学位論文

Studies on Tate-Hochschild cohomology for Frobenius algebras and eventually periodic Gorenstein algebras

(フロベニウス多元環及び終局周期ゴレンシュタイン多元環に対するテイト・ホッホシルトコホモロジーの研究)

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Chapter 1

Introduction

This thesis is based on [**31**], [**44**] and [**45**].

Throughout this thesis, let k denote a field. By an algebra A, we mean a finite dimensional associative and unital k-algebra, and all modules are assumed to be finitely generated left modules.

The *singularity category* $\mathcal{D}_{\mathrm{sg}}(A)$ of an algebra A, introduced by Buchweitz [**12**], is defined by the Verdier quotient of the bounded derived category of A-modules by the full subcategory of perfect complexes. The category $\mathcal{D}_{\mathrm{sg}}(A)$ measures homological singularity of A in the following sense: the global dimension of A is finite if and only if $\mathcal{D}_{\mathrm{sg}}(A) = 0$. Therefore, singularity categories can be considered as homological invariants for algebras of infinite global dimension.

The notion of *Tate cohomology groups* was also introduced by Buchweitz [**12**]. For each integer i, the i-th Tate cohomology group of an A-module M with coefficients in an A-module N is defined to be

$$\widehat{\mathrm{Ext}}_A^i(M, N) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A)}(M, N[i]).$$

He observed in [**12**] that there is an isomorphism $\widehat{\mathrm{Ext}}_{\mathbb{Z}G}^*(\mathbb{Z}, N) \cong \widehat{\mathrm{H}}^*(G, N)$, where $\widehat{\mathrm{H}}^*(G, N)$ stands for the original Tate cohomology group of a finite group G with coefficients in a $\mathbb{Z}G$-module N. This justifies the terminology "Tate cohomology". Recently, Wang [**47**] defined the i-th *Tate-Hochschild cohomology group* of the algebra A as

$$\widehat{\mathrm{HH}}^i(A) := \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^i(A, A) = \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^{\mathrm{e}})}(A, A[i])$$

for any integer i. Then the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^\bullet(A) := \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{HH}}^i(A)$ naturally carries a structure of a graded ring, where the multiplication is given by the Yoneda product. We call such a graded ring the *Tate-Hochschild cohomology ring* of A. It was proved by Wang [**47**] that the Tate-Hochschild cohomology ring of any algebra is graded commutative. Furthermore, Dotsenko, Gélinas and Tamaroff [**17**] showed that, for a monomial Gorenstein algebra A, its Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A)$ is isomorphic to $\widehat{\mathrm{HH}}^{\geq 0}(A)[\chi^{-1}]$, where $\widehat{\mathrm{HH}}^{\geq 0}(\Lambda) := \bigoplus_{i \geq 0} \widehat{\mathrm{HH}}^{i}(A)$, and χ is an invertible homogeneous element of positive degree. We point out that such an invertible element was obtained from the fact that any minimal projective resolution of a monomial Gorenstein algebra becomes periodic from some step. On the other hand, Wang [**47**] showed that $\widehat{\mathrm{HH}}^{\bullet}(A)$ carries a *Gerstenhaber structure*. Roughly speaking, it is a structure of a Lie algebra on the Tate-Hochschild cohomology ring of A. Recently, there are studies on a *Batalin-Vilkovisky (BV) structure* on $\widehat{\mathrm{HH}}^{\bullet}(A)$, because if a BV structure exists, then it generates a Gerstenhaber structure on the ring $\widehat{\mathrm{HH}}^{\bullet}(A)$. It was proved in Wang [**47**] that if A is a symmetric algebra, then the Tate-Hochschild cohomology ring of A has a BV structure generating Wang's Gerstenhaber structure.

In this thesis, we first consider the existence of a BV structure on $\widehat{\mathrm{HH}}^{\bullet}(A)$ in the case of Frobenius algebras. Let A be a Frobenius algebra. Then, for every integer i, $\widehat{\mathrm{HH}}^{i}(A)$ is isomorphic to the i-th cohomology group $\mathrm{H}^i(\mathrm{Hom}_{A^{\mathrm{e}}}(T_{\bullet}, A))$ of the cochain complex $\mathrm{Hom}_{A^{\mathrm{e}}}(T_{\bullet}, A)$, where $T_{\bullet}$ is a complete resolution of A over A^{e}, that is, an (unbounded) acyclic complex of projective A^{e}-modules with $\mathrm{Cok}(d_1^T : T_1 \to T_0) \cong A$. The group $\mathrm{H}^i(\mathrm{Hom}_{A^{\mathrm{e}}}(T_{\bullet}, A))$ is called the *i-th complete cohomology group* of A and denoted by $\mathrm{CH}^i(A)$. Thus, letting $\mathrm{CH}^{\bullet}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{CH}^i(A)$, we have an isomorphism $\mathrm{CH}^{\bullet}(A) \cong \widehat{\mathrm{HH}}^{\bullet}(A)$ of graded vector spaces. Together with this isomorphism, we aim at providing a sufficient condition for $\widehat{\mathrm{HH}}^{\bullet}(A)$ to have a BV structure generating Wang's Gerstenhaber structure. On the other hand, inspired by a result of Dotsenko, Gélinas and Tamaroff [**17**], we also investigate the ring structure on Tate-Hochschild cohomology, and our second aim is to give a necessary and sufficient condition for the Tate-Hochschild cohomology ring of A to have an invertible homogeneous element of positive degree in the case that A is Gorenstein.

The organization of this thesis is as follows. In Chapter 2, we recall basic terminology and facts related to Tate-Hochschild cohomology and Gorenstein algebras.

In Chapter 3, we try to clarify when the Tate-Hochschild cohomology ring of a Frobenius algebra A has a BV structure generating Wang's Gerstenhaber structure. For this purpose, we first fix a complete resolution $T_{\bullet}$ of the A^{e}-module A and set $\mathcal{D}^{\bullet}(A, A) := \mathrm{Hom}_{A^{\mathrm{e}}}(T_{\bullet}, A)$. We then recall from [**47**] a differential graded algebra structure on $\mathcal{D}^{\bullet}(A, A)$ whose cohomology ring $\mathrm{H}^{\bullet}(\mathcal{D}^{\bullet}(A, A)) = \mathrm{CH}^{\bullet}(A)$ is isomorphic to the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A)$. Let $\nu : A \to A$ be the Nakayama automorphism of the Frobenius algebra A, and assume that the set Λ of eigenvalues of the Nakayama automorphism ν as a k-linear map is contained in k. Then, for each product μ of finitely many elements in Λ, we define a graded subspace $\mathcal{D}^{\bullet}_{(\mu)}(A, A)$ of $\mathcal{D}^{\bullet}(A, A)$, and it is proved that $\mathcal{D}^{\bullet}_{(1)}(A, A)$ becomes a differential graded subalgebra of $\mathcal{D}^{\bullet}(A, A)$. Letting $\mathrm{CH}^{\bullet}_{(1)}(A)$ denote the cohomology ring for $\mathcal{D}^{\bullet}_{(1)}(A, A)$, we will show that if the Nakayama automorphism $\nu : A \to A$ is diagonalizable, then the induced ring homomorphism $\mathrm{CH}^{\bullet}_{(1)}(A) \to \mathrm{CH}^{\bullet}(A)$ is an isomorphism, and $\mathrm{CH}^{\bullet}_{(1)}(A) = \mathrm{CH}^{\bullet}(A)$ has a BV structure such that the induced Gerstenhaber structure coincides with Wang's

Gerstenhaber structure on $\widehat{\mathrm{HH}}^{\bullet}(A)$. Namely, we prove the following main result of Chapter 3:

Main Result 1 (Theorem 3.3.7)**.** *Let A be a Frobenius algebra. If the Nakayama automorphism of A is diagonalizable, then the complete cohomology ring $\mathrm{CH}^{\bullet}(A)$ is a BV algebra such that the induced Gerstenhaber algebra is isomorphic to the Gerstenhaber algebra $\widehat{\mathrm{Ext}}^{\bullet}_{A^{\mathrm{e}}}(A, A)$.*

We remark that this result generalizes Wang's result for symmetric algebras, because the Nakayama automorphism of a symmetric algebra is the identity.

Moreover, for certain three self-injective Nakayama algebras with diagonalizable Nakayama automorphisms, we compute their Tate-Hochschild cohomology rings, the BV structures constructed above and the induced Gerstenhaber structures.

In Chapter 4, under the assumption that the ground field k is algebraically closed, we study the Tate-Hochschild cohomology rings themselves and decide when they have invertible homogeneous elements of positive degree in the case of Gorenstein algebras. It is shown that, for any module M over a Gorenstein algebra A, it is eventually periodic if and only if there exists an invertible homogeneous element of positive degree in the Tate cohomology ring of M. Since the enveloping algebra of a Gorenstein algebra is also Gorenstein, we obtain the following main result of Chapter 4.

Main Result 2 (Theorem 4.2.3)**.** *Let A be a Gorenstein algebra. Then the following conditions are equivalent.*

(1) *The Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A)$ has an invertible homogeneous element of positive degree.*

(2) *A is an eventually periodic algebra.*

In this case, there exists an isomorphism $\widehat{\mathrm{HH}}^{\bullet}(A) \cong \widehat{\mathrm{HH}}^{\geq 0}(A)[\chi^{-1}]$ of graded algebras, where the degree of an invertible homogeneous element χ equals the period of the periodic syzygy $\Omega^{n}_{A^{\mathrm{e}}}(A)$ of A for some $n \geq 0$.

As an application, we show that the property of being eventually periodic Gorenstein is invariant under derived equivalence. It turns out that this result is new only for eventually periodic Gorenstein algebras of infinite global dimension that are not periodic. Taking this into account, we provide a method of giving such algebras.

Moreover, we describe the Tate-Hochschild cohomology rings of connected periodic algebras. More concretely, for a connected periodic algebra A, we determine the Tate-Hochschild cohomology ring modulo nilpotence $\widehat{\mathrm{HH}}^{\bullet}(A)/\widehat{\mathcal{N}}$ and the graded subring $\widehat{\mathrm{HH}}^{\geq 0}(A)$. These results enable us to calculate the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A)$ whenever the Hochschild cohomology ring of A has been computed.

Chapter 2

Preliminaries

In this chapter, we recall basic terminology and facts which are used in this thesis. Let us first fix some conventions. We write $\otimes$ for $\otimes_k$ and Hom for Hom_k. For an algebra A, we denote by A-mod the category of A-modules, by A-proj the category of projective A-modules, by $\mathrm{gl.dim}\, A$ the global dimension of A and by A^{e} the enveloping algebra $A \otimes A^{\mathrm{op}}$ of A. Here, we denote by A^{op} the opposite algebra of A. Remark that we can identify an A-bimodule M with a left (right) A^{e}-module M whose structure is given by $(a \otimes b^{\circ})m := amb$ $(m(a \otimes b^{\circ}) := bma)$ for $m \in M$ and $a \otimes b^{\circ} \in A^{\mathrm{e}}$. We denote by $\overline{A}$ the quotient space of A by the subspace $k1_A$ generated by unit 1_A. Let $\sigma : A \to A$ be an algebra automorphism of A and $\pi : A \to \overline{A}$ the canonical epimorphism of k-vector spaces. We denote by $\overline{a}$ the image of $a \in A$ under the epimorphism $\pi : A \to \overline{A}$. We write $a_{1,m} \in A^{\otimes m}$ for $a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$, $\overline{b}_{1,n} \in \overline{A}^{\otimes n}$ for $\overline{b}_1 \otimes \cdots \otimes \overline{b}_n \in \overline{A}^{\otimes n}$ and $\overline{\sigma c}_{1,l} \in \overline{A}^{\otimes l}$ for $\overline{\sigma(c_1)} \otimes \overline{\sigma(c_2)} \otimes \cdots \otimes \overline{\sigma(c_l)} \in \overline{A}^{\otimes l}$ when no confusion occurs. For an A-module M, we denote by $\mathrm{inj.dim}_A M$ (resp. $\mathrm{proj.dim}_A M$) the injective (resp. projective) dimension of M.

2.1. Gerstenhaber algebras and Hochschild (co)homology

In this section, we review the definition of Gerstenhaber algebras and Hochschild (co)homology and some related facts. Let us start with the definition of Gerstenhaber algebras.

Definition 2.1.1. *A Gerstenhaber algebra is a graded k-vector space $\mathcal{H}^{\bullet} = \bigoplus_{r \in \mathbb{Z}} \mathcal{H}^r$ equipped with two bilinear maps: a cup product*

$$\smile : \mathcal{H}^{|\alpha|} \otimes \mathcal{H}^{|\beta|} \to \mathcal{H}^{|\alpha|+|\beta|}, \quad (\alpha, \beta) \longmapsto \alpha \smile \beta$$

and a Lie bracket, called the Gerstenhaber bracket,

$$[\ ,\ ] : \mathcal{H}^{|\alpha|} \otimes \mathcal{H}^{|\beta|} \to \mathcal{H}^{|\alpha|+|\beta|-1}, \quad (\alpha, \beta) \longmapsto [\alpha, \beta]$$

such that

(i) $(\mathcal{H}^\bullet, \smile)$ *is a graded commutative algebra with unit* $1 \in \mathcal{H}^0$*, in particular,* $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$;

(ii) $(\mathcal{H}^\bullet[1], [\ ,\ ])$ *is a graded Lie algebra with components* $(\mathcal{H}^\bullet[1])^r = \mathcal{H}^{r+1}$*, that is,*

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]$$

and

$$(-1)^{(|\alpha|-1)(|\gamma|-1)}[[\alpha, \beta], \gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)}[[\beta, \gamma], \alpha] \\ +(-1)^{(|\gamma|-1)(|\beta|-1)}[[\gamma, \alpha], \beta] = 0;$$

(iii) The Lie bracket $[\ ,\ ]$ *is compatible with the cup product* $\smile$ *:*

$$[\alpha, \beta \smile \gamma] = [\alpha, \beta] \smile \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \smile [\alpha, \gamma],$$

where α, β, γ *are homogeneous elements in* $\mathcal{H}^\bullet$*, and we denote by* $|\alpha|$ *the degree of a homogeneous element* α *in* $\mathcal{H}^\bullet$*.*

We now recall that the Hochschild cohomology of an algebra A carries a structure of a Gerstenhaber algebra. There is a projective resolution $\mathrm{Bar}_\bullet(A)$ of A over A^{e}, which is the so-called *normalized bar resolution* of A:

$$\cdots \to A \otimes \overline{A}^{\otimes r} \otimes A \xrightarrow{d_r} A \otimes \overline{A}^{\otimes r-1} \otimes A \to \cdots \to A \otimes \overline{A} \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \to 0,$$

where we set

$$\begin{aligned} d_r(a_0 \otimes \overline{a}_{1,r} \otimes a_{r+1}) &= a_0a_1 \otimes \overline{a}_{2,r} \otimes a_{r+1} \\ &\quad + \sum_{i=1}^{r-1} (-1)^i a_0 \otimes \overline{a}_{1,i-1} \otimes \overline{a_ia_{i+1}} \otimes \overline{a}_{i+2,r} \otimes a_{r+1} \\ &\quad + (-1)^r a_0 \otimes \overline{a}_{1,r-1} \otimes a_ra_{r+1}, \\ d_0(a_0 \otimes a_1) &= a_0a_1. \end{aligned}$$

We denote $\overline{\Omega}^r(A) := \operatorname{Im} d_r$ for all $r \geq 0$. For an A-bimodule M, consider the cochain complex $C^\bullet(A, M) := \mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{Bar}_\bullet(A), M)$ with differential $\mathrm{Hom}_{A^{\mathrm{e}}}(d_\bullet, M)$. Note that for any $r \geq 0$, we have

$$C^r(A, M) = \mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{Bar}_r(A), M) = \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes \overline{A}^{\otimes r} \otimes A, M) \cong \mathrm{Hom}(\overline{A}^{\otimes r}, M).$$

We identify $C^0(A, M)$ with M. Thus, the cochain complex $C^\bullet(A, M)$ is of the form

$$0 \to M \xrightarrow{\delta^0} \mathrm{Hom}(\overline{A}, M) \to \cdots \to \mathrm{Hom}(\overline{A}^{\otimes r}, M) \xrightarrow{\delta^r} \mathrm{Hom}(\overline{A}^{\otimes r+1}, M) \to \cdots$$

whose each δ^r is defined by

$$\delta^r(f)(\overline{a}_{1,\,r+1}) = a_1 f(\overline{a}_{2,\,r+1}) + \sum_{i=1}^{r} (-1)^{i+1} f(\overline{a}_{1,\,i-1} \otimes \overline{a_i a_{i+1}} \otimes \overline{a}_{i+2,\,r+1}) + (-1)^{r+1} f(\overline{a}_{1,\,r}) a_{r+1}$$

for any $f \in \mathrm{Hom}(\overline{A}^{\otimes r}, M)$ and $\overline{a}_{1,\,r+1} \in \overline{A}^{\otimes r+1}$. Then the r-th cohomology group

$$\mathrm{H}^r(A, M) := \mathrm{H}^r(C^\bullet(A, M), \delta^\bullet)$$

is said to be the r-th *Hochschild cohomology group* of A with coefficients in M. We will write $\mathrm{HH}^r(A) := \mathrm{H}^r(A, A)$. Since A is projective over k, we get $\mathrm{H}^r(A, M) \cong \mathrm{Ext}^r_{A^e}(A, M)$. Namely, Hochschild cohomology groups do not depend on the choice of a projective resolution of A over A^e. For two A-bimodules M and N, the cup product

$$\smile : C^m(A, M) \otimes C^n(A, N) \to C^{m+n}(A, M \otimes_A N)$$

is defined by

$$(\alpha \smile \beta)(\overline{a}_{1,\,m+n}) := \alpha(\overline{a}_{1,\,m}) \otimes_A \beta(\overline{a}_{m+1,\,m+n})$$

for all $\alpha \in C^m(A, M), \beta \in C^n(A, N)$ and $\overline{a}_{1,\,m+n} \in \overline{A}^{\otimes m+n}$. The cup product $\smile$ induces a well-defined operator

$$\smile : \mathrm{H}^m(A, M) \otimes \mathrm{H}^n(A, N) \to \mathrm{H}^{m+n}(A, M \otimes_A N).$$

The Gerstenhaber bracket on the Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is defined as follows: let $\alpha \in C^m(A, A)$ and $\beta \in C^n(A, A)$. We define a k-bilinear map

$$[\ ,\ ] : C^m(A, A) \otimes C^n(A, A) \to C^{m+n-1}(A, A)$$

as

$$[\alpha, \beta] := \alpha \circ \beta - (-1)^{(m-1)(n-1)} \beta \circ \alpha \in C^{m+n-1}(A, A),$$

where we determine $\alpha \circ \beta$ by

$$\alpha \circ \beta(\overline{a}_{1,\,m+n-1}) := \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} \alpha(\overline{a}_{1,\,i-1} \otimes \overline{\beta}(\overline{a}_{i,\,i+n-1}) \otimes \overline{a}_{i+n,\,m+n-1})$$

with $\overline{\beta} := \pi \circ \beta$. This k-bilinear map $[\ ,\ ]$ induces a well-defined operator

$$[\ ,\ ] : \mathrm{HH}^m(A) \otimes \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n-1}(A).$$

Gerstenhaber proved the following result.

Theorem 2.1.2 ([**25**, page 267])**.** *The Hochschild cohomology* $\mathrm{HH}^\bullet(A)$ *equipped with the cup product* $\smile$ *and the Lie bracket* $[\ ,\ ]$ *is a Gerstenhaber algebra.*

For an A-bimodule M, consider a complex $C_\bullet(A, M) := M \otimes_{A^e} \mathrm{Bar}_\bullet(A)$ with differential $\mathrm{id}_M \otimes_{A^e} d_\bullet$. Note that for any $r \geq 0$, we have

$$C_r(A, M) = M \otimes_{A^e} \mathrm{Bar}_r(A) = M \otimes_{A^e} (A \otimes \overline{A}^{\otimes r} \otimes A) \cong M \otimes \overline{A}^{\otimes r}.$$

We identify $C_0(A, M)$ with M. Thus, the complex $C_\bullet(A, M)$ is of the form

$$\cdots \to M \otimes \overline{A}^{\otimes r+1} \xrightarrow{\partial_{r+1}} M \otimes \overline{A}^{\otimes r} \to \cdots \to M \otimes \overline{A} \xrightarrow{\partial_1} M \to 0,$$

where ∂_{r+1} sends $m \otimes \overline{a}_{1,r+1} \in M \otimes \overline{A}^{\otimes r+1}$ to

$$ma_1 \otimes \overline{a}_{2,r+1} + \sum_{i=1}^{r} (-1)^i m \otimes \overline{a}_{1,i-1} \otimes \overline{a_i a_{i+1}} \otimes \overline{a}_{i+2,r+1} + (-1)^{r+1} a_{r+1} m \otimes \overline{a}_{1,r}$$

Then the r-th homology group

$$\mathrm{H}_r(A, M) := \mathrm{H}_r(C_\bullet(A, M), \partial_\bullet)$$

is called the r-th *Hochschild homology group* of A with coefficients in M. We denote $\mathrm{HH}_r(A) := \mathrm{H}_r(A, A)$. Since A is projective over k, we get $\mathrm{H}_r(A, M) \cong \mathrm{Tor}^r_{A^e}(A, M)$, which means that Hochschild homology groups are independent of projective resolutions of A.

There is an action of Hochschild cohomology on Hochschild homology, called the cap product. For two A-bimodules M, N and integers $r, p \geq 0$ with $r \geq p$, a k-bilinear map

$$\frown : C_r(A, M) \otimes C^p(A, N) \to C_{r-p}(A, M \otimes_A N)$$

is defined by

$$(m \otimes \overline{a}_{1,r}) \frown \alpha := m \otimes_A \alpha(\overline{a}_{1,p}) \otimes \overline{a}_{p+1,r}$$

for all $m \otimes \overline{a}_{1,r} \in C_r(A, M)$ and $\alpha \in C^p(A, N)$. The k-bilinear map $\frown$ induces a well-defined operator

$$\frown : \mathrm{H}_r(A, M) \otimes \mathrm{H}^p(A, N) \to \mathrm{H}_{r-p}(A, M \otimes_A N).$$

2.2. Tate-Hochschild cohomology and its Gerstenhaber structure

This section is devoted to recalling Tate-Hochschild cohomology groups and a Gerstenhaber structure on the Tate-Hochschild cohomology. For more details, we refer the reader to [**46**, Section 3 and 4].

Let A be an algebra. The *singularity category* $\mathcal{D}_{\mathrm{sg}}(A)$ of A is the Verdier quotient of the bounded derived category $\mathcal{D}^{\mathrm{b}}(A) = \mathcal{D}^{\mathrm{b}}(A\text{-mod})$ of the module category A-mod of A by the full subcategory formed by those complexes quasi-isomorphic to bounded complexes of projective A-modules. Recall that the shift functors [1] on $\mathcal{D}^{\mathrm{b}}(A)$ and $\mathcal{D}_{\mathrm{sg}}(A)$ are induced by shift of complexes. Let M and N be A-modules. Following [**12**], we define the *i-th Tate cohomology group of M with coefficients in N* to be $\widehat{\mathrm{Ext}}{}^i_A(M, N) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A)}(M, N[i])$ for any $i \in \mathbb{Z}$. Then $\widehat{\mathrm{Ext}}{}^i_{A^{\mathrm{e}}}(A, A)$ is called the *i-th Tate-Hochschild cohomology group* of A and denoted by $\widehat{\mathrm{HH}}{}^i(A)$.

The Tate cohomology $\widehat{\mathrm{Ext}}{}^\bullet_A(M, M) := \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{Ext}}{}^i_A(M, M)$ of a A-module M carries a graded algebra structure, where the multiplication is given by the *Yoneda product*

$$\smile : \widehat{\mathrm{Ext}}{}^i_A(M, M) \otimes \widehat{\mathrm{Ext}}{}^j_A(M, M) \to \widehat{\mathrm{Ext}}{}^{i+j}_A(M, M); \quad \alpha \otimes \beta \mapsto \alpha[j] \circ \beta.$$

We call the graded algebra $\widehat{\mathrm{Ext}}{}^\bullet_A(M, M)$ equipped with the Yoneda product $\smile$ the *Tate cohomology ring* of M, which is called the "stabilized Yoneda Ext algebra" of M by Buchweitz [**12**]. Although Tate cohomology ring $\widehat{\mathrm{Ext}}{}^\bullet_A(M, M)$ is not necessarily graded commutative, Wang [**47**] showed that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}{}^\bullet(A) := \widehat{\mathrm{Ext}}{}^\bullet_{A^{\mathrm{e}}}(A, A)$ of any algebra A is graded commutative. On the other hand, since $\mathrm{Ext}^i_A(M, M) \cong \mathrm{Hom}_{\mathcal{D}^b(A)}(M, M[i])$ for $i \geq 0$, using the canonical triangle functor $\mathcal{D}^{\mathrm{b}}(A) \to \mathcal{D}_{\mathrm{sg}}(A)$, we obtain a morphism $\mathrm{Ext}^\bullet_A(M, M) \to \widehat{\mathrm{Ext}}{}^\bullet_A(M, M)$ of graded algebras.

We now recall another description of Tate-Hochschild cohomology and the Gerstenhaber structure on Tate-Hochschild cohomology based on the description. Recall that $\overline{\Omega}^p(A) = \mathrm{Im}\, d_p$, where $d_p : \mathrm{Bar}_p(A) \to \mathrm{Bar}_{p-1}(A)$ is the p-th differential of the normalized bar resolution $\mathrm{Bar}_\bullet(A)$. We fix an integer m and put $\mathrm{I}_{(m)} := \{p \in \mathbb{Z} \,|\, p \geq 0, m + p \geq 0\}$. Consider an inductive system

$$\left\{X^{(m)}_p,\ \theta_{m+p,\,p} : X^{(m)}_p \to X^{(m)}_{p+1}\right\}_{p\,\in\, \mathrm{I}_{(m)}},$$

where

$$X^{(m)}_p = \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A)),$$

and $\theta_{m+p,\,p} : X^{(m)}_p \to X^{(m)}_{p+1}$ is the connecting homomorphism

$$\theta_{m+p,\,p} : \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A)) \to \mathrm{Ext}^{m+p+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p+1}(A)) \tag{2.1}$$

induced by the short exact sequence

$$0 \longrightarrow \overline{\Omega}^{p+1}(A) \longrightarrow A \otimes \overline{A}^{\otimes p} \otimes A \longrightarrow \overline{\Omega}^{p}(A) \longrightarrow 0.$$

Here, we regard $\mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A))$ as $\mathrm{H}^{m+p}(A, \overline{\Omega}^{p}(A))$, or equivalently, any element of $\mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A))$ is represented by an element in $\mathrm{Hom}_k(\overline{A}^{\otimes m+p}, \overline{\Omega}^{p}(A))$. Note that the inductive system above has the form

$$\mathrm{Ext}^{m+i}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{i}(A)) \xrightarrow{\theta_{m+i,\,i}} \mathrm{Ext}^{m+i+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{i+1}(A)) \xrightarrow{\theta_{m+i+1,\,i+1}} \mathrm{Ext}^{m+i+2}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{i+2}(A)) \to \cdots,$$

where $i \geq 0$ is the least integer such that $m + i \geq 0$.

Remark 2.2.1. Using the explicit description of the connecting homomorphism (2.1) in [**46**, page 16], we see that, for any $m \in \mathbb{Z}$ and $p \in \mathrm{I}_{(m)}$, the connecting homomorphism

$$\theta_{m+p,\,p} : \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A)) \to \mathrm{Ext}^{m+p+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p+1}(A))$$

sends an element $[f] \in \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A))$ represented by $f \in \mathrm{Hom}_k(\overline{A}^{\otimes m+p}, \overline{\Omega}^{p}(A))$ to the element $[\theta_{m+p,\,p}(f)] \in \mathrm{Ext}^{m+p+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p+1}(A))$. Here, $[\theta_{m+p,\,p}(f)]$ is represented by the k-linear map

$$\theta_{m+p,\,p}(f) : \overline{A}^{\otimes m+p+1} \to \overline{\Omega}^{p+1}(A)$$

taking an element $\overline{a}_{1,\,m+p+1} \in \overline{A}^{\otimes m+p+1}$ into

$$(-1)^{m+p}\, d_{p+1}(f(\overline{a}_{1,\,m+p}) \otimes \overline{a}_{m+p+1} \otimes 1) \in \mathrm{Im}\, d_{p+1} = \overline{\Omega}^{p+1}(A),$$

where $d_{p+1} : \mathrm{Bar}_{p+1}(A) \to \mathrm{Bar}_{p}(A)$ is the $(p+1)$-th differential of $\mathrm{Bar}_{\bullet}(A)$.

Proposition 2.2.2 ([**46**, Proposition 3.1 and Remark 3.3])**.** *For any $m \in \mathbb{Z}$, there is an isomorphism*

$$\varinjlim_{p \in \mathrm{I}_{(m)}} \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p}(A)) \cong \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^{\mathrm{e}})}(A, A[m]) = \widehat{\mathrm{Ext}}^{m}_{A^{\mathrm{e}}}(A, A).$$

We now define a Gerstenhaber structure on Tate-Hochschild cohomology defined by Wang ([**46**]). Let m, n, p and q be integers such that $m, n, p, q \geq 0$. A cup product

$$\smile_{\mathrm{sg}} : C^{m}(A, \overline{\Omega}^{p}(A)) \otimes C^{n}(A, \overline{\Omega}^{q}(A)) \to C^{m+n}(A, \overline{\Omega}^{p+q}(A))$$

is defined by

$$f \smile_{\mathrm{sg}} g(\overline{b}_{1,\,m+n}) := \Phi_{p+q}(f(\overline{b}_{1,\,m}) \otimes_A g(\overline{b}_{m+1,\,m+n})),$$

where $f \otimes g \in C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A))$ and $\Phi_{p+q} : \overline{\Omega}^p(A) \otimes_A \overline{\Omega}^q(A) \to \overline{\Omega}^{p+q}(A)$ is an isomorphism of A-bimodules determined by

$$\Phi_{p+q}(a_0 \otimes \overline{a}_{1,\,p} \otimes a_{p+1} \otimes_A b_0 \otimes \overline{b}_{1,\,q} \otimes b_{q+1}) = a_0 \otimes \overline{a}_{1,\,p} \otimes \overline{a_{p+1}b_0} \otimes \overline{b}_{1,\,q} \otimes b_{q+1}$$

for $a_0 \otimes \overline{a}_{1,\,p} \otimes a_{p+1} \in \overline{\Omega}^p(A)$ and $b_0 \otimes \overline{b}_{1,\,q} \otimes b_{q+1} \in \overline{\Omega}^q(A)$, which is given in [**47**, Lemma 2.6].

