Ph.D. Thesis

Studies on properties of Brauer-friendly modules and slash functors

（ブラウアーフレンドリー加群とスラッシュ関手の性質に関する研究）

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Chapter 1

Introduction

This thesis is based on [23] and [22].

Let p be a prime number and $(K, \mathcal{O}, k)$ a p-modular system such that k is algebraically closed. Throughout this thesis, RG-modules mean finitely generated RG-lattices, for $R \in \{\mathcal{O}, k\}$. In the modular representation theory of finite groups, the following philosophy exists such as the local-global principle : Representations of a finite group are controlled by representations of p-local subgroups of the group. One of the specific formulations of this philosophy is known to be the following Broué's conjecture.

Conjecture (Broué's conjecture)**.** *Let G be a finite group, b a block of RG with a defect group P, and c the Brauer correspondent of b in $RN_G(P)$. If P is abelian, then the block algebras RGb and $RN_G(P)c$ are derived equivalent.*

This conjecture is one of the most important problems and has been studied by many researchers, in the modular representation theory of finite groups. It is known that the conjecture holds in many blocks(see [18, 5.2.2]). In many cases, Okuyama's method, introduced in [17], played an important role. It is a method of constructing a derived equivalence from a stable equivalence of Morita type. The details of the method may be found in [17]. From the method, constructing a stable equivalence of Morita type between the block algebras RGb and $RN_G(P)c$ can be used to prove Broué's conjecture. We review the gluing principle of constructing stable equivalences of Morita type for principal blocks and general blocks.

First, we consider the case where b is the principal block of RG. In this case, M. Broué introduced the following method which is useful for constructing a stable equivalence of Morita type.

Theorem 1.0.1 (Broué's gluing principle [6, 6.3. Theorem])**.** *Let G and H be finite groups having a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(H)$. Let b and c be the principal blocks of RG and RH, respectively. For any subgroup Q of P, let b_Q and c_Q be the principal* *blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, and $M = S(G\times H, \Delta P)$ the Scott $R(G\times H)$-module with vertex ΔP. Then the following are equivalent.*

(i) The bimodule M and its dual M^* induce a stable equivarence of Morita type between RGb and RHc.

(ii) For each non-trivial subgroup Q of P, the bimodule $Br_Q(M)$ and its dual $Br_Q(M)^*$ induce a Morita equivalence between $kC_G(Q)b_Q$ and $kC_H(Q)c_Q$.

In [11], R. Kessar, N. Kunugi, and N. Mitsuhashi introduced the notation of Brauer indecomposability, which plays a key role when we apply the principle to principal blocks.

Definition 1.0.2 ([11])**.** Let M be an indecomposable RG-module. We say that M is *Brauer indecomposable* if $\mathrm{Res}^{N_G(Q)/Q}_{QC_G(Q)/Q}(\mathrm{Br}_Q(M))$ is indecomposable or 0, for any p-subgroup Q of G.

In [9], H. Ishioka and N. Kunugi gave an equivalent condition for Scott modules to be Brauer indecomposable.

Theorem 1.0.3 ([9, Theorem 1.3])**.** *Let G be a finite group and P a p-subgroup of G. Let $M = S(G, P)$ and suppose that $\mathcal{F} = \mathcal{F}_P(G)$ is saturated. Then the following conditions are equivalent.*

(i) M is Brauer indecomposable.

(ii) $Res^{N_G(Q)}_{QC_G(Q)}(S(N_G(Q), N_P(Q)))$ is indecomposable, for each fully $\mathcal{F}$-normalized subgroup Q of P.

If one of the equivalent conditions is satisfied, then $Br_Q(M) \cong S(N_G(Q), N_P(Q))$ for each fully $\mathcal{F}$-normalized subgroup Q of P.

Next, we consider the case where b is a general block of RG. M. Linckelmann generalized Broué's gluing principle to general blocks.

Theorem 1.0.4 (Linckelmann's gluing principle [14, Theorem 1.2])**.** *Let G and H be finite groups and b and c blocks of RG and RH, respectively, with a common defect group P. Let $i \in (RGb)^{\Delta P}$ and $j \in (RH)^{\Delta P}$ be almost source idempotents. For any subgroup Q of P, denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $Br_{\Delta Q}(i)e_Q \neq 0$ and $Br_{\Delta Q}(j)f_Q \neq 0$. Denote by $\hat{e}_Q$ and $\hat{f}_Q$ the unique blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ lifting e_Q and f_Q, respectively. Suppose that $\mathcal{F}_{(P,\hat{e}_P)}(G, b) = \mathcal{F}_{(P,\hat{f}_P)}(H, c)$, and write $\mathcal{F} = \mathcal{F}_{(P,\hat{e}_P)}(G, b)$. Let V be an $\mathcal{F}$-stable indecomposable endo-permutation RP-module with vertex P, viewed as an $R\Delta P$-module through the canonical isomorphism $\Delta P \cong P$. Let M be an indecomposable direct summand of the RGb-RHc-bimodule*

$$RGi \otimes_{RP} \mathrm{Ind}^{P\times P}_{\Delta P}(V) \otimes_{RP} jRH.$$

Suppose that M has ΔP as a vertex as an $R[G \times H]$-module. Then for any non-trivial subgroup Q of P, there is a canonical $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-module M_Q satisfying $End_k(M_Q) \cong \mathrm{Br}_{\Delta Q}(\mathrm{End}_R(\hat{e}_Q M \hat{f}_Q))$. Moreover, if for all non-trivial subgroups Q of P the bimodule M_Q induces a Morita equivalence between $kC_G(Q)e_Q$ and $kC_H(Q)f_Q$, then M and its dual M^* induce a stable equivalence of Morita type between RGb and RHc.

In [7], E. C. Dade introduced slash constructions for endo-permutation modules. In [3], E. Biland defined Brauer-friendly modules and generalized slash constructions to slash functors for Brauer-friendly modules. Brauer-friendly modules are generalizations of (endo-)p-permutation modules. The module M which appears in the theorem above is a Brauer-friendly module, and the module M_Q which appears in the theorem can be represented as $Sl_{(\Delta Q, \hat{e}_Q \otimes \hat{f}_Q)}(M)$ by using a $(\Delta Q, \hat{e}_Q \otimes \hat{f}_Q)$-slash functor $Sl_{(\Delta Q, \hat{e}_Q \otimes \hat{f}_Q)}$. For Brauer-friendly modules, the slash indecomposability can be defined in a similar way as Brauer indecomposability (For Frobenius-friendly modules (*i.e.* endo-p-permutation modules), the slash indecomposability has been defined by Feng-Li [8]). The slash indecomposability plays an important role in Linckelmann's gluing principle.

Our first result is that we generalize Ishioka-Kunugi's equivalent condition to an equivalent condition for Brauer-friendly modules to be slash indecomposable.

We have the following relation:

$$\begin{array}{ccc} & {}_{\mathcal{O}G}\mathbf{Mod} & \\ {\scriptstyle K\otimes_{\mathcal{O}}-}\swarrow & & \searrow{\scriptstyle k\otimes_{\mathcal{O}}-} \\ {}_{KG}\mathbf{Mod} & & {}_{kG}\mathbf{Mod} \end{array}$$

where ${}_A\mathbf{Mod}$ is the category of all A-modules. In general, the functor on the right side is not essentially surjective. There exist kG-modules which have corresponding $\mathcal{O}G$-modules. These modules are said to be liftable, *i.e.* a kG-module M is said to be *liftable* if there exists an $\mathcal{O}G$-module $\widehat{M}$ such that $k \otimes_{\mathcal{O}} \widehat{M} \cong M$. In this situation, $\widehat{M}$ is called a lift of M. Liftability is one of the properties that we want to be satisfied in order to study the structure of a kG-module. If a module is liftable, then we can construct the ordinary character corresponding to a lift of the module, from the above relation. Then we can examine the structure of the module using the ordinary character. Therefore, in the modular representation theory of finite groups, it is important to find a class of liftable modules. A few classes of liftable modules are known. For example, any p-permutation kG-module is liftable, in particular, any projective kG-module is liftable. Moreover, they lift to a p-permutation $\mathcal{O}G$-module and a projective $\mathcal{O}G$-module, respectively. In addition, any endo-permutation kG-module is liftable to an endo-permutation $\mathcal{O}G$-module. More details on these examples may be found in [13, 1. Introduction]. In [21], J.-M. Urfer introduced endo-p-permutation modules, which are generalizations of endo-permutation modules. In [13], C. Lassueur and J. Thévenaz proved that any endo-p-permutation kG-module is liftable to an endo-p-permutation $\mathcal{O}G$-module.

Theorem 1.0.5 ([13, Theorem 4.2])**.** *Let M be an indecomposable endo-p-permutation kG-module and P a vertex of M. Then there exists an indecomposable endo-p-permutation $\mathcal{O}G$-module $\widehat{M}$ with vertex P such that $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$.*

By [21, Theorem 1.5], any endo-p-permutation RG-module has a G-stable endo-permutation source. Hence, in [13, Remark 4.3], C. Lassueur and J. Thévenaz raised the question of whether or not kG-modules with an endo-permutation source which is not necessarily G-stable are liftable. In [3], E. Biland introduced Brauer-friendly modules, which are generalizations of endo-p-permutation modules. Any Brauer-friendly module has an endo-permutation source. From the question and since Brauer-friendly modules may induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ by Linckelmann's gluing principle, we want to know the liftability of Brauer-friendly modules.

Our second result is that we show that any indecomposable Brauer-friendly module satisfying certain condition is liftable to an indecomposable Brauer-friendly module, which is a generalization of the main theorem of [13].

Chapter 2

Preliminary

2.1 Notation

Throughout this thesis, we use the following notation and terminology.

