学位申請論文

Noncommutative deformation of Kähler manifolds and its application to transformation from instanton to Hermitian Ricci-flat metric

（ケーラー多様体の非可換変形とそれのインスタントンからリッチ平坦なエルミート計量への変換に対する応用）

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Contents

Chapter 1

Introduction

In this thesis, we study deformation quantization of locally symmetric Kähler manifolds and Einstein metrics derived from gauge curvature on a quantized locally symmetric Kähler manifolds.

Locally symmetric Kähler manifolds include an important class of Kähler manifolds. We have discovered a way to construct noncommutative products on locally symmetric Kähler manifolds.

Einstein manifold is an important object in Riemannian geometry. We have found that the Einstein metric is related to the gauge curvature of the quantized $\mathbb{R}^4$

Deformation quantization is a quantization that defines a noncommutative product on the function algebra $C^\infty(M)$ of a symplectic manifold or Poisson manifold M. Deformation quantizations were introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3] as a method to quantize spaces. After [3], several ways of deformation quantization were proposed [8, 9, 20, **?**]. Specifically, the existence of deformation quantization is proved by [8, 9, 26] when the manifold is a symplectic manifold and proved by [20] when it is a Poisson manifold. In particular, deformation quantizations of Kähler manifolds were provided in [6, 7, 22, 23]. The deformation quantization with separation of variables is one of the methods to construct noncommutative Kähler manifolds given by Karabegov [75, 41, 16]. Star products on the fuzzy $\mathbb{C}P^N$ which is a typical Kähler manifold are investigated in [1, 2, 12, 13, 18, 28]. A deformation quantization of the hyperbolic plane which is also a Kähler manifold was provided in [4, 28].

In gauge theories, the Yang-Mills equation of the non-Abelian gauge theory is important in particle physics, and there is the Instanton equation as a method for obtaining its exact solution. The Yang Mills equation is a differential equation that appears in gauge theory. You can construct algebraically a solution called instanton solution with ADHM construction method. There is a method called ADHM construction [77] method as a method for obtaining a solution of the instanton equation. Nekrasov and Schwarz [38] discovered noncommutative ADHM equations and constructed noncommutative instantons using the ADHM construction . This work initiated the study of noncommutative ADHM instantons, and at present there is a large body of works on this problem [38]. Later, several noncommutative instanton solutions were created using the ADHM construction method [?, ?, ?, ?, ?, 39, ?, ?, 66, ?, ?, ?, ?, ?]. After those achievements, research on noncommutative instanton solutions advanced. For example, it was shown that the instanton number is equal to the dimension of the vector space that appears in the ADHM construction method [78, 89, 86, 87, 88]. This discovery is a generalization of the proposition that a natural number called an instanton number is equal to a topological invariant called a Pontryagin number in gauge theory on a commutative space.

The noncommutative $U(1)$ instanton solutions are written in an operator form acting on a Fock space. The Fock space is defined by Heisenberg algebra generated by noncommutative complex coordinates. For example, in [29], the Fock representation for noncommutative $\mathbb{C}P^N$ is constructed. There is a dictionary between the linear operators acting on the Fock space and usual functions [40].

It has been known that there is a close relationship between noncommutative geometry and string theory. Especially since Seiberg and Witten clarified this relationship in 1999 [34], great progress has been made in research. The Seiberg-Witten transformation is the relationship between the gauge curvature on a commutative space and the gauge curvature on a noncommutative space.

The original Seiberg-Witten map is a map from noncommutative gauge fields to commutative gauge fields with a background B-field [34]. On the other hand, it has been interpreted in [35, 33, 36] as a map from a noncommutative gauge field to a Kähler metric. It has been conjectured in [42, 43] that noncommutative $U(1)$ gauge theory is the fundamental description of Kähler gravity at all scales including the Planck scale and provides a quantum gravity description such as quantum gravitational foams. Recently it was shown in [44, 45, 46] that the electromagnetism in noncommutative spacetime can be realized as a theory of gravity, and the symplectization of spacetime geometry is the origin of gravity. Such picture is called emergent gravity and it proposes a candidate of the origin of spacetime. See also related works in Refs. [55, 52, 54, 47, 53, 48, 49, 50, 51]. As a bottom-up approach of the emergent gravity formulated in [56], the Eguchi-Hanson metric [57, 58] in four-dimensional Euclidean gravity is used to construct anti-self-dual symplectic $U(1)$ gauge fields, and $U(1)$ gauge fields corresponding to the Nekrasov-Schwarz instanton [38] are reproduced by the reverse process [59]. As a top-down approach of the emergent gravity, the $U(1)$ instanton found by Braden and Nekrasov [60] derives a corresponding gravitational metric.

Based on the above, this thesis aims to clarify the following. In the previous studies, the correspondence between Hermitian-Einstein metric and anti-selfdual 2 form based on the Seiberg-Witten map was shown as a concrete example [59]. When considering Eguchi-Hanson's metric as Hermitian-Einstein metric, it was shown that an anti-self-dual two-form satisfying the second kind of Abel differential equation was derived [59]. If so, it is natural to see if the correspondence between them is more general. As a result of our research, it has been clarified that when the anti-self dual condition is imposed on the two forms corresponding to the gauge curvature, the metric derived by the correspondence becomes the Hermitian-Einstein metric. This result is a generalization of the previous research. We also gave results for the opposite of this proposition. We have shown that when the metric is Hermitian-Einstein metric and the two forms corresponding to the gauge curvature are asymptotically zero, the two forms are anti-selfdual.

Previous studies have shown the existence and construction of the deformation quantizations with separation of variables of the Kähler manifold [41]. Noncommutative products were obtained explicitly in complex projective spaces, complex hyperbolic spaces and so on. But in other cases it is not so easy. We succeeded in finding a recurrence formula that determines noncommutative products under the locally symmetric condition. By using this method, in addition to rederiving noncommutative products of complex projective spaces and complex hyperbolic spaces, we can specifically obtain noncommutative products of compact Riemann surfaces with arbitrary genus.

The organization of this thesis is as follows. In Chapter 2, we first explain complex manifolds and Hermitian manifolds. The structure required for manifolds in deformation quantization, which is the center of this paper, is a Poisson structure, not a complex structure.

In Section **??**, we explain deformation quantization motivated by quantum mechanics, which is one of the basis of modern physics. In deformation quantization, a Poisson structure is required for a manifold as described above. To explain the Poisson structure in more detail, the Poisson algebra is an algebra whose elements are functions on manifolds. It is known that the algebra has deformation quantization based on the definition derived from the relationship between classical mechanics and quantum mechanics. A particularly famous algebraic structure is the Moyal product.

In Chapter **??**, main subject is gauge theory. Gauge theory is a theory of physics and explains various phenomena in this universe. In the standard model, there are gauge fields whose gauge groups are $U\left(1\right)$, $SU\left(2\right)$, and $SU\left(3\right)$, which explain electromagnetics, neutrino observation, nuclear physics, etc. In mathematics, after defining the Yang-Mills connection, the curvature can be considered in the same way as the Levi-Civita connection in Riemannian geometry, and the curvature is called the gauge curvature.

It is known that gauge theory can be developed even on the noncommutative manifold constructed in the previous chapter. It is mathematically known that topological invariants can be constructed in gauge theory, but such things have also been studied in noncommutative manifolds. It is important to specifically find solutions to equations that appear in the gauge theory. In general, it is naturally difficult to find all solutions, but it is known that some special solutions are derived algebraically. They are the local minimum points found by completing the action squarely, and are called instanton solutions. The algebraic construction method is called the ADHM construction method, and has recently attracted attention in the study of integrable systems. We explain that this construction method can also be applied to noncommutative manifolds.

Gauge theories on noncommutative manifolds are considered to be physically meaningful, and their relation to gauge theories on ordinary commutative manifolds has also been investigated. Specifically, there are research results that two forms corresponding to gauge fields are anti-selfdual when Eguchi-Hanson metric is assumed as the metric [59]. Eguchi-Hanson metric is an important example of Kähler-Einstein metric called gravitational instanton.

In Chapter 4 we will explain the relationship between gauge curvature and metric we found. If the two forms corresponding to the gauge curvature are anti-selfdual in that relationship, the Hermitian metric derived from that relationship is always Einstein metric. Naturally, the Eguchi-Hanson metric described above is an example of this. In addition, we specifically constructed Hermitian-Einstein metrics using the ADHM construction method described above.

In addition, we study the inverse of this proposition. In other words, it is a question of what kind of property the two forms corresponding to the gauge curvature have when the two forms implies Hermitian-Einstein metric. The partial answer is given. If we impose an asymptotic boundary condition on two forms corresponding to gauge curvature after imposing that it is Hermitian-Einstein metric, the two forms should be anti-selfdual.

In Chapter **??**, we will explain in more detail of the deformation quantization of the Kähler manifold that is mentioned a little in Chapter 2. The deformation quantization of the Kähler manifold mainly dealt with in this paper is the deformation quantizations with separation of variables given by Karabegov. In addition to conditions to be Hermitian manifolds, Kähler manifolds have a function called Kähler potential, and noncommutative products can be determined by using their properties. In fact, in previous researches, deformation quantizations with separation of variables of important Kähler manifolds such as complex projective space and complex hyperbolic space has been obtained [27]. The complex projective space and the complex hyperbolic space have common properties. Both are manifolds called Riemannian symmetric spaces. They give a family of basic Riemannian manifolds, including constant curvature spaces, projective spaces, Grassmann manifolds, and compact Lie groups. One of the properties of Riemann symmetric space is that the covariant derivative of Riemann curvature is zero. This property itself is called locally symmetric.

In Chapter 10, we gave a better construction method for deformation quantizations with separation of variables of locally symmetric Kähler manifolds. However, even if this better method is used, a noncommutative product is not necessarily explicitly obtained. Since the recurrence formula appears there, it can be difficult to find a solution as in a differential equation. However, in the case of the recurrence formula, it is possible to obtain the first few terms. It can be said that it is sufficient in the approximate calculation when the Planck constant is small.

Specific results obtained by using this better method are as follows. Compact complex one-dimensional manifolds are called compact Riemann surfaces and are important objects in complex analytics and algebraic geometry, but their noncommutative products have been completely determined. The same noncommutative product was derived again in the complex projective space derived in the previous study [27]. It is not obvious whether it is possible to find a recurrence solution for complex Grassmann manifolds. Other locally symmetric Kähler manifolds will be left for further study.

In this thesis, we use the Einstein summation convention over repeated indices.

Chapter 2

Review of the deformation quantization

The deformation quantization required for this paper is briefly reviewed in this chapter.

2.1 Complex manifold

The definitions of complex manifolds, Hermitian manifolds, etc. are given below, which also serve as fixed notations.

A complex manifold is a manifold with an atlas of charts to the open unit disk in $\mathbb{C}^n$ on which you can consider holomorphic functions. The definition of complex manifold is given below.

Definition 2.1 (Complex manifold)**.** Assume that M is a Hausdorff space and $\{U_\lambda\}_{\lambda\in\Lambda}$ are open covers of M. M is a complex manifold if the homeomorphism

$$\phi_\lambda : U_\lambda \longrightarrow \phi_\lambda\left(U_\lambda\right) \subset \mathbb{C}^n$$

exists and $\phi_\alpha \circ \left(\phi_\beta\right)^{-1}$ is a biholomorphism for any $\alpha, \beta \in \Lambda$.

As similar to the Riemannian manifolds in Riemannian geometry we study Hermitian manifolds equipped Hermitian metrics in the complex geometry.

Definition 2.2 (Hermitian metric and Hermitian manifold)**.** Assume that M is a complex manifold, $p \in M$, and $z^1, \cdots, z^n$ are local holomorphic coordinates of M. If $\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l}$ are defined as

$$\frac{\partial}{\partial z^k} := \frac{1}{2}\frac{\partial}{\partial x^k} - \frac{\mathrm{i}}{2}\frac{\partial}{\partial y^k} \in T_pM \otimes_{\mathbb{R}} \mathbb{C}, \quad \frac{\partial}{\partial \bar{z}^l} := \frac{1}{2}\frac{\partial}{\partial x^l} + \frac{\mathrm{i}}{2}\frac{\partial}{\partial y^l} \in T_pM \otimes_{\mathbb{R}} \mathbb{C}$$

where $z^k = x^k + \mathrm{i}y^k$, $\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^l} \in \Gamma\left(TM\right)$ then T'_pM and T''_pM are define as

$$T'_pM \oplus T''_pM := T_pM \otimes_{\mathbb{R}} \mathbb{C} = span_{\mathbb{C}}\left[\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\right]$$

where

$$T'M := span_{\mathbb{C}}\left[\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\right], \quad T''M := span_{\mathbb{C}}\left[\frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n}\right].$$

A map $h : T'_pM \times T''_pM \to \mathbb{C}$ is called Hermitian metric if

1. $h(x, y + az) = h(x, y) + ah(x, z)$,
2. $h(ax + w, z) = \bar{a}h(x, z) + h(w, z)$,
3. $h(x, \bar{x}) = 0 \Longrightarrow x = 0$,
4. $x \neq 0 \Longrightarrow h(x, \bar{x}) > 0$,

where $x, w \in T'_pM,\ z, y \in T''_pM$ and $a \in \mathbb{C}$. $h(y, x)$ is defined as $h(y, x) := \overline{h(x, y)}$. The complex manifold with Hermitian metric (M, h) is called Hermitian manifold. Components $h_{k,\bar{l}}(z)$ is defined as

$$h_{k,\bar{l}}(z) := h\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k}\right).$$

As a special case of Hermitian manifolds, Kähler manifold is introduced in the context of symplectic geometry.

Definition 2.3 (Kähler manifold)**.** Assume that M is a Hermitian manifold and $h_{k\bar{l}}$ is a Hermitian metric. Kähler form ω is defined as

$$\omega := \frac{\sqrt{-1}}{2}\sum_{k,l} h_{k\bar{l}}(z)\, \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l.$$

M is called a Kähler manifold if

$$\mathrm{d}\omega = 0.$$

Hamilton vector fields appearing in symplectic geometry, Poisson geometry and analytical mechanics are defined in the same way.

Definition 2.4 (Hamiltonian vector field)**.** Assume that M is a Kähler manifold, $f \in C^\infty(M), X, Y$ are vectors fields on M and ω is a Kähler form. The interior product is defined as

$$(i_X\omega)(Y) := 2\omega(X, Y),$$

and Hamiltonian vector field X_f is defined by

$$i_{X_f}\omega = \mathrm{d}f.$$

2.2 Deformation quantization

Since algebras of functions on manifolds are transformed into noncommutative algebras in noncommutative geometry, we review algebraic structures here.

Definition 2.5 (Associative algebra)**.** Let $\mathbb{K}$ be a field. For $\forall a, b \in \mathcal{A}$, and $\forall \lambda \in \mathbb{K}$ an associative algebra $(\mathcal{A}, +, \ \cdot\ )$ on the field $\mathbb{K}$ is defined as follows.

1. $(\mathcal{A}, +)$ is a vector space over a field $\mathbb{K}$.
2. $(\mathcal{A}, +, \ \cdot\ )$ is a ring.
3. $\lambda\left(a \cdot b\right) = \left(\lambda a\right) \cdot b = a \cdot \left(\lambda b\right)$.

Example 2.6. $\left(C^{\infty}\left(\mathbb{R}^{n}\right), +, \ \cdot\ \right)$ is an associative algebra with the following definition of + and $\cdot$.

1. If $f, g \in C^{\infty}\left(\mathbb{R}^{n}\right)$ then the addition $+ : C^{\infty}\left(\mathbb{R}^{n}\right) \times C^{\infty}\left(\mathbb{R}^{n}\right) \longrightarrow C^{\infty}\left(\mathbb{R}^{n}\right)$ is defined as
$$\left(f+g\right)\left(x\right) := f\left(x\right) + g\left(x\right).$$
2. If $f, g \in C^{\infty}\left(\mathbb{R}^{n}\right)$ then the multiplication $\cdot : C^{\infty}\left(\mathbb{R}^{n}\right) \times C^{\infty}\left(\mathbb{R}^{n}\right) \longrightarrow C^{\infty}\left(\mathbb{R}^{n}\right)$ is defined as
$$\left(f \cdot g\right)\left(x\right) := f\left(x\right) \cdot g\left(x\right).$$

Definition 2.7 (Lie algebra)**.** Let $(\mathfrak{g}, +)$ be a vector space over a field $\mathbb{K}$. $(\mathfrak{g}, +, \{\cdot\ ,\ \cdot\})$ is called a Lie algebra if

1. $\{\cdot\ ,\ \cdot\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a bilinear map,
2. $\{a, b\} = -\{b, a\} \quad \left(\forall a, b \in \mathfrak{g}\right),$
3. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad (\forall a, b, c \in \mathfrak{g})$.

If a function algebra has a Poisson structure defined below, the differentiable manifold is called a Poisson manifold.

Definition 2.8 (Poisson algebra)**.** Let $(\mathcal{A}, +, \{\cdot\ ,\ \cdot\})$ be a Lie algebra and $(\mathcal{A}, +, \ \cdot\ )$ be a commutative algebra. $(\mathcal{A}, +, \ \cdot\ , \{\cdot\ ,\ \cdot\})$ is called a Poisson algebra if

$$\{a, (b \cdot c)\} = \{a, b\} \cdot c + b \cdot \{a, c\}\,,\ for\ \forall a, b, c \in \mathcal{A}.$$

Example 2.9. Let M be a Kähler manifold and $\left(C^{\infty}\left(M\right), +, \ \cdot\ \right)$ is a Poisson algebra if

$$\{f, g\} := -\omega\left(X_{f}, X_{g}\right)\ for\ \forall f, g \in C^{\infty}\left(M\right).$$

Using above definitions, deformation quantization is introduced as follows. In the following, $\mathbb{C}[[x]]$ is a set of formal power series of x.

Definition 2.10. [Deformation quantization] Assume that $(\mathcal{A}, +, \cdot, \{\cdot, \cdot\})$ is a Poisson algebra and $f, g, h \in \mathcal{A}$. $(\mathcal{A}, *)$ is called a deformation quantization if

1. $* : \mathcal{A}[[\hbar]] \times \mathcal{A}[[\hbar]] \longrightarrow \mathcal{A}[[\hbar]]$ is a bilinear map, a $\hbar$-bilinear map and $f * (g * h) = (f * g) * h$.
2. If $f * g = \sum_{r=0}^{\infty} C_r(f, g)\hbar^r$ then $C_0(f, g) = f \cdot g$, and

$$C_1(f, g) = \frac{1}{2}\{f, g\}. \tag{2.2.1}$$

where C_k is a bidifferential operator.

In particular, if we consider Poisson algebra consisting of functional algebra of Poisson manifold as $\mathcal{A}$, deformation quantization can be regarded as noncommutative geometry of Poisson manifold.

Example 2.11. For $f, g \in C^\infty(\mathbb{R}^4)[[\hbar]]$, the product $* : \{C^\infty(\mathbb{R}^4)[[\hbar]]\}^2 \longrightarrow C^\infty(\mathbb{R}^4)[[\hbar]]$ is defined as

$$f(x) * g(x) := f(x) \exp\left(\frac{\mathrm{i}}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right) g(x),$$

where $\theta^{\mu\nu} \in \mathbb{R}$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. This is called Moyal product.

Remark 2.12. Sometimes, in addition to Definition **??**, the following conditions may be added. If $f * g = \sum_{r=0}^{\infty} C_r(f, g)\hbar^r$ then

1. $C_r(f, g)(x) = \sum_{IJ} a_{IJ}(x)\partial_I f(x)\partial_J g(x)$ where $I := (i_1, \cdots, i_n), \partial_I := \partial_1^{i_1} \cdots \partial_n^{i_n}$
2. $C_r(f, g)(x) = (-1)^r C_r(g, f)(x)$
3. $1 * f = f * 1 = f$.

2.3 Deformation quantizations with separation of variables

We summarize deformation quantization for Kähler manifolds. In particular we focus on Karabegov's deformation quantization for Kähler manifolds [75, 41]. He showed that there exists deformation quantization for arbitrary Kähler manifolds, and his method is called deformation quantization with separation of variables. Note that (**??**) is replaced by

$$C_1(f, g) - C_1(g, f) = \{f, g\}, \tag{2.3.1}$$

in Karabegov's deformation quantization.

Definition 2.13 (A star product with separation of variables)**.** Let $*$ be a star product on a Kähler manifold as a Poisson manifold. The $*$ is called a star product with separation of variables on a Kähler manifold when

$$a * f = af \tag{2.3.2}$$

for an arbitrary holomorphic function a and

$$f * b = fb \tag{2.3.3}$$

for an arbitrary anti-holomorphic function b.

The detail of how to construct the deformation quantization with separation of variables is reviewed in Chapter **??**. Here we see an example of $*$-product in $\mathbb{C}^2$. The star product on $\mathbb{C}^2$ constructed by Karabegov's deformation quantization is given as

$$f * g = \sum_{n=0}^{\infty} \sum_{k_1,\cdots,k_n,\ l_1,\cdots,l_n} \frac{\zeta_{k_1}\cdots\zeta_{k_n}}{n!} \delta^{k_1 l_1} \cdots \delta^{k_n l_n} \left(\partial_{\bar{k}_1} \cdots \partial_{\bar{k}_n} f\right) \left(\partial_{l_1} \cdots \partial_{l_n} g\right), \tag{2.3.4}$$

where $\zeta_i (i = 1 \cdots \dim_{\mathbb{C}} M)$ are noncommutative parameters. In Chapter 4 we made Ricci-flat metrics from (anti-)self-dual two-forms on a noncommutative manifold.

2.4 Fock space representation of $\mathbb{C}^2$

In this section we construct Fock representation of $\mathbb{C}^2$. For Karabegov's deformation quantization all noncommutative Kähler manifolds have Fock representation [40, 76]. Here we make the Fock representation of $\mathbb{C}^2$ by using $*$-product (**??**).

Consider a noncommutative algebra $\left(C^\infty\left(\mathbb{C}^2\right)[[\hbar]], *\right)$ led by (**??**) in Section **??**. The star product induces a Heisenberg algebra

$$\left[z^k, \bar{z}^l\right]_* = -\zeta_k \delta_{kl}, \qquad \left[z^k, z^l\right]_* = 0, \qquad \left[\bar{z}^k, \bar{z}^l\right]_* = 0, \tag{2.4.1}$$

where $[x, y]_* := x * y - y * x$. We represent it by creation and annihilation operators given by

$$a_k := \frac{\bar{z}^k}{\sqrt{\zeta_k}}, \qquad a_k^\dagger := \frac{z^k}{\sqrt{\zeta_k}},$$

then

$$\left[a_k, a_l^\dagger\right]_* = \delta_{kl}, \qquad \left[a_k^\dagger, a_l^\dagger\right]_* = 0, \qquad \left[a_k, a_l\right]_* = 0.$$

In the following $\zeta_1 = \zeta_2 = \zeta > 0$ is assumed.

Note that the choice of a noncommutative parameter has the freedom associated with a choice of a background two-form [34]. Here the ζ in (6.1) is regarded as the only noncommutative parameter.

The algebra $\mathcal{F}$ on $\mathbb{C}$ is defined as follows. The Fock space $\mathcal{H}$ is a linear space spanned by the bases generated by acting $a_l^\dagger$'s on $|0,0\rangle$:

$$\frac{1}{\sqrt{m_1!m_2!}}\left(a_1^\dagger\right)_*^{m_1} * \left(a_2^\dagger\right)_*^{m_2} |0,0\rangle = |m_1, m_2\rangle\,, \tag{2.4.2}$$

where m_1 and m_2 are positive integers and $(a)_*^m$ stands for $\overbrace{a * \cdots * a}^{m}$. The ground state $|0,0\rangle$ satisfies $a_l\,|0,0\rangle = 0,\ \forall\ l$. Here, we define the basis of a dual vector space by acting a_l's on $\langle 0,0|$ as

$$\frac{1}{\sqrt{n_1!n_2!}}\langle 0,0|\,(a_1)_*^{n_1} * (a_2)_*^{n_2} = \langle n_1, n_2|\,,$$

where $\langle 0,0|$ satisfies $\langle 0,0|\,a_l^\dagger = 0,\ \forall\ l$. Then we define a set of linear operators as

$$\mathcal{F} := span_{\mathbb{C}}\left(|m_1,m_2\rangle\,\langle n_1,n_2|\,; m_1,m_2,n_1,n_2 = 0,1,2,\cdots\right) \tag{2.4.3}$$

where $(|m_1,m_2\rangle\,\langle n_1,n_2|)\,|k_1,k_2\rangle = \delta_{k_1n_1}\delta_{k_2n_2}\,|m_1,m_2\rangle$ and $\langle k_1,k_2|\,(|m_1,m_2\rangle\,\langle n_1,n_2|) = \delta_{k_1m_1}\delta_{k_2m_2}\,\langle n_1,n_2|$. The product on $\mathcal{F}$ is defined as

$$(|j_1,j_2\rangle\,\langle k_1,k_2|)\circ(|m_1,m_2\rangle\,\langle n_1,n_2|) := \delta_{k_1m_1}\delta_{k_2m_2}\,|j_1,j_2\rangle\,\langle n_1,n_2|\,,$$

so, $\mathcal{F}$ is an algebra.

There is a one to one correspondence between $\mathcal{F}$ and some subalgebra of $C^\infty\left(\mathbb{C}^2\right)$. For arbitrary noncommutative Kähler manifold obtained by deformation quantization with separation of variables [41], we can find the similar correspondence [40]. The following is the simplest example of the correspondence.

Definition 2.14. (Twisted Fock representation). The linear map $\iota : \mathcal{F} \longrightarrow C^\infty\left(\mathbb{C}^2\right)$ is defined as

$$\iota\left(|m_1,m_2\rangle\,\langle n_1,n_2|\right) = e_{(m_1,m_2,n_1,n_2)} := \frac{z_1^{m_1} z_2^{m_2} \mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}\,\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{\sqrt{m_1!m_2!n_1!n_2!}\left(\sqrt{\zeta}\right)^{m_1+m_2+n_1+n_2}}, \tag{2.4.4}$$

especially $\iota\left(|0,0\rangle\,\langle 0,0|\right) = e_{(0,0,0,0)} = \mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}$.