Let $m \in \mathbb{Z}_{>0},\ p \in \mathbb{Z}_{\geq 0}$ and $f \in C^m(A, \overline{\Omega}^p(A))$ and let $\pi : A \to \overline{A}$ be the canonical epimorphism. We set

$$\begin{aligned}
\pi_p^{(l)} &:= \pi \otimes \mathrm{id}_{\overline{A}}^{\otimes p-1} \otimes \mathrm{id}_A : A \otimes \overline{A}^{\otimes p-1} \otimes A \to \overline{A}^{\otimes p} \otimes A,\\
\pi_p^{(r)} &:= \mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes p-1} \otimes \pi : A \otimes \overline{A}^{\otimes p-1} \otimes A \to A \otimes \overline{A}^{\otimes p},\\
\pi_p^{(b)} &:= \pi \otimes \mathrm{id}_{\overline{A}}^{\otimes p-1} \otimes \pi : A \otimes \overline{A}^{\otimes p-1} \otimes A \to \overline{A}^{\otimes p+1}
\end{aligned}$$

and then denote

$$f^{(l)} := \pi_p^{(l)} f, \quad f^{(r)} := \pi_p^{(r)} f, \quad f^{(b)} := \pi_p^{(b)} f.$$

Let m, n, p and q be integers such that $m, n > 0$ and $p, q \geq 0$. We now define a bilinear map

$$[\ ,\ ]_{\mathrm{sg}} : C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A)) \to C^{m+n-1}(A, \overline{\Omega}^{p+q}(A)).$$

as follows: let

$$f \in C^m(A, \overline{\Omega}^p(A)) = \mathrm{Hom}_k(\overline{A}^{\otimes m}, \overline{\Omega}^p(A))$$

and

$$g \in C^n(A, \overline{\Omega}^q(A)) = \mathrm{Hom}_k(\overline{A}^{\otimes n}, \overline{\Omega}^q(A)).$$

We first define a k-linear map $f \bullet_i g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ for each integer i with $1 \leq i \leq m$. Consider the following four k-linear maps:

(1) $\left(\mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes g^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-i}\right) : \overline{A}^{\otimes m+n-1} \to \overline{A}^{\otimes m+q}$ is given by

$$\overline{a}_{1,\,m+n-1} \mapsto \overline{a}_{1,\,i-1} \otimes g^{(b)}(\overline{a}_{i,\,i+n-1}) \otimes \overline{a}_{i+n,\,m+n-1};$$

(2) $\left(f^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q}\right) : \overline{A}^{\otimes m+q} \to A \otimes \overline{A}^{\otimes p+q}$ is given by

$$\overline{a}_{1,\,m+q} \mapsto f^{(r)}(\overline{a}_{1,\,m}) \otimes \overline{a}_{m+1,\,m+q};$$

(3) $\left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes p+q} \otimes 1\right) : A \otimes \overline{A}^{\otimes p+q} \to A \otimes \overline{A}^{\otimes p+q} \otimes A$ is given by

$$a_0 \otimes \overline{a}_{1,\,p+q} \mapsto a_0 \otimes \overline{a}_{1,\,p+q} \otimes 1;$$

(4) $d_{p+q} : A \otimes \overline{A}^{\otimes p+q} \otimes A \to A \otimes \overline{A}^{\otimes p+q-1} \otimes A$ is the $(p+q)$-th differential of the normalized bar resolution $\mathrm{Bar}_\bullet(A)$.

We then define a k-linear map $f \bullet_i g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ by the composition of the above four maps

$$\begin{aligned} f \bullet_i g : &= d_{p+q} \circ \left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes p+q} \otimes 1\right) \circ \left(f^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q}\right) \circ \left(\mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes g^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-i}\right) \\ &= d_{p+q}((f^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q})(\mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes g^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-i}) \otimes 1) \end{aligned}$$

for $1 \leq i \leq m$. On the other hand, we assume that $q > 0$. We also define a k-linear map $f \bullet_{-i} g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ for each integer i with $1 \leq i \leq q$. Consider the following four k-linear maps:

(1) $\left(g^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-1}\right) : \overline{A}^{\otimes m+n-1} \to A \otimes \overline{A}^{\otimes m+q-1}$ is given by

$$\overline{a}_{1,\,m+n-1} \mapsto g^{(r)}(\overline{a}_{1,\,n}) \otimes \overline{a}_{n+1,\,m+n-1};$$

(2) $\left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes f^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q-i}\right) : A \otimes \overline{A}^{\otimes m+q-1} \to A \otimes \overline{A}^{\otimes p+q}$ is given by

$$a_0 \otimes \overline{a}_{1,\,m+q-1} \mapsto a_0 \otimes \overline{a}_{1,\,i-1} \otimes f^{(b)}(\overline{a}_{i,\,i+m-1}) \otimes \overline{a}_{i+m,\,m+q-1};$$

(3) $\left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes p+q} \otimes 1\right) : A \otimes \overline{A}^{\otimes p+q} \to A \otimes \overline{A}^{\otimes p+q} \otimes A$ is the same as above;

(4) $d_{p+q} : A \otimes \overline{A}^{\otimes p+q} \otimes A \to A \otimes \overline{A}^{\otimes p+q-1} \otimes A$ is the same as above.

Then we define a k-linear map $f \bullet_{-i} g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ by the composition of the above four maps

$$\begin{aligned} f \bullet_{-i} g : &= d_{p+q} \circ \left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes p+q} \otimes 1\right) \circ \left(\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes f^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q-i}\right) \circ \left(g^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-1}\right) \\ &= d_{p+q}((\mathrm{id}_A \otimes \mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes f^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q-i})(g^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-1}) \otimes 1) \end{aligned}$$

for $1 \leq i \leq q$. So far, the k-linear map $f \bullet_i g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ has been defined in the following way:

$$f \bullet_i g$$
$$= \begin{cases} d_{p+q}((f^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q})(\mathrm{id}_{\overline{A}}^{\otimes i-1} \otimes g^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-i}) \otimes 1) & \text{if } 1 \leq i \leq m; \\ d_{p+q}((\mathrm{id} \otimes \mathrm{id}_{\overline{A}}^{\otimes -i-1} \otimes f^{(b)} \otimes \mathrm{id}_{\overline{A}}^{\otimes q+i})(g^{(r)} \otimes \mathrm{id}_{\overline{A}}^{\otimes m-1}) \otimes 1) & \\ & \text{if } q > 0 \text{ and } -q \leq i \leq -1. \end{cases}$$

Now, we define a k-linear map $f \bullet g \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ by

$$f \bullet g := \begin{cases} \sum_{i=1}^{m} (-1)^{r(m,p;n,q;i)} f \bullet_i g + \sum_{i=1}^{q} (-1)^{s(m,p;n,q;i)} f \bullet_{-i} g & \text{if } q > 0; \\ \sum_{i=1}^{m} (-1)^{r(m,p;n,q;i)} f \bullet_i g & \text{if } q = 0, \end{cases}$$

where $r(m,p;n,q;i)$ and $s(m,p;n,q;i)$ are determined by

$$r(m,p;n,q;i) := p + q + (i-1)(q-n-1) \text{ for } 1 \leq i \leq m,$$
$$s(m,p;n,q;i) := p + q + (i-1)(q-n-1) \text{ for } 1 \leq i \leq q.$$

Finally, we are able to define a k-linear map $[f,g]_{\mathrm{sg}} \in C^{m+n-1}(A, \overline{\Omega}^{p+q}(A))$ as

$$[f,g]_{\mathrm{sg}} := f \bullet g - (-1)^{(m-p-1)(n-q-1)} g \bullet f.$$

Wang [**46**] showed that the cup product $\smile_{\mathrm{sg}}$ and the bilinear map $[\ ,\ ]_{\mathrm{sg}}$ induce well-defined operators, still denoted by $\smile_{\mathrm{sg}}$ and $[\ ,\ ]_{\mathrm{sg}}$, on a graded k-vector space

$$\bigoplus_{\substack{m \in \mathbb{Z},\, p \in \mathbb{Z}_{\geq 0}, \\ m+p \geq 0}} \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p}(A, \overline{\Omega}^{p}(A))$$

with grading

$$\left(\bigoplus_{m,p} \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p}(A, \overline{\Omega}^{p}(A)) \right)^{i} = \bigoplus_{\substack{l \geq 0, \\ i+l \geq 0}} \mathrm{Ext}_{A^{\mathrm{e}}}^{i+l}(A, \overline{\Omega}^{l}(A))$$

for $i \in \mathbb{Z}$, which make it into a Gerstenhaber algebra. Furthermore, he proved that the two induced operators $\smile_{\mathrm{sg}}$ and $[\ ,\ ]_{\mathrm{sg}}$ are compatible with the connecting homomorphisms $\theta_{m,p} : \mathrm{Ext}_{A^{\mathrm{e}}}^{m}(A, \overline{\Omega}^{p}(A)) \to \mathrm{Ext}_{A^{\mathrm{e}}}^{m+1}(A, \overline{\Omega}^{p+1}(A))$. Therefore, we have the following result.

Theorem 2.2.3 ([**46**, Theorem 4.1]). *Let A be a finite dimensional algebra over a field k. Then the graded k-vector space*

$$\bigoplus_{m \in \mathbb{Z}} \varinjlim_{p \in \mathrm{I}_{(m)}} \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p}(A, \overline{\Omega}^{p}(A))$$

equipped with the cup product $\smile_{\mathrm{sg}}$ and the Lie bracket $[\ ,\ ]_{\mathrm{sg}}$ is a Gerstenhaber algebra.

Remark 2.2.4. The Gerstenhaber brackets on $\bigoplus_{m,p} \mathrm{Ext}_{A^e}^{m+p}(A, \overline{\Omega}^p(A))$ involving elements of degree zero are defined via the connecting homomorphisms

$$\theta_{0,*} : \mathrm{Ext}^0_{A^e}(A, \overline{\Omega}^*(A)) \to \mathrm{Ext}^1_{A^e}(A, \overline{\Omega}^{*+1}(A)),$$

that is, for $f \in \mathrm{Ext}_{A^e}^{m+p}(A, \overline{\Omega}^p(A))$ and $\alpha \in \mathrm{Ext}^0_{A^e}(A, \overline{\Omega}^q(A))$, we define

$$[f, \alpha]_{\mathrm{sg}} := [f, \theta_{0,q}(\alpha)]_{\mathrm{sg}}.$$

Remark 2.2.5. The induced cup product $\smile_{\mathrm{sg}}$ in Theorem 2.2.3 commutes with the Yoneda product on $\widehat{\mathrm{HH}}^{\bullet}(A)$ via the isomorphism in Proposition 2.2.2 (see [**47**, Proposition 4.7]).

2.3. Gorenstein algebras

In this section, we recall the definition of Gorenstein algebras and facts related to those algebras from [**6, 11, 12**]. Let A be an algebra. Recall that the *stable category* A-$\underline{\mathrm{mod}}$ of A-modules is the category whose objects are the same as A-mod and morphisms are given by

$$\underline{\mathrm{Hom}}_A(M, N) := \mathrm{Hom}_A(M, N)/\mathcal{P}(M, N),$$

where $\mathcal{P}(M, N)$ is the space of morphisms factoring through a projective module. We denote by $[f]$ the element of $\underline{\mathrm{Hom}}_\Lambda(M, N)$ represented by a morphism $f : M \to N$. There exists a canonical functor $F : A\text{-}\underline{\mathrm{mod}} \to \mathcal{D}_{\mathrm{sg}}(A)$ making the following square commute:

$$\begin{array}{ccc} A\text{-mod} & \longrightarrow & \mathcal{D}^{\mathrm{b}}(A\text{-mod}) \\ \downarrow & & \downarrow \\ A\text{-}\underline{\mathrm{mod}} & \xrightarrow{F} & \mathcal{D}_{\mathrm{sg}}(A) \end{array}$$

where the two vertical functors are the canonical ones, and the upper horizontal functor is the one sending a module M to the complex M concentrated in degree 0. Further, the functor F satisfies $F \circ \Omega_\Lambda \cong [-1] \circ F$, where Ω_A is the syzygy functor on A-$\underline{\mathrm{mod}}$ (i.e. the functor sending a module M to the kernel of a projective cover of M). On the other hand, let $\underline{\mathrm{APC}}(A)$ be the homotopy category of acyclic complexes of projective A-modules. For a complex $X_\bullet$ and an integer i, we denote by $\Omega_i(X_\bullet)$ the cokernel $\mathrm{Cok}\, d^X_{i+1}$ of the differential d^X_{i+1} and by $X_\bullet[i]$ the complex given by $(X_\bullet[i])_j = X_{j-i}$ and $d^{X[i]} = (-1)^i d^X$. Then taking the cokernel $\Omega_0(X_\bullet) = \mathrm{Cok}\, d^X_1$ of the differential d^X_1 for a complex $X_\bullet$ defines a functor $\Omega_0 : \underline{\mathrm{APC}}(A) \to A\text{-}\underline{\mathrm{mod}}$ satisfying $\Omega_0 \circ [-1] \cong \Omega_A \circ \Omega_0$.

Recall that an algebra A is *Gorenstein* if $\mathrm{inj.dim}_A A < \infty$ and $\mathrm{inj.dim}_{A^{\mathrm{op}}} A < \infty$. Since the two dimensions coincide by [**49**, Lemma A], we call a Gorenstein algebra A with $\mathrm{inj.dim}_A A = d$ a *d-Gorenstein algebra*. In the rest of this section, we assume that A is a d-Gorenstein algebra unless otherwise specified. We call an A-module

M *Cohen-Macaulay* if $\mathrm{Ext}^i_A(M,A)=0$ for all $i>0$. It is clear that projective A-modules are Cohen-Macaulay. We denote by $\mathrm{CM}(A)$ the category of Cohen-Macaulay A-modules. It is well-known that $\mathrm{CM}(A)$ is a Frobenius category, that is, an exact category with enough projective objects and injective objects such that the classes of projective objects and of injective objects coincide. Note that projective objects of $\mathrm{CM}(A)$ are precisely projective A-modules. Thus the stable category $\underline{\mathrm{CM}}(A)$ carries a structure of a triangulated category (see [**12, 28**]). In particular, the syzygy functor Ω_A on A-$\underline{\mathrm{mod}}$ agrees with the inverse of the shift functor Σ on $\underline{\mathrm{CM}}(A)$. The following result is due to Buchweitz [**12**].

Theorem 2.3.1 ([**12**, Theorem 4.4.1])**.** *Let A be a Gorenstein algebra. Then there exist equivalences of triangulated categories*

$$\underline{\mathrm{APC}}(A) \xrightarrow{\Omega_0} \underline{\mathrm{CM}}(A) \xrightarrow{\iota_A} \mathcal{D}_{\mathrm{sg}}(A),$$

where the equivalence ι_A is given by the restriction of $F: A\text{-}\underline{\mathrm{mod}} \to \mathcal{D}_{\mathrm{sg}}(A)$ to $\underline{\mathrm{CM}}(A)$.

Thanks to the theorem, we can associate to any A-module M an object $T_\bullet = T_\bullet^M$ in $\underline{\mathrm{APC}}(A)$, uniquely determined up to isomorphism, satisfying that $\Omega_0(T_\bullet)\cong M$ in $\mathcal{D}_{\mathrm{sg}}(A)$. Thus the triangle equivalence $\iota_A : \underline{\mathrm{CM}}(A) \to \mathcal{D}_{\mathrm{sg}}(A)$ induces an isomorphism

$$\widehat{\mathrm{Ext}}^i_\Lambda(M,M) \cong \underline{\mathrm{Hom}}_A(\Omega_0(T_\bullet), \Sigma^i\Omega_0(T_\bullet))$$

for all $i\in\mathbb{Z}$. We identify these via this isomorphism.

Recall that, for an algebra A, the *Gorenstein dimension* $\text{G-dim}_A M$ of an A-module M is defined by the shortest length of a resolution of M by A-modules X with $X\cong X^{**}$ and $\mathrm{Ext}^i_A(X,A)=0=\mathrm{Ext}^i_{A^{\mathrm{op}}}(X^*,A)$ for all $i>0$, where we set $(-)^* := \mathrm{Hom}_A(-,A)$ (see [**2**] for its original definition). The next proposition is easily obtained from the results in [**6**] applied to the case of Gorenstein algebras: (1), (2) and (3) follow from [**6**, Theorems 3.1 and 3.2], [**6**, Lemma 2.4 and Theorem 3.1] and [**6**, Theorem 5.2], respectively.

Proposition 2.3.2. *The following hold for a module M over a d-Gorenstein algebra A.*

(1) *The Gorenstein dimension* $\text{G-dim}_A M$ *of M satisfies* $\text{G-dim}_A M \le d$ *and is equal to the smallest integer $r\ge 0$ for which $\Omega^r_A(M)$ is Cohen-Macaulay.*

(2) *There exists a diagram $T_\bullet \xrightarrow{\theta} P_\bullet \xrightarrow{\varepsilon} M$ satisfying the following conditions:*

   (i) $T_\bullet \in \underline{\mathrm{APC}}(A)$ and $P_\bullet \xrightarrow{\varepsilon} M$ is a projective resolution of M.

   (ii) $\theta : T_\bullet \to P_\bullet$ is a chain map with θ_i an isomorphism for any $i \gg 0$.

(3) *We have that* $\mathrm{Ext}^i_\Lambda(M,M)\cong \widehat{\mathrm{Ext}}^i_\Lambda(M,M)$ *for all* $i > \text{G-dim}_A M$.

We call such a diagram as in Proposition 2.3.2 (2) a *complete resolution* of M (see [**6**] for its definition in a general setting). A complete resolution is unique in the sense of [**6**, Lemma 5.3] (when it exists).

The canonical triangle functor $\mathcal{D}^{\mathrm{b}}(A) \to \mathcal{D}_{\mathrm{sg}}(A)$ induces a morphism

$$\Phi^i : \mathrm{Ext}^i_A(M,N) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(M, N[i]) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A)}(M, N[i]) = \widehat{\mathrm{Ext}}^i_A(M,N)$$

for any integer i. Note that $\Phi^i = 0$ for all $i < 0$. It follows from [**12**, Remarks (b) on page 41] that $\Phi^i : \mathrm{Ext}^i_A(M,N) \to \widehat{\mathrm{Ext}}^i_A(M,N)$ is surjective if $i = d$ and bijective if $i > d$. In particular, for a 0-Gorenstein algebra A, we have an epimorphism $\Phi^{\geq 0} = (\Phi^i)_{i\geq 0} : \mathrm{HH}^{\bullet}(A) \to \widehat{\mathrm{HH}}^{\geq 0}(A)$ of graded rings, where $\mathrm{HH}^{\geq 0}(A) = \bigoplus_{i\geq 0} \widehat{\mathrm{HH}}^i(A)$.

Next, we explain how we find the corresponding object $T^M_{\bullet}$ in $\underline{\mathrm{APC}}(A)$ for any A-module M. If $T_{\bullet} \to P_{\bullet} \to M$ is a complete resolution of M, then $T_{\bullet}$ in $\underline{\mathrm{APC}}(A)$ is the object corresponding to M via the triangle equivalence $\iota \circ \Omega_0 : \underline{\mathrm{APC}}(A) \to \mathcal{D}_{\mathrm{sg}}(A)$. Indeed, the morphism $\Omega_0(T_{\bullet}) \to M$ induced by the chain map $\theta_{\geq 0} : T_{\geq 0} \to P_{\bullet}$ is an isomorphism in $\mathcal{D}_{\mathrm{sg}}(A)$. Here, $T_{\geq 0}$ stands for the following truncated complex of $T_{\bullet}$:

$$T_{\geq 0} = \cdots \to T_2 \xrightarrow{d_2^T} T_1 \xrightarrow{d_1^T} T_0 \to 0 \to 0 \to \cdots .$$

Thus constructing a complete resolution of M is equivalent to finding the corresponding object $T^M_{\bullet}$ of $\underline{\mathrm{APC}}(A)$.

It follows from [**9**, Lemma 6.1] that if A is an n-Gorenstein algebra, then the enveloping algebra A^{e} is a $(2n)$-Gorenstein algebra. Hence, by Proposition 2.3.2, there exists a complete resolution $T_{\bullet} \xrightarrow{\theta} P_{\bullet} \xrightarrow{\varepsilon} A$ of A over A^{e}. The following is the definition of the complete cohomology groups of a Gorenstein algebra.

Definition 2.3.3 (cf. [**11**, page 911])**.** *Let A be a Gorenstein algebra and $T_{\bullet} \xrightarrow{\theta} P_{\bullet} \xrightarrow{d_0} A$ a complete resoluiton of A over A^{e}. For $r \in \mathbb{Z}$, the r-th complete cohomology group of A with coefficients in an A-bimodule N is defined by* $\mathrm{CH}^r(A,N) := \mathrm{H}^i(\mathrm{Hom}_A(T_{\bullet}, N))$. *We write* $\mathrm{CH}^r(A) := \mathrm{CH}^r(A,A)$.

Bergh and Jorgensen [**11**] defined the complete cohomology under the name "the Tate-Hochschild cohomology", and, in this thesis, we use the terminology "Tate-Hochschild" for the cohomology groups defined by Wang [**46, 47**], which are described in the previous section. We remark that both of these cohomology groups are isomorphic for any Gorenstein algebra.

Chapter 3

Batalin-Vilkovisky structures on complete cohomology rings for Frobenius algebras

In this chapter, we give a sufficient condition for the Tate-Hochschild cohomology ring of a Frobenius algebra to have a BV structure such that the induced Gerstenhaber structure coincides with the one of Wang. Moreover, we consider certain three self-injective Nakayama algebras and compute their Tate-Hochschild cohomology rings, their BV structures, and the induced Gerstenhaber structures.

3.1. Frobenius algebras and complete resolutions

Let A be an algebra with $\dim_k A = d$, and let σ be an algebra automorphism of A. For any A-bimodule M, we denote by M_σ the A-bimodule which is M as a k-vector space and whose A-bimodule structure is defined by $a \cdot m \cdot b := am\sigma(b)$ for $m \in M_\sigma$ and $a, b \in A$. We also denote by $A^\vee$ the right A^{e}-module $\mathrm{Hom}_{A^{\mathrm{e}}}({}_{A^{\mathrm{e}}}A, {}_{A^{\mathrm{e}}}A^{\mathrm{e}})$ whose structure is given by the multiplication of A^{e} on the right hand side. Note that we have an isomorphism of right A^{e}-modules

$$\begin{aligned} A^\vee &\xrightarrow{\sim} (A \otimes A)^A \\ &:= \left\{ \sum_i x_i \otimes y_i \,\middle|\, \sum a x_i \otimes y_i = \sum x_i \otimes y_i a \text{ for any } a \in A \right\}; f \mapsto f(1), \end{aligned}$$

where a right A^{e}-module structure of $(A \otimes A)^A$ is defined by the multiplication of A^{e} on the right hand side. Recall that A is a *Frobenius algebra* if there is an associative and non-degenerate bilinear form $\langle\ ,\ \rangle : A \otimes A \to k$. The associativity means that $\langle ab, c\rangle = \langle a, bc\rangle$ for all a, b and $c \in A$. If $(u_i)_{i=1}^d$ is a k-basis of A, then there is a k-basis $(v_i)_{i=1}^d$ of A such that $\langle v_i, u_j\rangle = \delta_{ij}$ with δ_{ij} Kronecker's delta. In such a case, we call $(u_i)_{i=1}^d, (v_i)_{i=1}^d$ *dual bases* of A. There exists an algebra automorphism ν, up to inner automorphism, of A such that $\langle a,b\rangle = \langle b,\nu(a)\rangle$ for all $a,b\in A$, and the automorphism ν is said to be the *Nakayama automorphism* of A. In fact, we can write both the Nakayama automorphism ν and its inverse ν^{-1}, explicitly: for $x\in A$,

$$\nu(x) := \sum_{i=1}^{d}\langle x, v_i\rangle u_i, \quad \nu^{-1}(x) := \sum_{i=1}^{d}\langle u_i, x\rangle v_i.$$

Another definition of Frobenius algebras is that A is isomorphic to $D(A)$ as right or as left A-modules. Here, the left (right) A-module structure of $D(A)$ is defined by $(af)(x) := f(xa)$ $((fa)(x) := f(ax))$ for any $f\in D(A)$ and any $a\in A$. We can see that the bilinear form $\langle\ ,\ \rangle : A\otimes A\to k$ induces an isomorphism of left A-modules

$$\phi : A \xrightarrow{\sim} D(A);\ a\mapsto \langle -, a\rangle.$$

Moreover, this isomorphism gives rise to an isomorphism $A_\nu \xrightarrow{\sim} D(A)$ of A-bimodules.

The first statement of the next lemma appears in [**23**, Lemma 2.1.35]. However, we prove it again in order to get the explicit form of the isomorphism below.

Lemma 3.1.1. *Let A be a Frobenius algebra. With the same notation as above, we have the following assertions.*

(1) *There is an isomorphism $A_{\nu^{-1}} \cong A^\vee$ of right A^{e}-bimodules.*

(2) *If $(u_i)_i, (v_i)_i$ and $(u'_j)_j, (v'_j)_j$ are two dual bases of A, then we have $\sum_i u_i\otimes v_i = \sum_j u'_j\otimes v'_j$.*

(3) *An element $\sum_i u_i\otimes v_i$ of $A\otimes A$ has the following properties:*

  (a) $\sum_i u_i\otimes v_i = \sum_i v_i\otimes \nu^{-1}(u_i) = \sum_i \nu(v_i)\otimes u_i$*;*

  (b) $\sum_i au_ib\otimes v_i = \sum_i u_i\otimes \nu^{-1}(b)v_ia$ *for any $a,b\in A$.*

Proof. For (2) and (3), consider the composition $\eta : A_{\nu^{-1}}\otimes A\to \mathrm{Hom}_k(A,A)$ of isomorphisms

$$\begin{array}{ccccc} A_{\nu^{-1}}\otimes A & \longrightarrow & D(A)\otimes A & \longrightarrow & \mathrm{Hom}_k(A,A) \\ \sum_i x_i\otimes y_i & \longmapsto & \sum_i\langle -, x_i\rangle\otimes y_i & \longmapsto & \big[x\mapsto \sum_i\langle x, x_i\rangle y_i\big]. \end{array}$$

Since $x = \sum_i\langle x, u_i\rangle v_i$ for any dual bases $(u_i)_{i=1}^d, (v_i)_{i=1}^d$ of A and any $x\in A$, the statements (2) and (3) follow from the injectivity of η. On the other hand, we define

$$\varphi : A_{\nu^{-1}}\to A^\vee;\ x\mapsto \Big[a\mapsto \sum_i au_i\nu(x)\otimes v_i\Big],$$

$$\psi : A^\vee\to (A\otimes A)^A\to A_{\nu^{-1}};\ \alpha\mapsto \alpha(1_A) = \sum_i x_i\otimes y_i\mapsto \sum_i\langle 1, x_i\rangle y_i.$$

Then we get φ is a right A^{e}-module homomorphism. Indeed, if $x \in A_{\nu^{-1}}$ and $a \otimes b^{\circ} \in A^{\mathrm{e}}$, then we have $\varphi(x\,(a \otimes b^{\circ})) = \sum_i u_i\, \nu(bx) a \otimes v_i = \sum_i u_i \nu(x) a \otimes b v_i = (\sum_i u_i \nu(x) \otimes v_i)(a \otimes b^{\circ})$. One can easily check that $\varphi\psi = \mathrm{id}_{A^{\vee}}$ and $\psi\varphi = \mathrm{id}_{A_{\nu^{-1}}}$. □

As remarked in Section 2.3, if A is a Frobenius algebra A, then so is A^{e}. In particular, A^{e} is a 0-Gorenstein algebra. Therefore, A has a complete resolution over A^{e}. Note that we may take a projective resolution of A over A^{e} as the non-negative part of the complete resolution. In fact, Nakayama [**39**] constructed a complete resolution $T_{\bullet}$ of A in the following way: we set $T_r := \mathrm{Bar}_r(A) = A \otimes \overline{A}^{\otimes r} \otimes A$ for every $r \geq 0$ and $T_{-s} := D(\mathrm{Bar}_{s-1}(A))_{\nu^{-1}}$ for each $s \geq 1$. Then one gets an exact sequence

$$\begin{array}{ccccccccccccccc} \cdots \longrightarrow T_r \xrightarrow{d_r} T_{r-1} \longrightarrow \cdots \xrightarrow{d_1} & T_0 & \xrightarrow{d'_0} & T_{-1} & \xrightarrow{d_{-1}} \cdots \longrightarrow T_{-s} \xrightarrow{d_{-s}} T_{-s-1} \longrightarrow \cdots \\ & {\scriptstyle d_0}\downarrow & & \uparrow{\scriptstyle D(d_0)} & \\ & A & \underset{\phi}{\xrightarrow{\sim}} & D(A)_{\nu^{-1}} & \end{array}$$

where we put

$$D(d_0)(f) = f d_0 \quad (f \in D(A)_{\nu^{-1}}), \quad d'_0 = D(d_0)\phi d_0, \qquad d_{-s}(g) = g d_s \quad (g \in T_{-s}).$$

Sanada [**42**, Lemma 1.1] proved that there is an isomorphism $\mathrm{Hom}_{A^{\mathrm{e}}}(T_r, M) \cong M_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} T_{-r-1}$ for any A-bimodule M and any integer r, and this isomorphism is natural in M. Thus each T_r ($r \in \mathbb{Z}$) is projective over A^{e}. On the other hand, the isomorphisms $\mathrm{Hom}_{A^{\mathrm{e}}}(T_r, M) \cong M_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} T_{-r-1}$ induces an isomorphism of complexes between $\mathrm{Hom}_{A^{\mathrm{e}}}(T, M)$ and $M_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} T$. Therefore, the following complex $(\mathcal{D}^{\bullet}(A, M), \widehat{d^{\bullet}})$ has the same cohomology groups as $\mathrm{Hom}_{A^{\mathrm{e}}}(T, M)$:

$$\cdots \to C_2(A, M_{\nu^{-1}}) \xrightarrow{\partial_2} C_1(A, M_{\nu^{-1}}) \xrightarrow{\partial_1} M_{\nu^{-1}} \xrightarrow{\mu} M \xrightarrow{\delta^0} C^1(A, M) \xrightarrow{\delta^1} C^2(A, M) \to \cdots,$$

where we define $\mu : M_{\nu^{-1}} \to M$ by $\mu(m) := \sum_{i=1}^{d} u_i m v_i$ for $m \in M$ and set

$$\mathcal{D}^r(A, M) := \begin{cases} C^r(A, M) & \text{if } r \geq 0, \\ C_{-r-1}(A, M_{\nu^{-1}}) & \text{if } r \leq -1; \end{cases}$$

$$\widehat{d^r} := \begin{cases} \delta^r & \text{if } r \geq 0, \\ \mu & \text{if } r = -1, \\ \partial_{-r-1} & \text{if } r \leq -2. \end{cases}$$

We give the explicit forms of the 0-th and (-1)-th cohomology groups as follows:

$$\mathrm{CH}^0(A) \cong M^A / N_A(M), \quad \mathrm{CH}^{-1}(A) = {}_{N_A}M / I_A(M),$$

where we set

$$M^A := \{\, m \in M \mid am = ma \text{ for all } a \in A\},$$

$$N_A(M) := \operatorname{Im}(\mu) = \Big\{\, \sum_i u_i m v_i \mid m \in M \Big\},$$

$${}_{N_A}M := \{\, m \in M \mid \sum_i u_i m v_i = 0\},$$

$$I_A(M) := \Big\{\, \sum_i (m_i \nu^{-1}(a_i) - a_i m_i) \text{ (finite sum)} \mid a_i \in A, m_i \in M \Big\}.$$

Note that, for any $x \in A$, $\sum_i u_i x v_i = 0$ holds if and only if $\sum_i u_i \nu(x) v_i = 0$ holds.