Let p be a prime number, $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field k of characteristic p, and set $\mathfrak{p} = J(\mathcal{O})$ and $R \in \{\mathcal{O}, k\}$. Throughout this paper, blocks mean block idempotents. We fix a finite group G and a block b of RG. For any $x \in \mathcal{O}G$, we denote by $\overline{x}$ its image by the natural map $\mathcal{O}G \twoheadrightarrow kG$. By the lifting theorem of idempotents, for a primitive idempotent $i \in kG$, there exists a primitive idempotent $\hat{i} \in \mathcal{O}G$ such that $\overline{\hat{i}} = i$. We only use the symbol $\hat{-}$ to satisfy the property, for primitive idempotents. We denote by ${}_{RG}\mathbf{Mod}$ the category of all RG-modules. We set $\Delta G = \{(g,g) \mid g \in G\}$. We write $\overline{N}_G(H) = N_G(H)/H$ for a subgroup H of G. For any G-set X and any subgroup H of G, we set $X^H = \{x \in X \mid h \cdot x = x, h \in H\}$. For any indecomposable RG-module M, we denote by $\mathrm{vtx}(M)$ a vertex of M. For any two RG-modules M and N, we write $M \mid N$ if M is isomorphic to a direct summand of N. For any RG-module M and any subgroup H of G, the relative trace map $\mathrm{Tr}_H^G : M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) = \sum_{x \in G/H} x \cdot m$. For any RG-module M and any p-subgroup P of G, the Brauer construction of M with respect to P is the $k\overline{N}_G(P)$-module defined by

$$\mathrm{Br}_P(M) = M^P / (\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q) + J(R)M^P).$$

We denote by $\mathrm{br}_P^M : M^P \to \mathrm{Br}_P(M)$ the natural map and we call this map the Brauer morphism of M with respect to P. In particular, we write $\mathrm{br}_P = \mathrm{br}_P^{RG}$. For any $f \in \mathrm{Hom}_{RG}(L, M)$, $k\overline{N}_G(P)$-homomorphism $\mathrm{Br}_P(f) \in \mathrm{Hom}_{k\overline{N}_G(P)}(\mathrm{Br}_P(L), \mathrm{Br}_P(M))$ is naturally determined. Hence, Br_P induces a functor

$$\mathrm{Br}_P : {}_{RG}\mathbf{Mod} \to {}_{k\overline{N}_G(P)}\mathbf{Mod}.$$

We recall the definition of subpairs. A subpair of G is a pair (P, b_P) consisting of a p-subgroup P of G and a block b_P of $\mathcal{O}C_G(P)$. We call the subpair (P, b_P) a (G, b)-subpair if $\bar{b}_P \mathrm{br}_P(b) \neq 0$. For (G, b)-subpair (P, b_P), the block b_P is also a block of $\mathcal{O}H$ for a subgroup H such that $C_G(P) \leq H \leq N_G(P, b_P)$. The set of (G, b)-subpairs is a poset, and the group G acts on the set by conjugation.

We recall the definition of the Brauer functor with respect to (G, b)-subpair. Let (P, b_P) be a (G, b)-subpair, M an RGb-module. The Brauer construction of M with respect to the subpair (P, b_P) is the $k\overline{N}_G(P, b_P)\bar{b}_P$-module defined by $\mathrm{Br}_{(P,b_P)}(M) = \mathrm{Br}_P(b_P M)$, here we identify the block $\bar{b}_P$ of $kN_G(P, b_P)$ with an idempotent of $k\overline{N}_G(P, b_P)$. The $kN_G(P, b_P)$-epimorphism

$$\mathrm{br}^M_{(P,b_P)} : M^P \to \mathrm{Br}_{(P,b_P)}(M)$$

is defined by $m \mapsto \mathrm{br}^{b_P M}_P(b_P M)$. For any $f \in \mathrm{Hom}_{\mathcal{O}Gb}(L, M)$, we define

$$\mathrm{Br}_{(P,b_P)}(f) = \mathrm{Br}_P(b_P f b_P) \in \mathrm{Hom}_{k\overline{N}_G(P,b_P)\bar{b}_P}(\mathrm{Br}_{(P,b_P)}(L), \mathrm{Br}_{(P,b_P)}(M)).$$

So $\mathrm{Br}_{(P,b_P)}$ induces a functor

$$\mathrm{Br}_{(P,b_P)} : {}_{RGb}\mathbf{Mod} \to {}_{k\overline{N}_G(P,b_P)\bar{b}_P}\mathbf{Mod}.$$

We recall the definitions of Brauer categories and fusion systems. The Brauer category $\mathbf{Br}(G, b)$ is defined as follows: the objects of $\mathbf{Br}(G, b)$ are the (G, b)-subpairs, and for any two objects (P, b_P), (Q, b_Q), the morphism set $\mathrm{Hom}_{\mathbf{Br}(G,b)}((P, b_P), (Q, b_Q))$ is the set of all group homomorphisms $\phi : P \to Q$ such that there exists $g \in G$ satisfying ${}^g(P, b_P) \leq (Q, b_Q)$ and $\phi(x) = {}^g x$ for any $x \in P$. Let (P, b_P) be a (G, b)-subpair. Let (P, b_P) be a (G, b)-subpair. The fusion system $\mathcal{F}_{(P,b_P)}(G, b)$ is defined as follows: the objects of $\mathcal{F}_{(P,b_P)}(G, b)$ are the subgroup of P, and for any two objects Q and R, the morphism set $\mathrm{Hom}_{\mathcal{F}_{(P,b_P)}(G,b)}(Q, R)$ is the set of all group homomorphisms $\phi : Q \to R$ such that there exists $g \in G$ satisfying ${}^g(Q, b_Q) \leq (R, b_R)$ for $(Q, b_Q), (R, b_R) \leq (P, b_P)$ and $\phi(x) = {}^g x$ for any $x \in Q$. The Frobenius category $\mathbf{Fr}(G)$ is defined as follows: the objects of $\mathrm{Fr}(G)$ are all p-subgroups of G and for any two objects P, Q, the morphism set $\mathrm{Hom}_{\mathbf{Fr}(G)}(P, Q)$ is the set of all group homomorphisms $\phi : P \to Q$ such that there exists $g \in G$ satisfying $\phi(x) = {}^g x$ for any $x \in P$, and ${}^g P \leq Q$.

We review the definition of vertex subpairs and source triples from [3]. Let M be an indecomposable RGb-module. A (G, b)-subpair (P, b_P) is called a vertex subpair of M if for some indecomposable RP-module V, $M \mid bRGb_P \otimes_{RP} V$ and $P \leq_G \mathrm{vtx}(M)$ hold. For such V, it is called a source of M with respect to the vertex subpair (P, b_P). A triple (P, b_P, V) is called a source triple of M if V is a source of M with respect to the vertex subpair (P, b_P). If M has a source triple (P, b_P, V), then a vertex of M is P and a source of M is V, from [3, Lemma 1].

We can consider the Green correspondence with respect to source triple.

Theorem 2.1.1 ([3, Lemma 1, Definition 2])**.** *Let (P, b_P) be a (G, b)-subpair. If M is an indecomposable $\mathcal{O}Gb$-module with source triple (P, b_P, V), then there exists a unique indecomposable $\mathcal{O}N_G(P, b_P)$-direct summand $f^b_{b_P}(M)$ of $b_P M$ with source triple (P, b_P, V). Then $f^b_{b_P}$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) and the isomorphism classes of indecomposable $\mathcal{O}N_G(P, b_P)$-modules with source triple (P, b_P, V).*

The $f^b_{b_P}$ is called the *Green correspondence* with respect to (P, b_P).

2.2 Definitions and Properties of (p-)permutation modules and endo-(p-)permutation modules

We recall the definitions of (p-)permutation modules and endo-(p-)permutation modules.

Definition 2.2.1 (Permutation modules and p-Permutation modules)**.** Let M be an RG-module. We call M a *permutation RG-module* if $M \cong \bigoplus_{1 \le i \le n} \mathrm{Ind}^G_{H_i}(R_{H_i})$ for some subgroups H_i of G. Also, we call M a *p-permutation RG-module* if $M \mid N$ for some permutation RG-module N.

Remark 2.2.2. We see that M is a permutation RG-module if and only if M has a G-invariant R-basis. Moreover, M is a p-permutation RG-module if and only if for any p-subgroup P of G, $\mathrm{Res}^G_P(M)$ has a P-invariant R-basis.

Definition 2.2.3 (Endo-permutation modules and Endo-p-permutation modules)**.** Let M be an RG-module. We call M an *endo-permutation RG-module* if $\mathrm{End}_R(M)$ is a permutation RG-module. Also, we call M an *endo-p-permutation RG-module* if $\mathrm{End}_R(M)$ is a p-permutation RG-module.

Remark 2.2.4. Let G be a p-group. Then permutation RG-modules are just p-permutation RG-modules. Similarly, endo-permutation RG-modules are just endo-p-permutation RG-modules.

In the following, we review some important properties and propositions for (endo-p-)permutation modules. The next lemma follows from the definition immediately.

Lemma 2.2.5. *Let V, W be (p-)permutation RG-modules. Then $V \oplus W$, $V \otimes_R W$, V^*, and $\mathrm{Hom}_R(V, W)$ are (p-)permutation RG-modules.*

Definition 2.2.6 (Capped endo-permutation module)**.** Let P be a p-group and V an endo-permutation RP-module. We say that V is *capped* if it has an indecomposable direct summand W of V with vertex P.

Lemma 2.2.7 ([7, PROPOSITION 2.2])**.** *Let P be a p-group. Let V, W (capped) endo-permutation RP-module. Then $V \otimes_R W$, V^*, and $\mathrm{Hom}_R(V, W)$ are (capped) endo-permutation RP-modules.*

Definition 2.2.8 (Compatible for endo-permutation modules)**.** Let P be a p-group and let V and W be endo-permutation RP-modules. We say that V and W are *compatible* if $V \oplus W$ is an endo-permutation RP-module.

Lemma 2.2.9 ([7, PROPOSITION 2.3])**.** *Let P be a p-group and V, W endo-permutation RP-modules. The module V and W are compatible if and only if $Hom_R(V, W)$(or $Hom_R(W, V)$) is a permutation RP-module.*

Proposition 2.2.10 ([7, THEOREM 3.8])**.** *Let P be a p-group. Let V, W capped indecomposable endo-permutation RP-modules. Then V and W are compatible if and only if they are RP-isomorphic.*

By Proposition 2.2.10, any two indecomposable direct summands with vertex P of capped endo-permutation RP-module V are isomorphic. We denote by $\mathrm{Cap}(V)$ any RP-module isomorphic to one of those summands.