Proposition 2.15. *Let $\iota(\mathcal{F})$ be defined by*

$$\iota(\mathcal{F}) := span_{\mathbb{C}}\left(e_{(m_1,m_2,n_1,n_2)}; m_1, m_2, n_1, n_2 = 0, 1, 2, \cdots\right). \tag{2.4.5}$$

*Then $\{\iota(\mathcal{F}), *\}$ is an algebra where $*$ is in (**??**).*

Proof. *After a little algebra, one can deduce the following identity*

$$e_{(k_1,k_2,l_1,l_2)} * e_{(m_1,m_2,n_1,n_2)} = \delta_{l_1m_1}\delta_{l_2m_2}e_{(k_1,k_2,n_1,n_2)}. \tag{2.4.6}$$

Details are given in [40]. □

The identity (6.6) derives the following fact.

Proposition 2.16. *The algebras $(\mathcal{F}, \circ)$ and $\{\iota(\mathcal{F}), *\}$ are isomorphic.*

Chapter 3

Gauge theory and D-brane

In this chapter, we will review the gauge theories derived from D-branes. First, we will review the gauge theory on a normal flat spacetime. In particular, a special solution called an instanton solution that can be obtained algebraically is mentioned. After that, we will explain the gauge theory on the noncommutative space mentioned in Chapter 2. Next, we will explain the relationship between the gauge field on the commutative space and the gauge field on the noncommutative space suggested by the theory of physics, and further explain the Einstein metric suggested from there. Finally, the concrete configuration of the instanton solution on the noncommutative space is described.

3.1 Gauge theory

Assume that $\mathfrak{g} = T_e G$ is a Lie algebra associated with a Lie group G, $A := A_\mu \mathrm{d}x^\mu$ is a $\mathfrak{g}$ valued 1-form on $\mathbb{R}^4$.

Gauge theories originate in physics, especially electromagnetism. F defined below is a field corresponding to an electric field/magnetic field in electromagnetism, i.e. $G = U(1)$, and A corresponds to what is called a vector potential. Assume that A is a gauge potential on $\mathbb{R}^4$. $F := \mathrm{d}A - \mathrm{i}A \wedge A$ is called the field-strength form. And $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,[A_\mu, A_\nu] = \mathrm{i}\,[D_\mu, D_\nu]$ where $F = F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

In electromagnetism, following corresponds to Gauss's law for magnetic fields and Faraday's law for electromagnetic induction.

Proposition 3.1. *The following formula is called Bianchi identity.*

$$\mathrm{d}_A F := \mathrm{d}F - \mathrm{i}\,[A, F] = 0$$

Proof.

$$\mathrm{d}F - \mathrm{i}\,[A, F] = -\mathrm{i}\mathrm{d}A \wedge A + \mathrm{i}A \wedge \mathrm{d}A - \mathrm{i}A \wedge \mathrm{d}A - A \wedge A \wedge A + \mathrm{i}\mathrm{d}A \wedge A + A \wedge A \wedge A = 0$$

□

Definition 3.2. $\epsilon_{\mu\nu\alpha\beta}$ is called Levi-Civita symbol and defined by:

$$\varepsilon_{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is an even permutation of } (1,2,3,4) \\ -1 & \text{if } (\mu,\nu,\alpha,\beta) \text{ is an odd permutation of } (1,2,3,4) \\ 0 & \text{otherwise} \end{cases} .$$

For $F = F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$, Hodge star operator $\star$ in $\mathbb{R}^4$ is defined by:

$$\star F := \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}.$$

Definition 3.3. The action of the gauge theory S_{YM} is defined as

$$S_{YM} := \int_{\mathbb{R}^4} \left\{ -\frac{1}{2} tr \left(F \wedge \star F \right) \right\}.$$

Like other classical physics, the equation of motion for gauge theory can be expressed as a solution of variational problems.

Proposition 3.4. *Let G be a semisimple Lie group and $\mathfrak{g}$ be the Lie algebra of G. We denote T^j as a basis of $\mathfrak{g}$ such that*

$$\left[T^a, T^b \right] = f^a_{bc} T^c, \quad B\left(T^k, T^l \right) = \frac{\delta_{kl}}{2},$$

where $B(\cdot,\cdot)$ is a Killing form, and we put

$$\mathcal{L} := -\frac{1}{2} tr \left(F_{\mu\nu} F^{\mu\nu} \right).$$

From the Euler-Lagrange equation for S_{YM}:

$$\frac{\partial \mathcal{L}}{\partial A^d_\sigma} - \frac{\partial}{\partial x^\rho} \frac{\partial \mathcal{L}}{\partial \left(\partial_\rho A^d_\sigma \right)} = 0$$

the following equation is obtained.

$$D^\rho F_{\rho\sigma} := \partial^\rho F_{\rho\sigma} + \left[A^\rho, F_{\rho\sigma} \right] = 0.$$

Proof. *At first we express $\mathcal{L}$ by A_μ^a as*

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}\\ &= -\frac{1}{4}g^{\mu\kappa}g^{\nu\lambda}F^a_{\mu\nu}F^a_{\kappa\lambda}\\ &= -\frac{1}{4}g^{\mu\kappa}g^{\nu\lambda}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^a_{bc}A^b_\mu A^c_\nu\right)\left(\partial_\kappa A^a_\lambda - \partial_\lambda A^a_\kappa - f^a_{bc}A^b_\kappa A^c_\lambda\right)\end{aligned}$$

where

$$F_{\mu\nu} = \mathrm{i}T^a F^a_{\mu\nu}, \quad A_\kappa = \mathrm{i}T^a A^a_\kappa, \quad g^{\mu\nu} = \delta_{\mu\nu}.$$

Then

$$\frac{\partial\mathcal{L}}{\partial A^d_\sigma} - \frac{\partial}{\partial x^\rho}\frac{\partial\mathcal{L}}{\partial\left(\partial_\rho A^d_\sigma\right)} = \partial^\rho F_{\rho\sigma} + [A^\rho, F_{\rho\sigma}] = 0.$$

□

This equation is called Yang-Mills equations. The law of minimum action and Bianchi identity holds in electromagnetism. From these equations , we obtain Maxwell's equations.

Assume that F is the curvature or field-strength form. We denote the elements of Hodge dual of F by $\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$.

It is not easy to find all solutions to variational problems, but some special solutions are known to be found algebraically. There is a special solution in gauge theory, called instanton.

Definition 3.5. F is decomposed into selfdual 2-form and anti-selfdual 2-form as $F = F^+ + F^-$, where $F^\pm = \frac{1}{2}(F \pm \star F)$. If $F = F^+$ or equivalently

$$F^- = 0 \tag{3.1.1}$$

we call the corresponding gauge potential anti-instanton. Similarly, for $F^+ = 0$ we call the gauge potential instanton.

Proposition 3.6. *The Hodge dual of F satisfies Yang-Mills equations:*

$$\left[D^\rho, \tilde{F}_{\rho\sigma}\right] = \partial^\rho\tilde{F}_{\rho\sigma} + \left[A^\rho, \tilde{F}_{\rho\sigma}\right] = 0.$$

Proof. *Jacobi identity (for $\rho \neq \sigma,\ \rho \neq \tau,\ \sigma \neq \tau$) is*

$$[D^\rho, [D^\sigma, D^\tau]] + [D^\sigma, [D^\tau, D^\rho]] + [D^\tau, [D^\rho, D^\sigma]] = 0$$

and this is same as the main equation. □

Corollary 3.7. *An anti-instanton is a solution of Yang-Mills equations. In other words*

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \Longrightarrow \partial^{\rho} F_{\rho\sigma} + [A^{\rho}, F_{\rho\sigma}] = 0.$$

The instanton equation is derived from the action as follows. The action of the gauge theory S_{YM} is

$$\begin{aligned} S_{YM} &= \int_{\mathbb{R}^4} \left\{ -\frac{1}{2} tr\left(F \wedge \star F\right) \right\} \\ &= -\frac{1}{2} \int_{\mathbb{R}^4} \left\{ tr\left(\left(F^+ + F^-\right) \wedge \star \left(F^+ + F^-\right)\right) \right\} \\ &= -\frac{1}{2} \int_{\mathbb{R}^4} \left\{ tr\left(\left|F^+\right|^2 + \left|F^-\right|^2 \right) \right\} \end{aligned}$$

where we use $F^+ \wedge F^- = 0$. So we find that the condition $F^+ = 0$ or $F^- = 0$ minimize S_{YM}.

3.2 Gauge theory in noncommutative $\mathbb{R}^4$

The even-dimensional Euclidean space is a trivial Poisson manifold, and there is a Moyal product as an example of the deformation quantization defined there. Here, we simply replace multiplication that appeared in gauge theory with noncommutative products.

Moyal product introduced in Example **??** is defined as follows. For $f, g \in C^{\infty}\left(\mathbb{R}^4\right)[[\hbar]]$ the product $* : \left\{C^{\infty}\left(\mathbb{R}^4\right)[[\hbar]]\right\}^2 \longrightarrow C^{\infty}\left(\mathbb{R}^4\right)[[\hbar]]$ is

$$f(x) * g(x) := f(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial_{\mu}} \theta^{\mu\nu} \overrightarrow{\partial_{\nu}} \right) g(x),$$

where $\theta^{\mu\nu} \in \mathbb{R}$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. And the product derived from the Moyal product is also defined for A.

Definition 3.8. Let $A_{\mu}^{(l)} \mathrm{d}x^{\mu}$ be a $\mathfrak{g}$ valued 1-form on $\mathbb{R}^4$ and $A_{\mu} := \sum_{l=0}^{\infty} A_{\mu}^{(l)} \hbar^l$.

$$A_{\mu} * A_{\nu} := \sum_{l,m,n=0}^{\infty} \frac{\hbar^{l+m+n}}{l!} A_{\mu}^{(m)} \left(\overleftrightarrow{\Delta}\right)^{l} A_{\mu}^{(n)},$$

where $\overleftrightarrow{\Delta} := \frac{\mathrm{i}}{2} \partial_{\mu} \theta^{\mu\nu} \partial_{\nu}$. We define curvature two form by

$$\hat{F} := \frac{1}{2} \hat{F}_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \mathrm{d}A - \mathrm{i} A \wedge *A,$$

where $A \wedge *A := \frac{1}{2}\left(A_{\mu} * A_{\nu}\right) \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$.

3.3 Noncommutative instanton solutions

In this section, we briefly review instanton solutions in noncommutative $\mathbb{R}^4$. As mentioned earlier, the instanton solution is a solution of gauge theory. In a commutative space, linear algebraic derivation of instanton solutions is known[77]. The following introduces a generalized version of the noncommutative space.

Assume that τ_k $(k = 1,\ 2,\ 3)$ are Pauli matrices :

$$\tau_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \tau_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote σ_k and $\bar{\sigma}_l$ as the matrices defined as

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) := (-\mathrm{i}\tau_1, -\mathrm{i}\tau_2, -\mathrm{i}\tau_3, id)\,,\ \ (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4) := (\mathrm{i}\tau_1, \mathrm{i}\tau_2, \mathrm{i}\tau_3, id)\,.$$

And $\bar{\sigma}_{\mu\nu}$ is defined as

$$\bar{\sigma}_{\mu\nu} := \frac{1}{4}\left(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu\right).$$

Definition 3.9. Dirac operators $\mathcal{D}_A, \bar{\mathcal{D}}_A$ are defined as

$$\mathcal{D}_A := \sigma^\mu D_\mu, \quad \bar{\mathcal{D}}_A := \bar{\sigma}^\mu D_\mu^\dagger,$$

and

$$D_\mu * f := \partial_\mu f + A_\mu * f.$$

The instanton equation (??) is rewritten as

$$P_{\mu\nu,\rho\tau} F^{\rho\tau} = 0,$$

where

$$P_{\mu\nu,\rho\tau} := \frac{1}{4}\left(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\nu\rho}\delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}\right).$$

We replace this $F^{\rho\tau}$ by $\hat{F}^{\rho\tau}$, then the noncommutative version of the instanton equation is given as

$$P_{\mu\nu,\rho\tau}\hat{F}^{\rho\tau} = 0. \quad (l = 0,\ 1,\ 2,\ \cdots)$$

The solution of this equation $A_\tau^{(l)}$ is called a smooth noncommutative deformed (SNCD) instanton. From the instanton solution of commutative $\mathbb{R}^4$ the asymptotic behavior of the commutative instanton $A_\mu^{(0)}$ is given by

$$A_\mu^{(0)} = g\mathrm{d}g^{-1} + \mathcal{O}'\left(|x|^{-2}\right), \quad g\mathrm{d}g^{-1} = \mathcal{O}'\left(|x|^{-1}\right)$$

Let us define $I\left(q\right)$ and $C_{\sigma\tau}^{(l)}$ as

$$I\left(q\right) := \left\{(p;m,n)\in\mathbb{Z}^3 \mid p+m+n=q, p,m,n\geq 0, m\neq q, n\neq q\right\},$$

$$C_{\sigma\tau}^{(l)} := \sum_{(p;l,m)\in I(n+1)} \frac{\hbar^{p+m+n}}{p!}\left\{A_\sigma^{(m)}\left(\overleftrightarrow{\Delta}\right)^p A_\tau^{(n)} - A_\tau^{(m)}\left(\overleftrightarrow{\Delta}\right)^p A_\sigma^{(n)}\right\}.$$

Then the instanton equation is equivalent to

$$P^{\mu\nu,\rho\tau}\left(\partial_\rho A_\tau^{(l)} - \partial_\tau A_\rho^{(l)} + \mathrm{i}\left[A_\rho^{(0)}, A_\tau^{(l)}\right] + \mathrm{i}\left[A_\rho^{(l)}, A_\tau^{(0)}\right] + C_{\sigma\tau}^{(l)}\right) = 0. \quad (l=0,\ 1,\ 2,\ \cdots)$$

Using the facts

$$A_\mu^{(0)} = g\mathrm{d}g^{-1} + \mathcal{O}'\left(|x|^{-2}\right)$$

and

$$g\mathrm{d}g^{-1} = \mathcal{O}'\left(|x|^{-1}\right),$$

we conclude

$$C_{\rho\tau}^{(1)} = \mathcal{O}'\left(x^{-4}\right). \tag{3.3.1}$$

We impose the following condition for $A^{(l)}$

$$A - A^{(0)} = D_{A^{(0)}}^* B, \quad A^{(l)} = D_{A^{(0)}}^* B^{(l)},$$

where

$$\left(D_{A^{(0)}}^*\right)_\rho^{\mu\nu} B_{\mu\nu} := \delta_\rho^\nu D^{(0)\mu} B_{\mu\nu} - \delta_\rho^\mu D^{(0)\nu} B_{\mu\nu}.$$

From (**??**)

$$\left|B^{(1)}\right| < \mathcal{O}'\left(|x|^{-2}\right)$$

and

$$\left|A^{(1)}\right| < \mathcal{O}'\left(|x|^{-3}\right)$$

are derived.

Fact 3.10. As shown in [80], if $\bar{\mathcal{D}}_A * \bar{\psi}_i = 0$

$$\bar{\psi} = \mathcal{O}'\left(|x|^{-3}\right).$$

Theorem 3.11. *The order of the smooth noncommutative deformed instantons is given by*

$$\left|A^{(l)}\right| < \mathcal{O}'\left(|x|^{-3+\epsilon}\right).$$

This theorem is proved in [83].

Proposition 3.12. *Assume that*

$$T^{\mu} := \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{d}^4 x x^{\mu} * \bar{\psi}^{\dagger} * \bar{\psi} + \bar{\psi}^{\dagger} * \bar{\psi} * x^{\mu}.$$

If $\bar{\mathcal{D}}_A * \bar{\psi} = 0$ *then*

$$2\left[T^{\mu}, T^{\nu}\right]^{+} = tr\left(S^{\dagger} S \bar{\sigma}^{\mu\nu}\right) - 2\mathrm{i}\theta^{\mu\nu +}$$

where

$$\bar{\mathcal{D}}_A * := \bar{\sigma}^{\mu} D^{\dagger}_{\mu}, \quad \tilde{\psi} :=^{t} \bar{\psi}\sigma_2 = -\frac{g^{-1} S x^{\dagger}}{|x|^4} + \mathcal{O}'\left(|x|^{-4}\right)$$

and

$$\left[T^{\mu}, T^{\nu}\right]^{+} := P^{\mu\nu,\rho\tau}\left[T_{\rho}, T_{\tau}\right], \quad \theta^{\mu\nu +} := P^{\mu\nu,\rho\tau}\theta^{\rho\tau},$$

with

$$P_{\mu\nu,\rho\tau} := \frac{1}{4}\left(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\nu\rho}\delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}\right).$$

This proposition is proved in [83].

Definition 3.13. For

$$\Delta_A := D_{\mu} * D^{\mu}$$

and the Green function $\sum_{k=0}^{\infty} G_A^{(k)}(x, y)\hbar^k$ is defined by

$$\Delta_A * G_A(x, y) = \delta(x - y).$$

Proposition 3.14. *Assume that*

$$S^{\dagger} = \begin{pmatrix} I \\ J^{\dagger} \end{pmatrix}, \quad T^{\mu}\bar{\sigma}_{\mu} = \begin{pmatrix} -B_2 & -B_1 \\ B_1^{\dagger} & -B_2^{\dagger} \end{pmatrix}.$$

If

$$2\left[T^{\mu}, T^{\nu}\right]^{+} = tr\left(S^{\dagger} S \bar{\sigma}^{\mu\nu}\right) - 2\mathrm{i}\theta^{\mu\nu +}$$

then

$$\left[B_1, B_1^{\dagger}\right] + \left[B_2, B_2^{\dagger}\right] + II^{\dagger} - J^{\dagger}J = -\left[z_1, \bar{z}_1\right]_* - \left[z_2, \bar{z}_2\right]_*,$$

$$\left[B_1, B_2\right] + IJ = -\left[z_1, z_2\right]_*.$$

These equations are called ADHM equations [77].

These ADHM equations are the same as those given by Nekrasov and Schwarz [38]. This proposition is proved in [83].

Theorem 3.15. *[38] Assume* $B_1,\ B_2 \in \mathbb{C}^{k\times k},\ I \in \mathbb{C}^{k\times n},\ J \in \mathbb{C}^{n\times k}$ *satisfy conditions*

$$\left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = -\left[z_1, \bar{z}_1\right]_* - \left[z_2, \bar{z}_2\right]_*$$

$$\left[B_1, B_2\right] + IJ = -\left[z_1, z_2\right]_*$$

and $\Psi : \mathbb{C}^n \to \mathbb{C}^k \oplus \mathbb{C}^k \oplus \mathbb{C}^n$ *satisfy condition*

$$\mathcal{D}^\dagger * \Psi = 0, \quad \Psi^\dagger * \Psi = 1 \qquad (a,\ b = 1, \cdots, n)$$

where

$$\mathcal{D}^\dagger := \begin{pmatrix} \tau \\ \sigma^\dagger \end{pmatrix}, \ \tau = \left(B_2 - z_2,\ B_1 - z_1, I\right), \ \sigma^\dagger = \left(-B_1^\dagger + \bar{z}_1, \ -B_2^\dagger + \bar{z}_2, J^\dagger\right).$$

Then

$$A_\mu = \mathrm{i}\Psi^\dagger * \partial_\mu \Psi$$

is an instanton.

Proof. *What is needed is an anti-self duality of the gauge curvature. The gauge curvature* F *is*

$$\begin{aligned} F &= \mathrm{d}A - \mathrm{i}A \wedge *A = \mathrm{i}\left\{\mathrm{d}\left(\Psi^\dagger * \mathrm{d}\Psi\right)\right\} + \mathrm{i}\left(\Psi^\dagger * \mathrm{d}\Psi\right) \wedge * \left(\Psi^\dagger * \mathrm{d}\Psi\right) \\ &= \mathrm{i}\left(\mathrm{d}\Psi^\dagger \wedge *\mathrm{d}\Psi\right) - \mathrm{i}\left(\mathrm{d}\Psi^\dagger * \Psi * \Psi^\dagger \wedge *\mathrm{d}\Psi\right) = \mathrm{i}\left\{\mathrm{d}\Psi^\dagger * \left(1 - \Psi * \Psi^\dagger\right) \wedge *\mathrm{d}\Psi\right\}. \end{aligned}$$

Then $\Psi * \Psi^\dagger = Q$*, where*

$$Q := 1 - \mathcal{D} * \left(\mathcal{D}^\dagger * \mathcal{D}\right)^{-1} * \mathcal{D}^\dagger,$$

because $Q^2 = Q,\ \mathcal{D}^\dagger * Q = 0$*. Hence*

$$F = \mathrm{i}\left(\mathrm{d}\Psi^\dagger\right) * \mathcal{D} * \left(\mathcal{D}^\dagger * \mathcal{D}\right)^{-1} * \mathcal{D}^\dagger \wedge *\mathrm{d}\Psi = \mathrm{i}\Psi^\dagger * \left(\mathrm{d}\mathcal{D}\right) * \left(\mathcal{D}^\dagger * \mathcal{D}\right)^{-1} \wedge * \left(\mathrm{d}\mathcal{D}^\dagger\right) * \Psi$$

and

$$\frac{\partial}{\partial z_\mu}\mathcal{D}^\dagger = \frac{1}{\sqrt{2}}\left(-\bar{\sigma}_\mu\ 0\right), \ \frac{\partial}{\partial z_\mu}\mathcal{D} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma_\mu \\ 0 \end{pmatrix}\left(-\bar{\sigma}_\mu\ 0\right), \ \mathcal{D}^\dagger * \mathcal{D} = \begin{pmatrix} \Box & 0 \\ 0 & \Box \end{pmatrix}, \ \Box = \tau * \tau^\dagger = \sigma^\dagger * \sigma.$$

Hence we find that this A_μ *is an instanton.* □

3.4 Noncommutative $U(1)$ gauge theory on the Fock representation of $\mathbb{C}^2$

In this section, we summarize notations for $U(1)$ gauge theory on the Fock representation of $\mathbb{C}^2$ of noncommutative $\mathbb{C}^2$ introduced in Section **??**.

$U(1)$ gauge connection in the noncommutative space is defined as follows (see for example [66]).

Definition 3.16. Rescaled coordinates of $\mathbb{C}^2$ are defined as

$$\hat{\partial}_{z_l} := \frac{\bar{z}_l}{\zeta_l}.$$

This acts on $\mathcal{H}$ as a linear operator.

Using $\hat{\partial}_{z_l}, \hat{\partial}_{\bar{z}_m}$, let us introduce covariant derivatives and the gauge curvature as follows.

Definition 3.17. Covariant derivatives for a scalar field in fundamental representation $\phi \in \mathcal{F}$ on noncommutative $\mathbb{C}^2$ are defined as

$$\hat{\nabla}_{z_l}\hat{\phi} := \left[\hat{\partial}_{z_l}, \hat{\phi}\right]_* + \hat{A}_{z_l} * \hat{\phi} = -\hat{\phi} * \hat{\partial}_{z_l} + \hat{D}_{z_l} * \hat{\phi}$$

where we define a local gauge field $\hat{A}_{z_l} \in \mathcal{F}$ and

$$\hat{D}_{z_l} := \hat{\partial}_{z_l} + \hat{A}_{z_l}.$$

The gauge curvature is defined as

$$\begin{aligned} \hat{F}_{z_l\bar{z}_m} &:= \mathrm{i}\left[\hat{\nabla}_{z_l}, \hat{\nabla}_{\bar{z}_m}\right]_* = -\frac{\mathrm{i}\delta_{lm}}{\zeta_l} + \mathrm{i}\left[\hat{D}_{z_l}, \hat{D}_{\bar{z}_m}\right]_*, \\ \hat{F}_{z_l z_m} &:= \mathrm{i}\left[\hat{\nabla}_{z_l}, \hat{\nabla}_{z_m}\right]_* = \mathrm{i}\left[\hat{D}_{z_l}, \hat{D}_{z_m}\right]_*, \\ \hat{F}_{\bar{z}_l\bar{z}_m} &:= \mathrm{i}\left[\hat{\nabla}_{\bar{z}_l}, \hat{\nabla}_{\bar{z}_m}\right]_* = \mathrm{i}\left[\hat{D}_{\bar{z}_l}, \hat{D}_{\bar{z}_m}\right]_*. \end{aligned} \tag{3.4.1}$$

3.5 Noncommutative $U(1)$ instanton in the Fock space

In Section **??**, we make a short review of the method to make a $U(1)$ instanton solution in [66] and multi instanton solutions in [39].

In noncommutative $\mathbb{R}^4$, Nekrasov and Schwarz found how to construct instanton gauge fields [38] by using the ADHM construction [77]. Their work has encouraged studies of noncommutative ADHM instantons. (See, for example, [39, 78, ?, ?, ?, 66, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?].) Another method to construct noncommutative instantons as smooth deformations of commutative instantons was provided in [80, 81, 82]. The correspondence between the smooth deformation and the ADHM construction are discussed in [83]. On the other hand, there exist instanton solutions which are not smoothly connected to commutative instantons. The commutative limit of the noncommutative instantons are discussed in [?, ?, 85].

Noncommutative instantons are labeled by topological charge called instanton numbers. The topological number of the noncommutative instanton is studied in [78, 86, 87, 88, 89]. It is shown that the topological number coincides with the dimension of a vector space appearing in the ADHM construction. In [87], this identification is shown when the noncommutative parameter is self-dual for a $U(N)$ gauge theory. In [88], the equivalence is shown with no restrictions on the noncommutative parameters, but a noncommutative version of the Osborn's identity (Corrigan's identity) is assumed. In [85] the final version of the proof was announced.