Remark 3.1.2. If $M = A$, then $\mathrm{CH}^0(A)$ and $\mathrm{CH}^{-1}(A)$ are appeared in the following exact sequence:

$$0 \to \mathrm{CH}^{-1}(A) \to A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} A \xrightarrow{\eta} \mathrm{Hom}_{A^{\mathrm{e}}}(A, A) \to \mathrm{CH}^0(A) \to 0,$$

where the morphism $\eta(x \otimes_{A^{\mathrm{e}}} a)(b) = \sum_i b u_i \nu(x) a v_i$.

Suppose that A is a self-injective algebra. Recall that A is a *self-injective* algebra if A is injective as a left and as a right A-module. Note that the enveloping algebra A^{e} is also a self-injective algebra. Observe that if A is a self-injective algebra, then all of the connecting homomorphisms (2.1)

$$\theta_{m+p,\,p} : \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A)) \to \mathrm{Ext}^{m+p+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p+1}(A))$$

are isomorphisms except for the case $m + p = 0$, so that we have an isomorphism $\mathrm{Ext}^{r+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A)) \cong \widehat{\mathrm{Ext}}^{\,r}_{A^{\mathrm{e}}}(A, A)$ for all $r, p \in \mathbb{Z}$ such that $p \geq 0$ and $r + p > 0$. We need modification for the inductive system $\{X^{(m)}_p, \theta_{m+p,\,p}\}_{p \in \mathrm{I}_{(m)}}$ defined in Section 2.3. Let us recall that

$$\varinjlim_{p \in \mathrm{I}_{(m)}} \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A))$$

is the inductive limit of the inductive system $\{X^{(m)}_p, \theta_{m+p,\,p}\}_{p \in \mathrm{I}_{(m)}}$ of which the term $X^{(m)}_p$ is defined by $X^{(m)}_p := \mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A))$ and whose morphism $\theta_{m+p,\,p}$ is the connecting homomorphism $\mathrm{Ext}^{m+p}_{A^{\mathrm{e}}}(A, \overline{\Omega}^p(A)) \to \mathrm{Ext}^{m+p+1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{p+1}(A))$. Consider another inductive system

$$\{Y^{(m)}_p, \varphi_{m+p,\,p}\}_{p \in \mathrm{I}_{(m)}}$$

of which the term $Y^{(m)}_p$ is the same as $X^{(m)}_p$ and whose morphism $\varphi_{m+p,p}$ is given by

$$\varphi_{m+p,\,p} := \begin{cases} (-1)^{m+i} \theta_{m+i,\,i} & \text{if } p = i, \\ (-1)^m \theta_{m+p,\,p} & \text{if } p > i, \end{cases}$$

where an integer $i \geq 0$ is the least one belonging to $\mathrm{I}_{(m)}$. Then we can readily see

$$\varinjlim_{p \in \mathrm{I}_{(m)}} Y_p^{(m)} \cong \varinjlim_{p \in \mathrm{I}_{(m)}} \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p}(A, \overline{\Omega}^p(A)).$$

We will utilize the inductive system $\{Y_p^{(m)}, \varphi_{m+p,\,p}\}_{p \in \mathrm{I}_{(m)}}$ instead of $\{X_p^{(m)}, \theta_{m+p,\,p}\}_{p \in \mathrm{I}_{(m)}}$ and denote

$$\varphi_{m+p,\,p}^q := \varphi_{m+p+q-1,\,p+q-1} \circ \cdots \circ \varphi_{m+p,\,p} : \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p}(A, \overline{\Omega}^p(A)) \to \mathrm{Ext}_{A^{\mathrm{e}}}^{m+p+q}(A, \overline{\Omega}^{p+q}(A)).$$

Note that $\varphi_{m+p,\,p}^1 = \varphi_{m+p,\,p}$.

The following is a special case of [**12**, Corollary 6.4.1], which says that the Tate-Hochschild cohomology of a self-injective algebra can be written by using Ext and Tor.

Proposition 3.1.3. *Let A be a self-injective algebra. Denote $A^\vee = \mathrm{Hom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}})$. Then we have the following.*

(1) $\widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{r}(A, A) \cong \mathrm{Ext}_{A^{\mathrm{e}}}^{r}(A, A)$ *for all* $r \geq 1$.

(2) $\widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{-r}(A, A) \cong \mathrm{Tor}_{r-1}^{A^{\mathrm{e}}}(A, A^\vee)$ *for all* $r \geq 2$.

(3) *There exists an exact sequence of k-vector spaces*

$$0 \to \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{-1}(A, A) \to A^\vee \otimes_{A^{\mathrm{e}}} A \xrightarrow{\eta} \mathrm{Hom}_{A^{\mathrm{e}}}(A, A) \to \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{0}(A, A) \to 0,$$

where the morphism η is given by $\eta((\sum_i x_i \otimes y_i) \otimes_{A^{\mathrm{e}}} a)(b) = \sum_i b x_i a y_i$.

(4) $\widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{0}(A, A) = \underline{\mathrm{Hom}}_{A^{\mathrm{e}}}(A, A)$, *which is the set of A-bimodule homomorphisms from A to A modulo those homomorphisms passing through projective A-bimodules.*

In particular, for $r \geq 2$ and $p \geq 1$,

$$\begin{aligned}
&\kappa_{-1,\,p} : \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{-1}(A, A) = \mathrm{Ker}\,(\eta) \xrightarrow{\sim} \mathrm{Ext}_{A^{\mathrm{e}}}^{p}(A, \overline{\Omega}^{p+1}(A)) \cong \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{-1}(A, A),\\
&\varphi_{0,\,0}^p : \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{0}(A, A) = \mathrm{Coker}\,(\eta) \xrightarrow{\sim} \mathrm{Ext}_{A^{\mathrm{e}}}^{p}(A, \overline{\Omega}^{p}(A)) \cong \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{0}(A, A),\\
&\kappa_{r-1,\,p} : \mathrm{Tor}_{r-1}^{A^{\mathrm{e}}}(A, A^\vee) \xrightarrow{\sim} \mathrm{Ext}_{A^{\mathrm{e}}}^{p}(A, \overline{\Omega}^{r+p}(A)) \cong \widehat{\mathrm{Ext}}_{A^{\mathrm{e}}}^{-r}(A, A)
\end{aligned}$$

are defined, on the (co)chain level, as

$$\begin{aligned}
&\kappa_{-1,\,p}(\alpha \otimes_{A^{\mathrm{e}}} a)(\overline{b}_{1,\,p}) = \sum_i d_{p+1}(x_i a \otimes \overline{y}_i \otimes \overline{b}_{1,\,p} \otimes 1),\\
&\varphi_{0,\,0}^p(f)(\overline{b}_{1,\,p}) = d_p(f(1) \otimes \overline{b}_{1,\,p} \otimes 1),\\
&\kappa_{r-1,\,p}(\alpha \otimes_{A^{\mathrm{e}}} \overline{a}_{1,\,r-1})(\overline{b}_{1,\,p}) = \sum_i d_{r+p}(x_i \otimes \overline{a}_{1,\,r-1} \otimes \overline{y}_i \otimes \overline{b}_{1,\,p} \otimes 1),
\end{aligned}$$

where we write $\alpha(1) = \sum_i x_i \otimes y_i$. *We denote* $\varphi_{0,\,0}^1$ *by* $\varphi_{0,\,0}$.

The third isomorphism $\kappa_{r-1,p} : \mathrm{Tor}^{A^e}_{r-1}(A, A^\vee) \xrightarrow{\sim} \widehat{\mathrm{Ext}}^{-r}_{A^e}(A, A)$ in Proposition 3.1.3 is given by Wang [**46**, Remark 6.3].

Assume that A is a Frobenius algebra. Then the A-bimodule isomorphism $A_{\nu^{-1}} \cong A^\vee$ induces an isomorphism of complexes between $\mathcal{D}^\bullet(A, A)$ and the complex $\mathcal{C}^\bullet(A, A)$ defined by Wang [**47**]

$$\cdots \to C_2(A, A^\vee) \xrightarrow{\partial_2} C_1(A, A^\vee) \xrightarrow{\partial_1} A^\vee \xrightarrow{\mu} A \xrightarrow{\delta^0} C^1(A, A) \xrightarrow{\delta^1} C^2(A, A) \to \cdots$$

whose negative part is the Hochschild chain complex $(C_\bullet(A, A^\vee), \partial_\bullet)$ and of which the non-negative part is the Hochschild cochain complex $(C^\bullet(A, A), \delta^\bullet)$. Here, the map $\mu : A^\vee \to A$ is defined by the multiplication of A, that is, $\mu(\alpha) = \sum_i x_i y_i$ for $\alpha \in A^\vee$ with $\alpha(1) = \sum_i x_i \otimes y_i$.

Moreover, Wang [**47**, Section 6.2] defined a product on $\mathcal{C}^*(A, A)$, called *$\star$-product*, which extends the cup product on $C^*(A, A)$ and the cap product between $C^*(A, A)$ and $C_*(A, A^\vee)$. Although the $\star$-product is not associative on $\mathcal{C}^*(A, A)$ in general, the $\star$-product induces a graded commutative and associative product on $\mathrm{H}^*(\mathcal{C}^\bullet(A, A))$. The following is the product

$$\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \to \mathcal{D}^*(A, A)$$

on $\mathcal{D}^*(A, A)$ via the isomorphism $\mathcal{D}^\bullet(A, A) \cong \mathcal{C}^\bullet(A, A)$: let $f \in C^m(A, A), g \in C^n(A, A)$ and $\alpha = a_0 \otimes \overline{a}_{1,p} \in C_p(A, A_{\nu^{-1}})$, $\beta = b_0 \otimes \overline{b}_{1,q} \in C_q(A, A_{\nu^{-1}})$.

(1) $(m, n \geq 0)$ $\star : C^m(A, A) \otimes C^n(A, A) \to C^{m+n}(A, A)$ is given by

$$f \star g := f \smile g \ ;$$

(2) $(m \geq 0, p \geq 0, p \geq m)$

(a) $\star : C_p(A, A_{\nu^{-1}}) \otimes C^m(A, A) \to C_{p-m}(A, A_{\nu^{-1}})$ is given by

$$\alpha \star f := \alpha \frown f = a_0 \nu^{-1}(f(\overline{a}_{1,m})) \otimes \overline{a}_{m+1,p} \ ;$$

(b) $\star : C^m(A, A) \otimes C_p(A, A_{\nu^{-1}}) \to C_{p-m}(A, A_{\nu^{-1}})$ is given by

$$f \star \alpha := f(\overline{a}_{p-m+1,p}) a_0 \otimes \overline{a}_{1,p-m} \ ;$$

(3) $(m \geq 0, p \geq 0, p < m)$

(a) $\star : C^m(A, A) \otimes C_p(A, A_{\nu^{-1}}) \to C^{m-p-1}(A, A)$ is given by

$$(f \star \alpha)(\overline{b}_{1,m-p-1}) := \sum_i f(\overline{b}_{1,m-p-1} \otimes \overline{u_i \nu(a_0)} \otimes \overline{a}_{1,p}) v_i \ ;$$

(b) $\star : C_p(A, A_{\nu^{-1}}) \otimes C^m(A, A) \to C^{m-p-1}(A, A)$ is given by

$$(\alpha \star f)(\overline{b}_{1,\,m-p-1}) := \sum_i u_i \nu(a_0) f(\overline{a}_{1,\,p} \otimes \overline{v}_i \otimes \overline{b}_{1,\,m-p-1})\ ;$$

(4) $(p, q \geq 0)$ $\star : C_p(A, A_{\nu^{-1}}) \otimes C_q(A, A_{\nu^{-1}}) \to C_{p+q+1}(A, A_{\nu^{-1}})$ is given by

$$\alpha \star \beta := \sum_i v_i b_0 \otimes \overline{b}_{1,\,q} \otimes \overline{u_i \nu(a_0)} \otimes \overline{a}_{1,\,p}\ .$$

Dual bases of A are used in our definition of $\star$-product, but Lemma 3.1.1 (2) shows that the $\star$-product does not depend the choice of dual bases of A.

We summarize the above results in the following.

Proposition 3.1.4 ([**47**, Lemma 6.2, Propositions 6.5 and 6.9])**.** *Let A be a Frobenius algebra. Then the $\star$-product is compatible with the differential $\widehat{d}$ of the complex $\mathcal{D}(A, A)$. Moreover, the induced product on $\mathrm{CH}^*(A) = \mathrm{H}^*(\mathcal{D}(A, A))$, still denoted by $\star$, is graded commutative and associative. In particular, $(\mathrm{CH}^\bullet(A), \star) \cong (\widehat{\mathrm{Ext}}^\bullet_{A^e}(A, A), \smile_{\mathrm{sg}})$ as graded algebras.*

3.2. Decomposition of complete cohomology associated with the spectrum of the Nakayama automorphism

In this section, we define and study certain subcomplexes of $\mathcal{D}^\bullet(A, A)$, which play important roles in proving the main result of this chapter. For this purpose, we need to recall the subcomplexes of the (co)chain Hochschild complexes defined in [**35**]. Let A be a (not necessarily Frobenius) algebra, and let σ be an algebra automorphism of A. Let Λ be the set of eigenvalues of σ, and assume that $\Lambda \subset k$. We have $0_A \notin \Lambda$ and $1_A \in \Lambda$ because σ is a ring automorphism. Let $\widehat{\Lambda} := \langle \Lambda \rangle$ be the submonoid of $k^\times$ generated by Λ. We denote by A_λ the eigenspace $\mathrm{Ker}\,(\sigma - \lambda\,\mathrm{id})$ associated with an eigenvalue $\lambda \in \Lambda$. For $\lambda \in \Lambda$, we write $\overline{A}_\lambda = A_\lambda$ for $\lambda \neq 1$ and $\overline{A}_1 = A_1/(k \cdot 1_A)$ for $\lambda = 1$, and for every $\mu \in \widehat{\Lambda}$ and every integer $r \geq 0$, we put

$$C^{(\mu)}_r(A, A_\sigma) := \bigoplus_{\mu_i \in \Lambda,\, \prod \mu_i = \mu} A_{\mu_0} \otimes \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r},$$
$$C^r_{(\mu)}(A, A) := \big\{ f \in C^r(A, A) \big| f(\overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r}) \subset A_{\mu\mu_1 \cdots \mu_r},\ \text{for any } \mu_i \in \Lambda \big\}.$$

Since $\sigma(xy) = \sigma(x)\sigma(y)$ for $x, y \in A$, we have $A_\lambda \cdot A_{\lambda'} \subset A_{\lambda\lambda'}$ for $\lambda, \lambda' \in \Lambda$. It is understood that $A_{\lambda\lambda'} = 0$ when $\lambda\lambda' \notin \Lambda$. Then these subspaces $C^{(\mu)}_*(A, A_\sigma)$ and $C^*_{(\mu)}(A, A)$ are compatible with the differentials ∂_* and δ^* of the complexes $(C_\bullet(A, A_\sigma), \partial_\bullet)$ and $(C^\bullet(A, A), \delta^\bullet)$, respectively. Thus, we obtain subcomplexes $(C^{(\mu)}_\bullet(A, A_\sigma), \partial^{(\mu)}_\bullet)$ and $(C^\bullet_{(\mu)}(A, A), \delta^\bullet_{(\mu)})$. Then we put

$$\begin{aligned}\mathrm{H}_r^{(\mu)}(A, A_\sigma) &:= \mathrm{H}_r(C_\bullet^{(\mu)}(A, A_\sigma), \partial_\bullet^{(\mu)}),\\ \mathrm{H}^r_{(\mu)}(A, A) &:= \mathrm{H}^r(C^\bullet_{(\mu)}(A, A), \delta^\bullet_{(\mu)}).\end{aligned}$$

Hence, for all $r \geq 0$, we get morphisms $\mathrm{H}_r^{(\mu)}(A, A_\sigma) \to \mathrm{H}_r(A, A_\sigma)$ of k-vector spaces and $\mathrm{H}^r_{(\mu)}(A, A) \to \mathrm{HH}^r(A)$. Kowalzing and Krähmer [**33**] defined a k-linear map

$$B_r^\sigma : C_r(A, A_\sigma) \to C_{r+1}(A, A_\sigma)$$

by

$$B_r^\sigma(a_0 \otimes \overline{a}_{1,r}) = \sum_{i=1}^{r+1} (-1)^{ir} 1 \otimes \overline{a}_i \otimes \cdots \otimes \overline{a}_r \otimes \overline{a}_0 \otimes \overline{\sigma(a_1)} \otimes \cdots \otimes \overline{\sigma(a_{i-1})}.$$

Let $T : C_r(A, A_\sigma) \to C_r(A, A_\sigma)$ be the k-linear map defined by

$$T(a_0 \otimes \overline{a}_{1,r}) = \sigma(a_0) \otimes \overline{\sigma(a_1)} \otimes \cdots \otimes \overline{\sigma(a_r)}.$$

A direct calculation shows that $\partial_{r+1} B_r^\sigma - B_{r-1}^\sigma \partial_r = (-1)^{r+1}(\mathrm{id} - T)$ for all $r \geq 0$.

Proposition 3.2.1 ([**35**, Propositions 2.1, 2.2 and 2.5])**.** *The following assertions hold.*

(1) *For every $1 \neq \mu \in \widehat{\Lambda}$ and every $r \geq 0$, we get*

$$\mathrm{H}_r^{(\mu)}(A, A_\sigma) = 0.$$

(2) *For all $r \geq 0$, the restriction of the map $B_r^\sigma : C_r(A, A_\sigma) \to C_{r+1}(A, A_\sigma)$ to the subspaces $C_*^{(1)}(A, A_\sigma)$ induces a twisted Connes operator*

$$B_r^\sigma : \mathrm{H}_r^{(1)}(A, A_\sigma) \to \mathrm{H}_{r+1}^{(1)}(A, A_\sigma),$$

and it satisfies $B_{r+1}^\sigma B_r^\sigma = 0$.

(3) *If σ is diagonalizable, then we have*

$$\mathrm{H}_r(A, A_\sigma) \cong \mathrm{H}_r^{(1)}(A, A_\sigma)$$

for $r \geq 0$.

The following is an easy consequence of Proposition 3.2.1.

Corollary 3.2.2. *If the algebra automorphism σ of A is diagonalizable, then so is its inverse σ^{-1}. Furthermore, if this is the case, then we have two twisted Connes operators*

$$B_*^\sigma : \mathrm{H}_*^{(1)}(A, A_\sigma) \to \mathrm{H}_{*+1}^{(1)}(A, A_\sigma), \quad B_*^{\sigma^{-1}} : \mathrm{H}_*^{(1)}(A, A_{\sigma^{-1}}) \to \mathrm{H}_{*+1}^{(1)}(A, A_{\sigma^{-1}}).$$

From now on, we assume A to be a Frobenius algebra with Nakayama automorphism ν. Let $\Lambda = \{\lambda_1, \ldots, \lambda_t\}$ be the set of distinct eigenvalues of ν. Suppose that $\Lambda \subset k$. Let $\widehat{\Lambda} := \langle \Lambda \rangle$ be the submonoid of $k^\times$ generated by Λ. For any $\mu \in \widehat{\Lambda}$, we define a subspace $\mathcal{D}^*_{(\mu)}(A,A)$ of $\mathcal{D}^*(A,A)$ in the following way: for any $\mu \in \widehat{\Lambda}$,

$$\mathcal{D}^r_{(\mu)}(A,A) := \begin{cases} C^r_{(\mu)}(A,A) & \text{if } r \geq 0, \\ C^{(\mu)}_{-r-1}(A, A_{\nu^{-1}}) & \text{if } r \leq -1. \end{cases}$$

Lemma 3.2.3. *For any $\mu \in \widehat{\Lambda}$, the subspaces $\mathcal{D}^*_{(\mu)}(A,A)$ of $\mathcal{D}^*(A,A)$ are compatible with the differentials $\widehat{d}^*$ of the complex $(\mathcal{D}^\bullet(A,A), \widehat{d}^\bullet)$.*

Proof. It is sufficient to show that $\widehat{d}^{-1}(\mathcal{D}^{-1}_{(\mu)}(A,A)) \subset \mathcal{D}^0_{(\mu)}(A,A)$. If $x \in A_\mu = \mathcal{D}^{-1}_{(\mu)}(A,A)$, then we have

$$\begin{aligned} \nu(\widehat{d}^{-1}(x)) &= \sum_i \nu(u_i)\nu(x)\nu(v_i) = \sum_{i,j} \langle u_i, v_j \rangle u_j \cdot \nu(x)\nu(v_i) \\ &= \sum_j u_j \nu(x) \nu\Big(\sum_i \langle u_i, v_j \rangle v_i\Big) = \sum_j u_j \nu(x) \nu(\nu^{-1}(v_j)) \\ &= \sum_j u_j \nu(x) v_j. \end{aligned}$$

Since $0 = (\nu - \mu\,\mathrm{id})(x) = \nu(x) - \mu x$, we get

$$(\nu - \mu\,\mathrm{id})(\widehat{d}^{-1}(x)) = \nu(\widehat{d}^{-1}(x)) - \mu\,\widehat{d}^{-1}(x) = \sum_j u_j \nu(x) v_j - \mu \sum_j u_j x v_j = 0.$$

Therefore, we have $\widehat{d}^{-1}(x) \in \mathcal{D}^0_{(\mu)}(A,A)$. □

From Lemma 3.2.3, we obtain a subcomplex $(\mathcal{D}^\bullet_{(\mu)}(A,A), \widehat{d}^\bullet_{(\mu)})$ of $(\mathcal{D}^\bullet(A,A), \widehat{d}^\bullet)$. Put

$$\mathrm{CH}^r_{(\mu)}(A) := \mathrm{H}^r(\mathcal{D}^\bullet_{(\mu)}(A,A), \widehat{d}^\bullet_{(\mu)})$$

for all $r \in \mathbb{Z}$. Hence the inclusion $\mathcal{D}^\bullet_{(\mu)}(A,A) \to \mathcal{D}^\bullet(A,A)$ induces a morphism $\mathrm{CH}^r_{(\mu)}(A) \to \mathrm{CH}^r(A)$ of k-vector spaces for $r \in \mathbb{Z}$. Before stating the next proposition, let us recall a well-known duality between Hochschild cohomology and Hochschild homology: there is an isomorphism $\Theta : D(C_*(A, A_\nu)) \to C^*(A,A)$ given by

$$\begin{aligned} D(C_r(A, A_\nu)) &= \mathrm{Hom}(A_\nu \otimes_{A^{\mathrm{e}}} A \otimes \overline{A}^{\otimes r} \otimes A, k) \\ &\cong \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes \overline{A}^{\otimes r} \otimes A, \mathrm{Hom}(A_\nu, k)) \\ &\cong \mathrm{Hom}_{A^{\mathrm{e}}}(A \otimes \overline{A}^{\otimes r} \otimes A, A) = C^r(A,A), \end{aligned}$$

where $r \geq 0$ and the second isomorphism is induced by $A_\nu \cong D(A)$. Then $\Theta : D(C_*(A, A_\nu)) \to C^*(A, A)$ is a morphism of complexes and hence induces a duality $D(\mathrm{H}_r(A, A_\nu)) \cong \mathrm{HH}^r(A)$. In fact, we can write $\Theta : D(C_r(A, A_\nu)) \to C^r(A, A)$ and its inverse $\Theta^{-1} : C^r(A, A) \to D(C_r(A, A_\nu))$ as follows:

$$\Theta : D(C_r(A, A_\nu)) \to C^r(A, A); \quad \psi \mapsto \Big[\overline{b}_{1,r} \mapsto \sum_j \psi(u_j \otimes \overline{b}_{1,r}) v_j\Big],$$
$$\Theta^{-1} : C^r(A, A) \to D(C_r(A, A_\nu)); \quad f \mapsto \big[a_0 \otimes \overline{a}_{1,r} \mapsto \langle f(\overline{a}_{1,r}), a_0\rangle\big].$$

Proposition 3.2.4. *Let A be a Frobenius algebra. If the Nakayama automorphism ν of A is diagonalizable, then the following statements hold.*

(1) *The isomorphism $\Theta : D(C_\bullet(A, A_\nu)) \to C^\bullet(A, A)$ induces an isomorphism of complexes*
$$D(C_\bullet^{(\mu)}(A, A_\nu)) \cong C^\bullet_{(\mu^{-1})}(A, A)$$
for all $\mu \in \widehat{\Lambda}$.

(2) *For $r \in \mathbb{Z}$ and $\mu \neq 1 \in \widehat{\Lambda}$, we get*
$$\mathrm{CH}^r_{(\mu)}(A) = 0.$$

(3) *For each $r \in \mathbb{Z}$, there exists an isomorphism of k-vector spaces*
$$\mathrm{CH}^r_{(1)}(A) \cong \mathrm{CH}^r(A).$$

Proof. It follows from Lemma 3.2.5 below that the inverse of each eigenvalue $\lambda \in \Lambda$ is also an eigenvalue of the Nakayama automorphism ν of A. Since A is the (finite) direct sum of the eigenspaces $A_{\lambda_1}, \ldots, A_{\lambda_t}$, we have $D(C_r(A, A_\nu)) \cong \bigoplus_{\mu \in \widehat{\Lambda}} D(C_r^{(\mu)}(A, A_\nu))$ for all $r \geq 0$. For the first statement, it is sufficient to show that the inverse $\Theta^{-1} : C^\bullet(A, A) \to D(C_\bullet(A, A_\nu))$ induces an isomorphism $C^r_{(\mu)}(A, A) \cong D(C_r^{(\mu^{-1})}(A, A_\nu))$. Since $\Theta^{-1}(f) \in D(C_r(A, A_\nu))$ is a non-zero map for $0 \neq f \in C^r_{(\mu)}(A, A)$, there exist $\mu' \in \widehat{\Lambda}$ and $a_0 \otimes \overline{a}_{1,r} \in C_r^{(\mu')}(A, A_\nu)$ such that $\langle f(\overline{a}_{1,r}), a_0\rangle \neq 0$, so that we get $(\mu\mu' - 1)\langle f(\overline{a}_{1,r}), a_0\rangle = 0$ and hence $\mu' = \mu^{-1}$. As a result, we have shown that if $\lambda \in \widehat{\Lambda}$ with $\lambda \neq \mu^{-1}$, then the restriction of $\Theta^{-1}(f)$ to $C_r^{(\lambda)}(A, A_\nu)$ is the zero map. Thus, we have a monomorphism

$$\Theta^{-1}_{(\mu)} := \Theta^{-1}|_{C^r_{(\mu)}(A,A)} : C^r_{(\mu)}(A, A) \to D(C_r^{(\mu^{-1})}(A, A_\nu)).$$

Furthermore, we get $\Theta_{(\mu)}^{-1}$ is surjective. Indeed, for any $\psi \in D(C_r^{(\mu^{-1})}(A, A_\nu))$, there exists $f \in C^r(A, A)$ such that $\psi = \Theta^{-1}(f)$. Let $\mu_1, \dots, \mu_r \in \Lambda$ and $\overline{b}_{1,r} \in \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_r}$. It follows from $A = \bigoplus_i A_{\lambda_i}$ and $\psi|_{C_r^{(\lambda)}(A,A_\nu)} = 0$ for all $\lambda \neq \mu^{-1}$ that

$$\langle f(\overline{b}_{1,r}), a\rangle = \langle \nu(f(\overline{b}_{1,r})), \nu(a)\rangle = \langle \nu(f(\overline{b}_{1,r})), (\mu_1 \cdots \mu_r)^{-1}\mu^{-1}a\rangle$$

for any $a \in A$. Consequently, we get $\nu(f(\overline{b}_{1,r})) = \mu_1 \cdots \mu_r \mu f(\overline{b}_{1,r})$ and hence $f \in C^r_{(\mu)}(A, A)$. This shows that $\Theta_{(\mu)}^{-1} : C^r_{(\mu)}(A, A) \to D(C_r^{(\mu^{-1})}(A, A_\nu))$ is surjective.

For the second statement, let r be an integer and $\mu \in \widehat{\Lambda}$ such that $\mu \neq 1$. In the case $r \leq -2$, the desired result is a consequence of Proposition 3.2.1 (1). If $r \geq 1$, then the first statement (1) and Proposition 3.2.1 (1) imply that there is an isomorphism

$$\mathrm{CH}^r_{(\mu)}(A) = \mathrm{H}^r_{(\mu)}(A, A) \cong D(\mathrm{H}_r^{(\mu^{-1})}(A, A_\nu)) = 0.$$

We also have $\mathrm{CH}^{-1}_{(\mu)}(A) = 0$ and $\mathrm{CH}^0_{(\mu)}(A) = 0$ because $\mathrm{CH}^{-1}_{(\mu)}(A) \leq \mathrm{H}_0^{(\mu)}(A, A_{\nu^{-1}})$, and $\mathrm{CH}^0_{(\mu)}(A)$ is a quotient space of $\mathrm{H}^0_{(\mu)}(A, A)$.