Brauer-friendly modules defined in the next section have fusion-stable endo-permutation modules as sources. We recall the definition of fusion-stable endo-permutation modules.

Definition 2.2.11 ([15, Definition 9.9.1])**.** Let (P, b_P) be a (G, b)-subpair, V an endo-permutation RP-module, and set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$. We say that V is *$\mathcal{F}$-stable* if the endo-permutation $\mathcal{O}Q$-modules $\mathrm{Res}^P_Q(V)$ and $\mathrm{Res}_{\phi_{g^{-1}}}(V) = \mathrm{Res}^{{}^gP}_Q({}^gV)$ are compatible for any subgroup Q of P and any $\phi_{g^{-1}} \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. We call the triple (P, b_P, V) a *fusion-stable endo-permutation source triple* if V is an $\mathcal{F}$-stable capped indecomposable endo-permutation RP-module.

Remark 2.2.12. In particular, we say that V is G-stable if V is $\mathbf{Fr}(G)$-stable.

Every endo-p-permutation RG-module has a G-stable endo-permutation source, from the following theorem.

Theorem 2.2.13 ([21, Theorem 1.5])**.** *Let M be an indecomposable RG-module with vertex P and source S. Then M is an endo-p-permutation RG-module if and only if S is a G-stable endo-permutation source.*

Chapter 3

Brauer-friendly modules and slash functors

In this chapter, we review the definitions of Brauer-friendly modules and slash functors defined by E. Biland in [3]. This Biland's slash functors are generalizations of Dade's slash functors.

3.1 Definition of Brauer-friendly modules

Definition 3.1.1 ([3, Definition 6]). Let (P_1, b_1, V_1) and (P_2, b_2, V_2) be fusion-stable endo-permutation source triples in (G, b). We say that (P_1, b_1, V_1) and (P_2, b_2, V_2) are *compatible* if the endo-permutation $\mathcal{O}Q$-modules $\mathrm{Res}_{\phi_1}(V_1)$ and $\mathrm{Res}_{\phi_2}(V_2)$ are compatible for any (G, b)-subpair (Q, b_Q) and any morphism $\phi_i \in \mathrm{Hom}_{\mathrm{Br}(G,b)}((Q, b_Q), (P_i, b_{P_i}))$ for $i \in \{1, 2\}$.

Definition 3.1.2 ([3, Definition 8]). Let M be an $\mathcal{O}Gb$-module which admits the decomposition $M = \bigoplus_{1\le i\le n} M_i$ of M, where each M_i is indecomposable $\mathcal{O}Gb$-module with source triple (P_i, b_{P_i}, V_i). We say that $\mathcal{O}Gb$-module M is *Brauer-friendly* if (P_i, b_{P_i}, V_i) is a fusion-stable endo-permutation source triple for any $i \in \{1, \ldots, n\}$, and, (P_i, b_{P_i}, V_i) and (P_j, b_{P_j}, V_j) are compatible for every $i, j \in \{1, \ldots, n\}$.

Definition 3.1.3 ([3, Definition 8]). Let L and M be Brauer-friendly $\mathcal{O}Gb$-modules. We say that L and M are *compatible* if $L \oplus M$ is a Brauer-friendly $\mathcal{O}Gb$-module.

Definition 3.1.4 ([3, Definition 15]). Let ${}_{\mathcal{O}Gb}\mathbf{M}$ be a subcategory of the category ${}_{\mathcal{O}Gb}\mathbf{Mod}$. We say that ${}_{\mathcal{O}Gb}\mathbf{M}$ is *Brauer-friendly* if any object of ${}_{\mathcal{O}Gb}\mathbf{M}$ is a Brauer-friendly $\mathcal{O}Gb$-module, and any two objects of ${}_{\mathcal{O}Gb}\mathbf{M}$ are compatible.

Definition 3.1.5. Let (P, b_P, V) be a fusion-stable endo-permutation source triples in (G, b). We say that a Brauer-friendly category is *big enough with respect to* (P, b_P, V) if any finite direct sum of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) belongs to the Brauer-friendly category. Let $\mathcal{S}$ be a set of compatible source triples of G. Also we define *big enough with respect to* $\mathcal{S}$.

Remark 3.1.6. In the case of b is the principal block of RG, Brauer-friendly RGb-modules and endo-p-permutation RGb-modules are equal class. In general, indecomposable endo-p-permutation RGb-modules are indecomposable Brauer-friendly RGb-modules but the converse is not true(see [3, The sentences under Definition 8]). Hence we have the following relation.

$$\begin{array}{cccc} & \text{permutation mod.} & \subseteq & \text{endo-permutation mod.} \\ & \cap\vert & & \cap\vert \\ \text{Scott mod.} & \subseteq p\text{-permutation mod.} & \subseteq & \text{endo-}p\text{-permutation mod.} \\ & & & \cap\vert \\ & & & \text{Brauer-friendly mod.} \end{array}$$

3.2 Definition and Properties of slash functors

E. C. Dade in [7] defined slash constructions and slash functors for endo-permutation modules over p-groups.

Theorem 3.2.1 ([7, THEOREM 4.15 and COROLLARY 4.17] and [3, Beginning of Section 4])**.** *Let P be a p-group, Q a subgroup of P and V an endo-permutation $\mathcal{O}P$-module. There exists an endo-permutation $kN_P(Q)/Q$-module $V[Q]$ such that there is an isomorphism of $N_P(Q)/Q$-algebra $Br_{\Delta Q}(\mathrm{End}_{\mathcal{O}}(V)) \cong \mathrm{End}_k(V[Q])$. The module $V[Q]$ is unique up to isomorphism.*

We call the module $V[Q]$ the Q-slashed module relative to V.

In [3], E. Biland generalized slash constructions to slash functors for Brauer-friendly modules as follows.

Definition 3.2.2 ([3, Definition 14])**.** Let G be a finite group, b a block of $\mathcal{O}G$, and ${}_{\mathcal{O}Gb}\mathbf{M}$ a subcategory of the category ${}_{\mathcal{O}Gb}\mathbf{Mod}$ of all $\mathcal{O}Gb$-modules. Let (P, b_P) be a (G, b)-subpair, and H a subgroup of G such that $PC_G(P) \leq H \leq N_G(P, b_P)$. We write $\overline{H} = H/P$. An additive functor $Sl : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}b_P}\mathbf{Mod}$ defined by the following data is called a (P, b_P)-*slash functor* :

- for each $L, M \in {}_{\mathcal{O}Gb}\mathbf{M}$, there exists a map

$$Sl^{L,M} : \mathrm{Hom}_{\mathcal{O}P}(L, M) \longrightarrow \mathrm{Hom}_k(Sl(L), Sl(M))$$

satisfying the following conditions.

– for any $M \in {}_{\mathcal{O}Gb}\mathbf{M}$, $Sl^{M,M}(1_{\mathrm{End}_{\mathcal{O}}(M)}) = 1_{\mathrm{End}_k(Sl(M))}$;

– for any $L, M, N \in {}_{\mathcal{O}Gb}\mathbf{M}$, and any $f \in \mathrm{Hom}_{\mathcal{O}P}(L, M)$, any $g \in \mathrm{Hom}_{\mathcal{O}P}(M, N)$, $Sl^{L,N}(g \circ f) = Sl^{M,N}(g) \circ Sl^{L,M}(f)$;

– for any $L, M \in {}_{\mathcal{O}Gb}\mathbf{M}$, there exists a $k(C_G(P) \times C_G(P))\Delta H$-isomorphism

$$f_{L,M} : \mathrm{Br}_{\Delta P}(\mathrm{Hom}_{\mathcal{O}}(b_P L, b_P M))) \xrightarrow{\sim} \mathrm{Hom}_k(Sl(L), Sl(M))$$

such that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}P}(L, M) & \xrightarrow{\quad Sl^{L,M} \quad} & \mathrm{Hom}_k(Sl(L), Sl(M)) \\ & {\scriptstyle \mathrm{br}^{\mathrm{Hom}_{\mathcal{O}}(L,M)}_{(\Delta P, b_P \otimes b_P)}} \searrow \quad \nearrow {\scriptstyle f_{L,M}} & \\ & \mathrm{Br}_{\Delta P}(\mathrm{Hom}_{\mathcal{O}}(b_P L, b_P M))) & \end{array}$$

Biland proved that there exists a slash functor for Brauer-friendly categories in [3].

Theorem 3.2.3 ([3, Theorem 18])**.** *Let b be a block of $\mathcal{O}G$ and ${}_{\mathcal{O}Gb}\mathbf{M}$ a Brauer-friendly category of $\mathcal{O}Gb$-modules. Let (P, b_P) be a (G, b)-subpair, H a subgroup of G such that $PC_G(P) \leq H \leq N_G(P, b_P)$, and we write $\overline{C}_G(P) = PC_G(P)/P$. Then the following statements hold.*

(i) There exists a (P, b_P)-slash functor $Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}b_P}\mathbf{Mod}$.

(ii) If $Sl'_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{H}b_P}\mathbf{Mod}$ is another (P, b_P)-slash functor, then there exists a linear character $\chi : \overline{H}/\overline{C}_G(P) \to k^\times$ such that there exists an isomorphism $\chi_* Sl_{(P,b_P)} \cong Sl'_{(P,b_P)}$ of functors.

Example 3.2.4. *We denote by ${}_{\mathcal{O}Gb}\mathbf{Perm}$ the category of all p-permutation $\mathcal{O}Gb$-modules. Then ${}_{\mathcal{O}Gb}\mathbf{Perm}$ is a Brauer-friendly category. Moreover, the slash functor on ${}_{\mathcal{O}Gb}\mathbf{Perm}$ is the Brauer functor which is unique up to twisting by a linear character.*

For Brauer-friendly modules, the slash indecomposability can be defined as well as the Brauer indecomposability as follows (For Frobenius-friendly modules (*i.e.* endo-p-permutation modules), the slash indecomposability was defined in [8, Definition 5.1]).

