On the Spherical Reciprocation in Space of n Dimensions.

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1. In his memoir "Contacts of Systems of Circles," Proc. London Math. Soc., **23** (1887), A. Larmor introduced the circular reciprocation in the plane geometry of circles. Lachlan gave a similar statement in his Modern Pure Geometry (1893), p. 257.

In this paper, I will treat an extended transformation in space R_n of n dimensions.

Consider an imaginary sphere I

$$\sum_{i=1}^{n} x_i^2 + r^2 = 0,$$

and a pencil of spheres

$$\sum_{i=1}^{n} x_i^2 + r^2 + \lambda \sum_{i=1}^{n} (x_i - x_i')^2 = 0,$$

x_i' being constant and λ a parameter. Then the equation of the sphere S belonging to the pencil and cutting I orthogonally, will be

$$S = \left(\sum_{i=1}^{n} x_i'^2 - r^2\right)\left(\sum_{i=1}^{n} x_i^2 - r^2\right) + 4r^2 \sum_{i=1}^{n} x_i' x_i = 0. \qquad (1)$$

We shall call this sphere S the sphere-reciprocal of the point P (x', y') with respect to I or briefly the reciprocal of the point (x', y'); and the sphere I the sphere of reciprocation.

From this definition we see that the spherical reciprocation is a contact-transformation determined by the equations

$$\begin{cases} S = 0, \\ \dfrac{\partial S}{\partial x_i} + \dfrac{\partial x_n}{\partial x_i} \dfrac{\partial S}{\partial x_n} = 0, \\ \dfrac{\partial S}{\partial x_i'} + \dfrac{\partial x_n'}{\partial x_i'} \dfrac{\partial S}{\partial x_n'} = 0, \\ (i = 1, 2, \ldots\ldots, n-1). \end{cases}$$

2. If we introduce the $n+1$-polyspherical coordinates

$$\begin{cases} \sigma z_i = 2x_i\,, & (i=1,2,\ldots\ldots,n)\,, \\ \sigma z_{n+1} = \dfrac{\sum\limits_{i=1}^{n} x_i^2 - r^2}{r}\,, & \qquad (2) \end{cases}$$

σ being an arbitrary quantity, equation (1) will be transformed into

$$\sum_{i=1}^{n+1} z_i' \, z_i = 0\,, \qquad (3)$$

which expresses the polar plane of a point $(z_1', z_2', \ldots\ldots, z_{n+1}')$ with respect to a surface K of the second degree,

$$\sum_{i=1}^{n+1} z_i^2 = 0\,.$$

Hence we get the following theorem:

Theorem I. *The spherical reciprocation, in non-homogeneous coordinates $(x_1, x_2, \ldots\ldots, x_n)$, with respect to I, will be the polar reciprocation, in $n+1$-polysphericalco ordinates $(z_1, z_2, \ldots\ldots, z_{n+1})$, with respect to K.*

Or, in other words, if the polar reciprocal of a surface

$$f(z_1, z_2, \ldots\ldots, z_{n+1}) = 0$$

with respect to $\sum\limits_{i=1}^{n+1} z_i^2 = 0$ is given by $F(z_1, z_2, \ldots\ldots, z_{n+1}) = 0$, the sphere-reciprocal of a surface

$$f\Big(2x_1, 2x_2, \ldots\ldots, 2x_n, \frac{\sum\limits_{i=1}^{n} x_i^2 - r^2}{r}\Big) = 0$$

with respect to $\sum\limits_{i=1} x_i^2 + r^2 = 0$ is determined by

$$F\Big(2x_1, 2x_2, \ldots\ldots, 2x_n, \frac{\sum\limits_{i=1}^{n} x_i^2 - r^2}{r}\Big) = 0\,.$$

3. If, in space R_{n+1} of $n+1$ dimensions, we project a point $(X_1, X_2, \ldots\ldots, X_{n+1})$ on the spherical surface

$$\sum_{i=1}^{n+1} X_i^2 = r^2$$

stereographically, we obtain a point $(x_1, x_2, \ldots\ldots, x_n)$ in space R_n of n dimensions, and the relations

$$\begin{cases} X_i = \dfrac{2\,r^2 x_i}{\sum x_i^2 + r^2}\,, & (i=1, 2, \ldots\ldots, n)\,, \\ X_{n+1} = r\,\dfrac{\sum x_i^2 - r^2}{\sum x_i^2 + r^2}\,. \end{cases}$$