For the last statement, let r be an integer. For the case $r \leq -2$, the desired result is a consequence of Proposition 3.2.1 (3). If $r \geq 1$, then the first statement (1) and Proposition 3.2.1 (1) yield that there are isomorphisms

$$\mathrm{CH}^r(A) = \mathrm{HH}^r(A) \cong D(\mathrm{H}_r(A, A_\nu)) \cong D(\mathrm{H}_r^{(1)}(A, A_\nu)) \cong \mathrm{H}^r_{(1)}(A, A) = \mathrm{CH}^r_{(1)}(A).$$

Since $A_{\nu^{-1}} = \bigoplus_i A_{\lambda_i}$ as k-vector spaces, the differential $\widehat{d}^{-1}$ can be decomposed as $\widehat{d}^{-1} = [\widehat{d}^{-1}_{\lambda_1} \cdots \widehat{d}^{-1}_{\lambda_t}]$, where $\widehat{d}^{-1}_{\lambda_j} : A_{\lambda_j} \to A$ is the restriction of $\widehat{d}^{-1}$ to A_{λ_j}. Then we have

$$\mathrm{CH}^{-1}(A) \cong \bigoplus_{1 \leq i \leq t} \mathrm{CH}^{-1}_{(\lambda_i)}(A) = \mathrm{CH}^{-1}_{(1)}(A).$$

Similarly, we have $\mathrm{CH}^0(A) \cong \mathrm{CH}^0_{(1)}(A)$. This completes the proof. □

Lemma 3.2.5 ([**35**, Lemma 3.5])**.** *Let A be a Frobenius algebra such that its Nakayama automorphism ν is diagonalizable. Then we have the following statements.*

(1) *For any $\lambda \in \Lambda$, its inverse λ^{-1} belongs to Λ.*

(2) *The isomorphism $A_\nu \cong D(A)$ of A-bimodules induces an isomorphism $A_\lambda \cong D(A_{\lambda^{-1}})$ of vector spaces for any $\lambda \in \Lambda$.*

Suppose that the Nakayama automorphism ν is diagonalizable. For each $\lambda_i \in \Lambda = \{\lambda_1, \dots, \lambda_t\}$, we denote by m_i its algebraic multiplicity. Then we have a k-basis $(u_1^{\lambda_i}, \dots, u_{m_i}^{\lambda_i})$ of the eigenspace A_{λ_i} associated with λ_i. Thus d-tuple $(u_1^{\lambda_1}, \dots, u_{m_1}^{\lambda_1}, \dots, u_1^{\lambda_t}, \dots, u_{m_t}^{\lambda_t})$ forms a k-basis of A, and we obtain its dual basis $(v_1^{\lambda_1}, \dots, v_{m_1}^{\lambda_1}, \dots, v_1^{\lambda_t}, \dots,$

$v^{\lambda_t}_{m_t}$) of A with respect to the bilinear form $\langle\ ,\ \rangle$. It follows from Lemma 3.2.5 and $\langle v^{\lambda_i}_k, u^{\lambda_j}_l\rangle = \delta_{ij}\delta_{kl}$ that the dual basis vectors $v^{\lambda_i}_1, \ldots, v^{\lambda_i}_{m_i}$ belong to $A_{\lambda_i^{-1}}$ for each λ_i. We fix the dual bases $(u^{\lambda_i}_j)_{i,j}, (v^{\lambda_i}_j)_{i,j}$ of A. For simplifying the notation, we will write $(u_1, \ldots, u_d)$ and $(v_1, \ldots, v_d)$ for $(u^{\lambda_i}_j)_{i,j}$ and $(v^{\lambda_i}_j)_{i,j}$ when there is no danger of confusion.

Proposition 3.2.6. *Let A be a Frobenius algebra such that its Nakayama automorphism ν is diagonalizable. For any $\mu, \mu' \in \widehat{\Lambda}$, $\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \to \mathcal{D}^*(A, A)$ induces the restrictions $\star_{\mu,\mu'} : \mathcal{D}^*_{(\mu)}(A, A) \otimes \mathcal{D}^*_{(\mu')}(A, A) \to \mathcal{D}^*_{(\mu\mu')}(A, A)$.*

Proof. We only show that the $\star$-product $\star$ restricts to the subcomplexes in the cases (3) (i). Proofs of the other cases are similar to the proof of the case (3) (i). Let $\mu, \mu' \in \widehat{\Lambda}$ be arbitrary and $m, p \in \mathbb{Z}$ such that $m \geq 0, p \geq 0$ and $p > m$, and let $f \in C^m_{(\mu)}(A, A)$ and $\alpha = a_0 \otimes \overline{a}_{1,p} \in A_{\mu'_0} \otimes \overline{A}_{\mu'_1} \otimes \cdots \otimes \overline{A}_{\mu'_p} \subset C^{(\mu')}_p(A, A_{\nu^{-1}})$ with $\prod \mu'_i = \mu'$. We claim that

$$(f \star \alpha)(\overline{b}_{1,m-p-1}) = \sum_{\substack{1\leq i\leq t,\\ 1\leq j\leq m_i}} f(\overline{b}_{1,m-p-1} \otimes \overline{u^{\lambda_i}_j \nu(a_0)} \otimes \overline{a}_{1,p}) v^{\lambda_i}_j \in A_{\mu\mu'\mu_1\cdots\mu_{m-p-1}}$$

holds for any $\overline{b}_{1,m-p-1} \in \overline{A}_{\mu_1} \otimes \cdots \otimes \overline{A}_{\mu_{m-p-1}}$, where the μ_i are elements of Λ. Indeed, we have

$$\begin{aligned}
&\nu(\sum_{i,j} f(\overline{b}_{1,m-p-1} \otimes \overline{u^{\lambda_i}_j \nu(a_0)} \otimes \overline{a}_{1,p}) v^{\lambda_i}_j) \\
&= \sum_{i,j} \nu(f(\overline{b}_{1,m-p-1} \otimes \overline{u^{\lambda_i}_j \nu(a_0)} \otimes \overline{a}_{1,p})) \nu(v^{\lambda_i}_j) \\
&= \sum_{i,j} \mu\mu_1 \cdots \mu_{m-p-1} \lambda_i \mu' f(\overline{b}_{1,m-p-1} \otimes \overline{u^{\lambda_i}_j \nu(a_0)} \otimes \overline{a}_{1,p}) \lambda_i^{-1} v^{\lambda_i}_j \\
&= \mu\mu'\mu_1 \cdots \mu_{m-p-1} \sum_{i,j} f(\overline{b}_{1,m-p-1} \otimes \overline{u^{\lambda_i}_j \nu(a_0)} \otimes \overline{a}_{1,p}) v^{\lambda_i}_j
\end{aligned}$$

and therefore $f \star \alpha \in C^{m-p-1}_{(\mu\mu')}(A, A)$. □

We denote $\star_1 := \star_{1,1} : \mathcal{D}^\bullet_{(1)}(A, A) \otimes \mathcal{D}^\bullet_{(1)}(A, A) \to \mathcal{D}^\bullet_{(1)}(A, A)$. Then we have the following result.

Corollary 3.2.7. *Let A be a Frobenius algebra. Then $(\mathrm{CH}^\bullet_{(1)}(A), \star_1)$ is a graded commutative algebra. Furthermore, if the Nakayama automorphism ν of A is diagonalizable, then $(\mathrm{CH}^\bullet_{(1)}(A), \star_1)$ is isomorphic to $(\mathrm{CH}^\bullet(A), \star)$ as graded algebras.*

3.3. BV structure on the complete cohomology

In this section, we prove the main result of this chapter. Let us recall the definition of Batalin-Vilkovisky algebras.

Definition 3.3.1. *A graded commutative algebra* $(\mathcal{H}^\bullet = \bigoplus_{r\in\mathbb{Z}} \mathcal{H}^r, \smile)$ *with* $1 \in \mathcal{H}^0$ *is called a Batalin-Vilkovisky algebra (BV algebra, for short) if there exists an operator* $\Delta_* : \mathcal{H}^* \to \mathcal{H}^{*-1}$ *such that:*

(i) $\Delta_{r-1}\Delta_r = 0$ *for any* $r \in \mathbb{Z}$*;*

(ii) $\Delta_0(1) = 0$*;*

(iii) For homogeneous elements α, β *and* γ *in* $\mathcal{H}^\bullet$*,*

$$\begin{aligned}\Delta(\alpha \smile \beta \smile \gamma) = {} & \Delta(\alpha \smile \beta) \smile \gamma + (-1)^{|\alpha|}\alpha \smile \Delta(\beta \smile \gamma) \\ & + (-1)^{|\beta|(|\alpha|-1)}\beta \smile \Delta(\alpha \smile \gamma) - \Delta(\alpha) \smile \beta \smile \gamma \\ & - (-1)^{|\alpha|}\alpha \smile \Delta(\beta) \smile \gamma - (-1)^{|\alpha|+|\beta|}\alpha \smile \beta \smile \Delta(\gamma),\end{aligned}$$

where $|\alpha|$ *denotes the degree of a homogeneous element* $\alpha \in \mathcal{H}^\bullet$*.*

Remark 3.3.2. For each BV algebra $(\mathcal{H}^\bullet, \smile, \Delta)$, we can associate a graded Lie bracket $[\ ,\ ]$ of degree -1 as

$$[\alpha, \beta] := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|}\left((-1)^{|\alpha|+1}\Delta(\alpha \smile \beta) + (-1)^{|\alpha|}\Delta(\alpha) \smile \beta + \alpha \smile \Delta(\beta)\right),$$

where α, β are homogeneous elements of $\mathcal{H}^\bullet$. The equation is said to be the BV identity. It follows from [**26**, Proposition 1.2] that the bracket $[\ ,\ ]$ above makes $(\mathcal{H}^\bullet, \smile, [\ ,\ ])$ into a Gerstenhaber algebra.

Recall that a *symmetric algebra* A is a Frobenius algebra with a non-degenerate bilinear form $\langle\ ,\ \rangle : A \otimes A \to k$ satisfying $\langle a, b\rangle = \langle b, a\rangle$ for all $a, b \in A$. Wang [**46**] has proved the following result.

Theorem 3.3.3 ([**46**, Corollary 6.21])**.** *Let* A *be a symmetric algebra. Then the complete cohomology ring* $(\mathrm{CH}^\bullet(A), \star)$ *is a BV algebra together with an operator* $\widehat{\Delta}_* : \mathrm{CH}^*(A) \to \mathrm{CH}^{*-1}(A)$ *defined by*

$$\widehat{\Delta}_r = \begin{cases} \Delta_r & \text{if } r \geq 1, \\ 0 & \text{if } r = 0, \\ (-1)^r\ B_{-r-1} & \text{if } r \leq -1, \end{cases}$$

where B_r *is the Connes operator defined by*

$$B_r(a_0 \otimes \overline{a}_{1,r}) = \sum_{i=1}^{r+1} (-1)^{ir} 1 \otimes \overline{a}_{i,r} \otimes \overline{a}_0 \otimes \overline{a}_{1,i-1}$$

for any $a_0 \otimes \overline{a}_{1,r} \in C_r(A, A_{\nu^{-1}})$*, and* Δ_r *defined in [**43**] is the dual of the Connes operator* B_{r-1}*, which is equivalent to saying that* Δ_r *is given by a formula*

$$\langle \Delta_r(f)(\overline{a}_{1,\,r-1}), a_r\rangle = \sum_{i=1}^{r}(-1)^{i(r-1)}\langle f(\overline{a}_{i,\,r-1}\otimes \overline{a}_r\otimes \overline{a}_{1,\,i-1}), 1\rangle$$

for any $f \in C^r(A,A)$. In particular, the restrictions $\mathrm{CH}^{\geq 0}(A)$ and $\mathrm{CH}^{\leq 0}(A)$ are BV subalgebras of $\mathrm{CH}^{\bullet}(A)$.

Remark 3.3.4. Let A be a symmetric algebra. It follows from Remark 3.3.2 that the BV differential $\widehat{\Delta}$ in Theorem 3.3.3 gives rise to a Lie bracket $\{\ ,\ \}$ (of degree -1) defined by

$$\{\alpha,\beta\} := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|}\Big((-1)^{|\alpha|+1}\widehat{\Delta}(\alpha\smile\beta)+(-1)^{|\alpha|}\widehat{\Delta}(\alpha)\smile\beta+\alpha\smile\widehat{\Delta}(\beta)\Big)$$

for any homogeneous elements $\alpha,\beta\in \mathrm{CH}^{\bullet}(A)$. Moreover, the Gerstenhaber algebra $(\mathrm{CH}^{\bullet}(A),\star,\{\ ,\ \})$ is isomorphic to $(\widehat{\mathrm{Ext}}^{\bullet}_{A^e}(A,A),\smile_{\mathrm{sg}},[\ ,\ ]_{\mathrm{sg}})$ as Gerstenhaber algebras.

In the rest of this section, we show the following result on Frobenius algebras whose Nakayama automorphisms are diagonalizable.

Theorem 3.3.5. *Let A be a Frobenius algebra. If its Nakayama automorphism ν is diagonalizable, then the graded commutative ring $(\mathrm{CH}^{\bullet}_{(1)}(A),\star_1)$ is a BV algebra together with an operator $\widehat{\Delta}_*:\mathrm{CH}^{*}_{(1)}(A)\to \mathrm{CH}^{*-1}_{(1)}(A)$ defined by*

$$\widehat{\Delta}_r=\begin{cases}\Delta_r^{\nu} & \text{if } r\geq 1,\\ 0 & \text{if } r=0,\\ (-1)^i\ B^{\nu^{-1}}_{-r-1} & \text{if } r\leq -1,\end{cases}$$

where $B^{\nu^{-1}}_r$ is the twisted Connes operator defined by

$$B^{\nu^{-1}}_r(a_0\otimes \overline{a}_{1,\,r}) = \sum_{i=1}^{r+1}(-1)^{ir}1\otimes \overline{a}_{i,\,r}\otimes \overline{a}_0\otimes \overline{\nu^{-1}(a_1)}\otimes\cdots\otimes \overline{\nu^{-1}(a_{i-1})}$$

*for any $a_0\otimes\overline{a}_{1,\,r}\in C_r(A,A_{\nu^{-1}})$, and Δ^{ν}_r defined in [**35**] is the dual of the twisted Connes operator B^{ν}_{r-1}, which is equivalent to saying that Δ^{ν}_r is given by a formula*

$$\langle \Delta^{\nu}_r(f)(\overline{a}_{1,\,r-1}), a_r\rangle = \sum_{i=1}^{r}(-1)^{i(r-1)}\langle f(\overline{a}_{i,\,r-1}\otimes \overline{a}_r\otimes \overline{\nu(a_1)}\otimes\cdots\otimes\overline{\nu(a_{i-1})}), 1\rangle$$

for any $f\in C^r(A,A)$. In particular, the restrictions $\mathrm{CH}^{\geq 0}_{(1)}(A)$ and $\mathrm{CH}^{\leq 0}_{(1)}(A)$ are BV subalgebras of $\mathrm{CH}^{\bullet}_{(1)}(A)$. Furthermore, the induced Gerstenhaber algebra $(\mathrm{CH}^{\bullet}_{(1)}(A),\star_1,\{\ ,\ \})$ is isomorphic to the Gerstenhaber algebra $(\widehat{\mathrm{Ext}}^{\bullet}_{A^e}(A,A),\smile_{\mathrm{sg}},[\ ,\ ]_{\mathrm{sg}})$.

Remark 3.3.6. Each of the components Δ_*^{ν} and $B_*^{\nu^{-1}}$ is defined on the chain level. Corollary 3.2.2 and Lemma 3.2.5 imply that we can lift the two components Δ_*^{ν} and $B_*^{\nu^{-1}}$ to the cohomology level when we restrict them to $\mathcal{D}^*_{(1)}(A, A)$.

Using the isomorphism $\mathrm{CH}^\bullet_{(1)}(A) \cong \mathrm{CH}^\bullet(A)$ appeared in Corollary 3.2.7, we have our main result.

Theorem 3.3.7. *Let A be a Frobenius algebra. If the Nakayama automorphism of A is diagonalizable, then the complete cohomology ring $\mathrm{CH}^\bullet(A)$ is a BV algebra such that the induced Gerstenhaber algebra is isomorphic to the Gerstenhaber algebra $\widehat{\mathrm{Ext}}^\bullet_{A^e}(A, A)$.*

In order to prove Theorem 3.3.5, we claim that the bilinear map

$$\{\ ,\ \} : \mathrm{CH}^m_{(1)}(A) \otimes \mathrm{CH}^n_{(1)}(A) \to \mathrm{CH}^{m+n-1}_{(1)}(A)\ (m, n \in \mathbb{Z})$$

defined by

$$\{\alpha, \beta\} := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|} \Big((-1)^{|\alpha|+1} \widehat{\Delta}(\alpha \star_1 \beta) + (-1)^{|\alpha|} \widehat{\Delta}(\alpha) \star_1 \beta + \alpha \star_1 \widehat{\Delta}(\beta) \Big)$$

for any $\alpha \otimes \beta \in \mathrm{CH}^m_{(1)}(A) \otimes \mathrm{CH}^n_{(1)}(A)$ commutes with the Gerstenhaber bracket

$$[\ ,\ ]_{\mathrm{sg}} : \widehat{\mathrm{Ext}}^m_{A^e}(A, A) \otimes \widehat{\mathrm{Ext}}^n_{A^e}(A, A) \to \widehat{\mathrm{Ext}}^{m+n-1}_{A^e}(A, A).$$

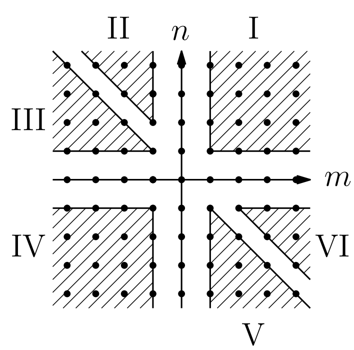


Figure 3.1: A plane with six regions

By considering whether $m+n-1$ is negative or not together with Figure 3.1, and by using the anti-commutativity of the Gerstenhaber bracket $[\ ,\ ]_{\mathrm{sg}}$, we see that it suffices to show our claim for a pair (m, n) of integers $n \leq m$ satisfying one of the following conditions:

(1) (m, n) is on the lines $m = 0$ or $n = 0$.

(2) (m, n) belongs to the regions I, IV, V or VI.

Thus our claim can be divided into the five cases Propositions 3.3.8, 3.3.9, 3.3.10 and 3.3.11 and Remark 3.3.12. In particular, Propositions 3.3.8, 3.3.9, 3.3.10 and 3.3.11 prove our claim for the pairs in the regions VI, V, IV and I, respectively. Further, we consider the case (1) in Remark 3.3.12. Among the four propositions, we prove only the first one. We also remark that, in the following propositions, the appearing integers m and n are independent of the above argument.

Proposition 3.3.8. *Let A be a Frobenius algebra with the Nakayama automorphism ν of A diagonalizable, and let m, n be integers such that $m > n \geq 1$, so $m-n-1 \geq 0$. Then we have the following commutative diagram.*

$$\begin{array}{ccc}
\mathrm{CH}^{m}_{(1)}(A) \otimes \mathrm{CH}^{-n}_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{m-n-1}_{(1)}(A) \\
\downarrow \cong & & \cong \downarrow \\
\mathrm{Ext}^{m}_{A^{\mathrm{e}}}(A, A) \otimes \mathrm{Tor}^{A^{\mathrm{e}}}_{n-1}(A, A_{\nu^{-1}}) & & \mathrm{Ext}^{m-n-1}_{A^{\mathrm{e}}}(A, A) \\
\mathrm{id}\otimes\kappa_{n-1,1} \downarrow \cong & & \cong \downarrow \varphi^{n+1}_{m-n-1,0} \\
\mathrm{Ext}^{m}_{A^{\mathrm{e}}}(A, A) \otimes \mathrm{Ext}^{1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{n+1}(A)) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \mathrm{Ext}^{m}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{n+1}(A)) \\
\downarrow \cong & & \cong \downarrow \\
\widehat{\mathrm{Ext}}^{m}_{A^{\mathrm{e}}}(A, A) \otimes \widehat{\mathrm{Ext}}^{-n}_{A^{\mathrm{e}}}(A, A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\mathrm{Ext}}^{m-n-1}_{A^{\mathrm{e}}}(A, A),
\end{array}$$

where $\{\ ,\ \} : \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A) \to \mathcal{D}^{m-n-1}_{(1)}(A, A)$ is defined by

$$\begin{aligned}
&\{f, z\} \\
&= (-1)^{|f||z|+|f|+|z|} \Big((-1)^{|f|+1} \Delta^{\nu}(f \star_1 z) + (-1)^{|f|} \Delta^{\nu}(f) \star_1 z + (-1)^{|z|} f \star_1 B^{\nu^{-1}}(z) \Big) \\
&= (-1)^{|f||z|+|f|+|z|} \Big((-1)^{|f|+1} \widehat{\Delta}(f \star_1 z) + (-1)^{|f|} \widehat{\Delta}(f) \star_1 z + f \star_1 \widehat{\Delta}(z) \Big)
\end{aligned}$$

for $f \otimes z \in \mathcal{D}^{m}_{(1)}(A, A) \otimes \mathcal{D}^{-n}_{(1)}(A, A)$.

Proof. It follows from the definition of the induced Gerstenhaber bracket on $\widehat{\mathrm{Ext}}^{\bullet}_{A^{\mathrm{e}}}(A, A)$ that the bottom square is commutative (see Theorem 2.2.3). It remains to show the commutativity of the top diagram. It suffices to prove that a formula

$$([\ ,\ ]_{\mathrm{sg}}(\mathrm{id} \otimes \kappa_{n-1,1}))(f \otimes z) = \varphi^{n+1}_{m-n-1,0}(\{\ ,\ \}(f \otimes z)) \tag{3.1}$$

holds in $\mathrm{Ext}^{m}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{n+1}(A))$ for $f \in \operatorname{Ker} \delta^{m}_{(1)}$ and $z := a_0 \otimes \overline{a}_{1,n-1} \in \operatorname{Ker} \partial^{(1)}_{n-1}$. Denote by $\overline{f}$ the composition of $f : \overline{A}^{\otimes m} \to A$ with the canonical epimorphism $\pi : A \to \overline{A}$. For the right hand side of the formula (3.1), we have, for $\bar{b}_{1,m} \in \overline{A}^{\otimes m}$,

$$
\begin{aligned}
&\varphi^{n+1}_{m-n-1,0}(\{f,z\})(\bar{b}_{1,m}) \\
&= (-1)^{mn+n+1}\varphi^{n+1}_{m-n-1,0}(\Delta^{\nu}(f \frown z))(\bar{b}_{1,m}) + (-1)^{mn+n}\varphi^{n+1}_{m-n-1,0}(\Delta^{\nu}(f) \frown z)(\bar{b}_{1,m}) \\
&\quad +(-1)^{mn+m}\varphi^{n+1}_{m-n-1,0}(f \frown \mathrm{B}^{\nu^{-1}}(z))(\bar{b}_{1,m}) \\
&= \sum_{j,k}\sum_{i=1}^{n-m}(-1)^{i(n-m+1)+n+1}d(\langle u_k\nu a_0 f(\bar{a}_{1,n-1}\otimes \bar{v}_k \otimes \bar{b}_{i,m-n-1}\otimes \bar{u}_j \otimes \overline{\nu b}_{1,i-1}),1\rangle v_j \\
&\quad \otimes \bar{b}_{m-n,m}\otimes 1) \\
&\quad +\sum_{j,k}\sum_{i=1}^{n}(-1)^{i(m+1)}d(u_j\nu a_0\langle f(\bar{a}_{i,n-1}\otimes \bar{v}_j \otimes \bar{b}_{1,m-n-1}\otimes \bar{u}_k \otimes \overline{\nu^{-1}a}_{1,i-1}),1\rangle v_k \\
&\quad \otimes \bar{b}_{m-n,m}\otimes 1) \\
&\quad +\sum_{j,k}\sum_{i=1}^{m-n}(-1)^{(i+n)(m+1)}d(u_j\nu a_0\langle f(\bar{b}_{i,m-n-1}\otimes \bar{u}_k \otimes \overline{\nu^{-1}a}_{1,n-1}\otimes \overline{\nu v}_j \\
&\quad \otimes \overline{\nu b}_{1,i-1}),1\rangle v_k \otimes \bar{b}_{m-n,m}\otimes 1) \\
&\quad +\sum_{j}\sum_{i=1}^{n}(-1)^{(m+i)(n+1)}d(f(\bar{b}_{1,m-n-1}\otimes \bar{u}_j \otimes \bar{a}_{i,n-1}\otimes \bar{a}_0 \otimes \overline{\nu^{-1}a}_{1,i-1})v_j \\
&\quad \otimes \bar{b}_{m-n,m}\otimes 1)
\end{aligned}
$$

in $\overline{\Omega}^{n+1}(A)$. On the other hand, for the left hand side of the formula (3.1), we get

$$
\begin{aligned}
&[f,\kappa_{n-1,1}(z)]_{\mathrm{sg}}(\bar{b}_{1,m}) \\
&= (f \bullet \kappa_{n-1,1}(z) - (-1)^{(m-1)(-n-1)}\kappa_{n-1,1}(z)\bullet f)(\bar{b}_{1,m}) \\
&= \sum_{j}\sum_{i=1}^{m-n}(-1)^{(n+1)(i+1)}d(f(\bar{b}_{1,i-1}\otimes \overline{u_j\nu a_0}\otimes \bar{a}_{1,n-1}\otimes \bar{v}_j \otimes \bar{b}_{i,m-n-1}) \\
&\quad \otimes \bar{b}_{m-n,m}\otimes 1) \qquad (3.2) \\
&\quad +\sum_{j}\sum_{i=m-n+1}^{m}(-1)^{(n+1)(i+1)}d(f(\bar{b}_{1,i-1}\otimes \overline{u_j\nu a_0}\otimes \bar{a}_{1,m-i})\otimes \bar{a}_{m-i+1,n-1}\otimes \bar{v}_j \\
&\quad \otimes \bar{b}_{i,m}\otimes 1) \qquad (3.3) \\
&\quad +\sum_{j}\sum_{i=1}^{n}(-1)^{i(m+1)}d(u_j\nu a_0 \otimes \bar{a}_{1,i-1}\otimes \overline{f}(\bar{a}_{i,n-1}\otimes \bar{v}_j \otimes \bar{b}_{1,m-n+i-1}) \\
&\quad \otimes \bar{b}_{m-n+i,m}\otimes 1). \qquad (3.4)
\end{aligned}
$$

We will transform $[f,\kappa_{n-1,1}(z)](\bar{b}_{1,m})$ to $\varphi^{n+1}_{m-n-1,0}(\{f,z\})(\bar{b}_{1,m})$ in $\overline{\Omega}^{n+1}(A)$, using some boundaries.