Definition 3.2.5. Let ${}_{\mathcal{O}Gb}\mathbf{M}$ be a Brauer-friendly category of $\mathcal{O}Gb$-modules, $Sl_{(Q,b_Q)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G(Q,b_Q)\bar{b}_Q}\mathbf{Mod}$ a (Q, b_Q)-slash functor for each (G, b)-subpair (Q, b_Q), and $M \in {}_{\mathcal{O}Gb}\mathbf{M}$. We say that M is *slash indecomposable* if for every (G, b)-subpair (Q, b_Q), $\mathrm{Res}^{N_G(Q,b_Q)/Q}_{QC_G(Q)/Q}(Sl_{(Q,b_Q)}(M))$ is indecomposable or zero.

Remark 3.2.6. The definition of the slash indecomposability is independent of the choice of Brauer-friendly categories and slash functors.

The following theorem is a generalization of [5, (3.2) THEOREM. (3)].

Theorem 3.2.7 ([3, Theorem 23])**.** *Let b be a block of $\mathcal{O}G$, (P, b_P, V) a fusion-stable endo-permutation source triple, ${}_{\mathcal{O}Gb}\mathbf{M}$ a Brauer-friendly category of $\mathcal{O}Gb$-modules that is big enough with respect to (P, b_P, V), and $Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k[\overline{N}_G(P,b_P)]\overline{b}_P}\mathbf{Mod}$ a (P, b_P)-slash functor. Then $Sl_{(P,b_P)}$ induces a one-to-one correspondence between the isomorphism classes of indecomposable $\mathcal{O}Gb$-modules with source triple (P, b_P, V) and the isomorphism classes of projective indecomposable $k[\overline{N}_G(P, b_P)]\overline{b}_P$-modules.*

By this theorem, Brauer-friendly modules can be presented as follows.

Definition 3.2.8. With the same notation as in Theorem 3.2.7, let $M \in {}_{\mathcal{O}Gb}\mathbf{M}$ be an indecomposable $\mathcal{O}Gb$-module with source triple (P, b_P, V). Then, by Theorem 3.2.7, there is up to isomorphism a unique simple $k[\overline{N}_G(P, b_P)]\overline{b}_P$-module S such that $Sl_{(P,b_P)}(M) \cong P(S)$. We denote the module M by $B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. In particular, if $S \cong k_{\overline{N}_G(P,b_P)\overline{b_P}}$, then we denote the module M by $BS(b, (P, b_P, V), Sl_{(P,b_P)})$. We call this module the *Brauer-friendly Scott $\mathcal{O}Gb$-module with respect to (P, b_P, V).*

Remark 3.2.9. (i) The above presentation of Brauer-friendly modules is unique up to twisted by a linear character.

(ii) The Scott $\mathcal{O}G$-module $S(G, P)$ is presented by

$$S(G, P) = BS(b, (P, b_P, \mathcal{O}_P), Sl_{(P,b_P)})$$

where b is the principal block of $\mathcal{O}G$.

Chapter 4

Slash indecomposability of Brauer-friendly modules

In this chapter, we give an equivalent condition for Brauer-friendly modules to be slash indecomposable.

4.1 Lemmas

In this section, we give lemmas for Brauer-friendly modules, Brauer-friendly Scott modules, and slash functors, which are analogies of lemmas for p-permutation modules, Scott modules, and Brauer functors respectively, which are used to prove the main theorem in [9].

Notation. Let M be a Brauer-friendly module and $\mathcal{S}_M$ be the set of source triples of any indecomposable summand of M. Hereinafter, we assume that M belongs to some Brauer-friendly categories that is big enough with respect to $\mathcal{S}_M$. Moreover, when we apply a slash functor to the Brauer-friendly module M, we assume that the domain of the slash functor is big enough with respect to $\mathcal{S}_M$.

Lemma 4.1.1. *Let (P, b_P) be a (G, b)-subpair, H a subgroup of G such that $PC_G(P) \leq H \leq N_G(P, b_P)$, M a Brauer-friendly $\mathcal{O}Gb$-module, and $Sl_{(P,b_P)}$ a (P, b_P)-slash functor. By [3, Lemma 10 (i)], we get a decomposition $b_P M = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}Hb_P$-module and L' is a direct sum of indecomposable $\mathcal{O}Hb_P$-modules with vertices that do not contain P. Then there exists an isomorphism*

$$\mathrm{Res}^{\overline{N}_G(P,b_P)}_{\overline{H}}(Sl_{(P,b_P)}(M)) \cong Sl'_{(P,b_P)}(L)$$

of $k\overline{H}\bar{b}_P$-modules for some (P,b_P)-slash functor $Sl'_{(P,b_P)}$. In particular, if $H = N_G(P,b_P)$ and M has the source triple (P,b_P,V), then there exists an isomorphism

$$Sl_{(P,b_P)}(M) \cong Sl'_{(P,b_P)}(f^b_{b_P}(M))$$

of $k\overline{H}\bar{b}_P$-modules, where $f^b_{b_P}$ is the Green correspondence with respect to (P,b_P).

Proof. Write $N_G = N_G(P,b_P)$. We have isomorphisms of $C_H(P)$-interior H-algebras

$$\begin{aligned}\mathrm{End}_k(\mathrm{Res}^{N_G}_H(Sl_{(P,b_P)}(M))) &\cong \mathbf{Res}^{N_G}_H(\mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(b_P M)))\\ &\cong \mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(b_P \mathrm{Res}^G_H(M)))\\ &\cong \mathrm{Br}_{\triangle P}(\mathrm{End}_{\mathcal{O}}(L))\\ &\cong \mathrm{End}_k(Sl''_{(P,b_P)}(L)),\end{aligned}$$

where **Res** is a restriction to H as algebras and $Sl''_{(P,b_P)}$ is a (P,b_P)-slash functor. By [4, Lemma 3 (ii)], there exists a linear character $\chi : H/PC_H(P) \longrightarrow k^\times$ such that $b_P\mathrm{Res}^{N_G}_H(Sl_{(P,b_P)}(M)) \cong \chi_* Sl''_{(P,b_P)}(L)$. Hence, setting $Sl'_{(P,b_P)} = \chi_* Sl''_{(P,b_P)}$, we obtain

$$b_P\mathrm{Res}^{N_G}_H(Sl_{(P,b_P)}(M)) \cong Sl'_{(P,b_P)}(L).$$

The rest follows from $b_P M = f^b_{b_P}(M) \oplus Z$, where Z is a direct sum of indecomposable $\mathcal{O}N_G b_P$-modules with vertices that do not contain P. □

The following lemma is an analogy of [5, (3.2) THEOREM. (1)].

Lemma 4.1.2 ([2, Corollary 3.17])**.** *Let M be an indecomposable Brauer-friendly $\mathcal{O}Gb$-module with source triple (P,b_P,V), (Q,b_Q) a (G,b)-subpair, and $Sl_{(Q,b_Q)}$ a (Q,b_Q)-slash functor. Then $Sl_{(Q,b_Q)}(M) \neq 0$ if and only if $(Q,b_Q) \leq_G (P,b_P)$.*

We define the conjugation of slash functors by an element of a group.

Definition 4.1.3. Let (P,b_P) be a (G,b)-subpair and $Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G(P,b_P)\bar{b}_P}\mathbf{Mod}$ a (P,b_P)-slash functor. For each $g \in G$, we denote by ${}^g(-)$ the conjugation functor by g, also we denote the functor ${}^g(-) \circ Sl_{(P,b_P)} : {}_{\mathcal{O}Gb}\mathbf{M} \to {}_{k\overline{N}_G({}^gP,{}^gb_P){}^g\bar{b}_P}\mathbf{Mod}$ by $g_\star Sl_{(P,b_P)}$. Then, by [3, Lemma 22 (ii)], the functor $g_\star Sl_{(P,b_P)}$ is a ${}^g(P,b_P)$-slash functor.

Lemma 4.1.4. *Let (P,b_P) be a (G,b)-subpair. For each element $g \in G$, we have an isomorphism*

$$B(b,(P,b_P,V),Sl_{(P,b_P)},S) \cong B(b,({}^gP,{}^gb_P,{}^gV),g_\star Sl_{(P,b_P)},{}^gS)$$

of $\mathcal{O}G$-modules.

Proof. Set $X = B(b, ({}^gP, {}^gb_P, {}^gV), g_\star Sl_{(P,b_P)}, {}^gS)$. Then X also has the source triple (P, b_P, V) and we have ${}^g(Sl_{(P,b_P)}(X)) = g_\star Sl_{(P,b_P)}(X) = {}^gP(S)$. Thus $Sl_{(P,b_P)}(X) = P(S)$. Hence we obtain

$$B(b, (P, b_P, V), Sl_{(P,b_P)}, S) \cong B(b, ({}^gP, {}^gb_P, {}^gV), g_\star Sl_{(P,b_P)}, {}^gS).$$

□

Lemma 4.1.5. *Let (P, b_P) be a (G, b)-subpair and $f^b_{b_P}$ the Green correspondence with respect to (P, b_P). Then there exists a (P, b_P)-slash functor $Sl'_{(P,b_P)}$ such that there exists an isomorphism*

$$f^b_{b_P}(B(b, (P, b_P, V), Sl_{(P,b_P)}, S)) \cong B(b_P, (P, b_P, V), Sl'_{(P,b_P)}, S)$$

of $\mathcal{O}N_G(P, b_P)b_P$-modules. In particular, we have an isomorphism

$$f^b_{b_P}(BS(b, (P, b_P, V), Sl_{(P,b_P)})) \cong BS(b_P, (P, b_P, V), Sl'_{(P,b_P)}).$$

Proof. Set $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. Then, by Lemma 4.1.1, there exists a (P, b_P)-slash functor $Sl'_{(P,b_P)}$ such that there exists an isomorphism

$$Sl'_{(P,b_P)}(f^b_{b_P}(M)) \cong Sl_{(P,b_P)}(M) \cong P(S)$$

of $k\overline{N}_G(P, b_P)\overline{b}_P$-modules. □

The following lemma is an analogy of [16, Chapter 4, Theorem 8.6 (ii)] for Brauer-friendly modules.

