tilting finite gentle algebras are representation-finite. *Pacific J. Math.* **302** (2019), no. 2, 709–716.

[Ric1] J. Rickard, Morita theory for derived categories. *J. London Math. Soc. (2)* **39** (1989), no. 3, 436–456.

[Ric2] J. Rickard, Derived categories and stable equivalence. *J. Pure Appl. Algebra* **61** (1989), no. 3, 303–317.

[RS] C. Riedtmann and A. Schofield , On a simplicial complex associated with tilting modules. *Comment. Math. Helv.* **66** (1991), no. 1, 70–78.

[Rin] C. M. Ringel, Tame algebras and integral quadratic forms. Lecture Notes in Mathematics, **1099**. *Springer-Verlag, Berlin*, 1984.

[SS] D. Simson and A. Skowroński, Elements of the representation theory of associative algebras. Vol. 2. Tubes and concealed algebras of Euclidean type. London Mathematical Society Student Texts, **71**. *Cambridge University Press, Cambridge*, 2007.

[S1] A. Skowroński, Selfinjective algebras : finite and tame type. *Trends in representation theory of Algebras and related topics*, 69–238. Contemp. Math., **406**, *Amer. Math. Soc.*, 2006.

[S2] A. Skowroński, Minimal representation-infinite Artin algebras. *Math. Proc. Cambridge Philos. Soc.* **116** (1994), no. 2, 229–243.

[W] Q. Wang, τ-tilting finite simply connected algebras. Preprint (2019), arXiv: 1910.01937.

[Zim] A. Zimmermann, A Noether–Deuring theorem for derived categories. *Glasg. Math. J.* **54** (2012), no. 3, 647–654.

[Zit] S. Zito, τ-tilting finite tilted and cluster-tilted algebras. Preprint (2019), arXiv: 1902.05866.