In Definition 44, a covariant derivative and gauge curvature are given as follows. Covariant derivatives for scalar field $\phi \in \mathcal{F}$ on noncommutative $\mathbb{C}^2$ are defined as $\hat{\nabla}_{z_l}\hat{\phi} := \left[\hat{\partial}_{z_l}, \hat{\phi}\right] + \hat{A}_{z_l}\hat{\phi} = -\hat{\phi}\hat{\partial}_{z_l} + \hat{D}_{z_l}\hat{\phi}$ where we define a local gauge field $\hat{A}_{z_l} \in \mathcal{F}$ and $\hat{D}_{z_l} := \hat{\partial}_{z_l} + \hat{A}_{z_l}$. The gauge curvature is defined as $\hat{F}_{z_l\bar{z}_m} := \mathrm{i}\left[\hat{\nabla}_{z_l}, \hat{\nabla}_{\bar{z}_m}\right] = -\frac{\delta_{lm}}{\zeta_l} + \mathrm{i}\left[\hat{D}_{z_l}, \hat{D}_{\bar{z}_m}\right]$.

Using this notation, we introduce the ADHM construction in the following.

3.5.1 Noncommutative ADHM construction

Definition 3.18. Let $B_1, B_2 \in \mathbb{C}^{k\times k},\ I \in \mathbb{C}^{k\times N},\ J \in \mathbb{C}^{N\times k}$ be matrices satisfying

$$\mu_{\mathbb{C}} := \left[B_1, B_2^{\dagger}\right] + IJ = 0, \quad \mu_{\mathbb{R}} := \left[B_1, B_1^{\dagger}\right] + \left[B_2, B_2^{\dagger}\right] + II^{\dagger} - J^{\dagger}J = (\zeta_1 + \zeta_2)\, E_k. \tag{3.5.1}$$

These equations are called the deformed ADHM equations. Here $\zeta_1,\ \zeta_2$ are noncommutative parameter in (6.1).

Let $E_k \in \mathbb{C}^{k\times k}$ be a unit matrix. We put $\beta_1, \beta_2 \in \mathbb{C}^{k\times k}, \tau \in \mathbb{C}^{k\times(2k+1)}, \sigma \in \mathbb{C}^{(2k+1)\times k}, \mathfrak{D} \in \mathbb{C}^{(2k+1)\times 2k}$ as

$$\beta_j := \frac{B_j}{\sqrt{\zeta_j}}, \quad \tau := \left(B_2 - z_2 E_k, B_1 - z_1 E_k, I\right), \quad \sigma := \left(-B_1 + z_1 E_k, B_2 - z_2 E_k, I\right)^T$$

$$\mathfrak{D}^{\dagger} := \begin{pmatrix} \tau \\ \sigma^{\dagger} \end{pmatrix} = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \\ -B_1^{\dagger} + \bar{z}_1 & B_2^{\dagger} - \bar{z}_2 & J^{\dagger} \end{pmatrix}.$$

The first step of the ADHM construction is solving the deformed ADHM equations (**??**).

The second step of the ADHM construction is solving the equation $\mathfrak{D}^\dagger * \Psi = 0,\ \Psi^\dagger * \Psi = 1$.

The third step of the ADHM construction is constructing gauge field $\hat{A}$ as $\hat{A}_{z_l} := \Psi^\dagger * \partial_{z_l}\Psi$, $\hat{A}_{\bar{z}_l} := \Psi^\dagger * \partial_{\bar{z}_l}\Psi$ where Ψ is a solution of the equations in the second step.

Then the curvature tensor $\tilde{F}_{z_l\bar{z}_m}$ constructed from $\hat{A}_{z_l},\ \hat{A}_{\bar{z}_m}$ is self-dual that means $\tilde{F}_{z_l\bar{z}_m}$ is an instanton curvature tensor.

For the $U(1)$ case, this construction process can be expressed more explicitly.

Assume

$$\Psi := \begin{pmatrix} \psi_+ \\ \psi_- \\ \xi \end{pmatrix} = \begin{pmatrix} \sqrt{\zeta_2}\left(\beta_2^\dagger - a_2\right) v \\ \sqrt{\zeta_1}\left(\beta_1^\dagger - a_1\right) v \\ \xi \end{pmatrix} \tag{3.5.2}$$

$$\hat{\Delta} := \zeta_1\left(\beta_1 - a_1^\dagger\right)\left(\beta_1^\dagger - a_1\right) + \zeta_2\left(\beta_2 - a_2^\dagger\right)\left(\beta_2^\dagger - a_2\right), \tag{3.5.3}$$

where $\xi \in \mathcal{F},\ v \in \mathbb{C}^k \otimes \mathcal{F}$. $\mathcal{F}$ is defined in (6.3), and $\left(\beta_l^\dagger - a_l\right) v := \left(\beta_l^\dagger \otimes id - E_k \otimes a_l\right) v$, where id is an identity mapping.

A vector space $\mathcal{H}$ is defined using (6.2) as

$$\mathcal{H} := span_{\mathbb{C}}\left(|0,0\rangle, |1,0\rangle, |0,1\rangle, |1,1\rangle, |2,2\rangle \cdots\right).$$

Definition 3.19. A linear operators P on $\mathcal{H}$ is defined as

$$P := I^\dagger\left(\exp\sum_\alpha \beta_\alpha^\dagger a_\alpha^\dagger\right)|0,0\rangle\, G^{-1}\,\langle 0,0|\left(\exp\sum_\alpha \beta_\alpha a_\alpha\right) I,$$

where

$$G := \langle 0,0|\left(\exp\sum_\alpha \beta_\alpha a_\alpha\right) II^\dagger\left(\exp\sum_\alpha \beta_\alpha^\dagger a_\alpha^\dagger\right)|0,0\rangle.$$

Fact 3.20. This linear operator is a projection operator, i. e. , $PP = P$.

A proposition below is true.

Proposition 3.21 (N. Nekrasov and A. S. Schwarz [38]). *Let* $\Psi,\ \hat{\Delta}v,\ \xi$ *be ones given above in (*??*). Then,*

$$\mathfrak{D}^\dagger\Psi = 0,\ \Psi^\dagger * \Psi = 1 \Longleftrightarrow \hat{\Delta}v + I\xi = 0,\ v^\dagger\hat{\Delta}v + \xi^\dagger\xi = 1.$$

Lemma 3.22 (N. Nekrasov and A. S. Schwarz [38])**.** *The operator S which satisfies $SS^{\dagger} = id, S^{\dagger}S = id - P$ exists. Let Λ be $id + I^{\dagger}\hat{\Delta}^{-1}I$. If we put*

$$\xi = \Lambda^{-1/2}S^{\dagger},\ v = -\hat{\Delta}^{-1}I\xi \tag{3.5.4}$$

then

$$\hat{\Delta}v + I\xi = 0,\ v^{\dagger}\hat{\Delta}v + \xi^{\dagger}\xi = 1. \tag{3.5.5}$$

This lemma means, if we find $\Lambda^{-1/2}$ and $\hat{\Delta}^{-1}$, then we can find a solution.

We define operators $\hat{\partial}_{z_l}$ and $\hat{D}_{z_l}$ on $\mathcal{H}$ in Section 6 as

$$\hat{\partial}_{z_l} := \frac{\bar{z}_l}{\zeta_l}, \qquad \hat{D}_{z_l} := \hat{\partial}_{z_l} + \hat{A}_{z_l}.$$

Noncommutative $U(1)$ instanton curvature in the Fock space is also defined as

$$\tilde{F}_{z_l\bar{z}_m} := \mathrm{i}\left[\hat{\partial}_{\bar{z}_m}, \hat{A}_{z_l}\right]_* - \mathrm{i}\left[\hat{\partial}_{z_l}, \hat{A}_{z_{\bar{m}}}\right]_* + \mathrm{i}\left[\hat{A}_l, \hat{A}_{\bar{m}}\right]_*.$$

Using $\hat{D}_{z_l}$, $\tilde{F}$ is rewritten as

$$\tilde{F}_{z_l\bar{z}_m} = \mathrm{i}\left[\hat{D}_{z_l}, \hat{D}_{\bar{z}_m}\right]_* + \frac{\mathrm{i}\delta_{lm}}{\zeta_l}. \tag{3.5.6}$$

Assume $\hat{A}_{z_l} := \Psi^{\dagger} * \partial_{z_l}\Psi,\ \hat{A}_{\bar{z}_l} := \Psi^{\dagger} * \partial_{\bar{z}_l}\Psi$ then

$$\hat{D}_{z_l} = -\frac{1}{\zeta_l}\Psi^{\dagger}\bar{z}_l\Psi, \qquad \hat{D}_{\bar{z}_l} = -\frac{1}{\zeta_l}\Psi^{\dagger}z_l\Psi.$$

Direct calculations derive the following results.

Theorem 3.23 (N. Nekrasov and A. S. Schwarz [38])**.** *If $\Lambda := id + I^{\dagger}\hat{\Delta}^{-1}I, \xi = \Lambda^{-1/2}S^{\dagger}, v = -\hat{\Delta}^{-1}I\xi$ then*

$$\hat{D}_{z_l} = -\frac{1}{\sqrt{\zeta_l}}S\Lambda^{-1/2}a_l\Lambda^{1/2}S^{\dagger}, \qquad \hat{D}_{\bar{z}_l} = \frac{1}{\sqrt{\zeta_l}}S\Lambda^{1/2}a_l^{\dagger}\Lambda^{-1/2}S^{\dagger}.$$

Theorem 3.24 (N. Nekrasov and A. S. Schwarz [38])**.** *If* $\tilde{F}^{-}_{z_k\bar{z}_l}$ *is given by* (**??**) *and* $\hat{D}_{z_l}, \hat{D}_{\bar{z}_l}$ *are defined in Theorem* **??***, then*

$$\begin{aligned}
\tilde{F}^{-}_{z_1\bar{z}_1}[k] &= \frac{\mathrm{i}}{\zeta_1} - \frac{\mathrm{i}}{\zeta_1} S\Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^{\dagger} S\Lambda^{\frac{1}{2}} a_1^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} + \frac{\mathrm{i}}{\zeta_1} S\Lambda^{\frac{1}{2}} a_1^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} S\Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^{\dagger},\\
\tilde{F}^{-}_{z_2\bar{z}_2}[k] &= \frac{\mathrm{i}}{\zeta_2} - \frac{\mathrm{i}}{\zeta_2} S\Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^{\dagger} S\Lambda^{\frac{1}{2}} a_2^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} + \frac{\mathrm{i}}{\zeta_2} S\Lambda^{\frac{1}{2}} a_2^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} S\Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^{\dagger},\\
\tilde{F}^{-}_{z_1\bar{z}_2}[k] &= -\frac{\mathrm{i}}{\sqrt{\zeta_1\zeta_2}} S\Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^{\dagger} S\Lambda^{\frac{1}{2}} a_2^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} + \frac{\mathrm{i}}{\sqrt{\zeta_1\zeta_2}} S\Lambda^{\frac{1}{2}} a_2^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} S\Lambda^{-\frac{1}{2}} a_1 \Lambda^{\frac{1}{2}} S^{\dagger},\\
\tilde{F}^{-}_{z_2\bar{z}_1}[k] &= -\frac{\mathrm{i}}{\sqrt{\zeta_1\zeta_2}} S\Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^{\dagger} S\Lambda^{\frac{1}{2}} a_1^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} + \frac{\mathrm{i}}{\sqrt{\zeta_1\zeta_2}} S\Lambda^{\frac{1}{2}} a_1^{\dagger} \Lambda^{-\frac{1}{2}} S^{\dagger} S\Lambda^{-\frac{1}{2}} a_2 \Lambda^{\frac{1}{2}} S^{\dagger}.
\end{aligned}$$

This curvature is an instanton curvature.

3.5.2 $U(1)$ k-instanton in the noncommutative $\mathbb{C}^2$

In this section we summarize $U(1)$ multi-instanton solutions on $\mathbb{C}^2$ in [39]. For simplicity, let us assume $\zeta_1 = \zeta_2 =: \zeta$.

Definition 3.25. Noncommutative instanton curvature in the noncommutative $\mathbb{C}^2$ is defined as

$$\hat{F}^{-}_{\mathbb{C}}[k] = \begin{pmatrix} \hat{F}^{-}_{z_1\bar{z}_1}[k] & \hat{F}^{-}_{z_1\bar{z}_2}[k] \\ -\hat{F}^{-}_{z_2\bar{z}_1}[k] & -\hat{F}^{-}_{z_1\bar{z}_1}[k] \end{pmatrix} := \begin{pmatrix} \iota\left(\tilde{F}^{-}_{z_1\bar{z}_1}[k]\right) & \iota\left(\tilde{F}^{-}_{z_1\bar{z}_2}[k]\right) \\ \iota\left(-\tilde{F}^{-}_{z_2\bar{z}_1}[k]\right) & \iota\left(-\tilde{F}^{-}_{z_1\bar{z}_1}[k]\right) \end{pmatrix}$$

where ι is defined in Definition 42.

We choose

$$B_1 = \sum_{l=1}^{k-1} \sqrt{2l\zeta} e_l e_{l+1}^{\dagger}, \ B_2 =, 0 \ , I = \sqrt{2k\zeta} e_k, \ J = 0$$

as a solution of the deformed ADHM equations (**??**). Here

$$e_l^{\dagger} = \begin{pmatrix} \delta_{1,l} & \delta_{2,l} & \cdots & \delta_{k-1,l} & \delta_{k,l} \end{pmatrix}.$$

In this case, the operator $S^{\dagger}$ in Lemma **??** is given by

$$S^{\dagger} = \sum_{n_1=0}^{\infty} |n_1 + k, 0\rangle \langle n_1, 0| + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} |n_1, n_2\rangle \langle n_1, n_2| \,. \tag{3.5.7}$$

From Theorem **??** and (**??**), a $U(1)$ k-instanton curvature in the noncommutative $\mathbb{C}^2$ is obtained as follows.

$$\tilde{F}^{-}_{z_1\bar{z}_1}[k] = \frac{\mathrm{i}}{\zeta} - \frac{\mathrm{i}}{\zeta}\sum_{n_2=0}^{\infty}|0,n_2\rangle\langle 0,n_2|\left(d_1(0,n_2;k)\right)^2$$

(3.5.8)
$$-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}|n_1,n_2\rangle\langle n_1,n_2|\left\{\left(d_1(n_1,n_2;k)\right)^2-\left(d_1(n_1-1,n_2;k)\right)^2\right\},$$

(3.5.9)

$$\tilde{F}^{-}_{z_2\bar{z}_2}[k] = -\tilde{F}^{-}_{z_1\bar{z}_1}[k]$$

(3.5.10)

$$\tilde{F}^{-}_{z_1\bar{z}_2}[k] = -\frac{\mathrm{i}}{\zeta}|k-1,1\rangle\langle 0,0|\,d_1(k-1,1;k)\,d_2(0,0;k)$$

$$-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{k-1}|n_1+k-1,1\rangle\langle n_1,0|\left\{d_1(n_1+k-1,1;k)\,d_2(n_1,0;k)-d_1(n_1-1,0;k)\,d_2(n_1-1,0;k)\right\}$$

$$-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}|n_1-1,n_2+1\rangle\langle n_1,n_2|$$

$$\times\left\{d_1(n_1-1,n_2+1;k)\,d_2(n_1,n_2;k)-d_1(n_1-1,n_2;k)\,d_2(n_1-1,n_2;k)\right\}$$

(3.5.11)

$$\tilde{F}^{-}_{z_2\bar{z}_1}[k] = \tilde{F}^{-}_{z_1\bar{z}_2}[k]^{\dagger},$$

where $d_1(n_1,n_2;k)$ and $d_2(n_1,n_2;k)$ are given by.

$$d_1(n_1,0;k) = \sqrt{n_1+k+1}\sqrt{\frac{\Lambda(n_1+k+1,0)}{\Lambda(n_1+k,0)}},$$

(3.5.12)
$$d_1(n_1,n_2;k) = \sqrt{n_1+1}\sqrt{\frac{\Lambda(n_1+1,n_2)}{\Lambda(n_1,n_2)}},$$

$$d_2(n_1,0;k) = \sqrt{\frac{\Lambda(n_1+k,1)}{\Lambda(n_1+k,0)}},$$

(3.5.13)
$$d_2(n_1,n_2;k) = \sqrt{n_2+1}\sqrt{\frac{\Lambda(n_1,n_2+1)}{\Lambda(n_1,n_2)}}.$$

Here

$$\Lambda\left[k\right]\left(n_1,n_2\right)=\frac{w_k\left[k\right]\left(n_1,n_2\right)}{w_k\left[k\right]\left(n_1,n_2\right)-2kw_{k-1}\left[k\right]\left(n_1,n_2\right)},$$

and

$$w_n\left[k\right]\left(n_1,n_2\right)=\sum_{l=0}^{n}\left\{\frac{n!}{l!}\frac{\left(n_1-n_2+k+l\right)!}{\left(n_1-n_2-k\right)!}\frac{2^{(n-l)}}{(n-l)!}\frac{\left(n_2+(n-l)\right)!}{n_2!}\right\}.$$

Next we change these curvature operators into functions on $\mathbb{C}^2$ using the isomorphism (6.4).

$$\begin{aligned}(3.5.14)\quad \hat{F}^{-}_{z_1\bar{z}_1}\left[k\right]:=&\iota\left(\tilde{F}^{-}_{z_1\bar{z}_1}\left[k\right]\right)\\ =&\frac{\mathrm{i}}{\zeta}-\frac{\mathrm{i}}{\zeta}\sum_{n_2=0}^{\infty}\frac{z_2^{n_2}\mathrm{e}^{-\frac{z_1\bar{z}_1+z_2\bar{z}_2}{\zeta}}\bar{z}_2^{n_2}}{n_2!\zeta^{n_2}}\left(d_1\left(0,n_2;k\right)\right)^2\\ &-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\frac{z_1^{n_1}z_2^{n_2}\mathrm{e}^{-\frac{z_1\bar{z}_1+z_2\bar{z}_2}{\zeta}}\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{n_1!n_2!\zeta^{n_1+n_2}}\left\{\left(d_1\left(n_1,n_2;k\right)\right)^2-\left(d_1\left(n_1-1,n_2;k\right)\right)^2\right\},\\ (3.5.15)\quad \hat{F}^{-}_{z_2\bar{z}_2}\left[k\right]:=&\iota\left(\tilde{F}^{-}_{z_2\bar{z}_2}\left[k\right]\right)=-\hat{F}^{-}_{z_1\bar{z}_1}\left[k\right],\end{aligned}$$

$$\begin{aligned}\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right]:=&\iota\left(\tilde{F}^{-}_{z_1\bar{z}_2}\left[k\right]\right)\\ (3.5.16)\qquad =&-\frac{\mathrm{i}}{\zeta}\frac{z_1^{k-1}z_2\mathrm{e}^{-\frac{z_1\bar{z}_1+z_2\bar{z}_2}{\zeta}}}{\sqrt{(k-1)!}\left(\sqrt{\zeta}\right)^k}d_1\left(k-1,1;k\right)d_2\left(0,0;k\right),\\ &-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{k-1}\frac{z_1^{n_1+k-1}z_2\mathrm{e}^{-\frac{z_1\bar{z}_1+z_2\bar{z}_2}{\zeta}}\bar{z}_1^{n_1}}{\sqrt{(n_1+k-1)!n_1!}\left(\sqrt{\zeta}\right)^{2n_1+k}}\\ &\times\left\{d_1\left(n_1+k-1,1;k\right)d_2\left(n_1,0;k\right)-d_1\left(n_1-1,0;k\right)d_2\left(n_1-1,0;k\right)\right\}\\ &-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{z_1^{n_1-1}z_2^{n_2+1}\mathrm{e}^{-\frac{z_1\bar{z}_1+z_2\bar{z}_2}{\zeta}}\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{\sqrt{(n_1-1)!\left(n_2+1\right)!n_1!n_2!}\left(\sqrt{\zeta}\right)^{2n_1+2n_2}}\\ &\times\left\{d_1\left(n_1-1,n_2+1;k\right)d_2\left(n_1,n_2;k\right)-d_1\left(n_1-1,n_2;k\right)d_2\left(n_1-1,n_2;k\right)\right\},\\ (3.5.17)\quad \hat{F}^{-}_{z_2\bar{z}_1}\left[k\right]=&\iota\left(\tilde{F}^{-}_{z_2\bar{z}_1}\left[k\right]\right)=-\overline{\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right]},\end{aligned}$$

where $\overline{a}$ is a complex conjugate of a.

In order to obtain Ricci-flat metrics in Section 6.3 and Section 6.4, we need the first three terms of the expansion for $\hat{F}^{-}_{\mathbb{C}}\left[k\right]$ in $\sqrt{\frac{1}{\zeta}}$.

$$
\begin{aligned}
\hat{F}^{-}_{z_1\bar{z}_1}\left[k\right] =& \frac{\mathrm{i}}{\zeta} - \frac{\mathrm{i}z_2\bar{z}_2}{\zeta^2}\left(d_1\left(0,1;k\right)\right)^2 - \frac{\mathrm{i}z_1\bar{z}_1}{\zeta^2}\left\{\left(d_1\left(1,0;k\right)\right)^2 - \left(d_1\left(0,0;k\right)\right)^2\right\} \\
&+ \mathrm{i}\frac{z_1\bar{z}_1z_2\bar{z}_2}{\zeta^3}\left(d_1\left(0,1;k\right)\right)^2 + \mathrm{i}\frac{\left(z_2\bar{z}_2\right)^2}{\zeta^3}\left(d_1\left(0,1;k\right)\right)^2 + \mathrm{i}\frac{\left(z_1\bar{z}_1\right)^2}{\zeta^3}\left\{\left(d_1\left(1,0;k\right)\right)^2 - \left(d_1\left(0,0;k\right)\right)^2\right\} \\
&+ \mathrm{i}\frac{z_1\bar{z}_1z_2\bar{z}_2}{\zeta^3}\left\{\left(d_1\left(1,0;k\right)\right)^2 - \left(d_1\left(0,0;k\right)\right)^2\right\} + \mathcal{O}\left(\zeta^{-4}\right), \\
d_1\left(0,0;k\right) =& \sqrt{\frac{\left(k+1\right)\Lambda\left(k+1,0\right)}{\Lambda\left(k,0\right)}}, d_1\left(1,0;k\right) = \sqrt{\frac{\left(k+2\right)\Lambda\left(k+2,0\right)}{\Lambda\left(k+1,0\right)}}, d_1\left(0,1;k\right) = \sqrt{\frac{\Lambda\left(1,1\right)}{\Lambda\left(0,1\right)}},
\end{aligned}
\tag{3.5.18}
$$

and

$$
\begin{aligned}
\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right] =& -\frac{\mathrm{i}}{\zeta\left(\sqrt{\zeta}\right)^k}\frac{z_1^{k-1}z_2}{\sqrt{(k-1)!}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right)\right)d_1\left(k-1,1;k\right)d_2\left(0,0;k\right) \\
&- \frac{\mathrm{i}}{\zeta\left(\sqrt{\zeta}\right)^k}\sum_{n_1=1}^{k-1}\frac{z_1^{n_1+k-1}z_2\bar{z}_1^{n_1}}{\sqrt{(n_1+k-1)!n_1!}\zeta^{n_1}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right)\right) \\
&\times\left\{d_1\left(n_1+k-1,1;k\right)d_2\left(n_1,0;k\right) - d_1\left(n_1-1,0;k\right)d_2\left(n_1-1,0;k\right)\right\} \\
&- \frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{z_1^{n_1-1}z_2^{n_2+1}\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{\sqrt{(n_1-1)!\left(n_2+1\right)!n_1!n_2!}\zeta^{n_1+n_2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta} + \mathcal{O}\left(\zeta^{-2}\right)\right) \\
&\times\left\{d_1\left(n_1-1,n_2+1;k\right)d_2\left(n_1,n_2;k\right) - d_1\left(n_1-1,n_2;k\right)d_2\left(n_1-1,n_2;k\right)\right\}.
\end{aligned}
\tag{3.5.19}
$$

It is useful to distinguish the cases for $k=1$ and $k>1$.

$$
k = 1 \Rightarrow \hat{F}^{-}_{z_1\bar{z}_2}\left[1\right] = -\frac{\mathrm{i}z_2}{\zeta^{3/2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right)d_1\left(0,1;1\right)d_2\left(0,0;1\right) + \mathcal{O}\left(\zeta^{-3}\right). \tag{3.5.20}
$$

$$
\begin{aligned}
k > 1 \Rightarrow \hat{F}^{-}_{z_1\bar{z}_2}\left[k\right] =& -\frac{\mathrm{i}z_1^{k-1}z_2}{\zeta\left(\sqrt{\zeta}\right)^k\sqrt{(k-1)!}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right)d_1\left(k-1,1;k\right)d_2\left(0,0;k\right) \\
&- \frac{\mathrm{i}z_1^kz_2\bar{z}_1}{\sqrt{k!}\zeta^2\left(\sqrt{\zeta}\right)^k}\left\{d_1\left(k,1;k\right)d_2\left(1,0;k\right) - d_1\left(0,0;k\right)d_2\left(0,0;k\right)\right\} \\
&- \frac{\mathrm{i}z_2^2\bar{z}_1\bar{z}_2}{\sqrt{2!}\zeta^3}\left\{d_1\left(0,2;k\right)d_2\left(1,1;k\right) - d_1\left(0,1;k\right)d_2\left(0,1;k\right)\right\} + \mathcal{O}\left(\zeta^{-4}\right).
\end{aligned}
\tag{3.5.21}
$$

Functions Λ, d_1, d_2 for $k=1$ are useful for Subsection 6.4 :

$$\begin{aligned}
\Lambda\left[1\right]\left(n_1,n_2\right) &= \frac{\omega_1\left(n_1,n_2\right)}{\omega_1\left(n_1,n_2\right)-2\omega_0\left(n_1,n_2\right)} = \frac{2+n_1+n_2}{n_1+n_2},\\
d_1\left(n_1,0;1\right) &= \sqrt{n_1+2}\sqrt{\frac{\Lambda\left[1\right]\left(n_1+2,0\right)}{\Lambda\left[1\right]\left(n_1+1,0\right)}} = \sqrt{\frac{\left(4+n_1\right)\left(1+n_1\right)}{\left(3+n_1\right)}},\\
d_1\left(n_1,n_2;1\right) &= \sqrt{n_1+1}\sqrt{\frac{\Lambda\left[1\right]\left(n_1+1,n_2\right)}{\Lambda\left[1\right]\left(n_1,n_2\right)}} = \sqrt{\frac{\left(n_1+1\right)\left(3+n_1+n_2\right)\left(n_1+n_2\right)}{\left(1+n_1+n_2\right)\left(2+n_1+n_2\right)}},\\
d_2\left(n_1,0;1\right) &= \left\{\frac{\Lambda\left[1\right]\left(n_1+1,1\right)}{\Lambda\left[1\right]\left(n_1+1,0\right)}\right\}^{\frac{1}{2}} = \sqrt{\frac{\left(n_1+4\right)\left(n_1+1\right)}{\left(n_1+2\right)\left(n_1+3\right)}},\\
d_2\left(n_1,n_2;1\right) &= \sqrt{\frac{\left(n_2+1\right)\Lambda\left[1\right]\left(n_1,n_2+1\right)}{\Lambda\left[1\right]\left(n_1,n_2\right)}} = \sqrt{\frac{\left(n_2+1\right)\left(n_1+n_2\right)\left(3+n_1+n_2\right)}{\left(n_1+n_2+1\right)\left(2+n_1+n_2\right)}}.
\end{aligned}$$

3.6 Seiberg-Witten map

The purpose in this section is to derive a relational expression between F and $\hat{F}$ that can be derived by assuming commutability between gauge transformation and quantization. This section is based on [34]. Here we define gauge transformations on normal commutative space and gauge transformations on noncommutative space. Let G be a gauge group and $\mathfrak{g}$ be the Lie algebra of G.