Lemma 4.1.6. *Let P be a p-subgroup of G, H a subgroup of G such that $PC_G(P) \leq H$, b' a block of $\mathcal{O}H$, and (P, b_P) a (G, b)-subpair. We assume that (P, b_P) is an (H, b')-subpair, and (P, b_P, V) is a fusion-stable endo-permutation source triple. Then there exist $t \in N_G(P, b_P)$, a (P, b_P)-slash functor $Sl''_{(P,b_P)}$, and a simple $k[\overline{N}_H(P, b_P)]\overline{b}_P$-module S' such that*

$$B(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}, S') \mid \mathrm{Res}^G_H(B(b, (P, b_P, V), Sl_{(P,b_P)}, S)).$$

In particular, we have

$$BS(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}) \mid \mathrm{Res}^G_H(BS(b, (P, b_P, V), Sl_{(P,b_P)})).$$

To prove Lemma 4.1.6, we need the following lemma.

Lemma 4.1.7 (Burry [16, Chapter 4, Theorem 4.8 (i)])**.** *Let H be a subgroup of G containing $PC_G(P)$, b' a block of $\mathcal{O}H$, and (P, b_P) a (G, b)-subpair. We assume that (P, b_P) is an (H, b')-subpair. Let $f^b_{b_P}$ and $f^{b'}_{b_P}$ be the Green correspondences with respect to (P, b_P). Then, for any indecomposable $\mathcal{O}Gb$-module V with vertex subpair (P, b_P) and any indecomposable $\mathcal{O}Hb'$-module W with vertex subpair (P, b_P), the following conditions are equivalent.*

(i) $W \mid \mathrm{Res}^G_H(V)$.

(ii) $f^{b'}_{b_P}(W) \mid \mathrm{Res}^{N_G(P,b_P)}_{N_H(P,b_P)}(f^b_{b_P}(V))$.

Proof. (Proof of Lemma 4.1.6) We prove Lemma 4.1.6 in a similar way as the proof of [16, Chapter 4, Theorem 8.6 (ii)]. Set $N_G = N_G(P, b_P)$ and $N_H = N_H(P, b_P)$. By Lemma 4.1.7, it is sufficient to show the following:

$$f^{b'}_{b_P}(B(b', (P, b_P, {}^tV), Sl''_{(P,b_P)}, S')) \mid \mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B(b, (P, b_P, V), Sl_{(P,b_P)}, S))).$$

Also, by Lemma 4.1.5, this statement is equivalent to the following:

$$B(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)}, S') \mid \mathrm{Res}^{N_G}_{N_H}(B(b_P, (P, b_P, V), Sl'_{(P,b_P)}, S)).$$

Set $B_G = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. It is equivalent to show that there exist an element $t \in N_G$, a (P, b_P)-slash functor $Sl'''_{(P,b_P)}$, a simple $k\overline{N}_H\bar{b}_P$-module S', and an indecomposable direct summand X of $b_P\mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G))$ such that X has a source triple $(P, b_P, {}^tV)$ and $Sl'_{(P,b_P)}(X) \cong P(S')$. By [3, Lemma 10 (i)], we get a decomposition $b_P\mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G)) = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}N_Hb_P$-module and L' is a direct sum of indecomposable $\mathcal{O}N_Hb_P$-modules with vertices that do not contain P. Since $f^b_{b_P}(B_G) = B(b_P, (P, b_P, V), Sl'_{(P,b_P)}, S)$, we obtain $f^b_{b_P}(B_G) \mid \mathrm{Ind}^{N_G}_P(V)$. The Mackey formula gives the relation

$$L \mid \mathrm{Res}^{N_G}_{N_H}(f^b_{b_P}(B_G)) \mid \bigoplus_{t \in N_H \backslash N_G / P} \mathrm{Ind}^{N_H}_P({}^tV).$$

Let $L = \oplus_{i \in I} L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}N_Hb_P$-modules. Then each L_i has the vertex subpair (P, b_P). Hence for each $i \in I$, there exists $t_i \in N_G$ such that $\mathrm{s}(L_i) = {}^{t_i}V$. By Lemma 4.1.1, there exists a (P, b_P)-slash functor $Sl'''_{(P,b_P)}$ such that

$$\mathrm{Res}^{N_G}_{N_H}(Sl'_{(P,b_P)}(f^b_{b_P}(B_G))) \cong Sl'''_{(P,b_P)}(L).$$

There exists a simple $k\overline{N}_H\bar{b}_P$-module S'_i such that $Sl'''_{(P,b_P)}(L_i) \cong P(S'_i)$, by the above argument and Theorem 3.2.7. This shows

$$L_i = B(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)}, S').$$

In particular, if $S = k_{\overline{N}_Gb_P}$, then $P(k_{\overline{N}_Hb_P}) \mid \mathrm{Res}^{N_G}_{N_H}(Sl'_{(P,b_P)}(f^b_{b_P}(B_G)))$. Thus there exists $i \in I$ such that $Sl'''_{(P,b_P)}(L_i) \cong P(k_{\overline{N}_Hb_P})$. This shows $L_i = BS(b_P, (P, b_P, {}^tV), Sl'''_{(P,b_P)})$. □

Lemma 4.1.8 (Burry-Carlson, Puig). *Let (P, b_P) be a (G, b)-subpair, $H := N_G(P, b_P)$, $f^b_{b_P}$ the Green correspondence with respect to (P, b_P), V an indecomposable $\mathcal{O}Gb$-module, and W an indecomposable summand of $b_P\mathrm{Res}^G_H(V)$. Then the following condition (i) implies (ii) and $f^b_{b_P}(V) = W$.*

(i) W has a vartex subpair (P, b_P).

(ii) V has a vartex subpair (P, b_P).

The following lemma is a generalization of H. Kawai [10, Theorem 1.7] for Brauer-friendly modules. We prove the lemma with a similar argument as [10, Theorem 1.7].

Lemma 4.1.9. *Let (P, b_P) be a (G, b)-subpair, $(Q, b_Q) \leq_G (P, b_P)$, and set $H = N_G(Q, b_Q)$ and $B_G = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$. If $R = {}^gP \cap H$ is a maximal element of $\{{}^iP \cap H \mid i \in G, (Q, b_Q) \leq {}^i(P, b_P)\}$, then there exist an (R, b_R)-slash functor $Sl_{(R,b_R)}$, an element $z \in G$, and a simple $k[\overline{N}_H(R, b_R)]\overline{b}_R$-module S' such that*

$$B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}^{{}^zP}_R({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}^G_H(B_G),$$

where b_R is the unique block satisfying $(R, b_R) \leq {}^g(P, b_P)$.

Proof. We prove this by induction on $|P|/|R|$.

If $|P|/|R| = 1$, *i.e.* ${}^gP = R$, then ${}^g(P, b_P)$ is a (G, b)-subpair. By $(Q, b_Q) \leq (R, b_R)$, $(R, b_R) = {}^g(P, b_P)$ is an (H, b_Q)-subpair. Hence, by Lemma 4.1.4 and Lemma 4.1.6, there exist an (R, b_R)-slash functor $Sl_{(R,b_R)}$ and an element $z \in N_G(R, b_R)$ such that

$$B(b_Q, (R, b_R, {}^zV), Sl_{(R,b_R)}, S') \mid \ \mathrm{Res}^G_H(B(b, ({}^gP, {}^gb_P, V), g_\star Sl_{(P,b_P)}, {}^gS),$$

and

$$\mathrm{Res}^G_H(B(b, ({}^gP, {}^gb_P, V), g_\star Sl_{(P,b_P)}, {}^gS) \cong \mathrm{Res}^G_H(B_G).$$

In this case, the statement follows.

Now suppose that $|P|/|R| \gneq 1$, *i.e.* $R \lneq_G P$. We set $H_1 = N_G(R, b_R)$ and $\Omega = \{{}^iP \cap H_1 | i \in G, (R, b_R) \leq {}^i(P, b_P)\}$. From $(R, b_R) \leq {}^g(P, b_P)$, we see $\Omega \neq \emptyset$. Let R_1 be a maximal element of Ω. Then H_1 and (R_1, b_{R_1}) satisfy the condition of the lemma. Therefore, by induction hypothesis, there exist an (R_1, b_{R_1})-slash functor $Sl_{(R_1,b_{R_1})}$, an element $x \in G$, and a simple $k[\overline{N}_{H_1}(R_1, b_{R_1})]\overline{b}_{R_1}$-module S_{R_1} such that

$$B(b_R, (R_1, b_{R_1}, \mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))), Sl_{(R_1,b_{R_1})}, S_{R_1}) \mid \mathrm{Res}^G_{H_1}(B_G).$$