These relations can be obtained from (2), by putting

$$\sigma = \frac{\sum x_i^2 + r^2}{r^2}\,, \qquad (i=1, 2, \ldots\ldots, n)\,,$$

and $$z_j = X_i\,, \qquad (j=1, 2, \ldots\ldots, n+1)\,.$$

But on the spherical surface in R_{n+1}, equation (3) or $\sum\limits_{i=1}^{n+1} X_i' X_i = 0$ may be considered as the polar of a point $(X_1', X_2', \ldots\ldots, X_{n+1}')$. Hence, the stereographical projection of a point and its polar on a sphere in R_{n+1} may be considered as a point and its sphere-reciprocal in R_n respectively. In other words, the spherical reciprocation in R_n is the stereographical projection of the polar reciprocation on a sphere in R_{n+1}.

Now according to Klein (¹), the geometry of reciprocal radii in R_n is the stereographical projection of the geometry of collineations on a sphere in R_{n+1}. Therefore we have the following theorem:

Theorem II. *The geometry of reciprocal radii adjoined by the spherical reciprocation in space of n dimensions is the stereographical projection of the projective geometry on a sphere in space of $n+1$ dimensions.*

4. From equation (1), we find that the centre $M\,(\xi_1, \xi_2, \ldots\ldots, \xi_n)$ of the sphere S is

$$\xi_i = -\frac{2\,r^2 x_i'}{\sum x_i'^2 - r^2}\,, \qquad (i=1, 2, \ldots\ldots, n).$$

If we introduce the homogeneous coordinates $(\zeta_1, \zeta_2, \ldots\ldots, \zeta_{n+1})$ defined by

$$\frac{\zeta_i}{\zeta_{n+1}} = -r\,\xi_i\,, \qquad (i=1, 2, \ldots\ldots, n),$$

we have, by (2),

$$\frac{\zeta_1}{z_1} = \frac{\zeta_2}{z_2} = \ldots\ldots\ldots = \frac{\zeta_{n+1}}{z_{n+1}}\,.$$

(¹) Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen (1872); Einleitung in die höhere Geometrie I (1892–93), p. 378. See also Bôcher, Ueber die Reihenentwickelungen der Potentialtheorie (1894), p. 22.

Hence, the locus of the centres of the reciprocals of points on a surface

$$f\left(2x_1,\ 2x_2, \ldots\ldots, 2x_n, \frac{\sum_{i=1}^{n} x_i^2 - r^2}{r}\right) = 0$$

will be a surface

$$f(\zeta_1, \zeta_2, \ldots\ldots, \zeta_n, \zeta_{n+1}) = 0 .$$

Next, equation (3) can be written

$$\sum_{i=1}^{n+1} \zeta_i' z_i = 0 . \qquad (4)$$

And since $z_i = 0$ $(i = 1, 2, \ldots, n+1)$ cut the sphere I orthogonally, all the spheres belonging to equation (4), ζ_i' being parameters, will form a system of ∞^n spheres cutting I orthogonally.

Therefore the reciprocal of the surface

$$f\left(2x_1,\ 2x_2, \ldots\ldots, 2x_n, \frac{\sum_{i=1}^{n} x_i^2 - r^2}{r}\right) = 0$$

with respect to $\sum_{i=1}^{n} x_i^2 + r^2 = 0$ will be an anallagmatic surface whose director-sphere and deferent are

$$\sum_{i=1}^{n} x_i^2 + r^2 = 0 ,$$

and

$$f(\zeta_1, \zeta_2, \ldots\ldots \zeta_{n+1}) = 0 ,$$

respectively.