Definition 3.26. The gauge transformations δ_λ are defined as

$$\begin{aligned}
\delta_\lambda A_k &:= \partial_k\lambda + \mathrm{i}\left[\lambda, A_k\right],\\
\delta_\lambda F_{kl} &:= \mathrm{i}\left[\lambda, F_{kl}\right],
\end{aligned}$$

where λ and A are $\mathfrak{g}$ valued scalar field and gauge field on gauge group G, respectively. The gauge transformations of noncommutative $U\left(N\right)$ gauge theory $\hat{\delta}_{\hat{\lambda}}$ are defined as

$$\begin{aligned}
\hat{\delta}_{\hat{\lambda}}\hat{A}_k &:= \partial_k\hat{\lambda} + \mathrm{i}\hat{\lambda} * \hat{A}_k - \mathrm{i}\hat{A}_k * \hat{\lambda}\\
\hat{\delta}_{\hat{\lambda}}\hat{F}_{kl} &:= \mathrm{i}\hat{\lambda} * \hat{F}_{kl} - \mathrm{i}\hat{F}_{kl} * \hat{\lambda}
\end{aligned}$$

where $\hat{F}_{kl} := \partial_k\hat{A}_l - \partial_l\hat{A}_k - \mathrm{i}\hat{A}_k * \hat{A}_l + \mathrm{i}\hat{A}_l * \hat{A}_k$. $\hat{A}_k$ and the gauge parameter field $\hat{\lambda}$ takes values in $\left(C^\infty\left(\mathbb{R}^4\right)\left[\left[\hbar\right]\right], *\right)$ tensored with $N \times N$ hermitian matrices, for some N.

The reason why Newtonian mechanics is sufficient in daily life is that Planck's constant is sufficiently small. Similarly, a case where the noncommutative parameter θ is small is considered.

Remark 3.27. From the above gauge transformation we find that

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} - \theta^{kl}\partial_k\hat{\lambda}\partial_l\hat{A}_k + \mathcal{O}\left(\theta^2\right),$$
$$\hat{\delta}_{\hat{\lambda}}\hat{F}_{ij} = -\theta^{kl}\partial_k\hat{\lambda}\partial_l\hat{F}_{ij} + \mathcal{O}\left(\theta^2\right),$$

and $\hat{F}_{ij} = \partial_i\hat{A}_j - \partial_j\hat{A}_i + \theta^{kl}\partial_k\hat{A}_i\partial_l\hat{A}_j + \mathcal{O}(\theta^2)$.

From the consistency of physical picture of D-brane theory discussed in Section **??** the following assumption is naturally obtained

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}(A) = \hat{A}(A + \delta_\lambda A). \tag{3.6.1}$$

This assumption means that there is a map $\hat{A}$, $\hat{\lambda}$ from commutative gauge theory to noncommutative gauge theory, and gauge transformations for commutative gauge theory and noncommutative gauge theory are compatible.

Proposition 3.28. *Assume $A'(A) := \hat{A} - A, \lambda'(\lambda, A) := \hat{\lambda}(\lambda, A) - \lambda$. Expanding (**??**) in powers of θ, we find that we need*

$$A'_k(A + \delta_\lambda A) - A'_k(A) - \partial_k\lambda' - \mathrm{i}[\lambda', A_k] - \mathrm{i}[\lambda, A'_k] = \frac{-\theta^{ij}}{2}(\partial_i\lambda\partial_j A_k + \partial_j A_k\partial_i\lambda) + \mathcal{O}\left(\theta^2\right). \tag{3.6.2}$$

This proposition is proved in [34] in P28.

Proposition 3.29. *If*

$$\hat{A}_k(A) = A_k - \frac{\theta^{ij}}{4}\{A_i, \partial_j A_k + F_{jk}\} + \mathcal{O}\left(\theta^2\right),$$
$$\hat{\lambda}(\lambda, A) = \lambda + \frac{\theta^{kl}}{4}\{\partial_k\lambda, A_l\} + \mathcal{O}\left(\theta^2\right),$$

where $\{A_j, \partial_k A_l\} := A_j \cdot \partial_k A_l + \partial_k A_l \cdot A_j$ as matrix products, then these are the solutions of (3.1). Hence

$$\hat{F}_{ij} = F_{ij} + \frac{\theta^{kl}}{4}(2\{F_{ik}, F_{jl}\} - \{A_k, D_l F_{ij} + \partial_l F_{ij}\}) + \mathcal{O}\left(\theta^2\right).$$

This proposition is also proved in [34] in P28.

The following lemma is used here for the next proposition, so it is described here.

Lemma 3.30.

$$\delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}(f * g) = \delta\theta^{kl}\frac{\partial f}{\partial x^k} * \frac{\partial g}{\partial x^l}$$

at $\theta = 0$.

This lemma is also proved in [34] in P29.

Proposition 3.31. *Assume that* $\delta\hat{A}_k(\theta), \delta\hat{\lambda}(\theta), \delta\hat{F}_{ij}(\theta)$ *are defined as*

$$\delta\hat{A}_k(\theta) := \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\hat{A}_k(\theta), \quad \delta\hat{\lambda}(\theta) := \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\hat{\lambda}(\theta), \quad \delta\hat{F}_{ij}(\theta) := \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\hat{F}_{ij}(\theta).$$

Then

$$\begin{aligned} &\delta\hat{A}_k(\theta) = -\frac{1}{4}\delta\theta^{kl}\left[\hat{A}_k * \left(\partial_l\hat{A}_i + \hat{F}_{li}\right) + \left(\partial_l\hat{A}_i + \hat{F}_{li}\right) * \hat{A}_k\right], \\ &\delta\hat{\lambda}(\theta) = \frac{1}{4}\delta\theta^{kl}\left(\partial_k\lambda * A_l + A_l * \partial_k\lambda\right), \\ &\delta\hat{F}_{ij}(\theta) = \frac{1}{4}\delta\theta^{kl}\left[2\hat{F}_{ik} * \hat{F}_{jl} + 2\hat{F}_{jl} * \hat{F}_{ik} - \hat{A}_k * \left(\hat{D}_l\hat{F}_{ij} + \partial_l\hat{F}_{ij}\right) - \left(\hat{D}_l\hat{F}_{ij} + \partial_l\hat{F}_{ij}\right) * \hat{A}_k\right]. \end{aligned} \tag{3.6.3}$$

This proposition is also proved in [34] in P29.

The differential equation (**??**) can be solved explicitly for the important case of a rank one gauge field with constant $\hat{F}$. In this case, the equation can be written

$$\delta\hat{F} = -\hat{F}\delta\theta\hat{F}. \tag{3.6.4}$$

The solution with the boundary condition $\hat{F}(\theta = 0) = F$ for (**??**) is

$$\hat{F}(x) = \left(\frac{1}{1 + F\theta}F\right)(x). \tag{3.6.5}$$

3.7 Dirac-Born-Infeld Action

The actual time evolution path followed by the system corresponds to the stationary point (usually the minimum point) of actions. The stationary point of the action is given by a variation on the action integral. This section is based on [34] and [**?**], so we use all symbols and notations as the same ones in [34] and [**?**] .

3.7.1 Approximate theory when B is small

Definition 3.32. For slowly varying fields on a single Dp-brane, the effective Lagrangian is the Dirac-Born-Infeld Lagrangian

$$\mathcal{L}_{DBI} := \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}\sqrt{\det\left(g + \kappa(F + B)\right)}$$

and for slowly varying $\hat{F}$ the effective Lagrangian of the noncommutative gauge fields is

$$\hat{\mathcal{L}}_{DBI} := \mathcal{L}\left(\hat{F}\right) = \frac{2\pi}{G_s \left(2\pi\kappa\right)^{\frac{p+1}{2}}} \sqrt{\det\left(G + \kappa\hat{F}\right)}$$

where $\kappa = 2\pi\alpha'$.

The following formula is easily obtained with $\mathrm{d}\left(\det A\right) = tr\left(A^{-1}\mathrm{d}A\right)\det A$.

$$\partial_l \det\left(G+F\right)^{\frac{1}{2}} = \frac{1}{2}\det\left(G+F\right)^{\frac{1}{2}} \left(\frac{1}{G+F}\right)_{ji} \partial_l F_{ij} \tag{3.7.1}$$

Proposition 3.33. *The following approximate expression can be considered.*

$$\sqrt{\det\left(1+M\right)} = 1 - \frac{1}{4} Tr M^2 - \frac{1}{8} Tr M^4 + \frac{1}{32}\left(Tr M^2\right)^2 + \mathcal{O}\left(M^6\right)$$

for antisymmetric M.

Proof. *This equation can be derived using* $\det\left(\exp A\right) = \exp\left(trA\right)$*.*

$$\begin{aligned}
&\sqrt{\det\left(1+M\right)} = \exp\left[\frac{Tr\left\{\log\left(1+M\right)\right\}}{2}\right] \\
&= \exp\left\{\frac{1}{2} Tr\left(M - \frac{M^2}{2} + \frac{M^3}{3} - \frac{M^4}{4} + \frac{M^5}{5} + \mathcal{O}\left(M^6\right)\right)\right\} \\
&= \exp\left(-\frac{Tr\left(M^2\right)}{4} - \frac{Tr\left(M^4\right)}{8} + \mathcal{O}\left(M^6\right)\right) \\
&= \exp\left(-\frac{Tr\left(M^2\right)}{4}\right) \exp\left(-\frac{Tr\left(M^4\right)}{8} + \mathcal{O}\left(M^6\right)\right) \\
&= \left(1 - \frac{Tr\left(M^2\right)}{4} + \frac{\left\{Tr\left(M^2\right)\right\}^2}{32}\right)\left(1 - \frac{Tr\left(M^4\right)}{8}\right) + \mathcal{O}\left(M^6\right).
\end{aligned}$$

□

We consider the boundary conditions:

$$G = g - \kappa^2 B g^{-1} B = \left(g - \kappa B\right) g^{-1} \left(g + \kappa B\right), \tag{3.7.2}$$

and

$$\theta^{ij} = -\kappa^2 \left(\left(g + \kappa B\right)^{-1} B \left(g - \kappa B\right)^{-1}\right)^{ij}. \tag{3.7.3}$$

Proposition 3.34 (N. Seiberg and E. Witten[34]). *If*

$$\mathcal{L}_{DBI}\left(F=0\right)=\hat{\mathcal{L}}_{DBI}\left(\hat{F}=0\right)$$

where $\hat{\mathcal{L}}_{DBI}$ *and* $\mathcal{L}_{DBI}$ *are as described above, then*

$$G_s=g_s\sqrt{\frac{\det G}{\det\left(g+\kappa B\right)}}=g_s\frac{1}{\sqrt{\det\left[\left\{\left(g+\kappa B\right)^{-1}-\frac{1}{\kappa}\theta\right\}\left(g+\kappa B\right)\right]}}.$$

Proof. *Since the boundary conditions (??) and (??),*

$$\left(g+\kappa B\right)^{-1}=G^{-1}+\frac{1}{\kappa}\theta.$$

$\square$

Approximations are made by ignore higher order terms of θ and B.

Remark 3.35. For small B and θ we obtained the following.

$$\begin{aligned}&G=\left(g+\kappa B\right)g^{-1}\left(g-\kappa B\right),\\&\theta=-\kappa^2g^{-1}Bg^{-1}+\mathcal{O}\left(B^3\right),\\&G_s=g_s\left(1-\frac{\kappa^2}{2}Tr\left(g^{-1}B\right)^2+\mathcal{O}\left(B^4\right)\right),\\&\hat{F}_{ij}=F_{ij}+\theta^{kl}\left(F_{ik}F_{jl}-A_k\partial_lF_{ij}\right)+\mathcal{O}\left(\theta^2\right).\end{aligned}$$

As a result of approximation ignoring higher order terms, $\hat{\mathcal{L}}_{DBI}$ can be approximated as follows and approximated to $\mathcal{L}_{DBI}$.

Lemma 3.36 (N. Seiberg and E. Witten[34]). $\hat{\mathcal{L}}_{DBI}$ *is expressed as*

$$\hat{\mathcal{L}}_{DBI}=\frac{2\pi}{g_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\det\left(g+\kappa F\right)^{\frac{1}{2}}\left(1+\frac{\kappa}{2}Tr\frac{1}{g+\kappa F}B\right)+\mathcal{O}\left(B^2\right)+\textit{total derivative.}\tag{3.7.4}$$

Proof. *First,* $\sqrt{\det\left(G+\kappa\hat{F}\right)} = \frac{g_s(2\pi\kappa)^{\frac{p+1}{2}}}{2\pi}\hat{\mathcal{L}}_{DBI}$ *is calculated with remark* **??**.

$$\begin{aligned}
&\sqrt{\det\left(G+\kappa\hat{F}\right)} = \sqrt{\det\left(G+\kappa\left(F_{ij}+\theta^{kl}\left(F_{ik}F_{jl}-A_k\partial_l F_{ij}\right)+\mathcal{O}\left(\theta^2\right)\right)\right)}\\
&= \sqrt{\det\left(G+\kappa F+\kappa\left(\theta^{kl}\left(F_{ik}F_{jl}-A_k\partial_l F_{ij}\right)+\mathcal{O}\left(\theta^2\right)\right)\right)}\\
&= \det\left\{(G+\kappa F)^{\frac{1}{2}}\left(1+\kappa\left(\frac{1}{G+\kappa F}\right)_{ji}\theta^{kl}\left(F_{ik}F_{jl}-A_k\partial_l F_{ij}\right)+\mathcal{O}\left(\theta^2\right)\right)\right\}\\
&= \det\left(G+\kappa F\right)^{\frac{1}{2}}\left(1-\kappa Tr\left(\frac{1}{G+\kappa F}F\theta F\right)+\frac{1}{2}Tr\theta F+\mathcal{O}\left(\theta^2\right)\right)+total\ derivative\\
&= \det\left(G+\kappa F\right)^{\frac{1}{2}}\left(1-\frac{1}{2\kappa}Tr\left(\frac{1}{G+\kappa F}G\theta G\right)+\mathcal{O}\left(\theta^2\right)\right)+total\ derivative
\end{aligned}$$

where we use $TrG\theta = 0$. ☐

By integrating both of (**??**) under the setting of zero at infinity, the two can be said to be approximately equal.

Theorem 3.37 (N. Seiberg and E. Witten[34])**.**

$$\int \mathrm{d}^4 y\hat{\mathcal{L}}_{DBI} = \int \mathrm{d}^4 x\mathcal{L}_{DBI} + \mathcal{O}\left(\theta^2\right).$$

Proof. *From Lemma* **??** *and*

$$\hat{\mathcal{L}}_{DBI} = \mathcal{L}_{DBI} + total\ derivative + \mathcal{O}\left(\theta^2\right),$$

and this derive the result. ☐

3.7.2 Dirac-Born-Infeld Action and Seiberg-Witten map

For simplicity the gauge group is assumed to be $U(1)$ from this point on.

Definition 3.38. The action is defined as the integral over the Lagrangian density in commutative $\mathbb{R}^4$ and noncommutative $\mathbb{R}^4$.

$$\begin{aligned}
S\left(g,g_s,A,B\right) &:= \frac{2\pi}{g_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4 y\sqrt{-\det\left(g+\kappa\left(F+B\right)\right)},\\
\hat{S}\left(G,G_s,A,\theta\right) &:= \frac{2\pi}{G_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4 x\sqrt{-\det\left(G+\kappa\hat{F}\right)}.
\end{aligned}$$

κ, g, G, A, B, θ, g_s, G_s are the same as those described above.

As we saw before, if

$$S\left(g,g_s,A,B\right)=\hat{S}\left(G,G_s,A,\theta\right)+\mathcal{O}\left(\sqrt{\kappa}\partial F\right)$$

for $F=\hat{F}=0$,

$$G_s=g_s\sqrt{\frac{\det G}{\det\left(g+\kappa B\right)}}.$$

And if

$$G^{-1}=\left(g+\kappa B\right)^{-1}g\left(g-\kappa B\right)^{-1},\theta=-\kappa^2\left(g+\kappa B\right)^{-1}B\left(g-\kappa B\right)^{-1},$$

then

$$\left(g+\kappa B\right)^{-1}=G^{-1}+\frac{1}{\kappa}\theta.$$

$\mathcal{G},\mathcal{G}_s$ are defined as

$$\mathcal{G}=G-\kappa\Phi$$

$$\mathcal{G}_s=g_s\sqrt{\frac{\det\left(\mathcal{G}+\kappa\Phi\right)}{\det\left(g+\kappa B\right)}}=G_s\sqrt{\frac{\det\left(g+\kappa B\right)}{\det G}}\sqrt{\frac{\det\left(\mathcal{G}+\kappa\Phi\right)}{\det\left(g+\kappa B\right)}}=G_s\sqrt{\frac{\det\left(\mathcal{G}+\kappa\Phi\right)}{\det G}}=G_s.$$

Proposition 3.39 (N. Seiberg and E. Witten[34])**.**

$$\int d^4x\sqrt{-\det\left\{\mathcal{G}+\kappa\left(\Phi+\hat{F}\right)\right\}}=\int d^4y\sqrt{\det\left(1+F\theta\right)}\sqrt{-\det\left\{\mathcal{G}+\kappa\left(\Phi+\boldsymbol{F}\right)\right\}}+\mathcal{O}\left(l_s\partial F\right),$$

where $\boldsymbol{F}_{\mu\nu}\left(x\right):=\left(\frac{1}{1+F\theta}F\right)_{\mu\nu}\left(x\right)$.

Proof. *At first* $S\left(g,g_s,A,B\right)=\hat{S}\left(G,G_s,A,\theta\right)+\mathcal{O}\left(\sqrt{\kappa}\partial F\right)$. *Because* $\mathcal{G}_s=g_s\sqrt{\frac{\det(\mathcal{G}+\kappa\Phi)}{\det(g+\kappa B)}}$, $\left(g+\kappa B\right)^{-1}=$

$G^{-1}+\frac{1}{\kappa}\theta$ and $G=\mathcal{G}+\kappa\Phi$ the left side is

$$\begin{aligned}
S\left(g,g_s,A,B\right) &= \frac{2\pi}{g_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{-\det\left(g+\kappa\left(F+B\right)\right)} \\
&= \frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{-\det\left(g+\kappa\left(F+B\right)\right)}\sqrt{\frac{\det G}{\det\left(g+\kappa B\right)}} \\
&= \frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{-\det\left\{1+\kappa F\left(g+\kappa B\right)^{-1}\right\}\det G} \\
&= \frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{-\det\left\{G+\kappa F\left(G^{-1}+\frac{1}{\kappa}\theta\right)G\right\}} \\
&= \frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{-\det\left\{\mathcal{G}+\kappa\Phi+\kappa F+F\theta\left(\mathcal{G}+\kappa\Phi\right)\right\}} \\
&= \frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4y\sqrt{\det\left(1+F\theta\right)}\sqrt{-\det\left\{\mathcal{G}+\kappa\left(\Phi+\boldsymbol{F}\right)\right\}}
\end{aligned}$$

where $\boldsymbol{F}_{\mu\nu}\left(x\right):=\left(\frac{1}{1+F\theta}F\right)_{\mu\nu}\left(x\right)$.
On the other side, the right hand side is

$$\hat{S}\left(G,G_s,A,\theta\right):=\frac{2\pi}{G_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4x\sqrt{-\det\left(G+\kappa\hat{F}\right)}=\frac{2\pi}{\mathcal{G}_s\left(2\pi\kappa\right)^{\frac{p+1}{2}}}\int \mathrm{d}^4x\sqrt{-\det\left(\mathcal{G}+\kappa\Phi+\kappa\hat{F}\right)}$$

because $\mathcal{G}_s=G_s$ and $\mathcal{G}=G-\kappa\Phi$.

□

3.8 Eguchi-Hanson metric and gauge theory

This section is based on [59]. As a summary of the results obtained in Section **??**, it was found that there is a correspondence between the commutative action and the noncommutative action such that

$$\int \mathrm{d}^4x\sqrt{\det\left\{\mathcal{G}+\kappa\left(\Phi+\hat{F}\right)\right\}}=\int \mathrm{d}^4y\sqrt{\det\left(1+F\theta\right)}\sqrt{\det\left\{\mathcal{G}+\kappa\left(\Phi+\boldsymbol{F}\right)\right\}}+\mathcal{O}\left(l_s\partial F\right),$$

and there is a correspondence between the gauge curvatures such that

$$\boldsymbol{F}_{\mu\nu}\left(x\right):=\left(\frac{1}{1+F\theta}F\right)_{\mu\nu}\left(x\right),$$

where $\kappa = 2\pi\alpha' = 2\pi l_s^2$ and with the ordinary $U(1)$ field strength defined by

$$F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x).$$

By comparing both sides of this equation, it can be seen that the relationship between commutative space and noncommutative space can be considered as follows.

$$\hat{F} = \left(\frac{1}{1+F\theta}F\right), \quad \mathrm{d}^4x = \mathrm{d}^4y\sqrt{\det(1+F\theta)},$$

where

$$y^\mu = x^\mu + \theta^{\mu\nu}\hat{A}_\nu.$$

This is consistent with (**??**). If $\tilde{G}$ is defined as $\tilde{G} := 1 + F\theta$ then

$$\int \mathrm{d}^4y\sqrt{\det(1+F\theta)}\sqrt{\det\{\mathcal{G}+\kappa(\Phi+\boldsymbol{F})\}} = \int \mathrm{d}^4y\sqrt{\det\tilde{G}}\sqrt{\det\{\mathcal{G}+\kappa(\Phi+\boldsymbol{F})\}}.$$

The right hand side of this equation looks like integration based on Riemann measures with $\tilde{G}$. If one identifies from the effective metric a gravitational metric defined by

$$\tilde{G}_{\mu\nu} = \frac{1}{2}(\delta_{\mu\nu} + \tilde{g}_{\mu\nu}).$$

Definition 3.40. Eguchi-Hanson metric is defined by

$$\mathrm{d}s^2 = \tilde{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \left\{\frac{\sqrt{r^4+t^4}\,(f(r)+1)\,\delta_{\mu\nu}}{2r^2} - \frac{\sqrt{r^4+t^4}\,(f(r)-1)\left(\eta^3\bar{\eta}^k\right)_{\mu\nu}T^k}{r^4}\right\}\mathrm{d}x^\mu\mathrm{d}x^\nu$$

where t is a free parameter and

$$r^2 = x_1^2+x_2^2+x_3^2+x_4^2,\ f(r) = 1-\frac{t^4}{r^4+t^4},\ \eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$T^1 = -x^1x^3 - x^2x^4, \quad T^2 = x^1x^4 - x^2x^3, \quad T^3 = \frac{1}{2}\left\{\left(x^1\right)^2+\left(x^2\right)^2-\left(x^3\right)^2-\left(x^4\right)^2\right\},$$

$$\bar{\eta}^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \bar{\eta}^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \bar{\eta}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Theorem 3.41. *Assume that $\tilde{g}_{\mu\nu}$ is Eguchi-Hanson metric. If*

$$\tilde{g}_{\mu\nu} = \delta_{\mu\nu} + 2\left(F\theta\right)_{\mu\nu}$$

then

$$F_{\mu\nu} = \frac{t^4 \bar{\eta}^k_{\mu\nu} T^k}{r^6 \sqrt{1+\frac{t^4}{r^4}}} - \frac{\left(\sqrt{1+\frac{t^4}{r^4}}-1\right)^2 \eta^3_{\mu\nu}}{2\sqrt{1+\frac{t^4}{r^4}}}, \hat{F}_{\mu\nu} = \frac{4}{r^2} \frac{\sqrt{1+\frac{t^4}{r^4}}-1}{\sqrt{1+\frac{t^4}{r^4}}+1} \bar{\eta}^k_{\mu\nu} T^k$$

and $\hat{F}$ is anti-selfdual.

This theorem is proved in [59].

Remark 3.42. If

$$\tilde{g}_{\mu\nu}\left(x\right) = \delta_{\mu\nu} + 2\left(F\theta\right)_{\mu\nu}$$

and

$$\hat{F} = \frac{1}{1+F\theta} F$$

then

$$\tilde{g}_{\mu\nu} = \delta_{\mu\nu} + 2\left\{\hat{F}\left(1-\theta\hat{F}\right)^{-1}\theta\right\}_{\mu\nu} = \left\{2\left(1-\hat{F}\theta\right)^{-1}\right\}_{\mu\nu} - \delta_{\mu\nu}. \tag{3.8.1}$$

Chapter 4

Transformation from Noncommutative gauge curvature to Hermitian Ricci flat metric

4.1 Introduction of this chapter

In this chapter, a linear map from differential two-forms to symmetric two-tensors in two-dimensional Hermitian manifolds introduced in [33] is studied. This mapping is a linear mapping from a differential two form to a symmetric two tensor on a two-dimensional Hermitian manifold, and illustrates another aspect of the Seiberg-Witten transformation. The original Seiberg-Witten transformation converts a gauge field on a noncommutative manifold into a gauge field on a commutative manifold with a background B field. On the other hand, it has been interpreted in [35, 33, 36] as a map from a noncommutative gauge field to a Kähler metric.