Set $N = B(b_R, (R_1, b_{R_1}, \mathrm{Cap}(\mathrm{Res}^{{}^xP}_{R_1}({}^xV))), Sl_{(R_1,b_{R_1})}, S_{R_1})$, $T = N_H(R, b_R)$. By [3, Lemma 10 (i)], we get a decomposition $b_R\mathrm{Res}^{H_1}_T(N) = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}Tb_R$-module and L' is a direct sum of indecomposable $\mathcal{O}Tb_R$-modules with vertices that do not contain R. Let $L = \oplus_{i\in I} L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}Tb_R$-modules. Then, for any $i \in I$, there exists a vertex of L_i which contains R. Here, the Mackey formula gives the relation

$$\begin{aligned}
&\bigoplus_{i\in I} L_i \ |\mathrm{Res}_T^{H_1}(\mathrm{Ind}_{R_1}^{H_1}(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV))))\\
&\qquad \cong \bigoplus_{h\in T\backslash H_1/R_1} \mathrm{Ind}_{{}^hR_1\cap T}^{T}(\mathrm{Res}_{{}^hR_1\cap T}^{{}^hR_1}({}^h(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV)))))\\
&\qquad \cong \bigoplus_{h\in T\backslash H_1/R_1} \mathrm{Ind}_{R}^{T}(\mathrm{Res}_{R}^{{}^hR_1}({}^h(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV))))),
\end{aligned}$$

where $R = {}^hR_1 \cap T$, for any element $h \in H_1$. Hence, for any $i \in I$, we have $\mathrm{vtx}(L_i) = R$. Therefore, for any $i \in I$, we can take a vertex subpair of L_i as (R, b_R). We may assume that

$$L_i \mid \mathrm{Ind}_R^H(\mathrm{Res}_R^{{}^{h_i}R_1}({}^{h_i}(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV))))),$$

for some element $h_i \in H_1$. Let $\mathrm{Res}_R^{{}^{h_i}R_1}({}^{h_i}(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV)))) = \bigoplus_{j\in J} Z_j$ be a decomposition as a direct sum of indecomposable $\mathcal{O}R$-modules. Then, there exists an element $j \in J$ such that $\mathrm{s}(L_i) = Z_j$. Since we can take a vertex of Z_j as R, we have

$$Z_j \cong \mathrm{Cap}(\mathrm{Res}_R^{{}^{h_i}R_1}({}^{h_i}(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV))))).$$

Moreover, we see that

$$\mathrm{Cap}(\mathrm{Res}_R^{{}^{h_i}R_1}({}^{h_i}(\mathrm{Cap}(\mathrm{Res}_{R_1}^{{}^xP}({}^xV))))) = \mathrm{Cap}(\mathrm{Res}_R^{{}^{h_ix}P}({}^{h_ix}V)).$$

From the above, for any $i \in I$, there exist an (R, b_R)-slash functor $Sl_{(R,b_R)}$ and a simple $k[\overline{N}_T(R, b_R)]\bar{b}_R$-module S'_i such that

$$L_i = B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^{hx}P}({}^{hx}V))), Sl_{(R,b_R)}, S').$$

We choose $i \in I$ and set $h = h_i$, $S' = S'_i$, $z = hx \in G$. Then we have

$$B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^zP}({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}_T^H(\mathrm{Res}_H^G(B_G)).$$

Therefore there exists a direct summand U of $\mathrm{Res}_H^G(B_G)$ such that

$$B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^zP}({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}_T^H(U).$$

By Lemma 4.1.8 and [3, Theorem 4], the module U has a vertex subpair (R, b_R) and lies in the block b_R of $\mathcal{O}H$ and

$$f_{b_R}^{b_Q}(U) = B(b_R, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^zP}({}^zV))), Sl_{(R,b_R)}, S').$$

Hence, by Lemma 4.1.5, we have

$$U \cong B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^zP}({}^zV))), Sl_{(R,b_R)}, S').$$

From the above, it follows that

$$B(b_Q, (R, b_R, \mathrm{Cap}(\mathrm{Res}_R^{{}^zP}({}^zV))), Sl_{(R,b_R)}, S') \mid \mathrm{Res}_H^G(B_G).$$

□

The following lemma is a generalization of J. Thévenaz [19, Exercises (27.4)] for Brauer-friendly modules.

Lemma 4.1.10. *Let (P, b_P) be a (G, b)-subpair and set $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ and $Q \le_G P$ and set $H = N_G(Q, b_Q)$. By [3, Lemma 10 (i)], we get a decomposition $b_Q Res_H^G(M) = L \oplus L'$, where L is a Brauer-friendly $\mathcal{O}Hb_Q$-module and L' is a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules with vertices that do not contain Q. Let $L = \oplus_{i \in I} L_i$ be a decomposition of L as a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules and we set $Z_i = \mathrm{vtx}(L_i)$. Then, for each $1 \le i \le n$ and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, there exist an element $g_i \in G$ and a simple $k[\overline{N}_H(Z_i, b_{Z_i})]\overline{b}_{Z_i}$-module S_i such that*

$$Sl_{(Q,b_Q)}(L_i) \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}_{Z_i}^{{}^{g_i}P}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j}),$$

where $X_{i,j}$ is an indecomposable Brauer-friendly kHb_Q-module with source triple

$$(\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}, \mathrm{s}(X_{i,j}))$$

such that

$$(Q, b_Q) \le (\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}) \le (Z_i, b_{Z_i})$$

and

$$\mathrm{s}(X_{i,j}) \mid \mathrm{Res}_{\mathrm{vtx}(X_{i,j})}^{Z_i}(\mathrm{Cap}(\mathrm{Res}_{Z_i}^{{}^{g_i}P}({}^{g_i}V)))[Q].$$

Therefore, we have

$$\begin{aligned} Sl_{(Q,b_Q)}(M) &\cong Sl_{(Q,b_Q)}(L) \\ &\cong \bigoplus_{1 \le i \le n} \big(B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}_{Z_i}^{{}^{g_i}P}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j})\big). \end{aligned}$$

Remark 4.1.11. If $Sl_{(Q,b_Q)}(L_i)$ is indecomposable, then we have

$$Sl_{(Q,b_Q)}(L_i) \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Proof. By Lemma 4.1.1, we have

$$Sl_{(Q,b_Q)}(M) \cong Sl_{(Q,b_Q)}(L) \cong \bigoplus_{1 \leq i \leq n} Sl_{(Q,b_Q)}(L_i).$$

First, we determine the structure of each L_i. By [3, Theorem 4], we see that there exists an element $g_i \in G$ such that $(Q, b_Q) \trianglelefteq (Z_i, b_{Z_i}) \leq {}^{g_i}(P, b_P)$ and $\mathrm{s}(L_i) = \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))$. Therefore, there exist an element $g_i \in G$, a (Z_i, b_{Z_i})-slash functor $Sl_{(Hb_Q, Z_i, b_{Z_i})}$, and a simple $k[\overline{N}_H(Z_i, b_{Z_i})]\overline{b}_{Z_i}$-module S_i such that

$$L_i \cong B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Next, we determine the structure of $Sl_{(Q,b_Q)}(L_i)$. Since we have $(Q, b_Q) \trianglelefteq (Z_i, b_{Z_i})$ by [2, Lemma 3.16 (i)], we see

$$P(S_i) \cong Sl_{(Z_i,b_{Z_i})}(L_i) \cong Sl_{(Z_i,b_{Z_i})} \circ Sl_{(Q,b_Q)}(L_i).$$

Thus, there exists the unique direct summand X_i of $Sl_{(Q,b_Q)}(L_i)$ such that $Sl_{(Z_i,b_{Z_i})}(X_i) \cong P(S_i)$. From [4, Lemma 3 (iii)] and Lemma 4.1.2, we see $\mathrm{vtx}(X_i) = \mathrm{vtx}(L_i)$ and $\mathrm{s}(X) = \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]$. Hence, we get

$$X_i = B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i).$$

Let $Sl_{(Q,b_Q)}(L_i) = X_i \oplus (\bigoplus_j X_{i,j})$ be a decomposition of $Sl_{(Q,b_Q)}(L_i)$ as a direct sum of indecomposable $\mathcal{O}Hb_Q$-modules. By [4, Lemma 3 (iii)], we have $(Q, b_Q) \leq (\mathrm{vtx}(X_{i,j}), b_{\mathrm{vtx}(X_{i,j})}) \leq (Z_i, b_{Z_i})$ and

$$\mathrm{s}(X_{i,j}) \mid \mathrm{Res}^{Z_i}_{\mathrm{vtx}(X_{i,j})}(\mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V)))[Q].$$

From the above, we have

$$\begin{aligned} Sl_{(Q,b_Q)}(M) &\cong Sl_{(Q,b_Q)}(L) \\ &\cong \bigoplus_{1 \leq i \leq n} \big(B(b_Q, (Z_i, b_{Z_i}, \mathrm{Cap}(\mathrm{Res}^{g_i P}_{Z_i}({}^{g_i}V))[Q]), Sl_{(Z_i,b_{Z_i})}, S_i) \oplus (\bigoplus_j X_{i,j})\big). \end{aligned}$$

□

The following lemma is the subpair version of [9, Lemma 3.1]. It can be proved in a similar way as the proof of [9, Lemma 3.1].

Lemma 4.1.12. *Let (P, b_P) be a (G, b)-subpair and Q a fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup of G. Assume that $(Q, b_Q) \leq (P, b_P)$. Then, $N_P(Q)$ is a maximal element of*

$$\{{}^gP \cap N_G(Q, b_Q) \mid g \in G, (Q, b_Q) \leq {}^g(P, b_P)\}.$$

The following lemma is the subpair version of [9, Lemma 3.2].