First, we deform the first term (3.2). A direct calculation shows

$$\sum_{j,k}\sum_{i=1}^{n-m}(-1)^{i(n-m+1)+n+1}d(\langle u_k\nu a_0 f(\overline{a}_{1,n-1}\otimes\overline{v}_k\otimes\overline{b}_{i,m-n-1}\otimes\overline{u}_j\otimes\overline{\nu b}_{1,i-1}),1\rangle v_j \otimes\overline{b}_{m-n,m}\otimes 1)$$
$$+\sum_{j,k}\sum_{i=1}^{m-n}(-1)^{(i+n)(m+1)}d(u_j\nu a_0\langle f(\overline{b}_{i,m-n-1}\otimes\overline{u}_k\otimes\overline{\nu^{-1}a}_{1,n-1}\otimes\overline{\nu v}_j\otimes\overline{\nu b}_{1,i-1}),1\rangle v_k \otimes\overline{b}_{m-n,m}\otimes 1)$$
$$=\sum_{j}\sum_{i=1}^{m-n-2}\sum_{l=i+2}^{m-n}(-1)^{i(m+1)+(n+1)l+1}\varphi^{n-1}_{m-n-1,0}\Big(\delta((\sum_{j,k}\langle f(\mathrm{id}_{\overline{A}}^{\otimes l-i-1}\otimes\overline{u_j\nu a_0} \otimes\overline{a}_{1,n-1}\otimes\overline{v}_j\otimes\mathrm{id}_{\overline{A}}^{\otimes m-n-l}\otimes\overline{u}_k\otimes\overline{\nu}^{\otimes i-1}),1\rangle v_k)\circ t^{i-1})\Big)(\overline{b}_{1,m})$$
$$+\sum_{j,k}\sum_{i=1}^{m-n-1}(-1)^{i(m+1)+(n+1)(i+1)+1}\varphi^{n-1}_{m-n-1,0}\Big(\delta((\sum_{j,k}\langle f(\overline{u_j\nu a_0}\otimes\overline{a}_{1,n-1}\otimes\overline{v}_j \otimes\mathrm{id}_{\overline{A}}^{\otimes m-n-i-1}\otimes\overline{u}_k\otimes\overline{\nu}^{\otimes i-1}),1\rangle v_k)\circ t^{i-1})\Big)(\overline{b}_{1,m})$$
$$+\sum_{j}\sum_{i=1}^{m-n}(-1)^{(n+1)(i+1)}d(f(\overline{b}_{1,i-1}\otimes\overline{u_j\nu a_0}\otimes\overline{a}_{1,n-1}\otimes\overline{v}_j\otimes\overline{b}_{i,m-n-1})\otimes\overline{b}_{m-n,m} \otimes 1),$$

where the k-linear map $t:\overline{A}^{\otimes m-n-2}\to\overline{A}^{\otimes m-n-2}$ is given by $t(\overline{b}_{1,m-n-2})=\overline{b}_{2,m-n-2}\otimes\overline{b}_1$ for $\overline{b}_{1,m-n-2}\in\overline{A}^{\otimes m-n-2}$. In particular, we have $t^{i-1}(\overline{b}_{1,m-n-2})=\overline{b}_{i,m-n-2}\otimes\overline{b}_{1,i-1}$. Note that the two maps

$$\varphi^{n-1}_{m-n-1,0}\left(\delta((\sum_{j,k}\langle f(\overline{u_j\nu a_0}\otimes\overline{a}_{1,n-1}\otimes\overline{v}_j\otimes\mathrm{id}_{\overline{A}}^{\otimes m-n-i-1}\otimes\overline{u}_k\otimes\overline{\nu}^{\otimes i-1}),1\rangle v_k)\circ t^{i-1})\right),$$

$$\varphi^{n-1}_{m-n-1,0}\Big(\delta((\sum_{j,k}\langle f(\mathrm{id}_{\overline{A}}^{\otimes l-i-1}\otimes\overline{u_j\nu a_0}\otimes\overline{a}_{1,n-1}\otimes\overline{v}_j\otimes\mathrm{id}_{\overline{A}}^{\otimes m-n-l}\otimes\overline{u}_k\otimes\overline{\nu}^{\otimes i-1}),1\rangle v_k) \circ t^{i-1})\Big)$$

are zero in $\mathrm{Ext}^m_{A^e}(A, \overline{\Omega}^{n+1}(A))$. Hence, we have

$$
\begin{aligned}
&\sum_j \sum_{i=1}^{m-n} (-1)^{(n+1)(i+1)} d(f(\overline{b}_{1,\,i-1} \otimes \overline{u_j \nu a_0} \otimes \overline{a}_{1,\,n-1} \otimes \overline{v}_j \otimes \overline{b}_{i,\,m-n-1}) \otimes \overline{b}_{m-n,\,m} \otimes 1) \\
&= \sum_j \sum_{i=1}^{m-n-2} \sum_{l=i+2}^{m-n} (-1)^{i(m+1)+(n+1)l} \varphi^{n-1}_{m-n-1,\,0}\Big(\delta((\sum_{j,\,k} \langle f(\mathrm{id}_{\overline{A}}^{\otimes l-i-1} \otimes \overline{u_j \nu a_0} \otimes \overline{a}_{1,\,n-1} \\
&\quad \otimes \overline{v}_j \otimes \mathrm{id}_{\overline{A}}^{\otimes m-n-l} \otimes \overline{u}_k \otimes \overline{\nu}^{\otimes i-1}), 1\rangle v_k) \circ t^{i-1})\Big)(\overline{b}_{1,\,m}) \\
&\quad + \sum_{j,\,k} \sum_{i=1}^{m-n-1} (-1)^{i(m+1)+(n+1)(i+1)} \varphi^{n-1}_{m-n-1,\,0}\Big(\delta((\sum_{j,\,k} \langle f(\overline{u_j \nu a_0} \otimes \overline{a}_{1,\,n-1} \otimes \overline{v}_j \\
&\quad \otimes \mathrm{id}_{\overline{A}}^{\otimes m-n-i-1} \otimes \overline{u}_k \otimes \overline{\nu}^{\otimes i-1}), 1\rangle v_k) \circ t^{i-1})\Big)(\overline{b}_{1,\,m}) \\
&\quad + \sum_{j,\,k} \sum_{i=1}^{n-m} (-1)^{i(n-m+1)+n+1} d(\langle u_k \nu a_0 f(\overline{a}_{1,\,n-1} \otimes \overline{v}_k \otimes \overline{b}_{i,\,m-n-1} \otimes \overline{u}_j \\
&\quad \otimes \overline{\nu b}_{1,\,i-1}), 1\rangle v_j \otimes \overline{b}_{m-n,m} \otimes 1) \\
&\quad + \sum_{j,\,k} \sum_{i=1}^{m-n} (-1)^{(i+n)(m+1)} d(u_j \nu a_0 \langle f(\overline{b}_{i,\,m-n-1} \otimes \overline{u}_k \otimes \overline{\nu^{-1} a}_{1,\,n-1} \otimes \overline{\nu v}_j \\
&\quad \otimes \overline{\nu b}_{1,\,i-1}), 1\rangle v_k \otimes \overline{b}_{m-n,\,m} \otimes 1).
\end{aligned}
\tag{3.5}
$$

Secondly, we deform the second term (3.3). A direct calculation shows

$$
\begin{aligned}
&\sum_j \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(f(\mathrm{id}_{\overline{A}}^{\otimes m-n+i-1} \otimes \overline{u}_j \otimes \overline{a}_{n+i-l+1,\,n-1} \otimes \overline{a}_0 \\
&\otimes \overline{\nu^{-1} a}_{1,\,n-l}) \otimes \overline{\nu^{-1} a}_{n-l+1,\,n+i-l} \otimes \overline{v}_j \otimes \mathrm{id}_{\overline{A}}^{\otimes n-i} \otimes 1))(\overline{b}_{1,\,m}) \\
&= \sum_j \sum_{i=m-n+1}^{m} (-1)^{(n+1)(i+1)} d(f(\overline{b}_{1,i-1} \otimes \overline{u_j \nu a}_0 \otimes \overline{a}_{1,\,m-i}) \otimes \overline{a}_{m-i+1,\,n-1} \otimes \overline{v}_j \\
&\quad \otimes \overline{b}_{i,\,m} \otimes 1) \\
&\quad + \sum_j \sum_{i=1}^{n} (-1)^{(m+i)(n+1)+1} d(f(\overline{b}_{1,\,m-n-1} \otimes \overline{u}_j \otimes \overline{a}_{i,\,n-1} \otimes \overline{a}_0 \otimes \overline{\nu^{-1} a}_{1,\,i-1}) v_j \\
&\quad \otimes \overline{b}_{m-n,\,m} \otimes 1) \\
&\quad + \sum_j \sum_{i=0}^{n-1} (-1)^{m(n+1)+i+1} d(f(\overline{b}_{1,\,m-n+i} \otimes \overline{u}_k \otimes \overline{\nu^{-1} a}_{i+1,\,n-1}) \nu a_0 \otimes \overline{a}_{1,\,i} \otimes \overline{v}_k \\
&\quad \otimes \overline{b}_{m-n+i+1,\,m} \otimes 1).
\end{aligned}
$$

Hence, we have

$$\sum_j \sum_{i=m-n+1}^{m} (-1)^{(n+1)(i+1)} d(f(\overline{b}_{1,i-1} \otimes \overline{u_j \nu a}_0 \otimes \overline{a}_{1,\,m-i}) \otimes \overline{a}_{m-i+1,\,n-1} \otimes \overline{v}_j \otimes \overline{b}_{i,\,m} \otimes 1)$$

$$\begin{aligned}
= \quad & \sum_j \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(f(\mathrm{id}_{\overline{A}}^{\otimes m-n+i-1} \otimes \overline{u}_j \otimes \overline{a}_{n+i-l+1,\,n-1} \\
& \otimes \overline{a}_0 \otimes \overline{\nu^{-1} a}_{1,\,n-l}) \otimes \overline{\nu^{-1} a}_{n-l+1,\,n+i-l} \otimes \overline{v}_j \otimes \mathrm{id}_{\overline{A}}^{\otimes n-i} \otimes 1))(\overline{b}_{1,\,m}) \\
& + \sum_j \sum_{i=1}^{n} (-1)^{(m+i)(n+1)} d(f(\overline{b}_{1,\,m-n-1} \otimes \overline{u}_j \otimes \overline{a}_{i,\,n-1} \otimes \overline{a}_0 \otimes \overline{\nu^{-1} a}_{1,\,i-1}) v_j \\
& \otimes \overline{b}_{m-n,\,m} \otimes 1) \\
& + \sum_j \sum_{i=0}^{n-1} (-1)^{m(n+1)+i} d(f(\overline{b}_{1,\,m-n+i} \otimes \overline{u}_k \otimes \overline{\nu^{-1} a}_{i+1,\,n-1}) \nu a_0 \otimes \overline{a}_{1,\,i} \otimes \overline{v}_k \\
& \otimes \overline{b}_{m-n+i+1,\,m} \otimes 1). \qquad (3.6)
\end{aligned}$$

Finally, we deform the last term (3.4). A direct calculation shows

$$\begin{aligned}
& \sum_j \sum_{i=0}^{n-1} \sum_{l=i+1}^{n} (-1)^{n(m+i+1)+(n+1)(l-i+1)} \delta(d(u_j \nu a_0 \otimes \overline{\nu^{-1} a}_{1,\,i} \otimes \langle f(\overline{\nu^{-1} a}_{n+i-l+1,\,n-1} \otimes \overline{v}_j \\
& \otimes \mathrm{id}_{\overline{A}}^{\otimes m-n+i-1} \otimes \overline{u}_k \otimes \overline{a}_{i+1,\,n-l+i}), 1\rangle \overline{v}_k \otimes \mathrm{id}_{\overline{A}}^{\otimes n-i} \otimes 1))(\overline{b}_{1,\,m}) \\
= \quad & \sum_j \sum_{i=1}^{n} (-1)^{i(m+1)} d(u_j \nu a_0 \otimes \overline{a}_{1,\,i-1} \otimes \overline{f}(\overline{a}_{i,\,n-1} \otimes \overline{v}_j \otimes \overline{b}_{1,\,m-n+i-1}) \otimes \overline{b}_{m-n+i,\,m} \\
& \otimes 1) \\
& + \sum_{j,\,k} \sum_{i=1}^{n} (-1)^{i(m+1)+1} d(u_j \nu a_0 \langle f(\overline{a}_{i,\,n-1} \otimes \overline{v}_j \otimes \overline{b}_{1,\,m-n-1} \otimes \overline{u}_k \otimes \overline{\nu^{-1} a}_{1,\,i-1}), 1\rangle v_k \\
& \otimes \overline{b}_{m-n,\,m} \otimes 1) \\
& + \sum_j \sum_{i=0}^{n-1} (-1)^{m(n+1)+i} d(f(\overline{b}_{1,\,m-n+i} \otimes \overline{u}_k \otimes \overline{\nu^{-1} a}_{i+1,\,n-1}) \nu a_0 \otimes \overline{a}_{1,\,i} \otimes \overline{v}_k \\
& \otimes \overline{b}_{m-n+i+1,\,m} \otimes 1).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\sum_{j}\sum_{i=1}^{n}(-1)^{i(m+1)}d(u_j\nu a_0\otimes \overline{a}_{1,\,i-1}\otimes \overline{f}(\overline{a}_{i,\,n-1}\otimes \overline{v}_j\otimes \overline{b}_{1,\,m-n+i-1})\otimes \overline{b}_{m-n+i,\,m}\otimes 1)\\
&=\ \sum_{j}\sum_{i=0}^{n-1}\sum_{l=i+1}^{n}(-1)^{n(m+i+1)+(n+1)(l-i+1)}\delta(d(u_j\nu a_0\otimes \overline{\nu^{-1}a}_{1,\,i}\otimes \langle f(\overline{\nu^{-1}a}_{n+i-l+1,\,n-1}\\
&\quad\otimes \overline{v}_j\otimes \mathrm{id}_{\overline{A}}^{\otimes m-n+i-1}\otimes \overline{u}_k\otimes \overline{a}_{i+1,\,n-l+i}),1\rangle \overline{v}_k\otimes \mathrm{id}_{\overline{A}}^{\otimes n-i}\otimes 1))(\overline{b}_{1,\,m})\\
&\quad+\sum_{j,\,k}\sum_{i=1}^{n}(-1)^{i(m+1)}d(u_j\nu a_0\langle f(\overline{a}_{i,\,n-1}\otimes \overline{v}_j\otimes \overline{b}_{1,\,m-n-1}\otimes \overline{u}_k\otimes \overline{\nu^{-1}a}_{1,\,i-1}),1\rangle v_k\\
&\quad\otimes \overline{b}_{m-n,\,m}\otimes 1)\\
&\quad+\sum_{j}\sum_{i=0}^{n-1}(-1)^{m(n+1)+i+1}d(f(\overline{b}_{1,\,m-n+i}\otimes \overline{u}_k\otimes \overline{\nu^{-1}a}_{i+1,\,n-1})\nu a_0\otimes \overline{a}_{1,\,i}\otimes \overline{v}_k\\
&\quad\otimes \overline{b}_{m-n+i+1,\,m}\otimes 1). \qquad (3.7)
\end{aligned}$$

Combining the formulas (3.6), (3.7) and (3.5), we obtain

$$[f,\kappa_{n-1,\,1}(z)]_{\mathrm{sg}}(\overline{b}_{1,\,m})+\delta(*)(\overline{b}_{1,\,m})+\varphi^{n-1}_{m-n-1,\,0}(\delta(*))(\overline{b}_{1,\,m})=\varphi^{n+1}_{m-n-1,\,0}(\{f,z\})(\overline{b}_{1,\,m})$$

in $\overline{\Omega}^{n+1}(A)$ for all $\overline{b}_{1,\,m}\in \overline{A}^{\otimes m}$ and therefore

$$[f,\kappa_{n-1,\,1}(z)]_{\mathrm{sg}}=\varphi^{n+1}_{m-n-1,\,0}(\{f,z\})$$

in $\mathrm{Ext}^m_{A^{\mathrm{e}}}(A,\overline{\Omega}^{n+1}(A))$ for $f\in \operatorname{Ker}\delta^m_{(1)}$ and $z=a_0\otimes \overline{a}_{1,\,n-1}\in \operatorname{Ker}\partial^{(1)}_{n-1}$. This completes the proof of the statement. □

Proposition 3.3.9. *Let A be a Frobenius algebra with the Nakayama automorphism ν of A diagonalizable, and let m,n be integers such that $n\geq m\geq 1$, so $m-n-1<0$. Then we have a commutative diagram*

$$\begin{array}{ccc}
\mathrm{CH}^m_{(1)}(A)\otimes \mathrm{CH}^{-n}_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{m-n-1}_{(1)}(A)\\
\downarrow\cong & & \cong\downarrow\\
\mathrm{Ext}^m_{A^{\mathrm{e}}}(A,A)\otimes \mathrm{Tor}^{A^{\mathrm{e}}}_{n-1}(A,A_{\nu^{-1}}) & & \mathrm{Tor}^{A^{\mathrm{e}}}_{n-m}(A,A_{\nu^{-1}})\\
\mathrm{id}\otimes\kappa_{n-1,1}\downarrow\cong & & \cong\downarrow\kappa_{n-m,m}\\
\mathrm{Ext}^m_{A^{\mathrm{e}}}(A,A)\otimes \mathrm{Ext}^1_{A^{\mathrm{e}}}(A,\overline{\Omega}^{n+1}(A)) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \mathrm{Ext}^m_{A^{\mathrm{e}}}(A,\overline{\Omega}^{n+1}(A))\\
\downarrow\cong & & \cong\downarrow\\
\widehat{\mathrm{Ext}}^m_{A^{\mathrm{e}}}(A,A)\otimes \widehat{\mathrm{Ext}}^{-n}_{A^{\mathrm{e}}}(A,A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\mathrm{Ext}}^{m-n-1}_{A^{\mathrm{e}}}(A,A),
\end{array}$$

where $\{\ ,\ \} : \mathcal{D}^{m}_{(1)}(A,A) \otimes \mathcal{D}^{-n}_{(1)}(A,A) \to \mathcal{D}^{m-n-1}_{(1)}(A,A)$ *is defined by*

$$
\begin{aligned}
&\{f,z\} \\
&= (-1)^{|f||z|+|f|+|z|}\Big((-1)^{|z|+1}B^{\nu^{-1}}(f \star_1 z) + (-1)^{|f|}\Delta^{\nu}(f) \star_1 z + (-1)^{|z|} f \star_1 B^{\nu^{-1}}(z)\Big) \\
&= (-1)^{|f||z|+|f|+|z|}\Big((-1)^{|f|+1}\widehat{\Delta}(f \star_1 z) + (-1)^{|f|}\widehat{\Delta}(f) \star_1 z + f \star_1 \widehat{\Delta}(z)\Big)
\end{aligned}
$$

for $f \otimes z \in \mathcal{D}^{m}_{(1)}(A,A) \otimes \mathcal{D}^{-n}_{(1)}(A,A)$.

Proposition 3.3.10. *Let A be a Frobenius algebra with the Nakayama automorphism ν of A diagonalizable, and let m, n be integers such that $m \geq 1$ and $n \geq 1$. Then we have the following commutative diagram:*

$$
\begin{array}{ccc}
\mathrm{CH}^{-m}_{(1)}(A) \otimes \mathrm{CH}^{-n}_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{-m-n-1}_{(1)}(A) \\
\downarrow \cong & & \cong \downarrow \\
\mathrm{Tor}^{A^{\mathrm{e}}}_{m-1}(A, A_{\nu^{-1}}) \otimes \mathrm{Tor}^{A^{\mathrm{e}}}_{n-1}(A, A_{\nu^{-1}}) & & \mathrm{Tor}^{A^{\mathrm{e}}}_{m+n}(A, A_{\nu^{-1}}) \\
\kappa_{m-1,1} \otimes \kappa_{n-1,1} \downarrow \cong & & \cong \downarrow \kappa_{m+n,1} \\
\mathrm{Ext}^{1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{m+1}(A)) \otimes \mathrm{Ext}^{1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{n+1}(A)) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \mathrm{Ext}^{1}_{A^{\mathrm{e}}}(A, \overline{\Omega}^{m+n+2}(A)) \\
\downarrow \cong & & \cong \downarrow \\
\widehat{\mathrm{Ext}}^{-m}_{A^{\mathrm{e}}}(A,A) \otimes \widehat{\mathrm{Ext}}^{-n}_{A^{\mathrm{e}}}(A,A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\mathrm{Ext}}^{-m-n-1}_{A^{\mathrm{e}}}(A,A),
\end{array}
$$

where $\{\ ,\ \} : \mathcal{D}^{-m}_{(1)}(A,A) \otimes \mathcal{D}^{-n}_{(1)}(A,A) \to \mathcal{D}^{-m-n-1}_{(1)}(A,A)$ *is defined by*

$$
\begin{aligned}
&\{w,z\} \\
&= (-1)^{|w||z|+|w|+|z|}\Big((-1)^{|z|+1}B^{\nu^{-1}}(w \star_1 z) + B^{\nu^{-1}}(w) \star_1 z + (-1)^{|z|} w \star_1 B^{\nu^{-1}}(z)\Big) \\
&= (-1)^{|w||z|+|w|+|z|}\Big((-1)^{|w|+1}\widehat{\Delta}(w \star_1 z) + (-1)^{|w|}\widehat{\Delta}(w) \star_1 z + w \star_1 \widehat{\Delta}(z)\Big)
\end{aligned}
$$

for $w \otimes z \in \mathcal{D}^{-m}_{(1)}(A,A) \otimes \mathcal{D}^{-n}_{(1)}(A,A)$.

The following is a consequence of Lambre-Zhou-Zimmermann.

Proposition 3.3.11 ([**35**, Corollary 3.8])**.** *Let A be a Frobenius algebra whose Nakayama automorphism ν is diagonalizable, and let m, n be integers such that $m > 0$ and $n > 0$.*

Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{CH}^m_{(1)}(A) \otimes \mathrm{CH}^n_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{m+n-1}_{(1)}(A) \\
\downarrow \cong & & \cong \downarrow \\
\mathrm{Ext}^m_{A^e}(A,A) \otimes \mathrm{Ext}^n_{A^e}(A,A) & \xrightarrow{[\ ,\ ]} & \mathrm{Ext}^{m+n-1}_{A^e}(A,A) \\
\downarrow \cong & & \cong \downarrow \\
\widehat{\mathrm{Ext}}^m_{A^e}(A,A) \otimes \widehat{\mathrm{Ext}}^n_{A^e}(A,A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\mathrm{Ext}}^{m+n-1}_{A^e}(A,A),
\end{array}$$

where $[\ ,\ ]$ *is the Gerstenhaber bracket on Hochschild cohomology and* $\{\ ,\ \} : \mathcal{D}^m_{(1)}(A,A) \otimes \mathcal{D}^n_{(1)}(A,A) \to \mathcal{D}^{m+n-1}_{(1)}(A,A)$ *is defined by*

$$\begin{aligned}
\{f,g\} &= (-1)^{|f||g|+|f|+|g|}\left((-1)^{|f|+1}\Delta^\nu(f \star_1 g) + (-1)^{|f|}\Delta^\nu(f) \star_1 g + f \star_1 \Delta^\nu(g)\right) \\
&= (-1)^{|f||g|+|f|+|g|}\left((-1)^{|f|+1}\widehat{\Delta}(f \star_1 g) + (-1)^{|f|}\widehat{\Delta}(f) \star_1 g + f \star_1 \widehat{\Delta}(g)\right)
\end{aligned}$$

for $f \otimes g \in \mathcal{D}^m_{(1)}(A,A) \otimes \mathcal{D}^n_{(1)}(A,A)$.

Remark 3.3.12. We have to consider the case of either $m = 0$ or $n = 0$. If $m \geq 0$ and $n = 0$, then we will prove that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^m_{(1)}(A) \otimes \mathrm{CH}^0_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{m-1}_{(1)}(A) \\
\downarrow \cong & & \cong \downarrow \\
\mathrm{Ext}^m_{A^e}(A,A) \otimes \mathrm{CH}^0(A) & & \mathrm{Ext}^{m-1}_{A^e}(A,A) \\
\mathrm{id}\otimes\varphi_{0,0} \downarrow \cong & & \cong \downarrow \varphi_{m-1,0} \\
\mathrm{Ext}^m_{A^e}(A,A) \otimes \mathrm{Ext}^1_{A^e}(A,\overline{\Omega}^1(A)) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \mathrm{Ext}^m_{A^e}(A,\overline{\Omega}^1(A)) \\
\downarrow \cong & & \cong \downarrow \\
\widehat{\mathrm{Ext}}^m_{A^e}(A,A) \otimes \widehat{\mathrm{Ext}}^0_{A^e}(A,A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\mathrm{Ext}}^{m-1}_{A^e}(A,A),
\end{array}$$

where the vertical isomorphism $\varphi_{0,0} : \mathrm{CH}^0(A) \to \mathrm{Ext}^1_{A^e}(A,\overline{\Omega}^1(A))$ is defined in Proposition 3.1.3 and $\{\ ,\ \}$ is defined by

$$\{f,g\} = (-1)^{|f|}\left((-1)^{|f|+1}\Delta^\nu(f \star_1 g) + (-1)^{|f|}\Delta^\nu(f) \star_1 g\right)$$

for $f \otimes g \in \mathcal{D}^m_{(1)}(A,A) \otimes \mathcal{D}^0_{(1)}(A,A)$. We must show that

$$\varphi_{m-1,0}(\{\ ,\ \}(f \otimes g)) = ([\ ,\ ]_{\mathrm{sg}}(\mathrm{id} \otimes \varphi_{0,0}))(f \otimes g)$$

in $\operatorname{Ext}^m_{A^e}(A, \overline{\Omega}^1(A))$ for $f \otimes g \in \operatorname{Ker} \delta^m_{(1)} \otimes \operatorname{Ker} \delta^0_{(1)}$. A direct calculation shows that we have

$$[f, \varphi_{0,0}(g)]_{\mathrm{sg}} = \varphi_{m-1,0}([f, g])$$

as maps, where $[\ ,\ ]$ is the Gerstenhaber bracket on Hochschild cohomology. It follows from [**35**, Corollary 3.8] that $[f, g] = -\Delta^\nu(f \star_1 g) + \Delta^\nu(f) \star_1 g$ in $\operatorname{Ext}^{m-1}_{A^e}(A, A)$. As a result, we obtain a formula in $\operatorname{Ext}^m_{A^e}(A, \overline{\Omega}^1(A))$:

$$[f, \varphi_{0,0}(g)]_{\mathrm{sg}} = \varphi_{m-1,0}([f, g]) = \varphi_{m-1,0}(-\Delta^\nu(f \star_1 g) + \Delta^\nu(f) \star_1 g) = \varphi_{m-1,0}(\{f, g\}).$$

Similarly, one can prove our claim in the other case $m = 0$ and $n \geq 0$.

We are now able to prove Theorem 3.3.5.

Proof of Theorem 3.3.5. It follows from Propositions 3.3.8, 3.3.9, 3.3.10 and 3.3.11 and Remark 3.3.12 that we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^m_{(1)}(A) \otimes \mathrm{CH}^{-n}_{(1)}(A) & \xrightarrow{\{\ ,\ \}} & \mathrm{CH}^{m-n-1}_{(1)}(A) \\ \downarrow \cong & & \cong \downarrow \\ \widehat{\operatorname{Ext}}^m_{A^e}(A, A) \otimes \widehat{\operatorname{Ext}}^{-n}_{A^e}(A, A) & \xrightarrow{[\ ,\ ]_{\mathrm{sg}}} & \widehat{\operatorname{Ext}}^{m-n-1}_{A^e}(A, A), \end{array}$$

where m, n are arbitrary integers. Since $(\widehat{\operatorname{Ext}}^\bullet_{A^e}(A, A), \smile_{\mathrm{sg}}, [\ ,\ ]_{\mathrm{sg}})$ is a Gerstenhaber algebra, we have

$$[f, g \smile_{\mathrm{sg}} h]_{\mathrm{sg}} = [f, g]_{\mathrm{sg}} \smile_{\mathrm{sg}} h + (-1)^{(|f|-1)|g|} g \smile_{\mathrm{sg}} [f, h]_{\mathrm{sg}}$$

for arbitrary homogeneous elements f, g and $h \in \widehat{\operatorname{Ext}}^\bullet_{A^e}(A, A)$. Since we have proved that $[\ ,\ ]_{\mathrm{sg}}$ commutes with $\{\ ,\ \}$, using the defining formula for $\{\ ,\ \}$ and the formula

$$\{f, g \star_1 h\} = (-1)^r \Big((-1)^{|f|+1} \widehat{\Delta}(f \star_1 g \star_1 h) + (-1)^{|f|} \widehat{\Delta}(f) \star_1 g \star_1 h + f \star_1 \widehat{\Delta}(g \star_1 h)\Big)$$

with $r = |f|(|g| + |h|) + |f| + |g| + |h|$, we obtain

$$\begin{aligned} \widehat{\Delta}(f \star_1 g \star_1 h) = {} & \widehat{\Delta}(f \star_1 g) \star h + (-1)^{|f|} f \star_1 \widehat{\Delta}(g \star_1 h) + (-1)^{|g|(|f|-1)} g \star_1 \widehat{\Delta}(f \star_1 h) \\ & - \widehat{\Delta}(f) \star_1 g \star_1 h + (-1)^{|f|} f \star_1 \widehat{\Delta}(g) \star_1 h + (-1)^{|f|+|g|} f \star_1 g \star_1 \widehat{\Delta}(h) \end{aligned}$$

for arbitrary homogeneous elements f, g and $h \in \mathrm{CH}^\bullet_{(1)}(A)$. Finally, by the definition of the operator $\widehat{\Delta}$, we get $\widehat{\Delta}^2 = 0$ and $\widehat{\Delta}_0(1) = 0$. □

Remark 3.3.13. Recall that the Nakayama automorphism ν of A is *semisimple* if the map $\nu \otimes \mathrm{id}_{\overline{k}} : A \otimes \overline{k} \to A \otimes \overline{k}$ is diagonalizable over the algebraic closure $\overline{k}$ of k. The results of Lambre-Zhou-Zimmermann [**35**, Section 4] and an easy calculation imply that the complete cohomology ring of a Frobenius algebra is a BV algebra when the Nakayama automorphism is semisimple.

3.4. Examples

Throughout this section, we assume that k is an algebraically closed field whose characteristic $\operatorname{char} k$ is p. Lambre-Zhou-Zimmermann [**35**] showed that there are many examples of Frobenius algebras with diagonalizable Nakayama automorphisms. This section is devoted to computing the graded commutative ring structure and the BV structure of the complete cohomology for three certain self-injective Nakayama algebras whose Nakayama automorphisms are diagonalizable. Lambre-Zhou-Zimmermann [**35**] also gave an useful and combinatorial criterion to check that the Nakayama automorphism is diagonalizable: let $A = kQ/I$ be the algebra given by a quiver with relations. Let Q_0 be the set of vertices in Q. It is well-known that we can choose a k-basis $\mathcal{B}$ of A such that $\mathcal{B}$ contains a k-basis of the socle of the right regular A-module A. Suppose that A is a Frobenius algebra. It follows from [**30**, Proposition 2.8] that we can construct an associative and non-degenerate bilinear form $\langle\ ,\ \rangle : A \otimes A \to k$ by defining $\langle a, b\rangle := tr(ab)$ for $a, b \in A$, where $tr : A \to k$ is given by

$$tr(p) = \begin{cases} 1 & \text{if } p \in \mathcal{B} \cap \operatorname{soc} A_A, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\mathcal{B}$ satisfies two additional conditions:

(i) For any two paths $p, q \in \mathcal{B}$, there exist a path $r \in \mathcal{B}$ and a constant $\lambda \in k$ such that $p \cdot q = \lambda r$ in A;

(ii) For every path $p \in \mathcal{B}$, there uniquely exists a path $p' \in \mathcal{B}$ such that $0 \neq p \cdot p' \in \operatorname{soc} A_A$.

Criterion 3.4.1 ([**35**, Criterion 5.1])**.** *Under the situation as above, assume that k is an algebraically closed field of characteristic zero or of characteristic p larger than the number of arrows of Q. Then the Nakayama automorphism of A associated with the bilinear form $\langle\ ,\ \rangle : A \otimes A \to k$ given above is diagonalizable over k.*

Suppose that $A = kQ/I$ is a self-injective Nakayama algebra. It is known that the ordinary quiver Q of A is a cyclic quiver with $|Q_0| = s$, and an admissible ideal I of kQ is of the form R_Q^N, where R_Q is the arrow ideal of kQ and $N \geq 2$. Obviously, we can take a k-basis $\mathcal{B}$ of A consisting of paths contains a k-basis of $\operatorname{soc} A_A$. Since any indecomposable projective A-module is uniserial, $\mathcal{B}$ satisfies the two condition (i) and (ii). Hence, we can rewrite Criterion 3.4.1 as follows:

Criterion 3.4.2. *Let $A = kQ/R_Q^N$ be a self-injective Nakayama algebra. If the characteristic of k is zero or p larger than the number of arrows of Q, then the Nakayama automorphism of A is diagonalizable over k.*

Remark 3.4.3. If $A = kQ/R_Q^N$ is a self-injective Nakayama algebra, then the exponent N does not affect Criterion 3.4.2, and only the number of arrows of Q is important.

We will compute BV algebras of Nakayama algebras $A = kQ/R_Q^N$ with $|Q_0| = s$ for three cases.

The case $s = 2, N = 2$.