This chapter clarifies the mapping of [35, 33, 36]. This mapping converts the (anti) self-dual two-form on $\mathbb{C}^2$ into a Hermitian-Einstein metric of a two-dimensional complex manifold. It might be worth noting that it is enough for these two-forms to be defined as a symplectic structure on a commutative manifold, although this map was developed in the context of Seiberg-Witten map in noncommutative gauge theory. However, this correspondence between the self-dual 2-form and the Hermitian-Einstein metric can be lifted into noncommutative space after quantization (canonical or deformation) [37].

The second purpose of this chapter is to construct explicit examples of Hermite-Einstein metrics from $U(1)$ instantons on noncommutative space. $U(1)$ instantons on noncommutative $\mathbb{C}^2$ were found by Nekrasov and Schwarz [38]. We construct two forms from $U(1)$ multi-instants on noncommutative space given in the form of operators acting on Fock space in [39]. A Fock space is defined by a Heisenberg algebra generated by a polynomial over a noncommutative manifold. There is a correspondence between linear operators acting on Fock space and ordinary functions [40]. This transformation can be applied to any noncommutative Kähler manifold obtained by deformation quantization with separation of variables [41]. Concrete Hermitian-Einstein metrics are obtained by transforming noncommutative instantons composed of linear operators into ordinary functions using the transformation of [40].

Here we refer to some research related to the subject of this chapter. In [42, 43], noncommutative $U(1)$ gauge theory is a fundamental description of Kähler gravity at all scales, including the Planck scale, and is speculated to emerge quantum gravity. Recently, [44, 45, 46] showed that electromagnetism in noncommutative space-time can be realized as a theory of gravity, and that the symplectization of the space-time geometry is the origin of gravity. Such a picture is called emergency gravity and suggests a candidate for the space-time origin. See also the related article in the bibliography.[55, 52, 54, 47, 53, 48, 49, 50, 51] As a bottom-up approach to emergent gravity formulated in [56], the Eguchi Hanson metric [57, 58] is used to construct an anti-self dual symplectic $U(1)$ gauge field. The $U(1)$ gauge field [38] corresponding to the Nekrasov-Schwarz instanton is reproduced by the reverse process [59]. As a top-down approach to emergent gravity, $U(1)$ instantons discovered by Braden and Nekrasov [60] derive corresponding gravity metrics.

This chapter is organized as follows: Section **??** provides some linear algebraic formulas for self-duality. In Section 5, the correspondence between the self-dual two-form and the Hermitian-Einstein metric is studied. In Section 6, the Hermite Einstein metric is explicitly constructed from noncommutative $U(1)$ instantons.

4.2 Self-duality

Definition 4.1 (Hodge star operator)**.** An automorphism $\star$ on the set of 4×4 alternative matrices is defined as

$$\star \left[\begin{pmatrix} 0 & \omega_{12} & \omega_{13} & \omega_{14} \\ -\omega_{12} & 0 & \omega_{23} & \omega_{24} \\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0 \end{pmatrix} \right] := \begin{pmatrix} 0 & \omega_{34} & -\omega_{24} & \omega_{23} \\ -\omega_{34} & 0 & \omega_{14} & -\omega_{13} \\ \omega_{24} & -\omega_{14} & 0 & \omega_{12} \\ -\omega_{23} & \omega_{13} & -\omega_{12} & 0 \end{pmatrix},$$

$$\left(i.e.,\ \omega_{12} \leftrightarrow \omega_{34},\ \omega_{13} \leftrightarrow \omega_{42},\ \omega_{14} \leftrightarrow \omega_{23} \right).$$

In other words, $\star\omega_{kl}$ is defined as

$$\star\omega_{kl} = \frac{1}{2} \sum_{m,n}^{4} \varepsilon_{klmn} \omega_{mn},$$

where ε_{klmn} is Levi-Civita symbol. The operator $\star$ is called the Hodge star operation in Euclidean $\mathbb{R}^4$.

Definition 4.2 (Anti-self-dual matrix)**.** A 4×4 alternative matrix $\omega^{\pm}$ is an (anti-)self-dual matrix if

$$\star\omega^{\pm} = \pm\omega^{\pm}. \tag{4.2.1}$$

An (anti-)self-dual matrix $\theta^{\pm}$ is defined as

$$\theta^{\pm} := \begin{pmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp\theta \\ 0 & 0 & \pm\theta & 0 \end{pmatrix} \tag{4.2.2}$$

where θ is a real number. Note that $\omega^{\pm}$ and $\theta^{\mp}$ commute each other:

$$\omega^{\pm}\theta^{\mp} = \theta^{\mp}\omega^{\pm}. \tag{4.2.3}$$

Definition 4.3 (Matrix $g^{\pm}$)**.** Let E_4 be the 4×4 unit matrix and $\omega^{\pm}$ be a 4×4 (anti-)self-dual matrix. Assume that $\det\left[E_4 - \omega^{\pm}\theta^{\mp}\right] \neq 0$, then 4×4 matrix $g^{\pm}$ is defined as

$$g^{\pm} := 2\left(E_4 - \omega^{\pm}\theta^{\mp}\right)^{-1} - E_4.$$

like (**??**).

Remark 4.4. $g^{\pm}$ is a symmetric matrix because of (4.3) and it can be inverted to

$$\omega^{\pm} = \left(g^{\pm} - E_4\right)\left(g^{\pm} + E_4\right)^{-1}\left(\theta^{\mp}\right)^{-1}.$$

The Remark 4 allows us to regard $g^{\pm}$ as a metric tensor since it is symmetric and nondegenerate.

Lemma 4.5. *For any* 4×4 *(anti-)self-dual matrix* $\omega^{\pm}$*,*

$$\star\omega^{\pm} = \pm\omega^{\pm} \Longrightarrow \det\left[g^{\pm}\right] = 1. \tag{4.2.4}$$

This lemma is proved by a direct calculation.

Definition 4.6. The map $\iota_{skew} : \left\{\omega_{\mathbb{C}} \in M_2[\mathbb{C}] \mid \omega_{\mathbb{C}}^{\dagger} = -\omega_{\mathbb{C}}\right\} \longrightarrow M_4[\mathbb{R}]$ is defined as

$$\iota_{skew}\left[\begin{pmatrix} \omega_{\mathbb{C}1\bar{1}} & \omega_{\mathbb{C}1\bar{2}} \\ \omega_{\mathbb{C}2\bar{1}} & \omega_{\mathbb{C}2\bar{2}} \end{pmatrix}\right] = \begin{pmatrix} 0 & 2\mathrm{i}\omega_{\mathbb{C}1\bar{1}} & \omega_{\mathbb{C}1\bar{2}} - \omega_{\mathbb{C}2\bar{1}} & \mathrm{i}\left(\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}}\right) \\ -2\mathrm{i}\omega_{\mathbb{C}1\bar{1}} & 0 & -\mathrm{i}\left(\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}}\right) & \omega_{\mathbb{C}1\bar{2}} - \omega_{\mathbb{C}2\bar{1}} \\ -\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}} & \mathrm{i}\left(\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}}\right) & 0 & 2\mathrm{i}\omega_{\mathbb{C}2\bar{2}} \\ -\mathrm{i}\left(\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}}\right) & -\omega_{\mathbb{C}1\bar{2}} + \omega_{\mathbb{C}2\bar{1}} & -2\mathrm{i}\omega_{\mathbb{C}2\bar{2}} & 0 \end{pmatrix}.$$

Note that $\omega_{\mathbb{C}1\bar{1}}$ and $\omega_{\mathbb{C}2\bar{2}}$ are pure imaginary.

If the coordinate transformation on the coordinate neighborhood is $z_1 := x^2 + \mathrm{i}x^1, z_2 := x^4 + \mathrm{i}x^3$, then the ι_{skew} is the pull-back of a two-form. This means

$$\sum_{k,l=1}^{2} \omega_{\mathbb{C}k\bar{l}} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_l = \frac{1}{2}\sum_{k,l=1}^{4} \omega_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l = \frac{1}{2}\sum_{k,l=1}^{4} \left(\iota_{skew}\left[\omega_{\mathbb{C}}\right]\right)_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

The above ι_{skew} is defined as satisfying this relation.

Remark 4.7. ι_{skew} satisfies the following relation

$$\det\left[\iota_{skew}\left[\omega_{\mathbb{C}}\right]\right] = 16\left(\det\left[\omega_{\mathbb{C}}\right]\right)^2.$$

Using this result, the following lemma can be deduced.

Lemma 4.8. *Suppose that the anti-Hermitian matrix* $\omega_{\mathbb{C}}$ *satisfies* $\omega_{\mathbb{C}2\bar{2}} = -\omega_{\mathbb{C}1\bar{1}}$, *i. e.* $\mathrm{tr}\omega_{\mathbb{C}} = 0$. *Then the two-form* $\iota_{skew}[\omega_{\mathbb{C}}]$ *is anti-self-dual, i. e. ,*

$$\star\left\{\iota_{skew}\left[\begin{pmatrix} \omega_{\mathbb{C}1\bar{1}} & \omega_{\mathbb{C}1\bar{2}} \\ \omega_{\mathbb{C}2\bar{1}} & \omega_{\mathbb{C}2\bar{2}} \end{pmatrix}\right]\right\} = -\iota_{skew}\left[\begin{pmatrix} \omega_{\mathbb{C}1\bar{1}} & \omega_{\mathbb{C}1\bar{2}} \\ \omega_{\mathbb{C}2\bar{1}} & \omega_{\mathbb{C}2\bar{2}} \end{pmatrix}\right].$$

4.3 Hermitian-Einstein metrics and (anti-)self-dual two-forms

In this section, we discuss how to make a Hermitian-Einstein metric from an anti-self-dual two-form. Let us define a $u(1)$-valued two-form on $\mathbb{R}^4$ by

$$\sum_{k,l=1}^{4} \omega_{kl} \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

where ω is an alternative matrix $(\omega)_{kl} := \omega_{kl}$. If ω is an anti-self-dual matrix, then the two-form is called anti-self-dual two-form.

Let M be a Hermitian manifold and h be its metric. As a well-known fact, Ricci curvature $R_{\bar{j}k}$ for a Hermitian manifold (M, h, ∇) with the Levi-Civita connection ∇ takes a simple form

$$R_{\bar{j}k} = \partial_{\bar{j}}\partial_k \log\left(\det\left[h\right]\right). \tag{4.3.1}$$

See, for example, [61, 62]. Let λ be a cosmological constant. When h satisfies the Einstein's equation.

$$R_{\bar{k}l} = \lambda h_{\bar{k}l}$$

then M is called an Einstein manifold. In this chapter we will focus on a Ricci flat manifold (i. e. $R_{\bar{k}l} = 0$ or $\lambda = 0$). We consider M as a real manifold with local coordinates x^μ $(\mu = 1, 2, 3, 4)$.

Definition 4.9. The map $\iota_{sym} : \left\{ h \in M_2[\mathbb{C}] \mid h^\dagger = h \right\} \longrightarrow M_4[\mathbb{R}]$ is defined as

$$\iota_{sym}\left[\left(\begin{array}{cc} h_{1\bar{1}} & h_{1\bar{2}} \\ h_{2\bar{1}} & h_{2\bar{2}} \end{array}\right)\right] = \left(\begin{array}{cccc} h_{1\bar{1}} & 0 & \frac{1}{2}\left(h_{1\bar{2}} + h_{2\bar{1}}\right) & \frac{1}{2\mathrm{i}}\left(h_{2\bar{1}} - h_{1\bar{2}}\right) \\ 0 & h_{1\bar{1}} & -\frac{1}{2\mathrm{i}}\left(h_{2\bar{1}} - h_{1\bar{2}}\right) & \frac{1}{2}\left(h_{1\bar{2}} + h_{2\bar{1}}\right) \\ \frac{1}{2}\left(h_{1\bar{2}} + h_{2\bar{1}}\right) & -\frac{1}{2\mathrm{i}}\left(h_{2\bar{1}} - h_{1\bar{2}}\right) & h_{2\bar{2}} & 0 \\ \frac{1}{2\mathrm{i}}\left(h_{2\bar{1}} - h_{1\bar{2}}\right) & \frac{1}{2}\left(h_{1\bar{2}} + h_{2\bar{1}}\right) & 0 & h_{2\bar{2}} \end{array}\right).$$

where h is a matrix and $(h)_{k\bar{l}} := h_{k\bar{l}}$.

Remark 4.10. Assume that h is a Hermitian metric. If the coordinate transformation on a coordinate neighborhood is $z^1 := x^2 + \mathrm{i}x^1, z^2 := x^4 + \mathrm{i}x^3$, the ι_{sym} is then the pull-back of the Hermitian metric given by

$$\sum_{k,l=1}^{2} h_{k\bar{l}} \mathrm{d}z_k \mathrm{d}\bar{z}_l = \sum_{k,l=1}^{4} \left(\iota_{sym}[h]\right)_{kl} \mathrm{d}x^k \mathrm{d}x^l.$$

Hence ι_{sym} squares the determinant:

$$\det\left[\iota_{sym}(h)\right] = \left(\det[h]\right)^2.$$

A Hermitian metric made with ι_{sym}^{-1} will be used below.

Definition 4.11. If $\tilde{h} \in C^\infty(U, M_2[\mathbb{C}])$ and $\tilde{h}^\dagger = \tilde{h}$, then

$$\tilde{h} > 0 \; in \; U \iff \forall u \in U, \; \tilde{h}(u) > 0$$

where $\tilde{h}(u) > 0$ means that $\tilde{h}$ is positive definite as a Hermitian matrix.

Lemma 4.12. *If $h \in C^\infty(U, M_2[\mathbb{C}])$ is a Hermitian matrix with $\det[h] = 1$ and h is positive (negative) at $^\exists p \in U$, then h is positive (negative) in U.*

Proof. *This follows from*

$$\begin{aligned} &\left\{ h \in M_2[\mathbb{C}] \;\middle|\; h = h^\dagger, \; \det[h] = 1 \right\} \\ &= \left\{ \left(\begin{array}{cc} a & b \\ \bar{b} & d \end{array}\right) \in M_2[\mathbb{C}] \;\middle|\; a, d \in \mathbb{R}, \; a > 0, \; d > 0, \; ad \geq 1, \; |b| = \sqrt{ad - 1} \right\} \\ &\coprod \left\{ \left(\begin{array}{cc} a & b \\ \bar{b} & d \end{array}\right) \in M_2[\mathbb{C}] \;\middle|\; a, d \in \mathbb{R}, \; a < 0, \; d < 0, \; ad \geq 1, \; |b| = \sqrt{ad - 1} \right\} \end{aligned}$$

which means two spaces are disconnected. □

From the above discussions, the following theorem is obtained.

Theorem 4.13. *Let $\omega^{\pm}$ be an (anti-)self-dual two-form on an open neighborhood U, i. e. $\star\omega^{\pm} = \pm\omega^{\pm}$, and*

$$h^{\pm} := \iota_{sym}^{-1}\left[2\left(E_4 - \omega^{\pm}\theta^{\mp}\right)^{-1} - E_4\right]. \tag{4.3.2}$$

Then $h^{\pm}$ gives a Ricci-flat Hermitian metric on U. So $(U, h^{\pm})$ is a local realization of an Einstein manifold.

Proof. *Because of Lemma* **??**, *if $\star\omega^{\pm} = \pm\omega^{\pm}$, then*

$$\det\left[h^{\pm}\right] = 1. \tag{4.3.3}$$

Because of Lemma **??** *and Remark 4, $h^{\pm}$ is a metric tensor. From equations (5.1) and (5.3), $R_{\bar{j}k} = \partial_{\bar{j}}\partial_k \log\left(\det\left[h^{\pm}\right]\right) = 0$.* □

Local complex coordinates can be arranged in such a way that the Jacobians of the transition functions on overlapping charts are one on all the overlaps. In that case, $\det[h^{\pm}]$ is a globally defined function and the Ricci-flat condition reduces to the Monge-Ampére equation [63]

$$\det[h^{\pm}] = \kappa, \tag{4.3.4}$$

where the constant κ is related to the volume of a Kähler manifold that depends only on the Kähler class. Therefore Theorem 5.2 implies that the self-duality for the two-form $\omega^{\pm}$ is equivalent to the Ricci-flat by the metric $h^{\pm}$ [**?**, **?**].

4.4 Hermitian-Einstein metric from noncommutative instanton on $\mathbb{C}^2$

In the previous section we found the way to construct a Hermitian-Einstein metric from an (anti-)self-dual two-form. To construct the Hermitian-Einstein metric, we will employ the instanton curvature on noncommutative $\mathbb{C}^2$ as the (anti-)self-dual two-form. There are many ways to obtain noncommutative $\mathbb{C}^2$ (see [**?**, **?**] for a review and references therein). We use the Fock representation of noncommutative $\mathbb{C}^2$ given in [40], which is based on the Karabegov's deformation quantization [41]. There is a simple dictionary between the Fock representation and ordinary functions. Using the dictionary, the Hermitian-Einstein metric is expressed in terms of usual functions.

In Section **??** we constructed Fock representation of noncommutative $\mathbb{C}^2$. We put $\zeta_1 = \zeta_2 = \zeta > 0$. Then the commutators of coordinates are

$$\left[z^k, \bar{z}^l\right]_* = -\zeta\delta_{kl}, \qquad \left[z^k, z^l\right]_* = 0, \qquad \left[\bar{z}^k, \bar{z}^l\right]_* = 0,$$

where $[x, y]_* := x * y - y * x$. The creation and annihilation operators are given by $a_k := \frac{\bar{z}^k}{\sqrt{\zeta}}, a_k^\dagger := \frac{z^k}{\sqrt{\zeta}}$, then

$$\left[a_k, a_l^\dagger\right]_* = \delta_{kl}, \qquad \left[a_k^\dagger, a_l^\dagger\right]_* = 0, \qquad [a_k, a_l]_* = 0.$$

Remember that the algebra $\mathcal{F}$ on $\mathbb{C}$ is defined as follows. The Fock space $\mathcal{H}$ is a linear space spanned by the bases generated by acting $a_l^\dagger$'s on $|0,0\rangle$:

$$\frac{1}{\sqrt{m_1!m_2!}} \left(a_1^\dagger\right)_*^{m_1} * \left(a_2^\dagger\right)_*^{m_2} |0,0\rangle = |m_1, m_2\rangle .$$

The ground state $|0,0\rangle$ satisfies $a_l |0,0\rangle = 0, \ \forall \ l$. The dual vector space was defined as

$$\frac{1}{\sqrt{n_1!n_2!}} \langle 0,0| (a_1)_*^{n_1} * (a_2)_*^{n_2} = \langle n_1, n_2| ,$$

where $\langle 0,0|$ satisfies $\langle 0,0| a_l^\dagger = 0, \ \forall \ l$. Using them, $\mathcal{F} := span_{\mathbb{C}} (|m_1, m_2\rangle \langle n_1, n_2|)$. Recall that $(|m_1, m_2\rangle \langle n_1, n_2|) |k_1, k_2\rangle = \delta_{k_1 n_1}\delta_{k_2 n_2} |m_1, m_2\rangle$ and $\langle k_1, k_2| (|m_1, m_2\rangle \langle n_1, n_2|) = \delta_{k_1 m_1}\delta_{k_2 m_2} \langle n_1, n_2|$.

There is a dictionary between $\mathcal{F}$ and some subalgebra of $C^\infty(\mathbb{C}^2)$. The following is an important correspondence. Using the linear map $\iota : \mathcal{F} \longrightarrow C^\infty(\mathbb{C}^2)$,

$$\iota(|m_1, m_2\rangle \langle n_1, n_2|) = e_{(m_1,m_2,n_1,n_2)} := \frac{z_1^{m_1} z_2^{m_2} \mathrm{e}^{-\frac{z^1\bar{z}^1 + z^2\bar{z}^2}{\zeta}} \bar{z}_1^{n_1} \bar{z}_2^{n_2}}{\sqrt{m_1!m_2!n_1!n_2!}\left(\sqrt{\zeta}\right)^{m_1+m_2+n_1+n_2}},$$

especially $\iota(|0,0\rangle \langle 0,0|) = e_{(0,0,0,0)} = \mathrm{e}^{-\frac{z^1\bar{z}^1 + z^2\bar{z}^2}{\zeta}}$.

Let us consider $\iota(\mathcal{F})$ as

$$\iota(\mathcal{F}) := span_{\mathbb{C}} \left(e_{(m_1,m_2,n_1,n_2)}; m_1, m_2, n_1, n_2 = 0, 1, 2, \cdots\right).$$

As we saw in Section **??** $\{\iota(\mathcal{F}), *\}$ is an algebra where $*$ is in (**??**), and the algebras $(\mathcal{F}, \circ)$ and $\{\iota(\mathcal{F}), *\}$ are isomorphic. This isomorphism ι is a "Fock space - function space" dictionary.

4.5 Ricci-flat metrics from noncommutative k-instantons

In this section, we make Ricci-flat metrics on a local neighborhood from noncommutative instantons on $\mathbb{C}^2$. As we saw in Section 5, (anti)-self-dual two-forms satisfying (4.1) derive Ricci-flat metrics. Nekrasov and Schwarz found in [38] how to construct noncommutative instantons on $\mathbb{C}^2$ by using the ADHM method and the general solutions for the $U(1)$ gauge theory are given in [66]. We introduce the commutation relation of complex coordinates as (6.1). As (anti)-self-dual two-forms in Section 5, we employ noncommutative instantons given in [39].

The general instanton solutions (see [39]) satisfy the (anti)-self-dual relation. An instanton curvature tensor is described by

$$\hat{F}_{\mathbb{C}}^{-}[k] := \begin{pmatrix} \hat{F}_{z_1\bar{z}_1}^{-}[k] & \hat{F}_{z_1\bar{z}_2}^{-}[k] \\ \hat{F}_{z_2\bar{z}_1}^{-}[k] & -\hat{F}_{z_1\bar{z}_1}^{-}[k] \end{pmatrix},$$

and satisfies (4.1):

$$\star\left(\iota_{skew}\left(\hat{F}_{\mathbb{C}}^{-}[k]\right)\right) = -\iota_{skew}\left(\hat{F}_{\mathbb{C}}^{-}[k]\right). \tag{4.5.1}$$

See Lemma **??** in Section 4. This fact leads to the following result.

Proposition 4.14. *If $\hat{F}_{\mathbb{C}}^{-}$ is a k-instanton curvature tensor of $U(1)$ gauge theory on noncommutative $\mathbb{C}^2$, and*

$$\begin{aligned} h[k] :=& \iota_{sym}^{-1}\left\{2\left(E_4 - \iota_{skew}\left(\hat{F}_{\mathbb{C}}^{-}[k]\right)\theta^{+}\right)^{-1} - E_4\right\} \\ =& \frac{1}{4\left|\hat{F}_{\mathbb{C}}^{-}[k]\right|\theta^2 - 1}\begin{pmatrix} -4\mathrm{i}\hat{F}_{z_1\bar{z}_1}^{-}[k]\theta - 2 & -4\mathrm{i}\hat{F}_{z_1\bar{z}_2}^{-}[k]\theta \\ -4\mathrm{i}\hat{F}_{z_2\bar{z}_1}^{-}[k]\theta & 4\mathrm{i}\hat{F}_{z_1\bar{z}_1}^{-}[k]\theta - 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{4.5.2}$$

then $h[k]$ is an Einstein (Ricci-flat) metric.

A concrete example of k-instanton curvature tensors is given in [39] and the curvature is written by using linear operators on a Fock space. It is known from (6.4) and Proposition 6.2 how to translate the operators into functions. (See also Subsection **??** and [40].) Then the k-instanton curvature tensor is expressed by concrete elementary functions as follows:

$$\begin{aligned} \hat{F}_{z_1\bar{z}_1}^{-}[k] =& \frac{\mathrm{i}}{\zeta} - \frac{\mathrm{i}}{\zeta}\sum_{n_2=0}^{\infty}\frac{z_2^{n_2}\mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}\bar{z}_2^{n_2}}{n_2!\zeta^{n_2}}\left(d_1(0,n_2;k)\right)^2 \\ & - \frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}\frac{z_1^{n_1}z_2^{n_2}\mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{n_1!n_2!\zeta^{n_1+n_2}}\left\{\left(d_1(n_1,n_2;k)\right)^2 - \left(d_1(n_1-1,n_2;k)\right)^2\right\}, \end{aligned}$$

$$
\begin{aligned}
&\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right] = -\frac{\mathrm{i}}{\zeta}\frac{z_1^{k-1}z_2\mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}}{\sqrt{(k-1)!}\left(\sqrt{\zeta}\right)^k}d_1\left(k-1,1;k\right)d_2\left(0,0;k\right)\\
&-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{k-1}\frac{z_1^{n_1+k-1}z_2\mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}\bar{z}_1^{n_1}}{\sqrt{(n_1+k-1)!n_1!}\left(\sqrt{\zeta}\right)^{2n_1+k}}\left\{d_1\left(n_1+k-1,1;k\right)d_2\left(n_1,0;k\right)-d_1\left(n_1-1,0;k\right)d_2\left(n_1-1,0;k\right)\right\}\\
&-\frac{\mathrm{i}}{\zeta}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{z_1^{n_1-1}z_2^{n_2+1}\mathrm{e}^{-\frac{z^1\bar{z}^1+z^2\bar{z}^2}{\zeta}}\bar{z}_1^{n_1}\bar{z}_2^{n_2}}{\sqrt{(n_1-1)!\left(n_2+1\right)!n_1!n_2!}\left(\sqrt{\zeta}\right)^{2n_1+2n_2}}\\
&\times\left\{d_1\left(n_1-1,n_2+1;k\right)d_2\left(n_1,n_2;k\right)-d_1\left(n_1-1,n_2;k\right)d_2\left(n_1-1,n_2;k\right)\right\},
\end{aligned}
$$

$$
\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right] = -\hat{F}^{-}_{z_2\bar{z}_1}\left[k\right]^{\dagger},
$$

where $n_2 \neq 0$ and

$$
d_1\left(n_1,0;k\right)=\sqrt{n_1+k+1}\sqrt{\frac{\Lambda\left(n_1+k+1,0\right)}{\Lambda\left(n_1+k,0\right)}},
$$

$$
d_1\left(n_1,n_2;k\right)=\sqrt{n_1+1}\sqrt{\frac{\Lambda\left(n_1+1,n_2\right)}{\Lambda\left(n_1,n_2\right)}}, \tag{4.5.3}
$$

$$
d_2\left(n_1,0;k\right)=\sqrt{\frac{\Lambda\left(n_1+k,1\right)}{\Lambda\left(n_1+k,0\right)}},
$$

$$
d_2\left(n_1,n_2;k\right)=\sqrt{n_2+1}\sqrt{\frac{\Lambda\left(n_1,n_2+1\right)}{\Lambda\left(n_1,n_2\right)}}. \tag{4.5.4}
$$

Here

$$
\Lambda\left[k\right]\left(n_1,n_2\right)=\frac{w_k\left[k\right]\left(n_1,n_2\right)}{w_k\left[k\right]\left(n_1,n_2\right)-2kw_{k-1}\left[k\right]\left(n_1,n_2\right)},
$$

and

$$
w_n\left[k\right]\left(n_1,n_2\right)=\sum_{l=0}^{n}\left\{\frac{n!}{l!}\frac{\left(n_1-n_2+k+l\right)!}{\left(n_1-n_2-k\right)!}\frac{2^{(n-l)}}{(n-l)!}\frac{\left(n_2+(n-l)\right)!}{n_2!}\right\}.
$$

Note that some notations are slightly changed from [39] and imaginary unit factor causes here. See also Section **??**.