Lemma 4.1.13. *Let (P, b_P) be a (G, b)-subpair and set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$. If Q is a fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive subgroup of P, then we have $N_{{}^gP}(Q) \leq_{N_G(Q,b_Q)} N_P(Q)$, for any element $g \in G$ such that $(Q, b_Q) \leq ({}^gP, {}^gb_P)$.*

Proof. Assume that $(Q, b_Q) \leq ({}^gP, {}^gb_P)$ for some element $g \in G$. Then ${}^{g^{-1}}Q$ and Q are $\mathcal{F}$-conjugate. Therefore, by [1, I, Lemma 2.6 (c)], there exists $\varphi_x \in \mathrm{Hom}_{\mathcal{F}}(N_P({}^{g^{-1}}Q), N_P(Q))$ such that $\varphi_x|_{{}^{g^{-1}}Q} \in \mathrm{Iso}_{\mathcal{F}}({}^{g^{-1}}Q, Q)$. Thus $xg^{-1} \in N_G(Q, b_Q)$ and

$$N_{{}^gP}(Q) = {}^gN_P({}^{g^{-1}}Q) =_{N_G(Q,b_Q)} {}^{(xg^{-1})g}N_P({}^{g^{-1}}Q) = {}^xN_P({}^{g^{-1}}Q) \leq N_P(Q).$$

□

4.2 Main theorem

Notation. Let (P, b_P) be a (G, b)-subpair, set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$, let Q be a fully $\mathcal{F}$-normalized subgroup of P, and $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ a Brauer-friendly $\mathcal{O}Gb$-module. Then, from Lemma 4.1.12, the subgroup $N_P(Q)$ is a maximal element of

$$\{{}^gP \cap N_G(Q, b_Q) \mid g \in G, (Q, b_Q) \leq {}^g(P, b_P)\}.$$

Therefore, by Lemma 4.1.9, there exist an element $n \in G$, an $(N_P(Q), b_{N_P(Q)})$-slash functor $Sl_{(N_P(Q),b_{N_P(Q)})}$, and a simple $k[\overline{N}_{N_G(Q,b_Q)}(N_P(Q), b_{N_P(Q)})]\bar{b}_{N_P(Q)}$-module S_Q such that

$$B(b_Q, (N_P(Q), b_{N_P(Q)}, W_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q) \mid \mathrm{Res}^G_{N_G(Q,b_Q)}(M),$$

where $W_Q = \mathrm{Cap}(\mathrm{Res}^{{}^nP}_{N_P(Q)}({}^nV))$. Also, by Lemma 4.1.10, for any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, we have

$$B(b_Q, (N_P(Q), b_{N_P(Q)}, V_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q) \mid Sl_{(Q,b_Q)}(M),$$

where $V_Q = W_Q[Q]$. In this section, we set

$$B_Q = B(b_Q, (N_P(Q), b_{N_P(Q)}, V_Q), Sl_{(N_P(Q),b_{N_P(Q)})}, S_Q).$$

The following theorem is the main theorem in this chapter, which is a generalization of [9, Theorem 1.3].

Theorem 4.2.1. *Let G be a finite group, b a block of $\mathcal{O}G$, and (P, b_P) a (G, b)-subpair. We set $M = B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$, $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$, $N_Q = N_G(Q, b_Q)$, and $H_Q = N_P(Q)$ for $Q \leq P$. Suppose that $\mathcal{F}$ is saturated and $\mathrm{Res}^{N_P}_{PC_G(P)}(S)$ is a simple $\mathcal{O}PC_G(P)$-module. Then the following conditions are equivalent.*

(i) M is slash indecomposable.

(ii) $\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q)$ is indecomposable for each fully $\mathcal{F}$-normalized subgroup Q of P.

If these conditions are satisfied, then for each fully $\mathcal{F}$-normalized subgroup Q of P and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, we have

$$Sl_{(Q,b_Q)}(M) \cong B_Q.$$

4.3 Proof of Theorem 4.2.1

In this section, we prove Theorem 4.2.1.

Proof. (Proof of Theorem 4.2.1) If (i) holds, *i.e.* $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable, for each fully $\mathcal{F}$-normalized subgroup Q of P and any (Q, b_Q)-slash functor $Sl_{(Q,b_Q)}$, then by the definition of B_Q, we have

$$\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q) \cong \mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M)).$$

Hence, $\mathrm{Res}^{N_Q}_{QC_G(Q)}(B_Q)$ is indecomposable. This shows (ii). Moreover, the module $Sl_{(Q,b_Q)}(M)$ is also indecomposable, since $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable. Therefore, we get

$$Sl_{(Q,b_Q)}(M) \cong B_Q.$$

Conversely, suppose that (ii) holds. It is sufficient to prove that $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is indecomposable, for each $Q \leq P$. We prove this by induction on $|P : Q|$.

If $|P : Q| = 1$, then this case is similar to the proof of [11, Lemma 4.3 (ii)], by the assumption of the theorem.

Now consider the case that $|P : Q| \gneq 1$. For some element $g \in G$, ${}^gQ \leq P$ and ${}^gQ$ is fully $\mathcal{F}$-normalized. We see that for any $({}^gQ, b_{{}^gQ})$-slash functor $Sl_{({}^gQ,b_{{}^gQ})}$,

$${}^g(\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))) \cong \mathrm{Res}^{N_{{}^gQ}}_{{}^gQC_G({}^gQ)}(Sl_{({}^gQ,b_{{}^gQ})}(M)).$$

Therefore, it is sufficient to prove that $\mathrm{Res}^{N_{{}^gQ}}_{{}^gQC_G({}^gQ)}(Sl_{({}^gQ,b_{{}^gQ})}(M))$ is indecomposable. Hence, without loss of generality, we may assume that Q is fully $\mathcal{F}$-normalized.

We set $N_1 = B_Q$. Let $Sl_{(Q,b_Q)}(M) = \underset{1\le i\le r}{\oplus} N_i$ be a decomposition of $Sl_{(Q,b_Q)}(M)$ as a direct sum of indecomposable kN_Qb_Q-modules. Then, by Lemma 4.1.10 and its proof, for N_i, there exists a direct summand $L_j \mid \mathrm{Res}^G_{N_Q}(M)$ and an element $g_i \in G$ such that

$$(Q, b_Q) \le (R, b_R) \le (\mathrm{vtx}(L_j), b_{\mathrm{vtx}(L_j)}) \le {}^{g_i}(P, b_P).$$

where $R = \mathrm{vtx}(N_i)$. By Lemma 4.1.2, $Sl_{(R,b_R)}(N_i) \neq 0$. Since Q is fully $\mathcal{F}$-normalized, Q is fully $\mathcal{F}$-automized and $\mathcal{F}$-receptive, and hence $N_{^{g_i}P}(Q) \le_{N_Q} H_Q$, from Lemma 4.1.13. Thus

$$R \le {}^{g_i}P \cap N_Q = N_{^{g_i}P}(Q) \le_{N_Q} H_Q$$

and $Sl_{(R,b_R)}(N_1) \neq 0$. Now we have

$$Sl_{(R,b_R)}(N_1) \oplus Sl_{(R,b_R)}(N_i) \mid Sl_{(R,b_R)}(Sl_{(Q,b_Q)}(M)) \cong \mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M)).$$

Thus $\mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M))$ is decomposable and $\mathrm{Res}^{N_R}_{RC_G(R)}(Sl_{(R,b_R)}(M))$ is decomposable, since $RC_G(R) \le N_R \cap N_Q$. If $Q = R$, then we see $P = Q$ from [4, Lemma 5] and Lemma 4.1.8. This is a contradiction. Hence $Q \lneqq R$ holds and we have that $|P : Q| \gneqq |P : R|$. By the induction hypothesis, the module $\mathrm{Res}^{N_R}_{N_R\cap N_Q}(Sl_{(R,b_R)}(M))$ is indecomposable. Hence $r = 1$, and we have that

$$Sl_{(Q,b_Q)}(M) \cong N_1 = B_Q.$$

Hence, $\mathrm{Res}^{N_Q}_{QC_G(Q)}(N_1)$ is indecomposable, and $\mathrm{Res}^{N_Q}_{QC_G(Q)}(Sl_{(Q,b_Q)}(M))$ is also indecomposable, by our hypothesis, . □

The following lemma can be proved in a similar way as [9, Lemma 4.3].

Lemma 4.3.1. *Let (P, b_P) be a (G, b)-subpair, $\mathcal{F} := \mathcal{F}_{(P,b_P)}(G, b)$, and Q a fully $\mathcal{F}$-automized subgroup of P. If there exists $N_P(Q) \le H_Q \le N_G(Q, b_Q)$ such that $|N_G(Q, b_Q) : H_Q| = p^a$ $(a \ge 0)$, then $N_G(Q, b_Q) = C_G(Q)H_Q$.*

The following proposition is a special analogy of [9, Theorem 1.4].

Proposition 4.3.2. *Let (P, b_P) be a (G, b)-subpair and Q a fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup of P. Suppose that $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$ is saturated. Moreover, we assume that the following two conditions:*

(i) $|N_G(Q, b_Q) : N_P(Q)| = p^a$ $(a \ge 0)$.

(ii) $\mathrm{Res}^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)$ *is indecomposable.*

Then $\mathrm{Res}^{N_G(Q,b_Q)}_{QC_G(Q)}(B_Q)$ *is indecomposable.*

Proof. We set $N_G = N_G(Q, b_Q)$. Since $\mathcal{F}$ is saturated, Q is a fully $\mathcal{F}$-automized subgroup of P. From the Mackey formula, Lemma 4.3.1, and the condition (i), we have

$$\mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q)) \cong \mathrm{Ind}^{QC_G(Q)}_{QC_G(Q)\cap N_P(Q)}(\mathrm{Res}^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)).$$

Hence, $\mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q))$ is indecomposable, by the condition (ii) and Green's indecomposability theorem, so

$$\mathrm{Res}^{N_G}_{QC_G(Q)}(B_Q) \cong \mathrm{Res}^{N_G}_{QC_G(Q)}(\mathrm{Ind}^{N_G}_{N_P(Q)}(V_Q))$$

is indecomposable. □

The following corollary is a consequence of Theorem 4.2.1 and Proposition 4.3.2.

Corollary 4.3.3. *Let (P, b_P) be a (G, b)-subpair, $B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ a Brauer-friendly $\mathcal{O}Gb$-module, and suppose that $\mathcal{F}_{(P,b_P)}(G, b)$ is saturated. If for every fully $\mathcal{F}_{(P,b_P)}(G, b)$-normalized subgroup Q of P, the subgroup $N_P(Q)$ and the module V_Q satisfy the conditions of Proposition 4.3.2, then the module $B(b, (P, b_P, V), Sl_{(P,b_P)}, S)$ is slash indecomposable.*

The following example is a generalization of [20, Lemma 2.2] to Brauer-friendly modules.

Example 4.3.4. *Let G be a p-group, $(P, 1_{C_G(P)})$ a $(G, 1_G)$-subpair, and suppose that $\mathcal{F} = \mathcal{F}_P(G)$ is saturated. Set $M = BS(1_G, (P, b_P, V), Sl_{(P,1_{C_G(P)})})$. Moreover, we assume that $Res^{N_P(Q)}_{QC_G(Q)\cap N_P(Q)}(V_Q)$ is indecomposable, for any fully $\mathcal{F}$-normalized subgroup Q of P. From Corollary 4.3.3, M is slash indecomposable.*

Chapter 5

Liftability of Brauer-friendly modules

In this chapter, we show that any indecomposable Brauer-friendly kGb-module satisfying certain condition is liftable to an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module.