Let Q be the following quiver

$$1 \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 2.$$

Consider the algebra $A := kQ/R_Q^2$. Thus, A is a self-injective Nakayama algebra and, moreover, a truncated algebra. It follows from Criterion 3.4.2 that the Nakayama automorphism of A is diagonalizable if and only if $\operatorname{char} k \neq 2$. Thus, we suppose that $\operatorname{char} k \neq 2$. Note that we need the assumption on $\operatorname{char} k$ only if we construct BV differential. However, we assume that $\operatorname{char} k \neq 2$ in advance. We denote by e_i the primitive idempotent of A corresponding to a vertex i of Q such that $e_i \alpha_i e_{i+1} = \alpha_i$ holds, where we regard the subscripts i of e_i and α_i modulo 2. Take a k-basis $\mathcal{B} = (u_1, u_2, u_3, u_4) = (e_1, e_2, \alpha_1, \alpha_2)$ of A. Clearly, it contains a k-basis $\{\alpha_1, \alpha_2\}$ of $\operatorname{soc} A_A$. We hence get an associative and non-degenerate bilinear form $\langle\ ,\ \rangle : A \otimes A \to k$ and the dual basis $\mathcal{B}^* = (v_1, v_2, v_3, v_4) = (\alpha_2, \alpha_1, e_1, e_2)$ of A such that $\langle v_i, u_j \rangle = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta. Under the basis $\mathcal{B}$, the representation matrix of the Nakayama automorphism ν of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and is similar to a diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Moreover, we have a decomposition $A = A_1 \oplus A_{-1}$ of A by two k-vector spaces

$$\begin{aligned} A_1 &= \operatorname{Ker}(\nu - \mathrm{id}) = k\, 1_A \oplus k\,(\alpha_1 + \alpha_2), \\ A_{-1} &= \operatorname{Ker}(\nu + \mathrm{id}) = k\,(e_1 - e_2) \oplus k\,(\alpha_1 - \alpha_2). \end{aligned}$$

Let us recall that a set $\{Ae_i \otimes e_j A \mid i, j \in Q_0\}$ is a complete set of pairwise non-isomorphic indecomposable projective A-bimodules, and we denote by $P(i,j)$ the indecomposable projective A-bimodule $Ae_i \otimes e_j A$. It follows from [**7**] that a minimal projective resolution $\mathcal{P}_\bullet$ of A as an A-bimodule is an exact sequence

$$\cdots \to P_{2r+1} \xrightarrow{\phi_{2r+1}} P_{2r} \xrightarrow{\phi_{2r}} P_{2r-1} \to \cdots \to P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} A \to 0,$$

where

$$P_n := \begin{cases} P(1,2) \oplus P(2,1) & \text{if } n \text{ is odd,} \\ P(1,1) \oplus P(2,2) & \text{if } n \text{ is even} \end{cases}$$

and A-bimodule homomorphisms $\phi_* : P_* \to P_{*-1}$ are defined as follows:

$$\begin{aligned} &\phi_0(e_i \otimes e_i) = e_i; \\ &\phi_{2r}(e_i \otimes e_i) = e_i \otimes \alpha_{i+1} + \alpha_i \otimes e_i; \\ &\phi_{2r+1}(e_i \otimes e_{i+1}) = \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i. \end{aligned}$$

For a k-vector space V and a k-basis $\mathcal{B}$ of V, given a basis vector $p \in \mathcal{B}$, we denote by p^* the k-linear map $V \to k$ sending $q \in \mathcal{B}$ to 1 if $q = p$ and to 0 otherwise. Applying the exact functor $D = \operatorname{Hom}(-, k)$ to $\mathcal{P}_\bullet$ and twisting each term of $D(\mathcal{P}_\bullet)$ by the automorphism ν^{-1} on the right hand side, we get an exact sequence $D(\mathcal{P}_\bullet)_{\nu^{-1}}$ as follows:

$$0 \to D(A)_{\nu^{-1}} \xrightarrow{D(\phi_0)} \cdots \to D(P_{2r-1})_{\nu^{-1}} \xrightarrow{D(\phi_{2r})} D(P_{2r})_{\nu^{-1}} \xrightarrow{D(\phi_{2r+1})} D(P_{2r+1})_{\nu^{-1}} \to \cdots,$$

where

$$D(P_n)_{\nu^{-1}} = \begin{cases} A(\alpha_2 \otimes \alpha_2)^* A \oplus A(\alpha_1 \otimes \alpha_1)^* A & \text{if } n \text{ is odd,} \\ A(\alpha_2 \otimes \alpha_1)^* A \oplus A(\alpha_1 \otimes \alpha_2)^* A & \text{if } n \text{ is even} \end{cases}$$

and A-bimodule homomorphisms $D(\phi_*) : D(P_{*-1})_{\nu^{-1}} \to D(P_*)_{\nu^{-1}}$ are defined as follows:

$$\begin{aligned} &D(\phi_0)(\langle -, 1_A \rangle) = \alpha_1(\alpha_2 \otimes \alpha_1)^* + (\alpha_2 \otimes \alpha_1)^* \alpha_1 + \alpha_2(\alpha_1 \otimes \alpha_2)^* + (\alpha_1 \otimes \alpha_2)^* \alpha_2; \\ &D(\phi_{2r})((\alpha_i \otimes \alpha_i)^*) = \alpha_{i+1}(\alpha_i \otimes \alpha_{i+1})^* + (\alpha_{i+1} \otimes \alpha_i)^* \alpha_i; \\ &D(\phi_{2r+1})((\alpha_i \otimes \alpha_{i+1})^*) = (\alpha_i \otimes \alpha_i)^* \alpha_{i+1} - \alpha_{i+1}(\alpha_{i+1} \otimes \alpha_{i+1})^*. \end{aligned}$$

Therefore, we obtain an exact sequence $X_\bullet$

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & P_2 & \xrightarrow{\phi_2} & P_1 & \xrightarrow{\phi_1} & P_0 & \xrightarrow{\mu} & D(P_0)_{\nu^{-1}} & \xrightarrow{D(\phi_1)} & D(P_1)_{\nu^{-1}} & \xrightarrow{D(\phi_2)} & D(P_2)_{\nu^{-1}} & \longrightarrow \cdots \\ & & & & & {\scriptstyle \phi_0}\downarrow & \circlearrowleft & \uparrow{\scriptstyle D(\phi_0)} & & & & & \\ & & & & & A & \xrightarrow{\cong} & D(A)_{\nu^{-1}} & & & & & \end{array}$$

of which the composition μ is defined by

$$\mu(e_i \otimes e_i) = \alpha_i \otimes e_i + \alpha_{i+1} \otimes e_{i+1}$$

and whose term P_n is of degree $n \geq 0$. Observe that there are A-bimodule isomorphisms

$$\begin{aligned} D(P(i,j))_{\nu^{-1}} &= D({}_{\nu^{-1}}Ae_i \otimes e_jA) \cong \mathrm{Hom}(e_jA, D({}_{\nu^{-1}}Ae_i)) \\ &\cong D(e_jA) \otimes D(Ae_i)_{\nu^{-1}} \cong Ae_{j+1} \otimes e_{i+1}A_{\nu^{-1}} \\ &\cong Ae_{j+1} \otimes e_iA = P(j+1,i), \end{aligned}$$

where the fourth isomorphism is induced by the A-bimodule isomorphism $A_\nu \cong D(A)$ and the fact that $\nu = \nu^{-1}$. Since A^{e} is injective as an A-bimodule, the contravariant functor $\mathrm{Hom}_{A^{\mathrm{e}}}(-, A^{\mathrm{e}})$ is exact, so that the exact sequence $X_\bullet$ is a complete resolution of A. Before applying the functor $\mathrm{Hom}_{A^{\mathrm{e}}}(-, A)$ to $X_\bullet$, we notice that there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_{A^{\mathrm{e}}}(D(P(i,j))_{\nu^{-1}}, A) &\cong \mathrm{Hom}_{A^{\mathrm{e}}}(D(P(i,j)), D(A)) \cong D(A \otimes_{A^{\mathrm{e}}} D(P(i,j))) \\ &\cong \mathrm{Hom}_{A^{\mathrm{e}}}(A, P(i,j)) \cong \mathrm{Hom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \otimes_{A^{\mathrm{e}}} P(i,j) \\ &\cong A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P(i,j) \end{aligned}$$

for any $i, j \in Q_0$. Using these isomorphisms, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Hom}_{A^{\mathrm{e}}}(D(P_{2r+1})_{\nu^{-1}}, A) & \xrightarrow{\mathrm{Hom}(D(\phi_{2r+1}),A)} & \mathrm{Hom}_{A^{\mathrm{e}}}(D(P_{2r})_{\nu^{-1}}, A) & \xrightarrow{\mathrm{Hom}(D(\phi_{2r}),A)} & \mathrm{Hom}_{A^{\mathrm{e}}}(D(P_{2r-1})_{\nu^{-1}}, A) \\ \downarrow\cong & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong \\ A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_{2r+1} & \xrightarrow{\mathrm{id}\otimes\phi_{2r+1}} & A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_{2r} & \xrightarrow{\mathrm{id}\otimes\phi_{2r}} & A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_{2r-1}, \end{array}$$

where the A-bimodules $A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_*$ are given by

$$\begin{aligned} A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_{2r} &= k\,(e_2 \otimes_{A^{\mathrm{e}}} e_1 \otimes e_2) \oplus k\,(e_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1); \\ A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_{2r+1} &= k\,(\alpha_2 \otimes_{A^{\mathrm{e}}} e_1 \otimes e_1) \oplus k\,(\alpha_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_2), \end{aligned}$$

and the k-linear maps $\mathrm{id} \otimes_{A^{\mathrm{e}}} \phi_*$ are given by

$$\begin{aligned} &\mathrm{id} \otimes \phi_{2r}(\alpha_i \otimes_{A^{\mathrm{e}}} e_i \otimes e_i) = 0; \\ &\mathrm{id} \otimes \phi_{2r+1}(e_i \otimes_{A^{\mathrm{e}}} e_{i+1} \otimes e_i) = \alpha_i \otimes_{A^{\mathrm{e}}} e_i \otimes e_i - \alpha_{i+1} \otimes_{A^{\mathrm{e}}} e_{i+1} \otimes e_{i+1}. \end{aligned}$$

Hence, the complex $\mathrm{Hom}_{A^{\mathrm{e}}}(X_\bullet, A)$ can be identified with a complex

$$\begin{aligned} \cdots \to A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_1 &\xrightarrow{\mathrm{id}\otimes\phi_1} A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_0 \\ &\xrightarrow{\mathrm{Hom}(\mu,A)} \mathrm{Hom}_{A^{\mathrm{e}}}(P_0, A) \xrightarrow{\mathrm{Hom}(\phi_1,A)} \mathrm{Hom}_{A^{\mathrm{e}}}(P_1, A) \to \cdots \end{aligned}$$

of which the remaining terms and differentials are given by

$$\begin{aligned} &\mathrm{Hom}_{A^{\mathrm{e}}}(P_n, A) \cong \begin{cases} e_1Ae_2 \oplus e_2Ae_1 = k\,\alpha_1 \oplus k\,\alpha_2 & \text{if } n \text{ is odd,} \\ e_1Ae_1 \oplus e_2Ae_2 = k\,e_1 \oplus k\,e_2 & \text{if } n \text{ is even;} \end{cases} \\ &\mathrm{Hom}_{A^{\mathrm{e}}}(\phi_{2r+1}, A)(e_i) = \alpha_{i+1} - \alpha_i; \\ &\mathrm{Hom}_{A^{\mathrm{e}}}(\phi_{2r}, A)(\alpha_i) = 0; \\ &\mathrm{Hom}_{A^{\mathrm{e}}}(\mu, A)(\alpha_i \otimes_{A^{\mathrm{e}}} e_i \otimes e_i) = 0 \end{aligned}$$

and whose term $\mathrm{Hom}_{A^e}(P_n, A)$ is of degree $n \geq 0$.

Therefore, the complete cohomology groups $\mathrm{CH}^*(A)$ are given as follows: for $n \geq 0$

$$\mathrm{CH}^n(A) = \begin{cases} k\,\overline{\alpha_1} & \text{if } n \text{ is odd,} \\ k\,\overline{1_A} & \text{if } n \text{ is even;} \end{cases} \tag{3.8}$$

$$\mathrm{CH}^{-n}(A) = \begin{cases} k\,\overline{\alpha_1 \otimes_{A^e} e_1 \otimes e_1} & \text{if } n \text{ is odd,} \\ k\,\overline{e_1 \otimes_{A^e} e_2 \otimes e_1 + e_2 \otimes_{A^e} e_1 \otimes e_2} & \text{if } n > 0 \text{ is even.} \end{cases} \tag{3.9}$$

Observe that we have $\mathrm{CH}^0(A) = \mathrm{HH}^0(A)$ and $\mathrm{CH}^{-1}(A) = \mathrm{H}_0(A, A_{\nu^{-1}})$.

From now on, we fix a k-basis

$$(u_1, u_2, u_3, u_4) = (1, \alpha_1 + \alpha_2, e_1 - e_2, \alpha_1 - \alpha_2)$$

of A consisting of eigenvectors associated with the eigenvalues of the diagonalizable Nakayama automorphism ν of A. Then we have its dual basis

$$(v_1, v_2, v_3, v_4) = \big((1/2)(\alpha_1 + \alpha_2), 1/2, (1/2)(\alpha_1 - \alpha_2), (1/2)(e_1 - e_2)\big)$$

of A. Following [**1**], we will construct comparison morphisms between the minimal projective resolution $\mathcal{P}_\bullet$ and the normalized bar resolution $\mathrm{Bar}_\bullet(A)$ of A (cf. [**40**] for monomial algebras in general). Let $\mathbf{F}_0$ be the canonical inclusion $P_0 \hookrightarrow A \otimes A$, and for each $n > 0$, we define $\mathbf{F}_n : P_n \to A \otimes \overline{A}^{\otimes n} \otimes A$ in the following way: if $n = 2r$, then let

$$\mathbf{F}_{2r}(e_i \otimes e_i) = 1 \otimes \overbrace{\overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1}}^{2r} \otimes 1,$$

where $\overline{\alpha}_i$ and $\overline{\alpha}_{i+1}$ appear each other. If $n = 2r + 1$, then let

$$\mathbf{F}_{2r+1}(e_i \otimes e_{i+1}) = 1 \otimes \overbrace{\overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \overline{\alpha}_i}^{2r+1} \otimes 1.$$

On the other hand, let $\mathbf{G}_0$ be the canonical projection $A \otimes A \to P_0$, and for each $n > 0$, $\mathbf{G}_n : A \otimes \overline{A}^{\otimes n} \otimes A \to P_n$ is given as follows: if $n = 2r$, then let

$$\mathbf{G}_{2r}(w) = \begin{cases} e_i \otimes e_i & \text{if } w = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $n = 2r + 1$, then let

$$\mathbf{G}_{2r+1}(w) = \begin{cases} e_i \otimes e_{i+1} & \text{if } w = 1 \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \cdots \otimes \overline{\alpha}_i \otimes \overline{\alpha}_{i+1} \otimes \overline{\alpha}_i \otimes 1, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that $\mathbf{F}$ and $\mathbf{G}$ are comparison morphisms. Using these comparison morphisms and the definition of the $\star$-product $\star$, we have the following result.

Proposition 3.4.4. *For every $i \in \mathbb{Z}$, the n-th complete cohomology group $\mathrm{CH}^n(A)$ of A is of dimension one, and the complete cohomology ring $(\mathrm{CH}^\bullet(A), \star)$ is isomorphic to*

$$k[\alpha, \beta, \gamma]/\langle \alpha\gamma - 1, \beta^2 \rangle$$

with $|\alpha| = 2, |\beta| = 1$ and $|\gamma| = -2$, where α, β and γ correspond to $\overline{1_A} \in \mathrm{CH}^2(A)$ in (3.8), $\overline{\alpha_1} \in \mathrm{CH}^1(A)$ in (3.8) and $\overline{e_1 \otimes_{A^e} e_2 \otimes e_1 + e_2 \otimes_{A^e} e_1 \otimes e_2} \in \mathrm{CH}^{-2}(A)$ in (3.9), respectively.

Remark 3.4.5. As we have seen before, the complete cohomology groups $\mathrm{CH}^n(A)$ with $n \geq 0$ of A coincide with the Hochschild cohomology groups $\mathrm{HH}^n(A)$ of A. Hence, the Hochschild cohomology ring $(\mathrm{HH}^\bullet(A), \smile)$ of A is a subring of the complete cohomology ring $(\mathrm{CH}^\bullet(A), \star)$.

Remark 3.4.6. We have another description of the complete cohomology ring above as follows:

$$k[\alpha, \beta, \alpha^{-1}]/\langle \beta^2 \rangle$$

where $|\alpha| = 2, |\beta| = 1$ and $|\alpha^{-1}| = -2$. Therefore, we will write α^{-1} for γ.

Following our main result, we now construct a BV operator $\widehat{\Delta}_i : \mathrm{CH}^i(A) \to \mathrm{CH}^{i-1}(A)$ for all $i \in \mathbb{Z}$. It follows from Proposition 3.4.4 that

$$\mathrm{CH}^{2l}(A) = k\,\alpha^l \quad \text{and} \quad \mathrm{CH}^{2l+1}(A) = k\,\beta\alpha^l$$

for all $l \in \mathbb{Z}$. Note that the number of the generators contained in the basis element of $\mathrm{CH}^i(A)$ is at least 3 except for $-4 \leq i \leq 4$. Thus, one can use the operators $\widehat{\Delta}_i : \mathrm{CH}^i(A) \to \mathrm{CH}^{i-1}(A)$ for $-4 \leq i \leq 4$ and the formulas in Definition 3.3.1 to obtain the remaining operators $\widehat{\Delta}_* : \mathrm{CH}^*(A) \to \mathrm{CH}^{*-1}(A)$. From this point of view, it suffices to construct $\widehat{\Delta}_i$ only for $i = -4, -2, -1, 1, 2, 3, 4$. We will show a way of constructing $\widehat{\Delta}_1$ and $\widehat{\Delta}_{-1}$. The others can be constructed in a similar way. Let us recall that every complete cohomology group has a decomposition associated with the product of eigenvalues and in particular, except for the cohomology associated with the product of eigenvalues equal to 1_A, the other vanish. Moreover, the BV operator defined on the chain level can be lifted to the cohomology level when we restrict it to the subcomplex associated with the product of eigenvalues equal to 1_A.

We first compute $\widehat{\Delta}_1 : \mathrm{CH}^1(A) \to \mathrm{CH}^0(A)$. Consider a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{A^e}(P_1, A) & \xrightarrow{\widehat{\Delta}_1} & \mathrm{Hom}_{A^e}(P_0, A) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_{A^e}(\mathbf{G}_1, A)} & & \Big\uparrow{\scriptstyle \mathrm{Hom}_{A^e}(\mathbf{F}_0, A)} \\
\mathrm{Hom}(\overline{A}, A) & \circlearrowleft & A \\
\Big\downarrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \cong} \\
D(A_\nu \otimes \overline{A}) & \xrightarrow{D(B_0^\nu)} & D(A_\nu).
\end{array}$$

Since $\mathrm{CH}^1(A) = k\,\overline{\alpha_1}$, we deal with only α_1. Put

$$f_{\alpha_1} := \mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{G}_1, A)(\alpha_1), \quad f_{u_2} := \mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{G}_1, A)(u_2), \quad f_{u_4} := \mathrm{Hom}_{A^{\mathrm{e}}}(\mathbf{G}_1, A)(u_4).$$

Namely, each of f_{α_1}, f_{u_2} and f_{u_4} sends $\overline{x} \in \overline{A}$ with $x \in \mathcal{B}$ to

$$f_{\alpha_1}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ 0 & \text{otherwise}, \end{cases} \quad f_{u_2}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ \alpha_2 & \text{if } \overline{x} = \overline{\alpha_2}, \\ 0 & \text{otherwise}, \end{cases} \quad f_{u_4}(\overline{x}) = \begin{cases} \alpha_1 & \text{if } \overline{x} = \overline{\alpha_1}, \\ -\alpha_2 & \text{if } \overline{x} = \overline{\alpha_2}, \\ 0 & \text{otherwise}. \end{cases}$$

Then we have $f_{\alpha_1} = (1/2)\, f_{u_2} + (1/2)\, f_{u_4}$, $f_{u_2} \in C^1_{(1)}(A, A)$ and $f_{u_4} \in C^1_{(-1)}(A, A)$. Since it is sufficient to only consider the image of $(1/2) f_{u_2}$, a direct computation shows that

$$\widehat{\Delta}_1(\beta) = \widehat{\Delta}_1(\overline{\alpha_1}) = (1/2)\,\overline{1_A} = 1/2$$

in $\mathrm{CH}^0(A)$. On the other hand, consider a diagram

$$\begin{array}{ccc} A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_0 & \xrightarrow{\widehat{\Delta}_{-1}} & A_{\nu^{-1}} \otimes_{A^{\mathrm{e}}} P_1 \\ {\scriptstyle \mathrm{id} \otimes_{A^{\mathrm{e}}} \mathbf{F}_0} \downarrow & \circlearrowright & \uparrow {\scriptstyle \mathrm{id} \otimes_{A^{\mathrm{e}}} \mathbf{G}_1} \\ A_{\nu^{-1}} & \xrightarrow{-B_0^{\nu^{-1}}} & A_{\nu^{-1}} \otimes \overline{A}. \end{array}$$

We know that $\mathrm{CH}^{-1}(A) = k\,\overline{\alpha_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1}$ holds and hence handle $\alpha_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1$. The element $(\mathrm{id} \otimes_{A^{\mathrm{e}}} \mathbf{F}_0)(\alpha_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1) = \alpha_1$ can be decomposed as $\alpha_1 = (1/2)\, u_2 + (1/2)\, u_4$ in $A_{\nu^{-1}}$, where $u_2 \in C_0^{(1)}(A, A_{\nu^{-1}})$ and $u_4 \in C_0^{(-1)}(A, A_{\nu^{-1}})$. Thus, a direct calculation gives us the formula

$$\widehat{\Delta}_{-1}(\overline{\alpha_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1}) = (-1/2)\,\overline{e_1 \otimes_{A^{\mathrm{e}}} e_2 \otimes e_1 + e_2 \otimes_{A^{\mathrm{e}}} e_1 \otimes e_2}$$

in $\mathrm{CH}^{-2}(A)$. Thus, we have $\widehat{\Delta}_{-1}(\alpha^{-1}\beta) = (-1/2)\,\alpha^{-1}$. Combining the formulas in Definition 3.3.1, we have the following result.

Proposition 3.4.7. *The nonzero BV differentials $\widehat{\Delta}_*$ on $\mathrm{CH}^\bullet(A)$ are*

$$\widehat{\Delta}_{2n+1}(\alpha^n\beta) = ((2n+1)/2)\,\alpha^n$$

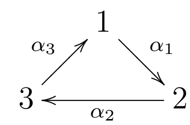
with $n \in \mathbb{Z}$. In particular, the nonzero Gerstenhaber brackets are induced by

$$\{\alpha, \beta\} = \alpha, \quad \{\beta, \alpha^{-1}\} = \alpha^{-1}.$$

Remark 3.4.8. Since the non-negative part $\mathrm{CH}^{\geq 0}(A)$ of the complete cohomology $\mathrm{CH}^\bullet(A)$ is the Hochschild cohomology of A, the non-negative BV differential $\widehat{\Delta}_{\geq 0}$ gives rise to a BV differential on the Hochschild cohomology ring of A, which means that there is a non-trivial example for our main theorem and for the theorem of Lambre-Zhou-Zimmermann [**35**, Theorem 4.1].

The case $s = 3, N = 2$.

Let Q be the following quiver



and A the algebra kQ/R_Q^2. It follows from Criterion 3.4.2 and the fact that a primitive root of a polynomial $x^3 - 1$ is not equal to $1 \in k$ when $\operatorname{char} k = 2$ that the Nakayama automorphism of A is diagonalizable if and only if $\operatorname{char} k \neq 3$. Hence we assume that $\operatorname{char} k \neq 3$. We see that A is a self-injective Nakayama algebra of which the representation matrix of the Nakayama automorphism ν is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

under a k-basis $(e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3)$ of A. This matrix is similar to a diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}$$

where the element $\omega \in k$ is one of roots of a polynomial $x^2 + x + 1$. Moreover, we can decompose $A = A_1 \oplus A_\omega \oplus A_{\omega^2}$, where

$$\begin{aligned} A_1 &= \operatorname{Ker}(\nu - \mathrm{id}) = k\, 1_A \oplus k\, (\sum_{i=1}^{3} \alpha_i), \\ A_\omega &= \operatorname{Ker}(\nu - \omega\, \mathrm{id}) = k\, (\omega^2 e_1 + \omega e_2 + e_3) \oplus k\, (\omega^2 \alpha_1 + \omega \alpha_2 + \alpha_3), \\ A_{\omega^2} &= \operatorname{Ker}(\nu - \omega^2\, \mathrm{id}) = k\, (\sum_{i=1}^{3} \omega^i e_i) \oplus k\, (\sum_{i=1}^{3} \omega^i \alpha_i). \end{aligned}$$

Let $l \geq 0$ be an integer. In a similar way to the first example, we have a complete resolution of A as follows:

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_2 & \xrightarrow{\phi_2} & P_1 & \xrightarrow{\phi_1} & P_0 & \xrightarrow{\mu} & D(P_0)_{\nu^{-1}} & \xrightarrow{D(\phi_1)} & D(P_1)_{\nu^{-1}} & \xrightarrow{D(\phi_2)} & D(P_2)_{\nu^{-1}} & \longrightarrow \cdots \\
 & & & & & {\scriptstyle \phi_0}\downarrow & \circlearrowleft & \uparrow{\scriptstyle D(\phi_0)} & & & & & \\
 & & & & & A & \xrightarrow{\cong} & D(A)_{\nu^{-1}} & & & & &
\end{array}$$

where each of the P_n and the $D(P_n)_{\nu^{-1}}$ is given by

$$P_n = \begin{cases} \bigoplus_{i=1}^{3} P(i,i) & \text{if } n = 3l, \\ \bigoplus_{i=1}^{3} P(i,i+1) & \text{if } n = 3l+1, \\ \bigoplus_{i=1}^{3} P(i,i+2) & \text{if } n = 3l+2, \end{cases}$$

$$D(P_n)_{\nu^{-1}} = \begin{cases} \bigoplus_{i=1}^{3} A(\alpha_i \otimes \alpha_{i+1})^* A & \text{if } n = 3l, \\ \bigoplus_{i=1}^{3} A(\alpha_i \otimes \alpha_{i+2})^* A & \text{if } n = 3l+1, \\ \bigoplus_{i=1}^{3} A(\alpha_i \otimes \alpha_i)^* A & \text{if } n = 3l+2, \end{cases}$$

each A-bimodule homomorphism $\phi_* : P_* \to P_{*-1}$ given by

$$\phi_{6l+1}(e_i \otimes e_{i+1}) = \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i; \quad \phi_{6l+2}(e_i \otimes e_{i+2}) = e_i \otimes \alpha_{i+1} + \alpha_i \otimes e_{i+2};$$
$$\phi_{6l+3}(e_i \otimes e_i) = \alpha_i \otimes e_i - e_i \otimes \alpha_{i+2}; \quad \phi_{6l+4}(e_i \otimes e_{i+1}) = e_i \otimes \alpha_i + \alpha_i \otimes e_{i+1};$$
$$\phi_{6l+5}(e_i \otimes e_{i+2}) = \alpha_i \otimes e_{i+2} - e_i \otimes \alpha_{i+1}; \quad \phi_{6l+6}(e_i \otimes e_i) = e_i \otimes \alpha_{i+2} + \alpha_i \otimes e_i,$$

each A-bimodule homomorphism $D(\phi_*) : D(P_{*-1})_{\nu^{-1}} \to D(P_*)_{\nu^{-1}}$ given by

$$\begin{aligned}
D(\phi_{6l+1})((\alpha_i \otimes \alpha_{i+1})^*) &= (\alpha_{i+2} \otimes \alpha_{i+1})^* \alpha_i - \alpha_{i+2}(\alpha_i \otimes \alpha_{i+2})^*; \\
D(\phi_{6l+2})((\alpha_i \otimes \alpha_{i+2})^*) &= \alpha_i(\alpha_i \otimes \alpha_i)^* + (\alpha_{i+2} \otimes \alpha_{i+2})^* \alpha_i; \\
D(\phi_{6l+3})((\alpha_i \otimes \alpha_i)^*) &= (\alpha_{i+2} \otimes \alpha_i)^* \alpha_i - \alpha_{i+1}(\alpha_i \otimes \alpha_{i+1})^*; \\
D(\phi_{6l+4})((\alpha_i \otimes \alpha_{i+1})^*) &= \alpha_{i+2}(\alpha_i \otimes \alpha_{i+2})^* + (\alpha_{i+2} \otimes \alpha_{i+1})^* \alpha_{i+2}; \\
D(\phi_{6l+5})((\alpha_i \otimes \alpha_{i+2})^*) &= (\alpha_{i+2} \otimes \alpha_{i+2})^* \alpha_i - \alpha_i(\alpha_i \otimes \alpha_i)^*; \\
D(\phi_{6l+6})((\alpha_i \otimes \alpha_i)^*) &= \alpha_{i+1}(\alpha_i \otimes \alpha_{i+1})^* + (\alpha_{i+2} \otimes \alpha_i)^* \alpha_i,
\end{aligned}$$

the A-bimodule homomorphism $\phi_0 : P_0 \to A$ given by the multiplication of A, the A-bimodule homomorphism $D(\phi_0) : D(A)_{\nu^{-1}} \to D(P_0)_{\nu^{-1}}$ given by

$$D(\phi_0)(\langle -, 1 \rangle) = \sum_{i=1}^{3} \alpha_{i+1}(\alpha_i \otimes \alpha_{i+1})^* + (\alpha_{i+2} \otimes \alpha_i)^* \alpha_i,$$

and the composition $\mu : P_0 \to D(P_0)_{\nu^{-1}}$ given by

$$\mu(e_i \otimes e_i) = \alpha_i(\alpha_{i+2} \otimes \alpha_i)^* + (\alpha_{i+1} \otimes \alpha_{i+2})^* \alpha_{i-1}.$$

A complex which is used to compute complete cohomology groups $\mathrm{CH}^*(A)$ is a complex

$$\begin{aligned} \cdots \to A_{\nu^{-1}} \otimes_{A^e} P_1 \xrightarrow{\mathrm{id}\otimes\phi_1} A_{\nu^{-1}} \otimes_{A^e} P_0 \\ \xrightarrow{\mathrm{Hom}(\mu,A)} \mathrm{Hom}_{A^e}(P_0, A) \xrightarrow{\mathrm{Hom}(\phi_1,A)} \mathrm{Hom}_{A^e}(P_1, A) \to \cdots \end{aligned}$$

of which the terms and the nonzero differentials of the non-negative part are determined by

$$\mathrm{Hom}_{A^e}(P_n, A) \cong \begin{cases} \bigoplus_{i=1}^3 e_i A e_i & \text{if } n = 3l, \\ \bigoplus_{i=1}^3 e_i A e_{i+1} & \text{if } n = 3l+1, \\ 0 & \text{if } n = 3l+2, \end{cases}$$

$$\mathrm{Hom}(\phi_{6l+1}, A)(e_i) = \alpha_{i+2} - \alpha_i; \quad \mathrm{Hom}(\phi_{6l+4}, A)(e_i) = \alpha_i + \alpha_{i+2}$$

and that of the negative part are given by

$$A_{\nu^{-1}} \otimes_{A^e} P_n = \begin{cases} 0 & \text{if } n = 3l, \\ \bigoplus_{i=1}^3 \alpha_{i+1} \otimes_{A^e} e_i \otimes e_{i+1} & \text{if } n = 3l+1, \\ \bigoplus_{i=1}^3 e_{i+2} \otimes_{A^e} e_i \otimes e_{i+2} & \text{if } n = 3l+2, \end{cases}$$

$$\mathrm{id} \otimes \phi_{6l+2}(e_{i+2} \otimes_{A^e} e_i \otimes e_{i+2}) = \alpha_{i+2} \otimes_{A^e} e_{i+1} \otimes e_{i+2} + \alpha_{i+1} \otimes_{A^e} e_i \otimes e_{i+1};$$
$$\mathrm{id} \otimes \phi_{6l+5}(e_i \otimes_{A^e} e_{i+1} \otimes e_i) = \alpha_{i+2} \otimes_{A^e} e_{i+1} \otimes e_{i+2} - \alpha_{i+1} \otimes_{A^e} e_i \otimes e_{i+1}.$$

Here the term $\mathrm{Hom}_{A^e}(P_n, A)$ is of degree $n \geq 0$. Note that the two morphisms $\mathrm{Hom}(\phi_{6l+4}, A)$ and $\mathrm{id} \otimes \phi_{6l+2}$ are isomorphisms when $\mathrm{char}\, k \neq 2$. We can see that the complete cohomology groups $\mathrm{CH}^*(A)$ of A are divided into two cases: for $l \geq 0$,

(1) $\mathrm{char}\, k \neq 2, 3$

$$\mathrm{CH}^n(A) = \begin{cases} k\,\overline{1_A} & \text{if } n \equiv 0 \pmod 6, \\ k\,\overline{\alpha_1} & \text{if } n \equiv 1 \pmod 6, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathrm{CH}^{-n}(A) = \begin{cases} k\,\overline{\alpha_2 \otimes_{A^e} e_1 \otimes e_2} & \text{if } n \equiv 5 \pmod 6, \\ k\,\overline{\sum_{i=1}^3 e_{i+2} \otimes_{A^e} e_i \otimes e_{i+2}} & \text{if } n \equiv 0 \pmod 6, n \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

(2) $\mathrm{char}\, k = 2$

$$\mathrm{CH}^n(A) = \begin{cases} k\,\overline{1_A} & \text{if } n = 3l, \\ k\,\overline{\alpha_1} & \text{if } n = 3l+1, \\ 0 & \text{if } n = 3l+2; \end{cases}$$

$$\mathrm{CH}^{-n}(A) = \begin{cases} 0 & \text{if } n = 3l+1, \\ k\,\overline{\alpha_2 \otimes_{A^e} e_1 \otimes e_2} & \text{if } n = 3l+2, \\ k\,\overline{\sum_{i=1}^{3} e_{i+2} \otimes_{A^e} e_i \otimes e_{i+2}} & \text{if } n = 3l+3. \end{cases}$$

As can be seen, the complete cohomology groups have the period six if $\mathrm{char}\, k \neq 2, 3$ and the period three if $\mathrm{char}\, k = 2$. We omit the constructions of two comparison morphisms between the minimal projective resolution and the normalized bar resolution of A. However, they are constructed in a similar way to the first example. We have the graded commutative ring structure and the BV structure on the complete cohomology of A.