Using these instanton curvatures, Hermitian-Einstein metrics can be constructed by concrete elementary functions according to the Theorem 5.2.

4.6 Einstein metric from finite N

The full noncommutative $U(1)$ instanton solution is very complicated. For simplicity, let us consider the ζ-expansion.

In the previous section, $\hat{F}^-$ is represented by an infinite series

$$\hat{F}^- = \sum_{n=1}^{\infty} \left(\frac{1}{\zeta}\right)^{\frac{n}{2}} \hat{F}^-_{\left(\frac{n}{2}\right)}. \tag{4.6.1}$$

The anti-self-dual condition $\star\hat{F}^- = -\hat{F}^-$ implies

$$\star\hat{F}^-_{\left(\frac{n}{2}\right)} = -\hat{F}^-_{\left(\frac{n}{2}\right)} \tag{4.6.2}$$

for each $n/2$. Therefore it is possible to employ an arbitrary partial sum of (6.12) determined by a subset $S \subset \dfrac{1}{2}\mathbb{Z}_{>0}$

$$\hat{F}^-_S = \sum_{\frac{n}{2}\in S} \left(\frac{1}{\zeta}\right)^{\frac{n}{2}} \hat{F}^-_{\left(\frac{n}{2}\right)} \tag{4.6.3}$$

for the anti-self-dual two-form to construct a Hermitian-Einstein metric h without losing rigorousness. [1] In the following we consider

$$\hat{F}^-_{\left\{\frac{N}{2}\right\}} := \sum_{n=1/2}^{N/2} \left(\frac{1}{\zeta}\right)^{\frac{n}{2}} \hat{F}^-_{\left(\frac{n}{2}\right)}. \tag{4.6.4}$$

Example 4.15. First let us make the Ricci-flat metric $h\left[k\right]_{\{1\}}$ from $\hat{F}^-_{\mathbb{C}}\left[k\right]_{\{1\}}$. The curvature tensor in this case is $\hat{F}^-_{\mathbb{C}}\left[k\right]_{\{1\}} = \begin{pmatrix} \frac{\mathrm{i}}{\zeta} & 0 \\ 0 & -\frac{\mathrm{i}}{\zeta} \end{pmatrix}$, and its determinant is $\det\left[\hat{F}^-_{\mathbb{C}}\left[k\right]_{\{1\}}\right] = \frac{1}{\zeta^2}$.

So the metric $h\left[k\right]_{\{1\}}$ is given by

$$\begin{aligned} h\left[k\right]_{\{1\}} &:= \frac{1}{4\ \det\left[\hat{F}^-_{\mathbb{C}}\left[k\right]_{\{1\}}\right]\theta^2 - 1} \begin{pmatrix} -4\mathrm{i}\hat{F}^-_{z_1\bar{z}_1}\left[k\right]_{\{1\}}\theta - 2 & -4\mathrm{i}\hat{F}^-_{z_1\bar{z}_2}\left[k\right]_{\{1\}}\theta \\ -4\mathrm{i}\hat{F}^-_{z_2\bar{z}_1}\left[k\right]_{\{1\}}\theta & 4\mathrm{i}\hat{F}^-_{z_1\bar{z}_1}\left[k\right]_{\{1\}}\theta - 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{1 - 4\zeta^{-2}\theta^2} \begin{pmatrix} 1 - 4\zeta^{-1}\theta + 4\zeta^{-2}\theta^2 & 0 \\ 0 & 1 + 4\zeta^{-1}\theta + 4\zeta^{-2}\theta^2 \end{pmatrix} = \begin{pmatrix} \frac{1-2\zeta^{-1}\theta}{1+2\zeta^{-1}\theta} & 0 \\ 0 & \frac{1+2\zeta^{-1}\theta}{1-2\zeta^{-1}\theta} \end{pmatrix}. \end{aligned}$$

This corresponds to the Euclidean metric essentially.

---

[1] One may choose even more loose condition than (6.14). One can choose a different subset S for each $\hat{F}^-_{z_1\bar{z}_1}, \hat{F}^-_{z_1\bar{z}_2}$ to obtain a Hermitian-Einstein metric.

Example 4.16. Let us make a Ricci-flat metric $h\left[k\right]_{\{2\}}$ from $\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}}$. From (**??**),(**??**),

$$\begin{aligned}\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}} = &\frac{\mathrm{i}}{\zeta}\left[1-\frac{z_2\bar{z}_2}{\zeta}\left(d_1\left(0,1;k\right)\right)^2-\frac{z_1\bar{z}_1}{\zeta}\left\{\left(d_1\left(1,0;k\right)\right)^2-\left(d_1\left(0,0;k\right)\right)^2\right\}\right]\left(\begin{array}{cc}1&0\\0&-1\end{array}\right)\\ &-\frac{\mathrm{i}d_1\left(k-1,1;k\right)d_2\left(0,0;k\right)}{\zeta^{1+k/2}\sqrt{(k-1)!}}\left(\begin{array}{cc}0&z_1^{k-1}z_2\\ \bar{z}_1^{k-1}\bar{z}_2&0\end{array}\right).\end{aligned}$$

Then its determinant is

$$\begin{aligned}\det\left[\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}}\right] = &\frac{1}{\zeta^2}\left[1-\frac{z_2\bar{z}_2}{\zeta}\left(d_1\left(0,1;k\right)\right)^2-\frac{z_1\bar{z}_1}{\zeta}\left\{\left(d_1\left(1,0;k\right)\right)^2-\left(d_1\left(0,0;k\right)\right)^2\right\}\right]^2\\ &+\frac{\left\{d_1\left(k-1,1;k\right)\right\}^2\left\{d_2\left(0,0;k\right)\right\}^2 z_1^{k-1}z_2\bar{z}_1^{k-1}\bar{z}_2}{\zeta^{2+k}\left(k-1\right)!}.\end{aligned}$$

So the metric $h\left[k\right]_{\{2\}}$ is given by

$$h\left[k\right]_{\{2\}} := \frac{1}{4\det\left[\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}}\right]\theta^2-1}\left(\begin{array}{cc}-4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_1}\left[k\right]_{\{2\}}\theta-2 & -4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_2}\left[k\right]_{\{2\}}\theta\\ -4\mathrm{i}\hat{F}^{-}_{z_2\bar{z}_1}\left[k\right]_{\{2\}}\theta & 4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_1}\left[k\right]_{\{2\}}\theta-2\end{array}\right)-\left(\begin{array}{cc}1&0\\0&1\end{array}\right),$$

which can be calculated concretely though its expression becomes complex. To simplify this we assume $k>3$, then

$$\begin{aligned}h\left[k\right]_{\{2\}} = &\left\{\frac{2}{1-4\det\left[\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}}\right]\theta^2}-1\right\}\left(\begin{array}{cc}1&0\\0&1\end{array}\right)+\frac{4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_1}\left[k\right]_{\{2\}}\theta}{1-4\det\left[\hat{F}^{-}_{\mathbb{C}}\left[k\right]_{\{2\}}\right]\theta^2}\left(\begin{array}{cc}1&0\\0&-1\end{array}\right)\\ = &\left\{\frac{2}{1-4\theta^2\zeta^{-2}\left[1-\frac{z_2\bar{z}_2}{\zeta}\left(d_1\left(0,1;k\right)\right)^2-\frac{z_1\bar{z}_1}{\zeta}\left\{\left(d_1\left(1,0;k\right)\right)^2-\left(d_1\left(0,0;k\right)\right)^2\right\}\right]^2}-1\right\}\left(\begin{array}{cc}1&0\\0&1\end{array}\right)\\ &-\frac{\frac{4\theta}{\zeta}\left[1-\frac{z_2\bar{z}_2}{\zeta}\left(d_1\left(0,1;k\right)\right)^2-\frac{z_1\bar{z}_1}{\zeta}\left\{\left(d_1\left(1,0;k\right)\right)^2-\left(d_1\left(0,0;k\right)\right)^2\right\}\right]}{1-4\theta^2\zeta^{-2}\left[1-\frac{z_2\bar{z}_2}{\zeta}\left(d_1\left(0,1;k\right)\right)^2-\frac{z_1\bar{z}_1}{\zeta}\left\{\left(d_1\left(1,0;k\right)\right)^2-\left(d_1\left(0,0;k\right)\right)^2\right\}\right]^2}\left(\begin{array}{cc}1&0\\0&-1\end{array}\right).\end{aligned}$$

In next section, we discuss a Hermitian-Einstein metric obtained from 1-instanton solution.

4.7 Hermitian-Einstein metric from a 1-instanton

For the simplest example of the Hermitian-Einstein metric given in the previous discussion, we describe a Hermitian-Einstein metric obtained from a single noncommutative $U(1)$ instanton. Now we pay attention to low order terms.

For $k=1, \hat{F}^{-}_{\mathbb{C}}\left[1\right]$ is

$$\begin{aligned}
\hat{F}^{-}_{z_1\bar{z}_1}\left[1\right] &= \frac{\mathrm{i}}{\zeta} - \frac{\mathrm{i}z_2\bar{z}_2}{\zeta^2}\left(d_1\left(0,1;1\right)\right)^2 - \frac{\mathrm{i}z_1\bar{z}_1}{\zeta^2}\left\{\left(d_1\left(1,0;1\right)\right)^2 - \left(d_1\left(0,0;1\right)\right)^2\right\} + \mathcal{O}\left(\zeta^{-3}\right)\\
&= \frac{\mathrm{i}}{\zeta} - \frac{2\mathrm{i}}{3}\frac{z_2\bar{z}_2}{\zeta^2} - \frac{\mathrm{i}z_1\bar{z}_1}{\zeta^2}\left\{\frac{5}{2} - \frac{4}{3}\right\} + \mathcal{O}\left(\zeta^{-3}\right) = \frac{\mathrm{i}}{\zeta} - \frac{\mathrm{i}}{6\zeta^2}\left(4z_2\bar{z}_2 + 7z_1\bar{z}_1\right) + \mathcal{O}\left(\zeta^{-3}\right)\\
\hat{F}^{-}_{z_1\bar{z}_2}\left[1\right] &= -\frac{\mathrm{i}z_2}{\zeta^{3/2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right)d_1\left(0,1;1\right)d_2\left(0,0;1\right) + \mathcal{O}\left(\zeta^{-3}\right)\\
&= -\frac{2\mathrm{i}z_2}{3\zeta^{3/2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right) + \mathcal{O}\left(\zeta^{-3}\right)\\
\hat{F}^{-}_{z_2\bar{z}_1}\left[1\right] &= -\frac{\mathrm{i}\bar{z}_2}{\zeta^{3/2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right)d_1\left(0,1;1\right)d_2\left(0,0;1\right) + \mathcal{O}\left(\zeta^{-3}\right)\\
&= -\frac{2\mathrm{i}\bar{z}_2}{3\zeta^{3/2}}\left(1 - \frac{z_1\bar{z}_1}{\zeta} - \frac{z_2\bar{z}_2}{\zeta}\right) + \mathcal{O}\left(\zeta^{-3}\right)
\end{aligned}$$

from (**??**),(**??**). Then

$$det\left[\hat{F}^{-}_{\mathbb{C}}\left[1\right]_{\{2\}}\right] = \frac{4z_2\bar{z}_2}{9\zeta^5}\left(\zeta - z_1\bar{z}_1 - z_2\bar{z}_2\right)^2 - \frac{1}{36\zeta^4}\left(6\zeta - 7z_1\bar{z}_1 - 4z_2\bar{z}_2\right)^2 \tag{4.7.1}$$

From this 1-instaoton curvature, the Hermitian-Einstein metric is given as

$$\begin{aligned}
&h\left[1\right]_{\{2\}} := \frac{1}{4\ \det\left[\hat{F}^{-}_{\mathbb{C}}\left[1\right]\right]_{\{2\}}\theta^2 - 1}\begin{pmatrix} -4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_1}\left[1\right]_{\{2\}}\theta - 2 & -4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_2}\left[1\right]_{\{2\}}\theta \\ -4\mathrm{i}\hat{F}^{-}_{z_2\bar{z}_1}\left[1\right]_{\{2\}}\theta & 4\mathrm{i}\hat{F}^{-}_{z_1\bar{z}_1}\left[1\right]_{\{2\}}\theta - 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\
&= \frac{4}{1 - 4\left\{\frac{4z_2\bar{z}_2}{9\zeta^5}\left(\zeta - z_1\bar{z}_1 - z_2\bar{z}_2\right)^2 - \frac{1}{36\zeta^4}\left(6\zeta - 7z_1\bar{z}_1 - 4z_2\bar{z}_2\right)^2\right\}\theta^2}\\
&\times\left\{\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{\zeta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\theta}{3\zeta^{3/2}}\begin{pmatrix} 0 & z_2 \\ \bar{z}_2 & 0 \end{pmatrix} + \frac{\theta}{6\zeta^2}\begin{pmatrix} -4z_2\bar{z}_2 - 7z_1\bar{z}_1 & 0 \\ 0 & 4z_2\bar{z}_2 + 7z_1\bar{z}_1 \end{pmatrix}\right.\\
&\left.+\frac{2\theta}{3\zeta^{5/2}}\begin{pmatrix} 0 & -z_2\left(z_1\bar{z}_1 + z_2\bar{z}_2\right) \\ -\bar{z}_2\left(z_1\bar{z}_1 + z_2\bar{z}_2\right) & 0 \end{pmatrix}\right\} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

4.8 Instantons from Ricci-flat metrics

Next, we will think of a converse of the last Theorem. That means "Is the $\hat{F}$ instanton if the metric is Ricci flat?".

First $g\left(\hat{F}(x)\right)$ needs to be a metric. Since the metric matrix is a symmetric matrix, therefore $g\left(\hat{F}(x)\right)$ should be a symmetric matrix, in which case a condition that is “fairly close” to an anti-self-dual condition is derived.

Then we consider “asymptotic to zero”. In conclusion, imposing a Ricci flat condition on the metric when $\hat{F}(x)$ satisfies this condition leads to $\hat{F}(x)$ being anti-self-dual.

An (anti-)self-dual matrix θ is defined as

$$\theta := \begin{pmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta \\ 0 & 0 & \theta & 0 \end{pmatrix}, \tag{4.8.1}$$

where θ is a real number.

Definition 4.17. Let E_4 be the 4×4 unit matrix and $\hat{F}$ be a 4×4 alternating matrix. Assume that $Det\left[E_4 - \hat{F}\theta\right] \neq 0$, then 4×4 matrix g is defined as

$$g\left(\hat{F}(x)\right) := 2\left(E_4 - \hat{F}\theta\right)^{-1} - E_4. \tag{4.8.2}$$

g should be a symmetric matrix because we assume $g\left(\hat{F}(x)\right)$ is a metric.

Lemma 4.18. *Assume that $g\left(\hat{F}(x)\right)$ is a symmetric matrix. If $g\left(\hat{F}(x)\right) = 2\left(E_4 - \hat{F}\theta\right)^{-1} - E_4$ then,*

$$\hat{F} = \begin{pmatrix} 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} \\ -\hat{F}_{12} & 0 & \hat{F}_{14} & -\hat{F}_{13} \\ -\hat{F}_{13} & -\hat{F}_{14} & 0 & \hat{F}_{34} \\ -\hat{F}_{14} & \hat{F}_{13} & -\hat{F}_{34} & 0 \end{pmatrix}.$$

This lemma is proved by a direct calculation. This means that if $\hat{F}_{12} + \hat{F}_{34} = 0$ then $\hat{F}$ is an anti-self-dual matrix. Now that the sufficient condition is “ Ricci flat ”, it is necessary to know how the Ricci curvature is calculated by the gauge field. Before that, $\hat{F}_{\mathbb{C}}$ is defined same as Definition 39 for convenience.

$$\hat{F}_{\mathbb{C}}(x) := \begin{pmatrix} \hat{F}_{1\bar{1}} & \hat{F}_{1\bar{2}} \\ \hat{F}_{2\bar{1}} & \hat{F}_{2\bar{2}} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \mathrm{i}\hat{F}_{12} & -\hat{F}_{13} + \mathrm{i}\hat{F}_{14} \\ \hat{F}_{13} + \mathrm{i}\hat{F}_{14} & \mathrm{i}\hat{F}_{34} \end{pmatrix}$$

Then determinant of the metric matrix $g\left(\hat{F}(x)\right)$ is calculated by the gauge field as below.

Proposition 4.19. *For above g, $\hat{F}$, $\hat{F}_{\mathbb{C}}$,*

$$Det\left[g\left(\hat{F}\left(x\right)\right)\right] = 1 + 8\mathrm{i}\theta Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right] - 32\theta^2\left(Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\right)^2 + \mathcal{O}\left(\eta^3\right).$$

The proof is given directly.

As is well known, the Ricci curvature is calculated from the determinant of the metric matrix.

Proposition 4.20. *Suppose that the Hermitian matrix g(Definition 40) satisfies (6.18). Then its Ricci curvature(5.1) is*

$$R_{\bar{j}k}\left(x\right) = \theta\partial_{\bar{j}}\partial_k\left(\hat{F}_{12} + \hat{F}_{34}\right) + \mathcal{O}\left(\theta^2\right).$$

Proof.

$$\begin{aligned} &R_{\bar{j}k}\left(x\right) = \partial_{\bar{j}}\partial_k\left[\log\left\{Det\left[g\left(\hat{F}\left(x\right)\right)\right]\right\}\right] \\ &= 2\theta\mathrm{i}\partial_{\bar{j}}\partial_k Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right] - 8\theta^2\partial_{\bar{j}}\partial_k\left(Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\right)^2 - 16\theta^2 Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\partial_{\bar{j}}\partial_k Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right] \\ &+ 16\theta^2\left\{\left(\partial_{\bar{j}} Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\right)\left(\partial_k Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\right)\right\} - 32\mathrm{i}\theta^2 Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right]\partial_{\bar{j}}\partial_k Tr\left[\hat{F}_{\mathbb{C}}\left(x\right)\right] + \mathcal{O}\left(\theta^3\right) \end{aligned}$$

□

In physics, "the field is almost zero at a far enough distance " is a natural setting.

Definition 4.21 (asymptotic to zero)**.** " $f\left(z_1, z_2\right)$ is asymptotically zero " is defined as

$$\lim_{\left|z_1\right|^2+\left|z_2\right|^2\to\infty} f\left(z_1, z_2\right) = 0.$$

The following "Maximum principle" is a well-known fact in Harmonic analysis.

Lemma 4.22 (Maximum principle)**.** *Harmonic functions satisfy the following maximum principle: if K is a nonempty compact subset of U, then f restricted to K attains its maximum and minimum on the boundary of K. If U is connected, this means that f cannot have local maxima or minima, other than the exceptional case where f is constant.*

This lemma leads to the following corollary.

Corollary 4.23. *Assume that $f \in C^\infty\left(\mathbb{C}^2, \mathbb{R}\right)$. If $f\left(z_1, z_2\right)$ is asymptotically zero and*

$$\partial_{\bar{j}}\partial_k f\left(z_1, z_2\right) = 0$$

then

$$f\left(z_1, z_2\right) = 0.$$

Proof.

$$\Delta f\left(z_1, z_2\right) = \partial_{\bar{1}}\partial_1 f\left(z_1, z_2\right) + \partial_{\bar{2}}\partial_2 f\left(z_1, z_2\right) = 0.$$

□

Remark 4.24. Because g is a symmetric tenser

$$\hat{F} = \begin{pmatrix} 0 & \hat{F}_{12} & \hat{F}_{13} & \hat{F}_{14} \\ -\hat{F}_{12} & 0 & \hat{F}_{14} & -\hat{F}_{13} \\ -\hat{F}_{13} & -\hat{F}_{14} & 0 & \hat{F}_{34} \\ -\hat{F}_{14} & \hat{F}_{13} & -\hat{F}_{34} & 0 \end{pmatrix}$$

and $\hat{F}_{ij}$ is asymptotically zero and $\partial_{\bar{j}}\partial_k\left(\hat{F}_{12} + \hat{F}_{34}\right) = 0$ means

$$\hat{F}_{12} = -\hat{F}_{34}.$$

Theorem 4.25. *If $\hat{F}_{ij}$ is asymptotically zero and $R_{\bar{j}k}\left(x\right) \equiv 0 \pmod{\eta^2}$ then $\hat{F}$ is an anti-self-dual matrix.*

Chapter 5

Deformation quantization for a Kähler manifold

Until the previous chapter, we focused on noncommutative $\mathbb{R}^4$ as a flat Kähler manifold. From this chapter, we consider noncommutative Kähler manifolds in general.

5.1 Karabegov's deformation quantization

In this section, we review the deformation quantization with separation of variables to construct noncommutative Kähler manifolds.

An N-dimensional Kähler manifold M is described by using a Kähler potential. Let Φ be a Kähler potential and ω be a Kähler 2-form:

$$\omega := i g_{k\bar{l}} dz^k \wedge d\bar{z}^l, \quad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \tag{5.1.1}$$

where $z^i, \bar{z}^i$ $(i = 1, 2, \ldots, N)$ are complex local coordinates.
The $g^{\bar{k}l}$ is the inverse of the Kähler metric tensor $g_{k\bar{l}}$. That means $g^{\bar{k}l} g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}$. In the following, we use

$$\partial_k = \frac{\partial}{\partial z^k}, \ \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \tag{5.1.2}$$

Deformation quantization is defined as follows.

Definition 5.1 (Deformation quantization)**.** Deformation quantization of Poisson manifolds is defined as follows. $\mathfrak{F}$ is defined as a set of formal power series: $\mathfrak{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k, \ f_k \in C^\infty(M) \right\}$. A star product is defined as

$$f * g = \sum_k C_k(f, g) \hbar^k \tag{5.1.3}$$

such that the product satisfies the following conditions.

1. $(\mathcal{F}, +, *)$ is a (noncommutative) algebra.
2. $C_k(\cdot, \cdot)$ is a bidifferential operator.
3. C_0 and C_1 are defined as

   $$C_0(f, g) = fg, \tag{5.1.4}$$
   $$C_1(f, g) - C_1(g, f) = \{f, g\}, \tag{5.1.5}$$

   where $\{f, g\}$ is the Poisson bracket.
4. $f * 1 = 1 * f = f$.

Karabegov introduced a method to obtain a deformation quantization of a Kähler manifold in [41]. His deformation quantization is called deformation quantizations with separation of variables.

Definition 5.2 (A star product with separation of variables)**.** $*$ is called a star product with separation of variables on a Kähler manifold when

$$a * f = af \tag{5.1.6}$$

for an arbitrary holomorphic function a and

$$f * b = fb \tag{5.1.7}$$

for an arbitrary anti-holomorphic function b.

We use

$$D^{\bar{l}} = g^{\bar{l}k}\partial_k = \mathrm{i}\{\bar{z}^l, \cdot\}$$

and introduce

$$\mathcal{S} := \Big\{ A \mid A = \sum_{\alpha} a_\alpha D^\alpha, \quad a_\alpha \in C^\infty(M) \Big\},$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We also use the Einstein summation convention over repeated multi-indices and $a_\alpha D^\alpha := \sum_\alpha a_\alpha D^\alpha$.

Example 5.3. $\mathbb{C}^2$

Remark 5.4. If L_f is defined as $f * g = L_f g$ then

$$L_f 1 = f, \quad \left[L_f, R_{\partial_{\bar{l}}\Phi}\right] = 0$$

where Φ is a Kähler potential.

Lemma 5.5. *There are some useful formulae.* $D^{\bar{l}}$ *satisfies the following equations.*

$$[D^{\bar{l}}, D^{\bar{m}}] = 0 \quad , \quad [D^{\bar{l}}, \partial_{\bar{m}}\Phi] = \delta^{\bar{l}}{}_{\bar{m}}, \quad \forall l, m, \tag{5.1.8}$$

where $[A,\ B] = AB - BA$.