5.1 Main theorem

The following theorem is the main theorem in this chapter.

Theorem 5.1.1. *Let G be a finite group, b a block of kG with a defect group D, and M an indecomposable Brauer-friendly kGb-module with a source triple (P, b_P, S). Suppose that $\mathcal{F} := \mathcal{F}_{(P,b_P)}(G, b)$ is saturated. Then there exists an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module $\widehat{M}$ with source triple $(P, \widehat{b}_P, \widehat{S})$ such that $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$ and $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$.*

5.2 Lemmas

The following lemma can be proved in the same way as the proof of [14, Proposition 3.2 (i)].

Lemma 5.2.1. *Let G be a finite group, b a block of $\mathcal{O}G$ with a defect group D, i a source idempotent of the block b, and P a subgroup of D. Set $A = i\mathcal{O}Gi$ and $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$, where b_P is the unique block of $\mathcal{O}C_G(P)$ such that $\bar{b}_P \mathrm{br}_P(i) \neq 0$. Let V be an $\mathcal{F}$-stable endo-permutation $\mathcal{O}P$-module having an indecomposable direct summand with vertex P. Set $U = A \otimes_{\mathcal{O}P} V$. Then, as an $\mathcal{O}P$-module, U is an endo-permutation module, and U has a direct summand isomorphic to V.*

The following lemma can be proved in a similar way as the proof of [12, Lemma 8.3].

Lemma 5.2.2. *Let G be a finite group, b a block of $\mathcal{O}G$ with a defect group D, i a source idempotent of the block b, and P a subgroup of D. Set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G,b)$, where b_P is the unique block of $\mathcal{O}C_G(P)$ such that $\bar{b}_P\mathrm{br}_P(i) \neq 0$. Let V be an indecomposable $\mathcal{F}$-stable endo-permutation $\mathcal{O}P$-module with vertex P. Set $X = OGi \otimes_{\mathcal{O}P} V$. The canonical algebra homomorphism*

$$\mathrm{End}_{\mathcal{O}G}(X) \to \mathrm{End}_{kG}(k \otimes_{\mathcal{O}} X)$$

is surjective. In particular, for any indecomposable direct summand M of $k \otimes_{\mathcal{O}} X$, there is an indecomposable direct summand $\widehat{M}$ of X such that $k \otimes_{\mathcal{O}} \widehat{M} \cong M$.

Proof. In the proof of Lemma 5.2.2, we use Lemma 5.2.1 instead of [14, Proposition 4.1] in the proof of [12, Lemma 8.3]. □

To prove the main theorem, we need the following lemma.

Lemma 5.2.3 ([12, Lemma 8.4])**.** *Let P be a finite p-group and $\mathcal{F}$ a saturated fusion system on P. The canonical map $\mathcal{D}_{\mathcal{O}}(P,\mathcal{F}) \to \mathcal{D}_k(P,\mathcal{F})$ is surjective.*

The following lemmas are over k version of [3, Lemma 3 (i), (ii)] and can be proved in a similar way as the proof of [3, Lemma 3 (i), (ii)].

Lemma 5.2.4 ([3, Lemma 3 (i)])**.** *Let M be an indecomposable kGb-module with a source triple (P, b_P, V). There exists a primitive idempotent i of the algebra $(kGb)^P$ such that $b_P\mathrm{br}_P(i) \neq 0$ and that M is isomorphic to a direct summand of the kGb-module $kGi \otimes_{kP} V$.*

Lemma 5.2.5 ([3, Lemma 3 (ii)])**.** *Let M be an indecomposable kGb-module with a source triple (P, b_P, V). There exists a defect group D of the block b such that $P \leq D$ and there exists a primitive idempotent j of the algebra $(kGb)^D$ such that $\mathrm{br}_D(j) \neq 0$ and $b_P\mathrm{br}_P(j) \neq 0$, and that M is isomorphic to a direct summand of the kGb-module $kGj \otimes_{kP} V$.*

5.3 Proof of Theorem 5.1.1

In this section, we prove Theorem 5.1.1.

Proof. (Proof of Theorem 5.1.1) By Lemma 5.2.3, there exists an indecomposable endo-permutation $\mathcal{O}P$-module $\widehat{S}$ such that $\widehat{S} \in \mathcal{D}_{\mathcal{O}}(P,\mathcal{F})$ and $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$. Let $bb_P = i_1 + \cdots + i_n$ be a decomposition of bb_P into mutually orthogonal primitive idempotents in the algebra $(kGb)^P$. By Lemma 5.2.4 and its proof, for some primitive idempotent $i_\ell \in (kGb)^P$, we have $b_P\mathrm{br}_P(i_\ell) \neq 0$ and a following relation

$$M \mid kGi_\ell \otimes_{kP} S \mid \bigoplus_{1\leq j\leq n} (kGi_j \otimes_{kP} S) = bkGb_P \otimes_{kP} S.$$

Set $i = i_\ell$. Also, by Lemma 5.2.5 and its proof, after suitable retake of a defect group, there exists a source idempotent j in $(kGb)^D$ such that $M \mid kGj \otimes_{kP} S$ and $b_P \mathrm{br}_P(j) \neq 0$, and there exists a decomposition $j = {}^x i + z_1 + \cdots + z_m$ of j into mutually orthogonal primitive idempotents in the algebra $(kGb)^P$, for some element $x \in ((kGb)^P)^\times$. By the lifting theorem of idempotents, there exists a decomposition $\hat{b}\hat{b}_P = \hat{i}_1 + \cdots + \hat{i}_n$ of $\hat{b}\hat{b}_P$ into mutually orthogonal primitive idempotents in the algebra $(\mathcal{O}G\hat{b})^P$. Also, by the lifting theorem of idempotents, there exists a source idempotent $\hat{j}$ in $(\mathcal{O}G\hat{b})^D$ and a decomposition $\hat{j} = \widehat{{}^x i} + \hat{z}_1 + \cdots + \hat{z}_m$ of $\hat{j}$ into mutually orthogonal primitive idempotents in the algebra $(\mathcal{O}G\hat{b})^P$. By the lifting theorem of idempotents, $\hat{i} := \hat{i}_\ell$ and $\widehat{{}^x i}$ are conjugate. Hence we get an isomorphism of $\mathcal{O}Gb{-}\mathcal{O}P$-bimodule

$$\mathcal{O}G\hat{i} \cong \mathcal{O}G\widehat{{}^x i}.$$

Therefore we have

$$\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} \cong (\mathcal{O}G\widehat{{}^x i} \otimes_{\mathcal{O}P} \widehat{S}) \mid \mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}. \tag{5.1}$$

By $\bar{\hat{j}} = j$ and $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$, we get

$$M \mid kGj \otimes_{kP} S \cong k \otimes_{\mathcal{O}} \mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}.$$

Then by Lemma 5.2.2, there exists a direct summand $\widehat{M}$ of $\mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}$ such that

$$k \otimes_{\mathcal{O}} \widehat{M} \cong \widehat{M}/\mathfrak{p}\widehat{M} \cong M.$$

In the following, we show that the module $\widehat{M}$ has a source triple $(P, \hat{b}_P, \widehat{S})$. Let

$$\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} = \bigoplus_{1 \leq i \leq m} \widehat{M'_i},$$

be a decomposition of $\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S}$ as a direct sum of indecomposable $\mathcal{O}G\hat{b}$-modules. Then we have an isomorphism

$$\bigoplus_{1 \leq i \leq m} (k \otimes_{\mathcal{O}} \widehat{M'_i}) \cong k \otimes_{\mathcal{O}} (\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S}) \cong kGi \otimes_{kP} S.$$

Here by Lemma 5.2.2 and the lifting theorem of idempotents, if $k \otimes_{\mathcal{O}} \widehat{M'_i} \neq 0$, then $k \otimes_{\mathcal{O}} \widehat{M'_i}$ is indecomposable. Therefore by the Krull-Schmidt theorem, there exists some j such that

$$k \otimes_{\mathcal{O}} \widehat{M'_j} \cong M.$$

Therefore by (4.1) and the lifting theorem of idempotents, we get

$$\widehat{M} \cong \widehat{M'_j} \mid \mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} \mid \hat{b}\mathcal{O}G\hat{b}_P \otimes_{\mathcal{O}P} \widehat{S}.$$

By this relation, $\widehat{M}$ is relative P-projective. Also, if for some $Q \lneq_G P$, $\widehat{M}$ is a direct summand of $\mathrm{Ind}_Q^G(\mathrm{Res}_Q^G(\widehat{M}))$, then M is a direct summand of $\mathrm{Ind}_Q^G(\mathrm{Res}_Q^G(M))$, *i.e.* M is relative Q-projective. This is a contradiction. Hence, we get $\mathrm{vtx}(\widehat{M}) = P$. Since $(P, \hat{b}_P, \widehat{S})$ is a fusion-stable endo-permutati-on source triple, $\widehat{M}$ is an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module with a source triple $(P, \hat{b}_P, \widehat{S})$ such that $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$ and $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$. □

Remark 5.3.1. 1. In general, the lifts given by Theorem 5.1.1 are not necessarily unique.

2. Our proof of Theorem 5.1.1 depends on the classification of endo-permutation modules.

Remark 5.3.2. Let G, H be finite groups and b, c blocks of kG, kH with a defect group D, respectively. In [12], Kessar and Linckelmann proved that any indecomposable $kGb - kHc$-bimodule with a fusion-stable endo-permutation kD-source which induces a Morita equivalence (or a stable equivalence of Morita type) between kGb and kHc is liftable. Moreover, it lifts to an indecomposable $\mathcal{O}G\hat{b} - \mathcal{O}H\hat{c}$-bimodule with a fusion-stable endo-permutation $\mathcal{O}D$-source which induces a Morita equivalence (or a stable equivalence of Morita type) between $\mathcal{O}G\hat{b}$ and $\mathcal{O}\hat{c}$, under the assumption that k is a splitting field for all subgroups of $G \times H$.

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