Theorem III. *The reciprocal of any surface is an anallagmatic surface, its director-sphere being the sphere of reciprocation.*

5. Here, as an example, I will consider a cyclide f

$$f(z_1, z_2, \ldots, z_{n+1}) = \sum_{i\,k} a_{ik} z_i z_k = 0 ,$$

$$a_{ik} = a_{ki} , \qquad (i, k = 1, 2, \ldots, n+1) .$$

In this case, the deferent is, by Art. 4, a surface of the second degree

$$f(\zeta_1, \zeta_2, \ldots, \zeta_{n+1}) = \sum_{i\,k} a_{ik} \zeta_i \zeta_k = 0 ,$$

and the reciprocal of f is, by Art. 2, another cyclide F

$$F(z_1, z_2, \ldots, z_{n+1}) = \sum_{i\,k} A_{ik} z_i z_k = 0 ,$$

where A_{ik} denotes the minor of a_{ik} in the determinant

$$\Delta = |a_{ik}| .$$

Now the condition for which the cyclide f should degenerate into two spheres is

$$\Delta = 0 ;$$

and that for the other cyclide F is

$$|A_{ik}| = 0 ,$$

that is also

$$\Delta = 0 .$$

And if F degenerate into two spheres, F being anallagmatic with respect to I, they will constitute a pair of inverse spheres with respect to I; the same result takes place for f also. Hence we obtain the theorem:

Theorem IV. *The reciprocal of a cyclide with respect to its director-sphere is another cyclide having the same director-sphere.*

The reciprocals of two spheres, constituting a pair of inverse spheres with respect to the sphere of reciprocation, are also two spheres having the same property as the original ones (1).

As the second example, I will show a theorem due to Larmor.

It is easily seen that the reciprocal of a sphere will be two spheres constituting a pair of inverse spheres with respect to I. Now let the reciprocals of two spheres S_1 and S_2 be S_1', S_1'' and S_2', S_2'' respectively. If S_1 and S_2 touch each other, S_1' and S_2' (or S_2'') will also touch each other. In this case, since S_1'' and S_2'' (or S_2') are inverse figures of S_1' and S_2' (or S_2'') with respect to I respectively, S_1'' and S_2'' (or S_2') must touch each other. Therefore we arrive at Larmor's theorem: *When two spheres touch, their reciprocals will also touch in pairs.*

6. In the last place, I will consider the invariant surface with respect to the spherical reciprocation.

By Theorem III, such a surface must be an anallagmatic surface, its director-sphere being the sphere of reciprocation.

Now Appell (2) shows us that, if z_1, z_2, z_3 be the homogeneous coordinates on a plane, the autopolar curve with respect to

(1) The latter part of this theorem is well-known. See Lachlan, p. 258.

(2) Appell, Courbes autopolaires, Nouv. Ann. de Math. [3], 13 (1894). See also Loria, Spezielle algebraische und transcendente ebene Kurven (1902), p. 357.

$$z_1^2+z_2^2+z_3^2=0\,,$$

will be ∞^2 conics

$$2\,(c_1 z_1+c_2 z_2+c_3 z_3)^2-(c_1^2+c_2^2+c_3^2)\,(z_1^2+z_2^2+z_3^2)=0\,,$$

where c_1, c_2 and c_3 are constants, or an envelope of ∞^1 conics in which c_1, c_2 and c_3 are arbitrary functions of a single parameter.

Without altering Appell's method, we can extend this theorem to the case of space of n dimensions.

Hence by the application of Theorem I upon this extended theorem, we get the following theorem:

Theorem V. *If the sphere of reciprocation be*

$$\sum_{i=1}^{n} x_i^2+r^2=0\,,$$

the invariant surface with respect to the spherical reciprocation will be ∞^n *cyclides*

$$2\{2r\sum_{i=1}^{n} c_i x_i+c_{n+1}(\sum_{i=1}^{n} x_i^2-r^2)\}^2-\sum_{i=1}^{n+1} c_i^2\cdot(\sum_{i=1}^{n} x_i^2+r^2)^2=0\,,$$

where c_1, c_2, $\ldots c_{n+1}$ *are constants, or an envelope of* ∞^1 *cyclides in which* c_1, c_2, $\ldots c_{n+1}$ *are arbitrary functions of a single parameter.*