Proposition 3.4.9. *If* $\mathrm{char}\, k \neq 2, 3$*, then the complete cohomology ring* $(\mathrm{CH}^\bullet(A), \star)$ *is isomorphic to*

$$k[\alpha, \beta, \alpha^{-1}]/\langle \beta^2 \rangle$$

where $|\alpha| = 6, |\beta| = 1$ *and* $|\alpha^{-1}| = -6$*. Further, if this is the case, then the nonzero BV differentials* $\widehat{\Delta}_*$ *on* $\mathrm{CH}^\bullet(A)$ *are*

$$\widehat{\Delta}_{6l+1}(\alpha^l \beta) = ((6l+1)/3)\,\alpha^l, \quad \widehat{\Delta}_{-6l-5}(\alpha^{-l-1}\beta) = ((-6l-5)/3)\,\alpha^{-l-1}$$

with $l \geq 0$*. In particular, the nonzero Gerstenhaber brackets are induced by*

$$\{\alpha, \beta\} = 2\alpha, \quad \{\beta, \alpha^{-1}\} = 2\alpha^{-1}.$$

Proposition 3.4.10. *If* $\mathrm{char}\, k = 2$*, then the complete cohomology ring* $(\mathrm{CH}^\bullet(A), \star)$ *is isomorphic to*

$$k[\alpha, \beta, \alpha^{-1}]/\langle \beta^2 \rangle$$

where $|\alpha| = 3, |\beta| = 1$ *and* $|\alpha^{-1}| = -3$*. Further, if this is the case, then the nonzero BV differentials on* $\mathrm{CH}^\bullet(A)$ *are*

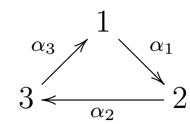
$$\widehat{\Delta}_{6l+1}(\alpha^{2l}\beta) = \alpha^{2l}, \quad \widehat{\Delta}_{-3l-2}(\alpha^{-l-1}\beta) = \alpha^{-l-1}$$

with $l \geq 0$*. In particular, the nonzero Gerstenhaber brackets are induced by*

$$\{\alpha, \beta\} = \alpha.$$

The case $s = 3, N = 3$.

Let Q be the following quiver



and A the algebra kQ/R_Q^3. It follows from Criterion 3.4.2 and Remark 3.4.3 that the Nakayama automorphism of A is diagonalizable if and only if $\operatorname{char} k \neq 3$. Hence, we assume that $\operatorname{char} k \neq 3$. We see that A is a self-injective Nakayama algebra of which the representation matrix of the Nakayama automorphism ν is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

under a k-basis $(e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1)$ of A. This matrix is similar to a diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}$$

where the element $\omega \in k$ is one of roots of a polynomial $x^2 + x + 1$. Moreover, we can decompose $A = A_1 \oplus A_\omega \oplus A_{\omega^2}$, where

$$
\begin{aligned}
A_1 &= \operatorname{Ker}(\nu - \mathrm{id}) = k\,1_A \oplus k\,(\sum_{i=1}^{3} \alpha_i) \oplus k\,(\sum_{i=1}^{3} \alpha_i\alpha_{i+1}),\\
A_\omega &= \operatorname{Ker}(\nu - \omega\,\mathrm{id}) = k\,(\sum_{i=1}^{3} \omega^i e_i) \oplus k\,(\sum_{i=1}^{3} \omega^i \alpha_i) \oplus k\,(\sum_{i=1}^{3} \omega^i \alpha_i\alpha_{i+1}),\\
A_{\omega^2} &= \operatorname{Ker}(\nu - \omega^2\,\mathrm{id})\\
&= k\,(\omega^2 e_1 + \omega e_2 + e_3) \oplus k\,(\omega^2\alpha_1 + \omega\alpha_2 + \alpha_3) \oplus k\,(\omega^2\alpha_1\alpha_2 + \omega\alpha_2\alpha_3 + \alpha_3\alpha_1).
\end{aligned}
$$

In a similar way to the first example, we have a complete resolution of A as follows:

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_2 & \xrightarrow{\phi_2} & P_1 & \xrightarrow{\phi_1} & P_0 & \xrightarrow{\mu} & D(P_0)_{\nu^{-1}} & \xrightarrow{D(\phi_1)} & D(P_1)_{\nu^{-1}} & \xrightarrow{D(\phi_2)} & D(P_2)_{\nu^{-1}} & \longrightarrow \cdots\\
 & & & & & {\scriptstyle\phi_0}\downarrow & \circlearrowright & \uparrow{\scriptstyle D(\phi_0)} & & & & & \\
 & & & & & A & \xrightarrow{\cong} & D(A)_{\nu^{-1}} & & & & &
\end{array}
$$

where each of the P_n and the $D(P_n)_{\nu^{-1}}$ is given by

$$
P_n = \begin{cases} \bigoplus_{i=1}^{3} P(i, i+1) & \text{if } n \text{ is odd,}\\ \bigoplus_{i=1}^{3} P(i, i) & \text{if } n \text{ is even;} \end{cases}
$$

$$
D(P_n)_{\nu^{-1}} = \begin{cases} \bigoplus_{i=1}^{3} A(\alpha_i\alpha_{i+1} \otimes \alpha_i\alpha_{i+1})^* A & \text{if } n \text{ is odd,}\\ \bigoplus_{i=1}^{3} A(\alpha_i\alpha_{i+1} \otimes \alpha_{i+2}\alpha_{i+3})^* A & \text{if } n \text{ is even,} \end{cases}
$$

each A-bimodule homomorphism $\phi_* : P_* \to P_{*-1}$ given by

$$
\begin{aligned}
\phi_{2r+1}(e_i \otimes e_{i+1}) &= \alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i;\\
\phi_{2r}(e_i \otimes e_i) &= e_i \otimes \alpha_{i+1}\alpha_{i+2} + \alpha_i \otimes \alpha_{i+2} + \alpha_i\alpha_{i+1} \otimes e_i,
\end{aligned}
$$

each A-bimodule homomorphism $D(\phi_*) : D(P_{*-1})_{\nu^{-1}} \to D(P_*)_{\nu^{-1}}$ given by

$$
\begin{aligned}
& D(\phi_{2r+1})((\alpha_i\alpha_{i+1} \otimes \alpha_{i+2}\alpha_{i+3})^*)\\
&\qquad = (\alpha_{i+2}\alpha_i \otimes \alpha_{i+2}\alpha_i)^*\alpha_{i+1} - \alpha_{i+1}(\alpha_i\alpha_{i+1} \otimes \alpha_i\alpha_{i+1})^*;\\
& D(\phi_{2r})((\alpha_i\alpha_{i+1} \otimes \alpha_i\alpha_{i+1})^*)\\
&\qquad = \alpha_{i+2}\alpha_{i+3}(\alpha_i\alpha_{i+1} \otimes \alpha_{i+2}\alpha_{i+3})^* + \alpha_{i+2}(\alpha_{i-1}\alpha_i \otimes \alpha_{i+1}\alpha_{i+2})^*\alpha_{i-2}\\
&\qquad\qquad + (\alpha_{i-2}\alpha_{i-1} \otimes \alpha_i\alpha_{i+1})^*\alpha_{i-3}\alpha_{i-2},
\end{aligned}
$$

the A-bimodule homomorphism $\phi_0 : P_0 \to A$ given by the multiplication of A, the A-bimodule homomorphism $D(\phi_0) : D(A)_{\nu^{-1}} \to D(P_0)_{\nu^{-1}}$ given by

$$
\begin{aligned}
D(\phi_0)(\langle -, 1\rangle) = \sum_{i=1}^{3} \big(&\alpha_i\alpha_{i+1}(\alpha_{i-2}\alpha_{i-1} \otimes \alpha_i\alpha_{i+1})^* + \alpha_{i+1}(\alpha_{i-2}\alpha_{i-1} \otimes \alpha_i\alpha_{i+1})^*\alpha_i\\
&+ (\alpha_{i-2}\alpha_{i-1} \otimes \alpha_i\alpha_{i+1})^*\alpha_{i-3}\alpha_{i-2}\big),
\end{aligned}
$$

and the composition $\mu : P_0 \to D(P_0)_{\nu^{-1}}$ given by

$$\mu(e_i \otimes e_i) = \alpha_i\alpha_{i+1}(\alpha_{i-2}\alpha_{i-1} \otimes \alpha_i\alpha_{i+1})^* + (\alpha_{i-1}\alpha_i \otimes \alpha_{i+1}\alpha_{i+2})^*\alpha_{i-2}\alpha_{i-1} \\ + \alpha_i(\alpha_{i-3}\alpha_{i-2} \otimes \alpha_{i-1}\alpha_i)^*\alpha_{i-4}.$$

Moreover, a complex which is used to compute complete cohomology groups is a complex

$$\cdots \to A_{\nu^{-1}} \otimes_{A^e} P_1 \xrightarrow{\mathrm{id}\otimes\phi_1} A_{\nu^{-1}} \otimes_{A^e} P_0 \\ \xrightarrow{\mathrm{Hom}(\mu,A)} \mathrm{Hom}_{A^e}(P_0, A) \xrightarrow{\mathrm{Hom}(\phi_1,A)} \mathrm{Hom}_{A^e}(P_1, A) \to \cdots$$

of which the terms and the nonzero differentials are determined by

$$\mathrm{Hom}_{A^e}(P_n, A) \cong \begin{cases} \bigoplus_{i=1}^3 k\,\alpha_i & \text{if } n \text{ is odd,} \\ \bigoplus_{i=1}^3 k\,e_i & \text{if } n \text{ is even;} \end{cases}$$

$$A_{\nu^{-1}} \otimes_{A^e} P_n = \begin{cases} \bigoplus_{i=1}^3 k\,e_{i+1} \otimes_{A^e} e_i \otimes e_{i+1} & \text{if } n \text{ is odd,} \\ \bigoplus_{i=1}^3 k\,\alpha_i \otimes_{A^e} e_i \otimes e_i & \text{if } n \text{ is even;} \end{cases}$$

$$\mathrm{Hom}(\phi_{2r+1}, A)(e_i) = \alpha_{i+1} - \alpha_i;$$

$$\mathrm{id} \otimes \phi_{2r+1}(e_i \otimes_{A^e} e_{i+1} \otimes e_i) = \alpha_i \otimes_{A^e} e_i \otimes e_i - \alpha_{i+1} \otimes_{A^e} e_{i+1} \otimes e_{i+1}$$

and whose term $\mathrm{Hom}_{A^e}(P_n, A)$ is of degree $n \geq 0$. Therefore, we have, for $n \geq 0$,

$$\mathrm{CH}^n(A) = \begin{cases} k\,\overline{\alpha_1} & \text{if } n \text{ is odd,} \\ k\,\overline{1_A} & \text{if } n \text{ is even;} \end{cases}$$

$$\mathrm{CH}^{-n}(A) = \begin{cases} k\,\overline{\alpha_1 \otimes_{A^e} e_1 \otimes e_1} & \text{if } n \text{ is odd,} \\ k\,\overline{\sum_{i=1}^3 e_{i+1} \otimes_{A^e} e_i \otimes e_{i+1}} & \text{if } n > 0 \text{ is even.} \end{cases}$$

We omit the description of comparison morphisms between the minimal projective resolution and the normalized bar resolution of A, because it is not easy to write the two comparison morphisms. However, a direct calculation shows the graded commutative ring structure and the BV structure on the complete cohomology of A.

Proposition 3.4.11. *The complete cohomology ring* $(\mathrm{CH}^\bullet(A), \star)$ *is isomorphic to*

$$k[\alpha, \beta, \alpha^{-1}]/\langle \beta^2 \rangle$$

where $|\alpha| = 2, |\beta| = 1$ *and* $|\alpha^{-1}| = -2$*. Moreover, the nonzero BV differentials on* $\mathrm{CH}^\bullet(A)$ *are*

$$\widehat{\Delta}_{2l+1}(\alpha^l\beta) = ((3l+2)/3)\alpha^l, \quad \widehat{\Delta}_{-2l-1}(\alpha^{-l-1}\beta) = \begin{cases} (-1/3)\,\alpha^{-1} & \text{if } l = 0, \\ ((-3l-2)/3)\,\alpha^{-l-1} & \text{if } l \neq 0 \end{cases}$$

with $l \geq 0$*. In particular, the nonzero Gerstenhaber brackets are induced by*

$$\{\alpha, \beta\} = \alpha, \quad \{\beta, \alpha^{-1}\} = \alpha^{-1}.$$

Chapter 4

Tate-Hochschild cohomology rings for eventually periodic Gorenstein algebras

In this chapter, for Gorenstein algebras, we decide when Tate-Hochschild cohomology rings have homogeneous invertible elements of positive degree. The key is eventual periodicity of algebras. Furthermore, for connected periodic algebras, we give a computation method of their Tate-Hochschild cohomology rings, using their Hochschild cohomology rings.

First, we define eventually periodic algebras and provide examples of them. Throughout this chapter, we assume that the ground field k is algebraically closed.

4.1. Eventually periodic algebras

As mentioned above, let us first define the eventual periodicity of algebras and provide examples of eventually periodic algebras. Before that, we make a remark on eventually periodic modules and eventually periodic algebras.

Eventually periodic modules have been studied over commutative Noetherian local ring and over artin algebras. In the first case, such modules are defined to have minimal free resolutions that eventually become periodic. On the other hand, over artin algebras, eventually periodic modules mean modules whose minimal projective resolutions have the same property as above. Eisenbud [**18**], Avramov, Gasharov and Peeva [**5**] and Croll [**16**] investigated eventually periodic modules in the commutative Noetherian local ring case, while Bergh [**10**] and the author [**45**] did in the artin algebra case.

By considering an algebra A as a one-sided module over the enveloping algebra A^{e}, Küpper [**34**, Definition 2.3] introduced eventually periodic algebras. Our definition of such algebras is slightly weaker than Küpper's one (since our eventually periodic algebras may have finite projective dimension as bimodules). Now, we define eventually periodic algebras precisely.

Definition 4.1.1. Let A be an algebra. An A-module M is called *periodic* if $\Omega_A^p(M) \cong M$ in A-mod for some $p > 0$. The smallest such p is said to be the *period* of M. We say that $M \in A$-mod is *eventually periodic* if $\Omega_A^n(M)$ is periodic for some $n \geq 0$. An algebra A is called *periodic* (resp. *eventually periodic*) if $A \in A^{\mathrm{e}}$-mod is periodic (resp. eventually periodic).

From the definition, periodic algebras are eventually periodic algebras. Periodic algebras have been studied for a long time (see [**19**]). We know from [**27**, Lemma 1.5] that periodic algebras are self-injective algebras (i.e. 0-Gorenstein algebras). On the other hand, it follows from the proof of [**17**, Corollary 6.4] that monomial Gorenstein algebras are eventually periodic algebras. It also follows from the formula $\mathrm{gl.dim}\, A = \mathrm{proj.dim}_{A^{\mathrm{e}}} A$ (see [**29**, Section 1.5]) that algebras of finite global dimension are eventually periodic algebras. As will be seen in Example 4.1.2 below, not all eventually periodic algebras are Gorenstein algebras.

Example 4.1.2. (1) Let A_1 be the algebra given by the following quiver with relation

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad \alpha\beta\alpha = 0.$$

Then A_1 is a monomial algebra that is not Gorenstein (since $\mathrm{inj.dim}_A Ae_1 = \infty$, where e_1 is the primitive idempotent corresponding to the vertex 1). Using Bardzell's minimal projective resolution of a monomial algebra (see [**7**]), we have that A_1 is an eventually periodic algebra having $\Omega^2_{A_1^{\mathrm{e}}}(A_1)$ as its first periodic syzygy.

(2) Let A_2 be the algebra given by the following quiver with relation

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \qquad \alpha^2 = 0.$$

Then the algebra A_2 is monomial 1-Gorenstein and hence eventually periodic. Bardzell's minimal projective resolution allows us to see that $\Omega^2_{A_2^{\mathrm{e}}}(A_2)$ is the first periodic syzygy of A_2.

Moreover, one can see that the algebras in [**14**, Example 4.3] are eventually periodic algebras.

4.2. Tate-Hochschild cohomology rings of eventually periodic Gorenstein algebras

This section is devoted to showing the main result of this chapter. We prove it after two propositions below. Before the first one, we prepare some terminology. Recall that we write $\Omega_i(X_\bullet) = \mathrm{Cok}\, d^X_{i+1}$ for a complex $X_\bullet$ and $i \in \mathbb{Z}$. For a module M over a

Gorenstein algebra A, we say that its complete resolution $T_\bullet \to P_\bullet \to M$ is *periodic* if there exists an integer $p > 0$ such that $\Omega_i(T_\bullet) \cong \Omega_{i+p}(T_\bullet)$ in A-mod for all $i \in \mathbb{Z}$. The least integer $p > 0$ with this property is called the *period* of the complete resolution.

Proposition 4.2.1. *Let A be a Gorenstein algebra and M an A-module. If there exists an integer $n \geq 0$ such that $\Omega_A^n(M)$ is periodic of period p, then M admits a periodic complete resolution of period p. Further, the period of the periodic complete resolution is independent of the choice of periodic syzygies.*

Proof. Assume that there exists a minimal projective resolution $P_\bullet \to M$ satisfying that $\Omega_A^n(M)$ is periodic of period p. Then, by using the periodicity of $\Omega_A^n(M)$, we can extend the truncated complex $P_{\geq n}$ to an (unbounded) complex $T_\bullet$ in $\underline{\mathrm{APC}}(A)$ having the following properties:

(i) $T_{\geq n} = P_{\geq n}$.

(ii) For each $i \in \mathbb{Z}$, there exists an integer $0 \leq j < p$ such that $\Omega_i(T_\bullet) \cong \Omega_A^{n+j}(M)$.

In particular, one sees that $\Omega_i(T_\bullet) \cong \Omega_{i+p}(T_\bullet)$ for all $i \in \mathbb{Z}$. Note that one may take $T_\bullet = 0$ if $\mathrm{proj.dim}_A M < \infty$. It follows from Theorem 2.3.1 that $\Omega_i(T_\bullet) = \Sigma^{-i}\Omega_0(T_\bullet)$ is Cohen-Macaulay for each $i \in \mathbb{Z}$, where Σ denotes the shift functor on $\underline{\mathrm{CM}}(A)$. Then it is easily checked that $\mathrm{Hom}_{\mathcal{K}(A)}(T_\bullet, A[i]) = 0$ for all $i \in \mathbb{Z}$, where $\mathcal{K}(A)$ is the homotopy category of A-modules. Hence, as in [**15**, Lemma 2.4], the family $\{\mathrm{id}_{T_j}\}_{j \geq n}$ can be extended uniquely up to homotopy to a chain map $\theta : T_\bullet \to P_\bullet$ with θ_j the identity for all $j \geq n$. Therefore, the chain map θ gives rise to the desired complete resolution. We remark that the period of the resulting complete resolution does not depend on the choice of n. Indeed, if we take the smallest integer $r \geq 0$ such that $\Omega_A^r(M)$ is periodic, then, for each $i \geq n$, the module $\Omega_A^i(M)$ is periodic and has the same period as $\Omega_A^r(M)$. □

Recall that the Yoneda product of the Tate cohomology ring $\widehat{\mathrm{Ext}}_A^\bullet(M, M)$ is denoted by $\smile$.

Proposition 4.2.2. *Let A be a Gorenstein algebra and M a A-module. Then the following conditions are equivalent.*

(1) *The Tate cohomology ring $\widehat{\mathrm{Ext}}_A^\bullet(M, M)$ has an invertible homogeneous element of positive degree.*

(2) *M is eventually periodic.*

Proof. It suffices to prove the statement for $M \in A$-mod with $\mathrm{proj.dim}_A M = \infty$. First, we assume that a A-module M satisfies that $\Omega_A^n(M)$ is periodic of period p for some $n \geq 0$. By Proposition 4.2.1, there exists a complete resolution $T_\bullet \to P_\bullet \to M$ such that $\Omega_0(T_\bullet)$ is periodic of period p, where p is the period of $\Omega_A^n(M)$. We fix this complete resolution. Then the shift functor Σ on $\underline{\mathrm{CM}}(A)$ satisfies $\Sigma^i\Omega_0(T_\bullet) = \Omega_{-i}(T_\bullet)$ for all $i \in \mathbb{Z}$. Let $f \in \mathrm{Hom}_A(\Omega_p(T_\bullet), \Omega_0(T_\bullet))$ be an isomorphism and consider two homogeneous elements

$$x := \Sigma^p[f] \in \widehat{\mathrm{Ext}}_A^p(M, M) \quad \text{and} \quad y := [f^{-1}] \in \widehat{\mathrm{Ext}}_A^{-p}(M, M).$$

Then we have $x \smile y = (\Sigma^{-p}x) \circ y = [f] \circ [f^{-1}] = 1$ and similarly $y \smile x = 1$, where we set $1 := [\mathrm{id}_{\Omega_0(T_\bullet)}]$.

Conversely, we let $T_\bullet \to P_\bullet \to M$ be a complete resolution of M and assume that there exists an isomorphism

$$x \in \underline{\mathrm{Hom}}_A(\Omega_0(T_\bullet), \Sigma^p\Omega_0(T_\bullet)) = \widehat{\mathrm{Ext}}_A^p(M, M)$$

of degree $p > 0$. From the definition of complete resolutions, we have

$$\begin{aligned} \underline{\mathrm{Hom}}_A(\Omega_0(T_\bullet), \Sigma^p\Omega_0(T_\bullet)) &\cong \underline{\mathrm{Hom}}_A(\Sigma^{-m-p}\Omega_0(T_\bullet), \Sigma^{-m}\Omega_0(T_\bullet)) \\ &\cong \underline{\mathrm{Hom}}_A(\Omega_A^{m+p}(M), \Omega_A^p(M)) \end{aligned}$$

for some sufficiently large $m > 0$. Hence we get $\Omega_A^{m+p}(M) \cong \Omega_A^m(M)$ in A-$\underline{\mathrm{mod}}$. This implies that $\Omega_A^{m+p}(M) \oplus P \cong \Omega_A^m(M) \oplus Q$ in A-mod for some P and $Q \in A$-proj. By applying the syzygy functor Ω_A to this isomorphism, we obtain an isomorphism $\Omega_A^{m+p+1}(M) \cong \Omega_A^{m+1}(M)$ in A-mod. This completes the proof. □

Using Proposition 4.2.2, we obtain our main result.

Theorem 4.2.3. *Let A be a Gorenstein algebra. Then the following conditions are equivalent.*

(1) *The Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^\bullet(A)$ has an invertible homogeneous element of positive degree.*

(2) *A is an eventually periodic algebra.*

In this case, there exists an isomorphism $\widehat{\mathrm{HH}}^\bullet(A) \cong \widehat{\mathrm{HH}}^{\geq 0}(A)[\chi^{-1}]$ of graded algebras, where the degree of an invertible homogeneous element χ equals the period of the periodic syzygy $\Omega_{A^e}^n(A)$ of A for some $n \geq 0$.

Proof. We know from [**3**, Proposition 2.2] that if A is a Gorenstein algebra, then so is the enveloping algebra A^e. Hence the former statement follows from Proposition 4.2.2 applied to $A \in A^e$-mod. On the other hand, suppose that the Gorenstein algebra A satisfies that $\Omega_{A^e}^n(A)$ is periodic for some $n \geq 0$. By the proof of Proposition 4.2.2, there exists an invertible homogeneous element $\chi \in \widehat{\mathrm{HH}}^\bullet(A)$ whose degree equals the period of the periodic A^e-module $\Omega_{A^e}^n(A)$. Then the fact that $\widehat{\mathrm{HH}}^\bullet(A)$ is a graded commutative algebra yields the desired isomorphism of graded algebras (cf. the proof of [**44**, Corollary 3.4]). □

Remark 4.2.4. From the definition of singularity categories, an algebra A has finite projective dimension as a A^{e}-module if and only if its Tate-Hochschild cohomology ring is the zero ring (cf. [**12**, Section 1]). Thus Theorem 4.2.3 makes essential sense for the case of Gorenstein algebras with infinite global dimension.

Remark 4.2.5. Applying Theorem 4.2.3 to monomial Gorenstein algebras and to periodic algebras, one obtains [**17**, Corollary 6.4] and [**44**, Corollary 3.4], respectively.

Remark 4.2.6. For an eventually periodic Gorenstein algebra A, one can obtain $\dim_k \widehat{\mathrm{HH}}^i(A)$ for all integers i by using Theorem 4.2.3 and the Hochschild cohomology $\mathrm{HH}^\bullet(A) := \bigoplus_{i\geq 0} \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A)$ of A (see Example 4.3.7). In Section 4.4, we explain how we compute the graded subring $\widehat{\mathrm{HH}}^{\geq 0}(A)$ when A is connected and periodic. However, it is open how we compute the ring structure of $\widehat{\mathrm{HH}}^{\geq 0}(A)$ in general.

Recall that two algebras A and B are *derived equivalent* if there exists a triangle equivalence between $\mathcal{D}^{\mathrm{b}}(A\text{-mod})$ and $\mathcal{D}^{\mathrm{b}}(B\text{-mod})$ (see [**41**]). The following shows that being eventually periodic Gorenstein is invariant under derived equivalence.

Proposition 4.2.7. *Assume that two algebras A and B are derived equivalent. If A is eventually periodic Gorenstein, then so is B. In particular, the periods of their periodic syzygies coincide.*

Proof. By [**48**, Theorem 1.1] and [**32**, Proposition 1.7], Tate-Hochschild cohomology ring and the property of being Gorenstein are invariant under derived equivalence. Thus the statement follows from Theorem 4.2.3.

Now, let p and q be the periods of some periodic syzygies of the regular bimodules A and B, respectively. Note that all the periodic syzygies of A and B has period p and q, respectively. Then it follows from the proof of Proposition 4.2.2 that there exist invertible homogeneous elements $\chi_A \in \widehat{\mathrm{HH}}^\bullet(A)$ and $\chi_B \in \widehat{\mathrm{HH}}^\bullet(B)$ with $\deg \chi_A = p$ and $\deg \chi_B = q$, where $\deg \chi_A$ denotes the degree of the homogeneous element χ_A. We claim that $p = q$. Since $\widehat{\mathrm{HH}}^\bullet(A) \cong \widehat{\mathrm{HH}}^\bullet(B)$ as graded rings, $\widehat{\mathrm{HH}}^\bullet(B)$ has an invertible homogeneous element of degree p, and an argument as in the proof of the implication from (2) to (1) in Proposition 4.2.2 shows that there exists an isomorphism $\Omega^{j+p}_{B^{\mathrm{e}}}(B) \cong \Omega^j_{B^{\mathrm{e}}}(B)$ in B^{e}-mod for some $j \gg 0$. Since the periodic syzygy $\Omega^j_{B^{\mathrm{e}}}(B)$ has period q, we obtain that q divides p. Since one can similarly show that p divides q, we conclude that $p = q$. □

Periodic algebras and algebras with finite global dimension are both eventually periodic Gorenstein, and being periodic and finiteness of global dimension are derived invariants (see [**32**] and [**24**] for example). Consequently, Proposition 4.2.7 gives a new reslt only for eventually periodic Gorenstein algebras of infinite global dimension that are not periodic. In the next section, we construct such algebras by means of tensor product of algebras.

4.3. Construction of eventually periodic Gorenstein algebras

In this section, we aim at describing a way of giving eventually periodic Gorenstein algebras. First, we show two propositions which will be used latter. Let us start with the following.

Proposition 4.3.1. *Any periodic A-module M over a d-Gorenstein algebra A is Cohen-Macaulay.*

Proof. Assume that M is a periodic A-module of period p. Since $\Omega_A^i(M) \in \mathrm{CM}(A)$ for $i \geq d$ by [**12**, Lemma 4.2.2], we have that $M \cong \Omega_A^{jp}(M) \in \mathrm{CM}(A)$ for some $j \gg 0$. □

We now show that, for an eventually periodic Gorenstein algebra A, the smallest integer $n \geq 0$ satisfying that $\Omega^n_{A^e}(A)$ is periodic has a lower bound.