Proof. *The following follows from the Jacobi identity of Poisson brackets.*

$$\begin{aligned} &\left[D^{\bar{l}}, D^{\bar{m}}\right] f \\ &= D^{\bar{l}} D^{\bar{m}} f - D^{\bar{m}} D^{\bar{l}} f \\ &= \mathrm{i}^2 \left(\left\{ \bar{z}^l, \left\{ \bar{z}^m, f \right\} \right\} - \left\{ \bar{z}^m, \left\{ \bar{z}^l, f \right\} \right\} \right) \\ &= - \left(\left\{ \bar{z}^l, \left\{ \bar{z}^m, f \right\} \right\} + \left\{ \bar{z}^m, \left\{ f, \bar{z}^l \right\} \right\} \right) \\ &= \left\{ f, \left\{ \bar{z}^l, \bar{z}^m \right\} \right\} \\ &= 0. \end{aligned}$$

The following is proved by direct calculation.

$$\begin{aligned} &[D^{\bar{l}}, \partial_{\bar{m}}\Phi] f \\ &= D^{\bar{l}} \left(\partial_{\bar{m}} \Phi \right) f - D^{\bar{l}} \left(\partial_{\bar{m}} \Phi \right) f \\ &= \left(g^{\bar{l}k} \partial_k \partial_{\bar{m}} \Phi \right) f \\ &= \left(g^{\bar{l}k} g_{k\bar{m}} \right) f \\ &= \delta^{\bar{l}}{}_{\bar{m}} f. \end{aligned}$$

□

Definition 5.6. A map from differential operators to formal polynomials is defined as

$$\sigma\left(A; \xi\right) := \sum_{\alpha} a_\alpha \xi^\alpha,$$

where

$$A = \sum_{\alpha} a_\alpha D^\alpha.$$

This map is called "twisted symbol". It becomes easier to calculate commutators by using the following theorem.

Proposition 5.7 (Karabegov [41])**.** *Let* $a(\xi)$ *be a twisted symbol of an operator* A*. Then the twisted symbol of the operator* $[A, \partial_{\bar{i}}\Phi]$ *is equal to* $\partial a/\partial \xi^{\bar{i}}$*;*

$$\sigma\left([A, \partial_{\bar{i}}\Phi]\right) = \frac{\partial}{\partial \xi^{\bar{i}}} \sigma\left(A\right).$$

Proof. *This proposition follows from* (9.8)*, i.e.*

$$\sigma\left([D^{\bar{l}}, \partial_{\bar{i}}\Phi]\right) = \delta^{\bar{l}}_{\bar{i}}.$$

$$\square$$

Using above lemmas, one can construct a star product as a differential operator L_f such that $f * g = L_f g$.

Lemma 5.8 (Karabegov [41])**.** *Assume that* $B_{\bar{l}} \in \mathcal{S}$*. If*

$$[B_{\bar{l}}, \partial_{\bar{m}}\Phi] = [B_{\bar{m}}, \partial_{\bar{l}}\Phi]$$

then the equation

$$[A, \partial_{\bar{m}}\Phi] = B_{\bar{l}}$$

is solvable.

Proof. *Define* $a := \sigma\left(A\right), b_{\bar{l}} := \sigma\left(B_{\bar{l}}\right)$ *then the equation is equivalent to*

$$\frac{\partial a}{\partial \xi^{\bar{l}}} = b_{\bar{l}}.$$

If

$$\frac{\partial b_{\bar{l}}}{\partial \xi^{\bar{m}}} = \frac{\partial b_{\bar{m}}}{\partial \xi^{\bar{l}}}$$

the equation is solvable. This condition is equivalent to

$$[A, \partial_{\bar{m}}\Phi] = B_{\bar{l}}.$$

$$\square$$

Proposition 5.9 (Karabegov [41])**.** *If* $B, \tilde{B} \in \mathcal{S}$

$$\left[[\partial_{\bar{l}}, B], \partial_{\bar{m}}\Phi\right] = \left[[\partial_{\bar{m}}, B], \partial_{\bar{l}}\Phi\right]$$

where $[B, \partial_{\bar{m}}\Phi] = \left[\partial_{\bar{m}}, \tilde{B}\right]$.

Proof. *From the Jacobi identity*

$$\begin{aligned}&[[\partial_{\bar{l}}, B], \partial_{\bar{m}}\Phi] \\ &= -[[B, \partial_{\bar{m}}\Phi], \partial_{\bar{l}}] - [[\partial_{\bar{m}}\Phi, \partial_{\bar{l}}], B] \\ &= \left[\partial_{\bar{l}}, \left[\partial_{\bar{m}}, \tilde{B}\right]\right] + [(\partial_{\bar{l}}\partial_{\bar{m}}\Phi), B] \\ &= \partial_{\bar{l}}\partial_{\bar{m}}\tilde{B} + [(\partial_{\bar{l}}\partial_{\bar{m}}\Phi), B] \\ &= \partial_{\bar{m}}\partial_{\bar{l}}\tilde{B} + [(\partial_{\bar{m}}\partial_{\bar{l}}\Phi), B] \\ &= [[\partial_{\bar{m}}, B], \partial_{\bar{l}}\Phi].\end{aligned}$$

□

Proposition 5.10 (Karabegov [41]). *An operator $\tilde{A}_f = \sum_{i=0}^{\infty} \hbar^i A_i \in \mathcal{S}[[\hbar]]$ exists and satisfies the following.*

$$\left[\tilde{A}_f, \bar{z}^l\right] = 0, \quad \left[\tilde{A}_f, R_{\partial_{\bar{l}}\Phi}\right] = 0, \quad \tilde{A}_f 1 = f, \quad A_0 = f$$

Proof. *The condition $\left[\tilde{A}_f, R_{\partial_{\bar{l}}\Phi}\right] = 0$ leads to the following relation.*

$$\begin{aligned}&[A_0, \partial_{\bar{l}}\Phi] = 0 \\ &[A_1, \partial_{\bar{l}}\Phi] = [\partial_{\bar{l}}, A_0] \\ &\vdots \\ &[A_n, \partial_{\bar{l}}\Phi] = [\partial_{\bar{l}}, A_{n-1}]\end{aligned}$$

Then $A_0 = f$ is consistent and $[A_m, \partial_{\bar{l}}\Phi] = [\partial_{\bar{l}}, A_{m-1}]$ is solvable. □

Theorem 5.11. *[Karabegov [41]]. For an arbitrary Kähler form ω, there exist a star product with separation of variables $*$ and it is constructed as follows. Let f be an element of $\mathcal{F}$ and $A_n \in \mathcal{S}$ be a differential operator whose coefficients depend on f i.e.*

$$A_n = a_{n,\alpha}(f) D^{\alpha}, \quad D^{\alpha} = \prod_{i=1}^{n} (D^{\bar{i}})^{\alpha_i}, \quad (D^{\bar{i}}) = g^{\bar{i}l}\partial_l, \tag{5.1.9}$$

where α is an multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then,

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n \tag{5.1.10}$$

is uniquely determined such that it satisfies the following conditions.

1. *For* $R_{\partial_{\bar{l}}\Phi} = \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}$,

   (5.1.11) $$\left[L_f, R_{\partial_{\bar{l}}\Phi}\right] = 0 \ .$$

2. (5.1.12) $$L_f 1 \quad = \quad f * 1 = f.$$

Then the star products are given by

(5.1.13) $$L_f g \quad := \quad f * g,$$

and the star products satisfy the associativity;

(5.1.14) $$L_h(L_g f) \quad = \quad h * (g * f) = (h * g) * f = L_{L_h g} f.$$

Recall that each two of $D^{\bar{i}}$ commute each other, so if a multi index α is fixed then the A_n is uniquely determined. (9.12)-(9.14) imply that $L_f g = f * g$ gives deformation quantization.

We want to construct L_f. Before that, let us investigate $L_{\bar{z}^l}$. In fact, L_f can be composed of $L_{\bar{z}^l}$.

Lemma 5.12 (Karabegov [41])**.** *Assume*

$$L_{\bar{z}^l} = \sum_{n=0}^{\infty} \hbar^n A_n.$$

Then

$$A_0 = \bar{z}^l, \quad A_1 = D^{\bar{l}}.$$

Proof. *Rewriting the associative property is as follows.*

$$(f * g) * h = f * (g * h) \Longleftrightarrow R_h\left(L_f g\right) = L_f\left(R_h g\right)$$

In other words, the above equation can also be expressed using a commutator as

$$(f * g) * h = f * (g * h) \Longleftrightarrow \left[L_f, R_h\right] g = 0.$$

In particular, when $f = \bar{z}^l$ *and* $h = \partial_{\bar{m}}\Phi$*, the following holds.*

$$\left[L_{\bar{z}^l}, R_{\partial_{\bar{m}}\Phi}\right] = 0$$

Since $R_{\partial_{\bar{m}}\Phi} = \partial_{\bar{m}}\Phi + \hbar\partial_{\bar{m}}$ *is known, the following holds for each* $\hbar$*.*

(5.1.15) $$\left[A_0, \partial_{\bar{m}}\Phi\right] = 0, \quad \left[A_1, \partial_{\bar{m}}\Phi\right] = \left[\partial_{\bar{m}}, A_0\right], \cdots, \left[A_n, \partial_{\bar{m}}\Phi\right] = \left[\partial_{\bar{m}}, A_{n-1}\right].$$

However, $A_0 = \bar{z}^l \times$ *by definition. If the polynomial* a_1 *is defined by twisted symbol as* $a_1(\xi) := \sigma(A_1, \xi)$*, the above* $[A_1, \partial_{\bar{m}}\Phi] = [\partial_{\bar{m}}, A_0]$ *is equal to*

$$\frac{\partial a_1(\xi)}{\partial \xi^{\bar{m}}} = \delta^{\bar{l}}_{\bar{m}},$$

because $\left[\partial_{\bar{m}}, \bar{z}^l\right] = \delta^{\bar{l}}_{\bar{m}}$*. Considering the solution of the above equation,* $A_1 = D^{\bar{l}}$*.* □

As mentioned above, we show that L_f can be constructed from $L_{\bar{z}^l}$.

Theorem 5.13 (Karabegov [41])**.**

$$L_f = \sum_\alpha \frac{1}{\alpha!}\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f\,(L_{\bar{z}} - \bar{z})^\alpha$$

Proof. *First,* A^f_α *and* B_α *are defined as*

$$A^f_\alpha := \left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f, \quad B_\alpha := \frac{1}{\alpha!}(L_{\bar{z}} - \bar{z})^\alpha$$

and this theorem is proved if

$$\left(\sum_\alpha A^f_\alpha B_\alpha\right)1 = f, \quad \left[\left(\sum_\alpha A^f_\alpha B_\alpha\right), R_{\partial_{\bar{l}}\Phi}\right] = 0$$

are proved. $\left(\sum_\alpha A^f_\alpha B_\alpha\right)1 = f$ *is easy because* $\alpha \neq 0 \Longrightarrow (L_{\bar{z}} - \bar{z})^\alpha 1 = 0$*. Next*

$$\left[A^f_\alpha, R_{\partial_{\bar{l}}\Phi}\right] = \left[\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha f, \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}\right] = -\hbar\left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha+(\bar{l})} f,$$

where $\alpha + \left(\bar{l}\right) := (\alpha_1, \cdots, \alpha_{l-1}, \alpha_l + 1, \alpha_{l+1}, \cdots, \alpha_n)$*. Finally* $\left[B_\alpha, R_{\partial_{\bar{l}}\Phi}\right]$ *is calculated from the equation*

$$[(L_{\bar{z}^m} - \bar{z}^m), \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}] = [-\bar{z}^m, \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}] = [-\bar{z}^m, \hbar\partial_{\bar{l}}] = \hbar\delta^{\bar{m}}_{\bar{l}}.$$

Then if $\alpha_l \neq 0$

$$\begin{aligned}
&\left[B_\alpha, R_{\partial_{\bar{l}}\Phi}\right] \\
&= \left[\frac{1}{\alpha!}\left(L_{\bar{z}} - \bar{z}\right)^\alpha, \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}\right] \\
&= \left[\frac{1}{\alpha!}\prod_{m=1}^{n}\left(L_{\bar{z}^m} - \bar{z}^m\right)_m^\alpha, \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}\right] \\
&= \frac{1}{\alpha!}\sum_{i=1}^{n}\left\{\prod_{m=1}^{i-1}\left(L_{\bar{z}^m} - \bar{z}^m\right)^{\alpha_m}\left[\left(L_{\bar{z}^i} - \bar{z}^i\right)^{\alpha_i}, \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}\right]\prod_{m=i+1}^{n}\left(L_{\bar{z}^m} - \bar{z}^m\right)^{\alpha_m}\right\} \\
&= \frac{1}{\alpha!}\sum_{i=1}^{n}\left\{\prod_{m=1}^{i-1}\left(L_{\bar{z}^m} - \bar{z}^m\right)^{\alpha_m}\left\{\hbar\alpha_i\delta_{\bar{l}}^{\bar{i}}\left(L_{\bar{z}^i} - \bar{z}^i\right)^{\alpha_i - 1}\right\}\prod_{m=i+1}^{n}\left(L_{\bar{z}^m} - \bar{z}^m\right)^{\alpha_m}\right\} \\
&= \frac{\hbar}{\left(\alpha - \left(\bar{l}\right)\right)!}\left(L_{\bar{z}} - \bar{z}\right)^{\alpha - \left(\bar{l}\right)},
\end{aligned}$$

where $\alpha - \left(\bar{l}\right) := \left(\alpha_1, \cdots, \alpha_{l-1}, \alpha_l - 1, \alpha_{l+1}, \cdots, \alpha_n\right)$ *and if* $\alpha_l = 0$ *then* $\left[B_\alpha, R_{\partial_{\bar{l}}\Phi}\right] = 0$. *Hence*

$$\begin{aligned}
&\left[\left(\sum_\alpha A_\alpha^f B_\alpha\right), R_{\partial_{\bar{l}}\Phi}\right] \\
&= \sum_\alpha\left(\left[A_\alpha^f, R_{\partial_{\bar{l}}\Phi}\right]B_\alpha + A_\alpha^f\left[B_\alpha, R_{\partial_{\bar{l}}\Phi}\right]\right) \\
&= \sum_\alpha\left(-\hbar\left(\frac{\partial}{\partial\bar{z}}\right)^{\alpha + \left(\bar{l}\right)} f \times \frac{1}{\alpha!}\left(L_{\bar{z}} - \bar{z}\right)^\alpha + \left(\frac{\partial}{\partial\bar{z}}\right)^\alpha f \times \frac{\hbar}{\left(\alpha - \left(\bar{l}\right)\right)!}\left(L_{\bar{z}} - \bar{z}\right)^{\alpha - \left(\bar{l}\right)}\right) \\
&= 0.
\end{aligned}$$

□

5.2 Explicit Formulae for Noncommutative Deformations of $\mathbb{C}P^N$

In this chapter, we construct star product specifically.

5.2.1 Projective space

Projective space appears in all fields such as topology, differential geometry, and algebraic geometry. Here we introduce the Fubini-Study metric, which is a Riemannian geometric aspect. This metric is a Kähler metric.

Definition 5.14. Kähler potential Φ on $\mathbb{C}P^N$ is defined as

$$\Phi := \log\left(1 + |z|^2\right)$$

where $z^1, z^2 \cdots z^N$ are inhomogeneous coordinates of $\mathbb{C}P^N$. The Kähler metric is realized as

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{\left(1 + |z|^2\right)\delta_{ij} - z^j \bar{z}^i}{\left(1 + |z|^2\right)^2}.$$

$\left(\mathbb{C}P^N, g\right)$ is called Projective space with the Fubini-Study metric.

Proposition 5.15 (A. Sako, T. Suzuki and H. Umetsu [27])**.** *Assume*

$$L_{\bar{z}^l} = \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n A_n$$

and

$$A_n = \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}}$$

for $\left(\mathbb{C}P^N, g\right)$*. Then*

$$a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1$$

and

$$a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1)\, a_m^{(n-1)}.$$

Proof. $[A_n, \partial_{\bar{i}}\Phi] = [\partial_{\bar{l}}, A_{m-1}]$ *(**??**) leads to the relation. Left hand side is*

$$\begin{aligned}
&[A_n, \partial_{\bar{i}}\Phi] \\
&= \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \left[D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}, \partial_{\bar{i}}\Phi\right] \\
&= \sum_{m=2}^{n-1} a_{m+1}^{(n)} \left\{m \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} + \delta_{il} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_m}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}}\right\} \\
&\qquad + a_2^{(n)} \left(\partial_{\bar{i}}\Phi D^{\bar{l}} + \delta_{il} \partial_{\bar{j}}\Phi D^{\bar{j}}\right) \qquad (5.2.1)
\end{aligned}$$

and right hand side is

$$
\begin{aligned}
&[\partial_{\bar{l}}, A_{m-1}] \\
&= \sum_{m=2}^{n-1} a_m^{(n-1)} \left[\partial_{\bar{i}}, \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}\right] \\
&= \sum_{m=2}^{n-1} \left(a_m^{(n-1)} + m a_{m+1}^{(n-1)}\right) \left\{m\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}} + \delta_{il}\partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_m}\Phi \partial_{\bar{i}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}}\right\} \\
&\qquad + a_2^{(n-1)} \left(\partial_{\bar{i}}\Phi D^{\bar{l}} + \delta_{il}\partial_{\bar{j}}\Phi D^{\bar{j}}\right). \qquad (5.2.2)
\end{aligned}
$$

*Comparing (**??**) and (**??**) we get the results we want.* □

Remark 5.16. Stirling number of the second kind $S(n, m)$ is defined as

$$S(n, m) = S(n-1, m-1) + mS(n-1, m)$$

and

$$x^n = \sum_{m=0}^{n} S(n, m) \times x(x-1) \cdots (x-m+1).$$

Then $a_{m+1}^{(n+1)}$ is a Stirling number of the second kind.

Example 5.17.

$$
\begin{aligned}
&1 = S(0,0) = S(1,1) = S(2,2) = \cdots \\
&0 = S(1,0) = S(2,0) = S(3,0) = \cdots \\
&1 = S(1,1) = S(2,1) = S(3,1) = \cdots \\
&S(3,2) = 3, S(4,2) = 7, S(4,3) = 6 \cdots
\end{aligned}
$$

Proposition 5.18 (A. Sako, T. Suzuki and H. Umetsu [27])**.** *Assume* $\alpha_m(t) := \sum_{n=m}^{\infty} t^n a_m^{(n)}$ *then*

$$L_{\bar{z}^l} = \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\hbar)\, \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}$$

where $\alpha_1(t) := t$.

Proof.

$$\begin{aligned}
L_{\bar{z}^l} &= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n A_n \\
&= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{n=2}^{\infty} \hbar^n \sum_{m=2}^{n} a_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} \\
&= \bar{z}^l + \hbar D^{\bar{l}} + \sum_{m=2}^{\infty} \left(\sum_{n=m}^{\infty} t^n a_m^{(n)} \right) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} \\
&= \bar{z}^l + \sum_{m=1}^{\infty} \alpha_m(\hbar) \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}}.
\end{aligned}$$

□

Proposition 5.19 (A. Sako, T. Suzuki and H. Umetsu [27]). *Assume $m \geq 2$ then*

$$\alpha_m(t) := \sum_{n=m}^{\infty} t^n a_m^{(n)} = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1-m+t^{-1})}{\Gamma(1+t^{-1})}.$$

Proof. *A recurrence formula of $\alpha_m(t)$ can be derived from recurrence formula of $a_m^{(n)}$ for $m > 2$.*

$$\alpha_m(t) = t\left\{\alpha_{m-1}(t) + (m-1)\alpha_m(t)\right\}$$

Hence

$$\alpha_m(t) = t^{m-2} \prod_{n=2}^{m-1} \frac{1}{1-nt} \times \alpha_2(t)$$

where

$$\alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1-t}.$$

As a result

$$\alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1-m+t^{-1})}{\Gamma(1+t^{-1})}.$$

□

Theorem 5.20 (A. Sako, T. Suzuki and H. Umetsu [27]). *Using Proposition* **??**, *for $\mathbb{C}P^N$, $L_{\bar{z}^l} = \bar{z}^l *$ is given by*

$$L_{\bar{z}^l} = \bar{z}^l + \sum_{m=1}^{\infty} \frac{\Gamma(1-m+t^{-1})}{\Gamma(1+t^{-1})} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}},$$

then $z^i * z^j = z^i z^j, z^i * \bar{z}^j = z^i \bar{z}^j, \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j$ *and*

$$\bar{z}^i * z^j = \bar{z}^i z^j + \frac{\hbar\left(1+|z|^2\right)^2 \bar{z}^i z^j}{|z|^2} \,{}_2F_1\left(1,2;1-\frac{1}{\hbar};-|z|^2\right) + \hbar\left(\delta_{ij} - \frac{{}^2\bar{z}^i z^j}{|z|^2}\right)\left(1+|z|^2\right) {}_2F_1\left(1,1;1-\frac{1}{\hbar};-|z|^2\right)$$

where ${}_2F_1\left(a_1, a_2; b; z\right)$ *is a hypergeometric function.*

5.2.2 Hyperbolic space

Definition 5.21. The open subset of $\mathbb{C}^N$

$$\mathbb{C}H^N := \left\{z \in \mathbb{C}^N \mid |z|^2 < 1\right\}$$

and the Kähler metric

$$g_{i\bar{j}} := \partial_i \partial_{\bar{j}} \Phi$$

is called hyperbolic space or Poincaré disk model $\left(\mathbb{C}H^N, g\right)$ where

$$\Phi := \frac{\left(1-|z|^2\right)\delta_{ij} + \bar{z}^i z^j}{\left(1-|z|^2\right)^2}.$$

Proposition 5.22 (A. Sako, T. Suzuki and H. Umetsu [27])**.** *Assume*

$$L_{\bar{z}^l} = \bar{z}^l + tD^{\bar{l}} + \sum_{n=2}^{\infty} t^n B_n$$

and

$$B_n = \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}}$$

for $\left(\mathbb{C}H^N, g\right)$. *Then*

$$b_2^{(n)} = b_2^{(n-1)} = \cdots = b_2^{(2)} = 1$$

and

$$b_m^{(n)} = b_{m-1}^{(n-1)} + (m-1)\, b_m^{(n-1)},$$

hence

$$L_{\bar{z}^l} = \bar{z}^l + \sum_{m=1}^{\infty} \beta_m(t)\, \partial_{\bar{j}_1}\Phi \cdots \partial_{\bar{j}_{m-1}}\Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} D^{\bar{l}},$$

where

$$\beta_m(t) = \frac{\Gamma\left(t^{-1}\right)}{\Gamma\left(n+t^{-1}\right)}.$$

It is possible to take a slightly different method as follows.

Proposition 5.23 (A. Sako, T. Suzuki and H. Umetsu [27]). *For* $\mathbb{C}P^N$

$$L_f = \sum_{n=0}^{\infty} c_n(t)\, g_{j_1\bar{k}_1}\cdots g_{j_n\bar{k}_n}\left(D^{j_1}\cdots D^{j_n} f\right) D^{\bar{k}_1}\cdots D^{\bar{k}_n}$$

where

$$c_n(t) = \frac{\Gamma(1-n+1/t)}{n!\Gamma(1+1/t)} = \frac{\alpha_n(t)}{n!}$$

Proof. *From associativity*

$$\begin{aligned} 0 &= \left[L_f, R_{\partial_{\bar{l}}\Phi}\right] \\ &= \sum_{n=0}^{\infty}\left[n\left\{1-t(n-1)\right\}c_n(t) - tc_{n-1}(t)\right] g_{j_1\bar{k}_1} g_{l\bar{i}}\cdots g_{j_{n-1}\bar{k}_{n-1}}\left(D^l D^{j_1}\cdots D^{j_{n-1}} f\right) D^{\bar{k}_1}\cdots D^{\bar{k}_{n-1}}. \end{aligned}$$

Hence if

$$n\left\{1-t(n-1)\right\}c_n(t) - tc_{n-1}(t) = 0$$

then

$$c_n(t) = \frac{\Gamma(1-n+1/t)}{n!\Gamma(1+1/t)}.$$

□

Corollary 5.24 (A. Sako, T. Suzuki and H. Umetsu [27]). *For* $\mathbb{C}H^N$

$$L_f = \sum_{n=0}^{\infty} c_n(t)\, g_{j_1\bar{k}_1}\cdots g_{j_n\bar{k}_n}\left(D^{j_1}\cdots D^{j_n} f\right) D^{\bar{k}_1}\cdots D^{\bar{k}_n}$$

where

$$c_n(t) = \frac{\Gamma(1/t)}{n!\Gamma(n+1/t)} = \frac{\beta_n(t)}{n!}.$$

This method will lead to later results.

Chapter 6

Deformation quantization with separation of variables for a locally symmetric Kähler manifold

In this chapter, explicit formulas to obtain star products on locally symmetric Kähler manifolds are constructed. A method of Karabegov in Chapter 2 and Chapter **??** is used for the constructing.

6.1 Deformation quantization with separation of variables for a locally symmetric Kähler manifold

At first we list notations used in this chapter. Let M be a N-dimensional Kähler manifold, $\partial_i := \dfrac{\partial}{\partial z_i}, \partial_{\bar{i}} := \dfrac{\partial}{\partial \bar{z}_i}$ $(i = 1, \ldots, N)$ be tangent vector fields on a coordinate chart $U \subset M$ with its local coordinates $\left(z^1, \cdots, z^N, \bar{z}^1, \cdots, \bar{z}^N\right), dz^i, d\bar{z}^i$ be cotangent vector fields on U and $Y^{\mu_1 \cdots \mu_k \bar{\mu}_1 \cdots \bar{\mu}_m}{}_{\nu_1 \cdots \nu_l \bar{\nu}_1 \cdots \bar{\nu}_n} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes \partial_{\bar{\mu}_1} \otimes \cdots \otimes \partial_{\bar{\mu}_m} \otimes dz^{\nu_1} \otimes \cdots \otimes dz^{\nu_l} \otimes d\bar{z}^{\bar{\nu}_1} \otimes \cdots \otimes d\bar{z}^{\bar{\nu}_n}$ $\in \Gamma\left[\left(T^{1,0}M\right)^{\otimes k} \otimes \left(T^{0,1}M\right)^{\otimes m} \otimes \left\{\left(T^{1,0}M\right)^*\right\}^{\otimes l} \otimes \left\{\left(T^{0,1}M\right)^*\right\}^{\otimes n}\right]$ be a $((k, m), (l, n))$-tensor field.