Proposition 4.3.2. *Let A be a d-Gorenstein algebra. Assume that there exists an integer $n \geq 0$ such that $\Omega^n_{A^e}(A)$ is periodic. Then the least such integer n satisfies $n \geq d$. In particular, an equality holds if and only if there exists a simple A-module S such that $\mathrm{Ext}^n_A(S, A) \neq 0$.*

Proof. Let A be an eventually periodic Gorenstein algebra and $P_\bullet \to A$ a minimal projective resolution of A over A^e satisfying that $\Omega^n_{A^e}(A)$ is the first periodic syzygy of period p. For any $M \in A$-mod, an exact sequence $P_\bullet \otimes_A M \to A \otimes_A M = M$ is a projective resolution of M and has the property that $\Omega_n(P_\bullet \otimes_A M) = \Omega^n_{A^e}(A) \otimes_A M \cong \Omega^{n+ip}_{A^e}(A) \otimes_A M = \Omega_{n+ip}(P_\bullet \otimes_A M)$ for all $i \geq 0$. In particular, as in Proposition 4.3.1, one concludes that $\Omega_n(P_\bullet \otimes_A M)$ is Cohen-Macaulay. This implies that $n \geq \mathrm{inj.dim}_A A = d$. Indeed, for any A-module M, we have $\mathrm{Ext}^{n+1}_A(M, A) \cong \mathrm{Ext}^1_A(\Omega_n(P_\bullet \otimes_A M), A) = 0$.

For the latter statement, we first suppose that $n = d$. Then it follows from [**17**, Proposition 2.4] that we have $n = \mathrm{G\text{-}dim}_A(A/\mathfrak{r})$, where $\mathfrak{r}$ denotes the Jacobson radical of A. This shows that $\mathrm{Ext}^n_A(A/\mathfrak{r}, A) \neq 0$, so that one obtains the desired simple A-module. Conversely, assume that $\mathrm{Ext}^n_A(S, A) \neq 0$ for some simple A-module S. Then one concludes that $\Omega^{n-1}_A(S) \notin \mathrm{CM}(A)$. However, since we know that $\Omega^n_A(S)$ is Cohen-Macaulay, we have $n = \mathrm{G\text{-}dim}_A S$ and hence $n \leq d$. Then the proof is completed since $n \geq d$ by the former statement. □

Now, we recall some facts on projective resolutions for tensor algebras. Let A and B be algebras and $P_\bullet \xrightarrow{\varepsilon_A} A$ and $Q_\bullet \xrightarrow{\varepsilon_B} B$ projective resolutions as bimodules. Then the tensor product $P_\bullet \otimes Q_\bullet \xrightarrow{\varepsilon_A \otimes \varepsilon_B} A \otimes B$ is a projective resolution of the tensor algebra $A \otimes B$ over $(A \otimes B)^e$ (see [**38**, Section X.7]). Here, we identify $(A \otimes B)^e$ with $A^e \otimes B^e$. It also follows from [**9**, Lemma 6.2] that if both $P_\bullet \to A$ and $Q_\bullet \to B$ are minimal, then so is $P_\bullet \otimes Q_\bullet \to A \otimes B$.

From now on, we assume that A is a periodic algebra of period p and that B is an algebra of finite global dimension n. Set $C := A \otimes B$. Since periodic algebras are self-injective algebras, it follows from [**9**, Lemma 6.1] that we have

$$\mathrm{inj.dim}\, C = \mathrm{inj.dim}\, A + \mathrm{inj.dim}\, B = 0 + n = n$$

as one-sided modules. Thus C is an n-Gorenstein algebra. Note that the same lemma also implies that the enveloping algebra C^{e} is a $(2n)$-Gorenstein algebra. We now show that the algebra C has an eventually periodic minimal projective resolution.

Proposition 4.3.3. *Let A and B be as above. Then $C = A \otimes B$ is an eventually periodic n-Gorenstein algebra with* $\operatorname{gl.dim} C = \infty$ *such that $\Omega^n_{C^{\mathrm{e}}}(C)$ is the first periodic syzygy of C.*

Proof. Let $P_\bullet \to A$ and $Q_\bullet \to B$ be minimal projective resolutions as bimodules. Recall that the r-th component of the total complex $P_\bullet \otimes Q_\bullet$ with $r \geq 0$ is given by

$$(P_\bullet \otimes Q_\bullet)_r = \bigoplus_{i=0}^{r} P_{r-i} \otimes Q_i.$$

Since $Q_i = 0$ for $i > n = \operatorname{gl.dim} B = \operatorname{proj.dim}_{B^{\mathrm{e}}} B$, we have

$$(P_\bullet \otimes Q_\bullet)_r = \bigoplus_{i=0}^{n} P_{r-i} \otimes Q_i$$

for all $r \geq n$. Moreover, the $(r+1)$-th differential

$$d^{P\otimes Q}_{r+1} : (P_\bullet \otimes Q_\bullet)_{r+1} \to (P_\bullet \otimes Q_\bullet)_r \quad (r \geq n)$$

can be written as the square matrix $(\partial^{ij}_{r+1})_{ij}$ of degree $n+1$ whose (i,j)-th entry

$$\partial^{ij}_{r+1} : P_{r+1-(j-1)} \otimes Q_{j-1} \to P_{r-(i-1)} \otimes Q_{i-1} \quad (1 \leq i, j \leq n+1)$$

is given by

$$\partial^{ij}_{r+1} = \begin{cases} d^P_{r-i+2} \otimes \mathrm{id}_{Q_{i-1}} & \text{if } i = j; \\ (-1)^{r-i+1}\, \mathrm{id}_{P_{r-i+1}} \otimes d^Q_i & \text{if } j = i+1; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\operatorname{Cok} d^{P\otimes Q}_{n+p+1} \cong \operatorname{Cok} d^{P\otimes Q}_{n+1}$. First, suppose that p is even. Since $\partial^{ij}_{n+p+1} = \partial^{ij}_{n+1}$ for all $1 \leq i, j \leq n+1$ because p is even and $d^P_l = d^P_{l+p}$ for any $l \geq 0$, we conclude that $d^{P\otimes Q}_{n+p+1} = d^{P\otimes Q}_{n+1}$, which implies the claim. Now, assume that p is odd. Consider the isomorphism of C^{e}-modules between $(P_\bullet \otimes Q_\bullet)_r$ and $(P_\bullet \otimes Q_\bullet)_{r+p}$ with $r \geq n$ induced by the diagonal matrix D of degree $n+1$ whose (i,i)-th entry is $(-1)^{n+i}$. Together with the fact that $p+1$ is even, a direct calculation shows that there exists a commutative diagram of C^{e}-modules with exact rows

$$\begin{array}{ccccccc} (P_\bullet \otimes Q_\bullet)_{n+p+1} & \xrightarrow{d^{P\otimes Q}_{n+p+1}} & (P_\bullet \otimes Q_\bullet)_{n+p} & \longrightarrow & \operatorname{Cok} d^{P\otimes Q}_{n+p+1} & \longrightarrow & 0 \\ \cong \big\downarrow D & & \cong \big\downarrow D & & & & \\ (P_\bullet \otimes Q_\bullet)_{n+1} & \xrightarrow{d^{P\otimes Q}_{n+1}} & (P_\bullet \otimes Q_\bullet)_{n} & \longrightarrow & \operatorname{Cok} d^{P\otimes Q}_{n+1} & \longrightarrow & 0 \end{array}$$

This implies the claim. Since the projective resolution $P_\bullet \otimes Q_\bullet \to C$ is minimal, we have that $\Omega^{n+p}_{C^e}(C) = \operatorname{Cok} d^{P\otimes Q}_{n+p+1} \cong \operatorname{Cok} d^{P\otimes Q}_{n+1} = \Omega^n_{C^e}(C)$. From Proposition 4.3.2 and the isomorphism, we conclude that the n-th syzygy $\Omega^n_{C^e}(C)$ is the first periodic syzygy of C. □

Remark 4.3.4. Proposition 2.3.2 allows us to get $\text{G-dim}_{C^e}C \leq 2n = \text{inj.dim}_{C^e}C^e$ and hence $\mathrm{HH}^i(C) \cong \widehat{\mathrm{HH}}^i(C)$ for all $i > 2n$. On the other hand, the i-th syzygy $\Omega^i_{C^e}(C)$ of C is Cohen-Macaulay for any $i \geq n$ by Propositions 4.3.1 and 4.3.3. Again, Proposition 2.3.2 yields that $\text{G-dim}_{C^e}C \leq n$. One of the advantages of this observation is that there exists an isomorphism $\mathrm{HH}^i(C) \cong \widehat{\mathrm{HH}}^i(C)$ for all $i > n$.

Remark 4.3.5. It follows from Theorem 4.2.3 and the proof of Proposition 4.3.3 that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^\bullet(C)$ of C is of the form $\widehat{\mathrm{HH}}^{\geq 0}(C)[\chi^{-1}]$, where the degree of χ divides the period p of A.

We end this section with the following two examples. Note that the tensor algebra C in Example 4.3.7 can be found in [**9**, Example 6.3].

Example 4.3.6. For an integer $n \geq 0$, let B_n be the algebra given by the following quiver with relations

$$0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} \cdots \longrightarrow n-1 \xrightarrow{\alpha_{n-1}} n \qquad \alpha_{i+1}\alpha_i = 0 \text{ for } i = 0, \ldots, n-2.$$

Then we have $\text{gl.dim}\, B_n = n$. By Proposition 4.3.3, any periodic algebra A gives us an eventually periodic n-Gorenstein algebra $C = A \otimes B_n$ with $\Omega^n_{C^e}(C)$ the first periodic syzygy of C.

Example 4.3.7. Let $A = k[x]/(x^2)$ and let B be the algebra B_1 defined in Example 4.3.6. Thanks to Bardzell's minimal projective resolution, we see that A is a periodic algebra whose period is equal to 1 if $\operatorname{char} k = 2$ and to 2 otherwise. On the other hand, the tensor algebra $C = A \otimes B$ is given by the following quiver with relations

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \circlearrowleft \gamma \qquad \alpha^2 = 0 = \gamma^2 \quad \text{and} \quad \beta\alpha = \gamma\beta.$$

Thus we see that C is a (non-monomial) eventually periodic Gorenstein algebra whose first periodic syzygy is $\Omega^1_{C^e}(C)$. Now, we compute $\dim_k \widehat{\mathrm{HH}}^i(C)$ for all $i \in \mathbb{Z}$. It follows from [**29**, Section 1.6] that the Hochschild cohomology ring $\mathrm{HH}^\bullet(B)$ is of the form

$$\mathrm{HH}^\bullet(B) = k.$$

According to [**8**, Section 5], the Hochschild cohomology ring $\mathrm{HH}^\bullet(A)$ is as follows:

$$\mathrm{HH}^\bullet(A) = \begin{cases} k[a_0,\, a_1]/(a_0^2) & \text{if } \operatorname{char} k = 2; \\ k[a_0,\, a_1,\, a_2]/(a_0^2,\, a_1^2,\, a_0a_1,\, a_0a_2) & \text{if } \operatorname{char} k \neq 2, \end{cases}$$

where the index i of a homogeneous element a_i denotes the degree of a_i. On the other hand, by [**36**, Lemma 3.1], there exists an isomorphism of graded algebras

$$\mathrm{HH}^\bullet(C) \cong \mathrm{HH}^\bullet(A) \otimes \mathrm{HH}^\bullet(B) = \mathrm{HH}^\bullet(A).$$

It follows from Remark 4.3.4 that $\mathrm{HH}^i(C) \cong \widehat{\mathrm{HH}}^i(C)$ for all $i > 1$. Hence, the fact that $\widehat{\mathrm{HH}}^*(C) \cong \widehat{\mathrm{HH}}^{*+p}(C)$ with p the period of A (see Remark 4.3.5) implies that, for any integer i, we have

$$\dim_k \widehat{\mathrm{HH}}^i(C) = \begin{cases} 2 & \text{if } \operatorname{char} k = 2; \\ 1 & \text{if } \operatorname{char} k \neq 2. \end{cases}$$

4.4. The case of connected periodic algebras

The aim of this section is to describe the Tate-Hochschild cohomology rings of connected periodic algebras. Throughout this section, all algebras are assumed to be connected.

We first remind the reader of two results, which are extended to Tate cases later. Let us begin with a result of Carlson [**13**] (see also [**27**, Proposition 1.3]).

Theorem 4.4.1 ([**13**]). *Let A be a self-injective algebra, M an indecomposable periodic A-module and $\mathcal{N}(M)$ the ideal of the Yoneda algebra $\mathrm{Ext}^\bullet_A(M,M)$ generated by the homogeneous nilpotent elements. Then we have*

$$\mathrm{Ext}^\bullet_A(M,M)/\mathcal{N}(M) \cong k[x],$$

where the degree of the homogeneous element x is equal to the period of M.

The following is a result due to Green, Snashall and Solberg [**27**].

Theorem 4.4.2 ([**27**, Theorem 1.6]). *Let A be an algebra satisfying $\Omega^n_{A^\mathrm{e}}(A) \cong {}_1\Lambda_\sigma$ for some $n \geq 1$ and some automorphism σ of A and $\mathcal{N}$ the ideal of $\mathrm{HH}^\bullet(A)$ generated by the homogeneous nilpotent elements. Then we have*

$$\mathrm{HH}^\bullet(A)/\mathcal{N} \cong \begin{cases} k[x] & \textit{if } A \textit{ is periodic} \\ k & \textit{otherwise.} \end{cases}$$

Remark that the algebra appearing in Theorem 4.4.2 is self-injective ([**27**, Lemma 1.5]). In particular, periodic algebras are self-injective algebras.

Now, we extend the two theorems above to Tate cases. Our proofs of the extended statements are based on that of the original statements. Let us first consider the Tate cohomology ring modulo nilpotence of a periodic module over a self-injective algebra.

Proposition 4.4.3. *Let A be a self-injective algebra, M an indecomposable periodic A-module of period d and $\widehat{\mathcal{N}}(M)$ the ideal of $\widehat{\mathrm{Ext}}_A^{\bullet}(M, M)$ generated by the homogeneous nilpotent elements. Then we have*

$$\widehat{\mathrm{Ext}}_A^{\bullet}(M, M)/\widehat{\mathcal{N}}(M) \cong k[x, y]/(xy - 1)$$

with $\deg x = d$ *and* $\deg y = -d$*, where* $\deg z$ *denotes the degree of a homogeneous element* z*.*

Proof. Let $f : \Omega_A^s(M) \to M$ be a morphism in A--mod, where s is an integer. We first show that if $s \not\equiv 0$ (-mod$d$), then $[f] \in \widehat{\mathrm{Ext}}_A^{s}(M, M)$ is nilpotent with respect to the Yoneda product. Let $r \geq 1$ and $q \neq 0$ be integers such that $rs = qd$, and consider $\alpha := [f]^r \in \widehat{\mathrm{Ext}}_A^{qd}(M, M) \cong \underline{\mathrm{End}}_A(M)$. Since $[f]$ is not an isomorphism because of the choice of s, then neither is α, so that we have $\alpha \in \operatorname{rad} \underline{\mathrm{End}}_A(M)$, which implies that it is a nilpotent element in the local algebra $\underline{\mathrm{End}}_A(M)$. Then, for any $n \geq 1$, the morphism $\Omega_A^{nd}(\alpha) \in \underline{\mathrm{End}}_A(M)$ is also a nilpotent element and hence in $\operatorname{rad} \underline{\mathrm{End}}_A(M)$. Since the ideal $\operatorname{rad} \underline{\mathrm{End}}_A(M)$ is nilpotent, we obtain $\alpha^l = \alpha \circ \Omega_A^d(\alpha) \circ \Omega_A^{2d}(\alpha) \circ \cdots \circ \Omega_A^{(l-1)d}(\alpha) = 0$ for some $l > 0$. This yields that $[f] \in \widehat{\mathcal{N}}(M)$.

Now, we claim that if $s \equiv 0$ (-mod$d$) and $[f] \in \underline{\mathrm{Hom}}_A(\Omega_A^s(M), M)$ is not an isomorphism, then $[f] \in \widehat{\mathcal{N}}(M)$. However, we are done by a similar discussion as above.

By assumption, there exists an isomorphism $\varphi : \Omega_A^d(M) \to M$ in A--mod. Observe that such an isomorphism form $\Omega_A^d(M)$ to M is uniquely determined up to scalar because $\mathrm{End}_A(M)/\operatorname{rad}\mathrm{End}_A(M) \cong k$. Set

$$x := [\varphi] \in \widehat{\mathrm{Ext}}_A^{d}(M, M), \quad y := \Omega_A^{-d}([\varphi^{-1}]) \in \widehat{\mathrm{Ext}}_A^{-d}(M, M).$$

Clearly, we have $x \smile y = 1 = y \smile x$, where $1 = [\mathrm{id}_M]$. Since the Yoneda product of $\widehat{\mathrm{Ext}}_A^{\bullet}(M, M)$ agrees with the one of $\mathrm{Ext}_A^{\bullet}(M, M)$ in positive degrees, it follows from Theorem 4.4.1 that the n-th power x^n of x is non-nilpotent for every $n \geq 1$. Then it is trivial that the n-th power y^n of the inverse y is also non-nilpotent for all $n \geq 1$. As a result, we have shown that the powers x^n and y^n with $n \geq 1$ are all the non-nilpotent homogeneous elements of $\widehat{\mathrm{Ext}}_A^{\bullet}(M, M)$. Since $\deg x = d, \deg y = -d$ and $d \geq 1$, we have $x - y \notin \widehat{\mathcal{N}}(M)$, because otherwise both x and y are nilpotent. Therefore, we obtain the desired isomorphism. □

Now, we extend Theorem 4.4.2 to a Tate-Hochschild case.

Theorem 4.4.4. *Let A be an algebra satisfying $\Omega_{A^{\mathrm{e}}}^n(A) \cong {}_1\Lambda_\sigma$ for some $n \geq 1$ and some automorphism σ of A and $\widehat{\mathcal{N}}$ the ideal of $\widehat{\mathrm{HH}}^{\bullet}(A)$ generated by the homogeneous nilpotent elements. Then we have*

$$\widehat{\mathrm{HH}}^{\bullet}(A)/\widehat{\mathcal{N}} \cong \begin{cases} k[x, y]/(xy - 1) & \text{if } A \text{ is periodic} \\ k & \text{otherwise.} \end{cases}$$

Proof. Let A be an algebra A which satisfies the assumption. By [**27**, Lemma 1.5], the algebra A is a self-injective algebra. It is known that the enveloping algebra A^{e} is self-injective as well (see [**3**, Proposition 2.2]).

Assume that A is a periodic algebra. Since A is connected, i.e, indecomposable as an A-bimodule, the statement follows from Proposition 4.4.3.

Suppose that A is non-periodic, i.e., $\Omega^{i}_{A^{\mathrm{e}}}(A) \ncong A$ for any $i \geq 0$. Clearly, this holds even for all negative integers. First, we claim that $[\eta] \in \widehat{\mathrm{HH}}^{np}(A)$ is nilpotent for any $\eta : \Omega^{np}_{A^{\mathrm{e}}}(A) \to A$, where p is a non-zero integer. It is clear that the indecomposable A-bimodule $\Omega^{inp}_{A^{\mathrm{e}}}(A)$ is isomorphic to ${}_1\Lambda_{\sigma^{ip}}$ for any $i \geq 1$. Hence each $\Omega^{inp}_{A^{\mathrm{e}}}(A)$ has the same length as the regular A-bimodule A. Since the induced morphism $\Omega^{(i-1)np}_{A^{\mathrm{e}}}([\eta]) : \Omega^{inp}_{A^{\mathrm{e}}}(A) \to \Omega^{(i-1)np}_{A^{\mathrm{e}}}(A)$ is not an isomorphism for every $i \geq 1$, it follows from Harada-Sai (see [**4**]) that there exists a positive integer N such that $[\eta]^N = [\eta] \circ \Omega^{np}_{A^{\mathrm{e}}}([\eta]) \circ \Omega^{2np}_{A^{\mathrm{e}}}([\eta]) \circ \cdots \circ \Omega^{(N-1)np}_{A^{\mathrm{e}}}([\eta]) = 0$. This implies that $[\eta] \in \widehat{\mathcal{N}}$. We now let $\eta : \Omega^{s}_{A^{\mathrm{e}}}(A) \to A$, where $0 \neq s \in \mathbb{Z}$ satisfies $s \not\equiv 0$ (-mod$n$). Taking integers $r \geq 1$ and $q \neq 0$ such that $rs = nq$, we have $[\eta]^r \in \widehat{\mathrm{HH}}^{nq}(A)$. The argument above shows that $[\eta]^r$ is nilpotent, and so is $[\eta]$. Hence we have proved that any homogeneous element of $\widehat{\mathrm{HH}}^{\bullet}(A)$ of non-zero degree is nilpotent. Since $\underline{\mathrm{End}}_A(M)/\operatorname{rad}\underline{\mathrm{End}}_A(M) \cong k$, we get the desired isomorphism. □

Let A be a periodic algebra of period d. We now apply the Hochschild cohomology ring $\mathrm{HH}^{\bullet}(A)$ in order to describe the non-negative subring $\widehat{\mathrm{HH}}^{\geq 0}(A)$. Recall from Section 2.3 that there exists an epimorphism $\Phi^{\bullet} : \mathrm{HH}^{\bullet}(A) \to \widehat{\mathrm{HH}}^{\geq 0}(A)$ of graded algebras such that Φ^0 is surjective with $\operatorname{Ker}\Phi^0 = \mathcal{P}(A, A)$ and $\Phi^{\geq 1}$ is bijective. We call $\mathcal{P}(A, A)$ the *projective center* of A and denote it by $Z^{\mathrm{pr}}(A)$. Remark that, in our setting, the non-negative part $\widehat{\mathrm{HH}}^{\geq 0}(A)$ coincides with the *stable Hochschild cohomology* (see [**37**]). So far, we obtain $\widehat{\mathrm{HH}}^{\geq 0}(A) \cong \mathrm{HH}^{\bullet}(A)/Z^{\mathrm{pr}}(A)$ as graded algebras. We now characterize the projective center in terms of the Yoneda product of the Hochschild cohomology ring. Recall from Theorems 4.4.2 and 4.4.4 that $\mathrm{HH}^{\bullet}(A)$ has a unique homogeneous non-nilpotent element χ of degree d such that it is invertible in $\widehat{\mathrm{HH}}^{\bullet}(A)$. Thus we have the following commutative square

$$\begin{array}{ccc} \mathrm{HH}^0(A) & \xrightarrow{-\cup\chi} & \mathrm{HH}^d(A) \\ {\scriptstyle\Phi^0}\downarrow & & \| {\scriptstyle\Phi^d} \\ \widehat{\mathrm{HH}}^0(A) & \xrightarrow{-\smile\chi} & \widehat{\mathrm{HH}}^d(A) \end{array}$$

where the lower horizontal k-linear map $- \smile \chi$ is bijective. Then, clearly, the k-linear map $- \cup \chi$ is surjective. Since $\chi \in \widehat{\mathrm{HH}}^d(A)$ is invertible, one easily shows that $Z^{\mathrm{pr}}(A)$ agrees with $\operatorname{Ker}(- \cup \chi) = \{\alpha \in \mathrm{HH}^0(A) \mid \alpha \cup \chi = 0\}$. Moreover, it is trivial that $Z^{\mathrm{pr}}(A) = 0$ if and only if $\dim_k \mathrm{HH}^0(A) = \dim_k \mathrm{HH}^d(A)$. Hence, we have proved the following proposition.

Proposition 4.4.5. *Let A be a periodic algebra of period d and $\chi \in \mathrm{HH}^d(A)$ a unique homogeneous non-nilpotent element of $\mathrm{HH}^\bullet(A)$. Then we have $Z^{\mathrm{pr}}(A) = \mathrm{Ker}(- \cup \chi)$ and hence $\widehat{\mathrm{HH}}^{\geq 0}(A) = \mathrm{HH}^\bullet(A)/\,\mathrm{Ker}(- \cup \chi)$. Furthermore, the following are equivalent.*

(1) *The projective center $Z^{\mathrm{pr}}(A)$ of A vanishes.*

(2) *The k-linear map $\Phi^0 : \mathrm{HH}^0(A) \to \widehat{\mathrm{HH}}^0(A)$ is bijective.*

(3) $\dim_k \mathrm{HH}^0(A) = \dim_k \mathrm{HH}^d(A)$.

In this case, we have $\widehat{\mathrm{HH}}^{\geq 0}(A) = \mathrm{HH}^\bullet(A)$ and hence $\widehat{\mathrm{HH}}^\bullet(A) = \mathrm{HH}^\bullet(A)[\chi^{-1}]$.

Next, we explain how we use Proposition 4.4.5 to compute the Tate-Hochschild cohomology of a periodic algebra. For this purpose, we deal with preprojective algebras of type A_n. Recall that the *preprojective algebra* Π_Q *associated with a Dynkin quiver* Q (i.e., the underlying graph of Q is one of the Dynkin diagrams $A_n (n \geq 1)$, $D_n (n \geq 4)$ and $E_n (n = 6, 7, 8)$) is defined by

$$\Pi_Q := k\overline{Q}/\left\langle \sum_{a \in Q_1} (a\overline{a} - \overline{a}a) \right\rangle,$$

where $\overline{Q}$ is the quiver obtained from Q by adding, for each arrow $a : i \to j$, an arrow $\overline{a} : j \to i$ having the opposite direction. By definition, the preprojective algebra Π_Q depends only on the underlying graph of Q. Hence we associate to each preprojective algebra one of the Dynkin diagrams. It was shown by Schofield that the preprojective algebra of Dynkin type is a periodic algebra whose period is dividing 6. In particular, the period of the preprojective algebra of type A_n is equal to 6 if $n \geq 3$ (cf. [**20**, Section 1]). On the other hand, the Hochschild cohomology ring for the preprojective algebras of Dynkin type has been obtained by Erdmann and Snashall [**21**] for type A_n and by Eu [**22**] for the other types.

Let Π_n denote the preprojective algebra of type A_n, and we assume that $n \geq 3$. Following the notation from [**21**], we set $m := (n-1)/2$ if n is odd, and $m := (n-2)/2$ if n is even. According to [**20, 21**], we have

$$\mathrm{HH}^0(\Pi_n) = k[z]/\langle z^{m+1} \rangle, \qquad \dim_k \mathrm{HH}^6(\Pi_n) = \begin{cases} m & \text{if } n \text{ is odd} \\ m+1 & \text{if } n \text{ is even.} \end{cases}$$

In view of Proposition 4.4.5, we see that $\widehat{\mathrm{HH}}^0(\Pi_n) = \mathrm{HH}^0(\Pi_n)$ if and only if Π_n has even vertices. On the other hand, if Π_n has odd vertices, then it follows from [**21**, Section 5] that $z^r X \neq 0$ for $1 \leq r < m$ and $z^m X = 0$, where X is a unique homogeneous non-nilpotent element of $\mathrm{HH}^\bullet(\Pi_n)$ of degree 6. Proposition 4.4.5 yields that $\mathrm{Ker}(- \cup X) =$

$\langle z^m \rangle$ and $\widehat{\mathrm{HH}}^0(\Pi_n) = \mathrm{HH}^0(\Pi_n)/\langle z^m \rangle = k[z]/\langle z^m \rangle$. Hence the 0-th Tate-Hochschild cohomology algebra $\widehat{\mathrm{HH}}^0(\Pi_n)$ is as follows:

$$\widehat{\mathrm{HH}}^0(\Pi_n) = \begin{cases} k[z]/\langle z^{m+1} \rangle & \text{if } n \text{ is even} \\ k[z]/\langle z^m \rangle & \text{if } n \text{ is odd.} \end{cases}$$

For the first case, we have $\widehat{\mathrm{HH}}^\bullet(\Pi_n) = \mathrm{HH}^\bullet(\Pi_n)[X^{-1}]$. As a result, using the description of $\widehat{\mathrm{HH}}^0(\Pi_n)$ and Theorem 4.2.3, one can completely describe the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^\bullet(\Pi_n)$ for all the preprojective algebras Π_n of type A_n with $n \geq 3$.

Example 4.4.6. Let Π_5 be the preprojective k-algebra of type A_5. For simplicity, we assume that $\operatorname{char} k = 2$. Since $n = 5$, we have $m = 2$ and $\widehat{\mathrm{HH}}^0(\Pi_5) = k[z]/\langle z^2 \rangle$. It follows from [**21**, Section 5] that the Hochschild cohomology ring $\mathrm{HH}^\bullet(\Pi_5)$ has generators

$$1, z, g_0, f_0, f_1, h_0, h_1, \psi_0, X$$

with $\deg z = 0$, $\deg g_0 = 1$, $\deg f_i = 2$, $\deg h_i = 3$, $\deg \psi_0 = 4$, $\deg X = 6$, and these generators satisfy the following relations

$$\begin{aligned} & z^3 = 0,\ g_0^2 = 0,\ f_0^3 = 0,\ f_1^2 = 0,\ \psi_0^2 = 0,\ zf_i = 0,\ z^2\psi_i = 0,\ z^2X = 0, \\ & f_0^2 = z\psi_0 = f_0f_1,\ f_i\psi_0 = 0,\ zh_i = 0,\ h_ih_j = 0,\ g_0h_i = 0,\ g_0z^2 = 0, \\ & g_0f_i = 0,\ h_if_j = \delta_{ij}zg_0\psi_0,\ h_i\psi_0 = 0, \end{aligned}$$

where $1 \leq i, j \leq 2$. Then one sees that the dimension of $\mathrm{HH}^i(\Pi_5)$ is equal to 3 if $i = 0$ and to 2 if $i \geq 1$. Theorem 4.2.3 implies that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^\bullet(\Pi_5) = \widehat{\mathrm{HH}}^{\geq 0}(\Pi_5)[X^{-1}]$ has generators

$$1, z, g_0, f_0, f_1, h_0, h_1, \psi_0, X, X^{-1}$$

with $\deg X^{-1} = -6$, and these generators satisfy the relations obtained from the ones of $\mathrm{HH}^\bullet(\Pi_5)$ by replacing z^3 by z^2. In particular, $\dim_k \widehat{\mathrm{HH}}^i(\Pi_5) = 2$ for all $i \in \mathbb{Z}$.

We remark that even if $\operatorname{char} k \neq 2$, we can calculate $\widehat{\mathrm{HH}}^\bullet(\Pi_5)$ from $\mathrm{HH}^\bullet(\Pi_5)$ in the same way as in the above. More concretely, after providing the generators and the relations of $\mathrm{HH}^\bullet(\Pi_5)$ based on [**21**, Section 5], we add X^{-1} to the generators and replace z^3 by z^2 in the relations to obtain the presentation of $\widehat{\mathrm{HH}}^\bullet(\Pi_5)$.

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