The classical style of covariant derivative $\nabla_i := \nabla_{\partial_i}$ acts on coefficients of tensor fields as

$$\begin{aligned}
&\nabla_i Y^{\mu_1\cdots\mu_k\bar{\mu}_1\cdots\bar{\mu}_m}{}_{\nu_1\cdots\nu_l\bar{\nu}_1\cdots\bar{\nu}_n}\\
&=\partial_i Y^{\mu_1\cdots\mu_k\bar{\mu}_1\cdots\bar{\mu}_m}{}_{\nu_1\cdots\nu_l\bar{\nu}_1\cdots\bar{\nu}_n}\\
&+\sum_{q=1}^{k}\Gamma^{\mu_q}_{i\rho_q}Y^{\mu_1\mu_2\cdots\rho_q\cdots\mu_k\bar{\mu}_1\cdots\bar{\mu}_m}{}_{\nu_1\cdots\nu_l\bar{\nu}_1\cdots\bar{\nu}_n}+\sum_{q=1}^{m}\Gamma^{\bar{\mu}_q}_{i\bar{\rho}_q}Y^{\mu_1\cdots\mu_k\bar{\mu}_1\bar{\mu}_2\cdots\bar{\rho}_q\cdots\bar{\mu}_m}{}_{\nu_1\cdots\nu_l\bar{\nu}_1\cdots\bar{\nu}_n}\\
&\qquad-\sum_{q=1}^{l}\Gamma^{\sigma_q}_{i\nu_q}Y^{\mu_1\cdots\mu_k\bar{\mu}_1\cdots\bar{\mu}_m}{}_{\nu_1\nu_2\cdots\sigma_q\cdots\nu_l\bar{\nu}_1\cdots\bar{\nu}_n}-\sum_{q=1}^{n}\Gamma^{\bar{\sigma}_q}_{i\bar{\nu}_q}Y^{\mu_1\cdots\mu_k\bar{\mu}_1\cdots\bar{\mu}_m}{}_{\nu_1\cdots\nu_l\bar{\nu}_1\bar{\nu}_2\cdots\bar{\sigma}_q\cdots\bar{\nu}_n}
\end{aligned}$$

where Γ^i_{jk} is the Christoffel symbol.

The Riemannian curvature of a Hermitian manifold M is defined as

$$R_{i\bar{j}k}{}^{l}=\partial_i\Gamma^l_{\bar{j}k}-\partial_{\bar{j}}\Gamma^l_{ik}+\Gamma^n_{\bar{j}k}\Gamma^l_{in}-\Gamma^n_{ik}\Gamma^l_{\bar{j}n}.$$

For Hermitian manifolds, the Christoffel symbols are given as

$$\Gamma^l_{jk}=g^{l\bar{q}}\frac{\partial g_{j\bar{q}}}{\partial z^k}.$$

The Riemannian curvature of a Hermitian manifold M is obtained as

$$R_{i\bar{j}k\bar{l}}=-\frac{\partial^2 g_{\bar{j}i}}{\partial z^k\partial\bar{z}^l}+g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{\bar{j}p}}{\partial\bar{z}^l}.$$

On a Kähler manifold, its metric is described by using Kähler potential Φ as (9.1). Then its Riemannian curvature is given by

$$R_{i\bar{j}k\bar{l}}=-\frac{\partial^4\Phi}{\partial z^i\partial\bar{z}^j\partial z^k\partial\bar{z}^l}+g^{p\bar{q}}\frac{\partial^3\Phi}{\partial z^i\partial\bar{z}^q\partial z^k}\frac{\partial^3\Phi}{\partial z^p\partial\bar{z}^j\partial\bar{z}^l}. \tag{6.1.1}$$

(See [61] P157.)

Operators $D^{\vec{\alpha_n}}$ and $D^{\vec{\beta_n^*}}$ are defined by using $D^k=g^{k\bar{m}}\partial_{\bar{m}}$ and $D^{\bar{j}}=g^{\bar{j}l}\partial_l$ as

$$D^{\vec{\alpha_n}}:=D^{\alpha_1^n}D^{\alpha_2^n}\cdots D^{\alpha_N^n}\qquad,\qquad D^{\vec{\beta_n}}:=D^{\beta_1}D^{\beta_2}\cdots D^{\beta_N}$$

where

$$D^{\alpha_k}:=\left(D^k\right)^{\alpha_k}\qquad,\qquad D^{\beta_j}:=\left(D^{\bar{j}}\right)^{\beta_j},$$

and $\vec{\alpha_n}$ and $\vec{\beta_n^*}$ are N-dimensional vectors whose summation of their all elements are set to be n;

$$\vec{\alpha}_n\in\left\{(\gamma_1^n,\gamma_2^n,\cdots,\gamma_N^n)\in\mathbb{Z}^N\;\middle|\;\sum_{k=1}^{N}\gamma_k^n=n\right\},\;\;\vec{\beta_n^*}\in\left\{(\gamma_1^n,\gamma_2^n,\cdots,\gamma_N^n)^*\in\mathbb{Z}^N\;\middle|\;\sum_{k=1}^{N}\gamma_k^n=n\right\}$$

i. e.

$$\vec{\alpha_n} := (\alpha_1^n, \alpha_2^n, \cdots, \alpha_N^n)\,, \quad |\vec{\alpha_n}| := \sum_{k=1}^{N} \alpha_k^n = n$$

$$\vec{\beta_n^*} := (\beta_1^n, \beta_2^n, \cdots, \beta_N^n)^*\,, \quad |\vec{\beta_n^*}| := \sum_{k=1}^{N} \beta_k^n = n.$$

For $\vec{\alpha_n} \notin \mathbb{Z}_{\geq 0}^N$ we define $D^{\vec{\alpha_n}} := 0$.

For example, $D^{(1,2,3)} = D^1 \left(D^2\right)^2 \left(D^3\right)^3$, $D^{(2,4,0)^*} = \left(D^{\bar{1}}\right)^2 \left(D^{\bar{2}}\right)^4$ and $D^{(5,-2,3)} = 0$ for a 3-dimensional manifolds case with $n = 6$.

$\vec{e_i}$ is used as a N-dimensional vector

$$\vec{e_i} = (\delta_{1i}, \delta_{2i}, \cdots, \delta_{Ni}). \tag{6.1.2}$$

From here to the end of this section, we make up recurrence relations to construct explicit expressions of star products on locally symmetric Kähler manifolds.

A Riemannian(Kähler) manifold (M, g) is called a locally symmetric Riemannian(Kähler) manifold when $\nabla_m R_{ijk}{}^l = 0$ $(\forall i, j, k, l, m)$. Only locally symmetric Kähler manifolds are studied in this article.

We assume that a star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold M has a form

$$L_f g = f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n} \vec{\beta_n^*}} T^n_{\vec{\alpha_n} \vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}} g\right), \tag{6.1.3}$$

where $T^n_{\vec{\alpha_n} \vec{\beta_n^*}}$ are covariantly constants. If $\vec{\alpha_n} \notin \mathbb{Z}_{\geq 0}^N$ or $\vec{\beta_n} \notin \mathbb{Z}_{\geq 0}^N$ then we define $T^n_{\vec{\alpha_n} \vec{\beta_n^*}} := 0$. $\sum\limits_{\vec{\alpha_n} \vec{\beta_n^*}}$ is defined by the summation over all $\vec{\alpha_n^*}$ and $\vec{\beta_n^*}$ satisfying $\left|\vec{\alpha_n^*}\right| = \left|\vec{\beta_n^*}\right| = n$. In brief,

$$n = \left|\vec{\alpha_n^*}\right| := \sum_{i=1}^{N} \alpha_i^n, \quad n = \left|\vec{\beta_n^*}\right| := \sum_{i=1}^{N} \beta_i^n\,, \quad \sum_{\vec{\alpha_n} \vec{\beta_n^*}} := \sum_{|\vec{\alpha_n}| = \left|\vec{\beta_n^*}\right| = n} .$$

Proposition 6.1. *For the star product on a locally symmetric Kähler manifold M as (10.3), $T^0_{\vec{\alpha_0} \vec{\beta_0^*}}$ and $T^1_{\vec{e_i}, \vec{e_j}}$ are given as*

$$T^0_{\vec{\alpha_0} \vec{\beta_0^*}} = 1, \quad T^1_{\vec{e_i}, \vec{e_j}} = \hbar g_{i\bar{j}}.$$

Proof. *From (10.3), the star product for smooth functions f and g on M is given as*

$$L_f g = T^0_{\vec{\alpha_0}\vec{\beta_0^*}} fg + \sum_{\vec{\alpha_1}\vec{\beta_1^*}} T^1_{\vec{\alpha_1}\vec{\beta_1^*}} \left(D^{\vec{\alpha_1}} f\right) \left(D^{\vec{\beta_1^*}} g\right) + \sum_{n=2}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}} g\right).$$

$T^0_{\vec{\alpha_0}\vec{\beta_0^*}} = 1$ *is trivial.* $C_1(f,g)$ *is expressed as*

$$C_1(f,g) = \sum_{\vec{\alpha_1}\vec{\beta_1^*}} T^1_{\vec{\alpha_1}\vec{\beta_1^*}} \left(D^{\vec{\alpha_1}} f\right) \left(D^{\vec{\beta_1^*}} g\right).$$

By the definition of the deformation quantization (9.5) the first term is related to the Poisson bracket:

$$\begin{aligned} &\hbar \sum_{i,j=1}^{n} g^{i\bar{j}} \left(\frac{\partial f}{\partial z^i} \frac{\partial g}{\partial \bar{z}^j} - \frac{\partial g}{\partial \bar{z}^j} \frac{\partial f}{\partial z^i} \right) \\ &= \sum_{\vec{\alpha_1}\vec{\beta_1^*}} (T^1_{\vec{\alpha_1}\vec{\beta_1^*}} \left(g^{\alpha_1 \bar{m}} \partial_{\bar{m}} f\right) \left(g^{\bar{\beta}_1 l} \partial_l g\right) - T^1_{\vec{\alpha_1}\vec{\beta_1^*}} \left(g^{\alpha_1 \bar{m}} \partial_{\bar{m}} g\right) \left(g^{\bar{\beta}_1 l} \partial_l f\right)). \end{aligned}$$

Then $T^1_{\vec{e_i},\vec{e_j}} = \hbar g_{i\bar{j}}$ *is shown.* □

The purpose of remained part of this section is to replace the recurrence relations as differential equations by those of algebraic equations. We need to calculate $[L_f, \partial_{\bar{i}}\Phi\cdot]$ and $[L_f, \partial_{\bar{i}}]$ in (9.11).

Proposition 6.2. *Let f and g be smooth functions on a locally symmetric Kähler manifold $M$and L_f be a left star product by f given as (10.3). Then*

$$\begin{aligned} \sigma\left([L_f, \partial_{\bar{i}}\Phi]\right) &= \frac{\partial \sigma\left(L_f\right)}{\partial \xi^{\bar{i}}} \\ &= \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(\xi^{\bar{1}\beta_1^n} \cdots \xi^{\bar{i}\beta_i^n - 1} \cdots \xi^{\bar{N}\beta_N^n}\right) & (\beta_i \neq 0) \\ 0 \qquad (\beta_i^n = 0) \end{cases}, \end{aligned}$$

or equivalently,

$$[L_f, \partial_{\bar{i}}\Phi]\, g = \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*} - \vec{e_i}} g\right) (\beta_i \neq 0) \\ 0 \qquad (\beta_i^n = 0) \end{cases}. \tag{6.1.4}$$

Proof. *By Proposition 9.2,*

$$\sigma\left([L_f, \partial_{\bar{i}}\Phi\cdot]\right) = \frac{\partial \sigma\left(L_f\right)}{\partial \xi^{\bar{i}}} = \frac{\partial}{\partial \xi^{\bar{i}}} \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(\xi^{\vec{\beta_n^*}}\right).$$

$\xi^{\vec{\beta_n^*}}$ *is explicitly written by* $\xi^{\vec{\beta_n^*}} = \xi^{\bar{1}^{\beta_1^n}} \xi^{\bar{2}^{\beta_2^n}} \cdots \xi^{\bar{N}^{\beta_N^n}}$, *then*

$$\begin{aligned}
&\sigma\left([L_f, \partial_{\bar{i}}\Phi\cdot]\right) \\
&= \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(\xi^{\bar{1}^{\beta_1^n}} \cdots \xi^{\bar{i}^{\beta_i^n - 1}} \cdots \xi^{\bar{N}^{\beta_N^n}}\right) & (\beta_i \neq 0) \\ 0 \qquad (\beta_i = 0) \end{cases} .
\end{aligned}$$

□

The following formulas are given in [28].

Fact 6.3. For smooth functions f and g on a locally symmetric Kähler manifold, the following formulas are given.

$$\begin{aligned}
&\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f = g_{l_1\bar{j}_1} \cdots g_{l_n\bar{j}_n} D^{l_1} \cdots D^{l_n} f \\
&\nabla_{k_1} \cdots \nabla_{k_n} g = g_{\bar{m}_1 k_1} \cdots g_{\bar{m}_n k_n} D^{\bar{m}_1} \cdots D^{\bar{m}_n} g \\
&D^{l_1} \cdots D^{l_n} f = g^{l_1\bar{j}_1} \cdots g^{l_n\bar{j}_n} \nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \\
&D^{\bar{m}_1} \cdots D^{\bar{m}_n} g = g^{\bar{m}_1 k_1} \cdots g^{\bar{m}_n k_n} \nabla_{k_1} \cdots \nabla_{k_n} g.
\end{aligned}$$

Fact 10.3 derives the following lemma.

Lemma 6.4. *Let f and g be smooth functions on a locally symmetric Kähler manifold M. Let L_f be a left star product by f given as (10.3). Then,*

$$\begin{aligned}
&[L_f, \hbar\partial_{\bar{i}}]g \\
&= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \sum_{k=1}^{N} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \frac{\beta_k^n\left(\beta_k^n - 1\right)}{2} R_{\bar{\rho}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*} + \vec{e_\rho} - \vec{e_k}} g\right) \\
&\quad + \hbar \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_k^n \beta_{k+l}^n R_{\bar{\rho}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*} + \vec{e_\rho} - \vec{e_k}} g\right) \\
&\quad - \hbar \sum_{n=1}^{\infty} \sum_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \sum_{d=1}^{N} g_{\bar{i}d} T^{n-1}_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \left(D^{\vec{\alpha_{n-1}} + \vec{e_d}} f\right) \left(D^{\vec{\beta_{n-1}^*}} g\right).
\end{aligned}$$

Proof. *We can calculate* $[L_f, \hbar\partial_{\bar{i}}]g$ *straightforwardly.*

$$\begin{aligned}
[L_f, \hbar\partial_{\bar{i}}]g =& \hbar[L_f, \nabla_{\bar{i}}]g \\
=& \hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}} T^n_{\vec{\alpha_n}\vec{\beta^*_n}} \left\{\left(D^1\right)^{\alpha^n_1}\cdots\left(D^N\right)^{\alpha^n_N} f\right\}\left[\left(D^{\bar{1}}\right)^{\beta^n_1}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N}, \nabla_{\bar{i}}\right] g \\
& - \hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}} T^n_{\vec{\alpha_n}\vec{\beta^*_n}} \left\{\nabla_{\bar{i}}\left(D^1\right)^{\alpha^n_1}\cdots\left(D^N\right)^{\alpha^n_N} f\right\}\left\{\left(D^{\bar{1}}\right)^{\beta^n_1}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N} g\right\}.
\end{aligned} \tag{6.1.5}$$

From Fact 10.3, the second term of (10.5) becomes

$$\begin{aligned}
&\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}} T^n_{\vec{\alpha_n}\vec{\beta^*_n}} \left\{\nabla_{\bar{i}}\left(D^1\right)^{\alpha^n_1}\cdots\left(D^N\right)^{\alpha^n_N} f\right\}\left\{\left(D^{\bar{1}}\right)^{\beta^n_1}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N} g\right\} \\
&= \hbar\sum_{n=1}^{\infty}\sum_{d=1}^{N}\sum_{\alpha_{\vec{n-1}}\beta^{\vec{*}}_{n-1}} g_{\bar{i}d} T^{n-1}_{\alpha_{\vec{n-1}}\beta^{\vec{*}}_{n-1}} \left(D^{\alpha_{\vec{n-1}}+\vec{e_d}} f\right)\left(D^{\beta^{\vec{*}}_{n-1}} g\right).
\end{aligned} \tag{6.1.6}$$

To calculate the first term of (10.5) we calculate $\left[D^{\vec{\beta^*_n}}, \nabla_{\bar{i}}\right] g$:

$$\begin{aligned}
&\left[\left(D^{\bar{1}}\right)^{\beta^n_1}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N}, \nabla_{\bar{i}}\right] g \\
&= \left\{\left[\left(D^{\bar{1}}\right)^{\beta^n_1}, \nabla_{\bar{i}}\right]\left\{\left(D^{\bar{2}}\right)^{\beta^n_2}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N}\right\} + \cdots + \left\{\left(D^{\bar{1}}\right)^{\beta^n_1}\cdots\left(D^{\overline{N-1}}\right)^{\beta^n_{N-1}}\right\}\left[\left(D^{\bar{N}}\right)^{\beta^n_N}, \nabla_{\bar{i}}\right]\right\} g.
\end{aligned}$$

For these terms, we evaluate

(6.1.7)

$$\left[\left(D^{\bar{a}}\right)^{\beta^n_a}, \nabla_{\bar{i}}\right]\left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N} g = \sum_{m=1}^{\beta^n_a}\left(D^{\bar{a}}\right)^{m-1}\left[D^{\bar{a}}, \nabla_{\bar{i}}\right]\left(D^{\bar{a}}\right)^{\beta^n_a-m}\left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N} g$$

by cases, Case1 $\beta^n_a = 1$, *Ace2* $\beta^n_a > 1$ *and* $\sum_{k=a+1}^{N}\beta^n_k > 0$, *and Cave3* $\beta^n_a > 1$ *and* $\sum_{k=a+1}^{N}\beta^n_k = 0$.

Case1. If $\beta_a^n = 1$ *the last line of (10.7) is written as*

$$\begin{aligned}
&\left[(D^{\bar{a}})^{\beta_a^n}, \nabla_{\bar{i}}\right] \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g \\
&= \sum_{j=a+1}^{N} \sum_{n_j=1}^{\beta_j^n} R_{\bar{i}}{}^{\bar{a}\bar{j}}{}_{\bar{c}}\, D^{\bar{c}} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{j}}\right)^{\beta_j^n - 1} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g \\
&= \sum_{j=a+1}^{N} \beta_j^n \beta_a^n R_{\bar{i}}{}^{\bar{a}\bar{j}}{}_{\bar{c}}\, D^{\bar{c}} \left(D^{\bar{a}}\right)^{\beta_a^n - 1} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{j}}\right)^{\beta_j^n - 1} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g. \qquad (6.1.8)
\end{aligned}$$

Recall that $(D^{\bar{j}})^n = 0$ *for negative* n *by definition.*

Case2. If $\beta_a^n > 1$ *and* $\sum_{k=a+1}^{N} \beta_k^n > 0$*, by using Fact 10.3, we obtain (10.7) as*

$$\begin{aligned}
&\sum_{m=1}^{\beta_a^n} \left(D^{\bar{a}}\right)^{m-1} \left[D^{\bar{a}}, \nabla_{\bar{i}}\right] \left(D^{\bar{a}}\right)^{\beta_a^n - m} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g \qquad (6.1.9) \\
=&\sum_{m=1}^{\beta_a^n} \left(D^{\bar{a}}\right)^{m-1} g^{\bar{a}b} g^{\bar{a}k_{a,1}} \cdots g^{\bar{a}k_{a,\beta_a^n - m}} \left[\nabla_b, \nabla_{\bar{i}}\right] \nabla_{k_{a,1}} \cdots \nabla_{k_{a,\beta_a^n - m}} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g \\
=&\sum_{m=1}^{\beta_a^n} \sum_{n_a=1}^{\beta_a^n - m} \left(D^{\bar{a}}\right)^{m-1} R_{\bar{i}}{}^{\bar{a}\bar{a}}{}_{\bar{c}}\, D^{\bar{c}} \left(D^{\bar{a}}\right)^{\beta_a^n - m} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g \\
&+ \sum_{m=1}^{\beta_a^n} \sum_{j=a+1}^{N} \sum_{\vec{\alpha_n} \vec{\beta_n^*}} \beta_j^n R_{\bar{i}}{}^{\bar{a}\bar{j}}{}_{\bar{c}}\, D^{\bar{c}} \left(D^{\bar{a}}\right)^{m} \left(D^{\overline{a+1}}\right)^{\beta_{a+1}^n} \cdots \left(D^{\bar{j}}\right)^{\beta_j^n - 1} \cdots \left(D^{\bar{N}}\right)^{\beta_N^n} g. \qquad (6.1.10)
\end{aligned}$$

Here, we used

$$\left[\nabla_i, \nabla_j\right] \nabla_{k_1} \cdots \nabla_{k_m} f = -\sum_{n=1}^{m} R_{ijk_n}{}^{l}\, \nabla_{k_1} \cdots \nabla_{k_{n-1}} \nabla_l \nabla_{k_{n+1}} \cdots \nabla_{k_m} f. \qquad (6.1.11)$$

for $m \geq 1$*.*

Case3. If $\sum_{k=a+1}^{N} \beta_k^n = 0$ *the R. H. S of (10.7) is written as*

$$\sum_{m=1}^{\beta_a^n} \left(D^{\bar{a}}\right)^{m-1} \left[D^{\bar{a}}, \nabla_{\bar{i}}\right] \left(D^{\bar{a}}\right)^{\beta_a^n - m} g = \sum_{m=1}^{\beta_a^n} \sum_{n_a=1}^{\beta_a^n - m} \left(D^{\bar{a}}\right)^{m-1} R_{\bar{i}}{}^{\bar{a}\bar{a}}{}_{\bar{c}}\, D^{\bar{c}} \left(D^{\bar{a}}\right)^{\beta_a^n - m - 1} g. \qquad (6.1.12)$$

Putting Case 1,2 and 3 into a shape and recalling that M is a locally symmetric Kähler manifold, (10.7) is rewritten as

$$
\begin{aligned}
&\frac{\beta^n_a\left(\beta^n_a-1\right)}{2}R_{\bar{i}}{}^{\bar{a}\bar{a}}{}_{\bar{c}}\,D^{\bar{c}}\left(D^{\bar{a}}\right)^{\beta^n_a-2}\left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N}g\\
&+\sum_{j=a+1}^{N}\beta^n_j\beta^n_aR_{\bar{i}}{}^{\bar{a}\bar{j}}{}_{\bar{c}}\,D^{\bar{c}}\left(D^{\bar{a}}\right)^{\beta^n_a-1}\left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}}\cdots\left(D^{\bar{j}}\right)^{\beta^n_j-1}\cdots\left(D^{\bar{N}}\right)^{\beta^n_N}g.
\end{aligned}
\tag{6.1.13}
$$

Then we find that the first term of (10.5) is expressed as

$$
\begin{aligned}
&\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{k=1}^{N}\frac{\beta^n_k\left(\beta^n_k-1\right)}{2}R_{\bar{c}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_c}-2\vec{e_k}}g\right)\\
&+\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{k=1}^{N}\sum_{l=1}^{N-k}\beta^n_k\beta^n_{k+l}R_{\bar{c}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_c}-\vec{e_k}-\vec{e}_{k+l}}g\right)
\end{aligned}
\tag{6.1.14}
$$

Finally , we get the result with substituting (10.6) and (10.14) into (10.5)

$$
\begin{aligned}
[L_f,\hbar\partial_{\bar{i}}]g=&\,\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{k=1}^{N}\frac{\beta^n_k\left(\beta^n_k-1\right)}{2}R_{\bar{c}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_c}-2\vec{e_k}}g\right)\\
&+\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{k=1}^{N}\sum_{l=1}^{N-k}\beta^n_k\beta^n_{k+l}R_{\bar{c}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_c}-\vec{e_k}-\vec{e}_{k+l}}g\right)\\
&-\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{d=1}^{N}g_{\bar{i}d}T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}+\vec{e_d}}f\right)\left(D^{\vec{\beta^*_n}}g\right).
\end{aligned}
$$

□

Theorem 6.5. *When the star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold is given as*

$$
f*g=\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}}g\right),
$$

these covariantly constants $T^n_{\vec{\alpha_n}\vec{\beta^*_n}}$ *are determined by the following recurrence relations*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\beta^n_i T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}-\vec{e_i}}g\right)-\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}g_{\bar{i}d}T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}+\vec{e_d}}f\right)\left(D^{\vec{\beta^*_n}}g\right)\\
&\quad+\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{k=1}^{N}\sum_{p=1}^{N}\frac{\beta^n_k\left(\beta^n_k-1\right)}{2}R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_p}-2\vec{e_k}}g\right)\\
&\quad+\hbar\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}\sum_{\rho=1}^{N}\sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\beta^n_k\beta^n_{k+l}R_{\bar{\rho}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}+\vec{e_\rho}-\vec{e_k}-\vec{e_{k+l}}}g\right)\\
&=0
\end{aligned}$$

Proof. $0=[L_f,\partial_{\bar{i}}\Phi+\hbar\partial_{\bar{i}}]\,g$ *is the condition that determines the star product.* $[L_f,\partial_{\bar{i}}\Phi]\,g$ *and* $[L_f,\hbar\partial_{\bar{i}}]\,g$ *were calculated in Proposition 10.2 and 10.4.* □

Theorem 6.6. *When the star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold is given as*

$$f*g=\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}}T^n_{\vec{\alpha_n}\vec{\beta^*_n}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta^*_n}}g\right),$$

these smooth functions $T^n_{\vec{\alpha_n}\vec{\beta^*_n}}$*, which are covariantly constants, are determined by the following recurrence relations for* $\forall i$:

$$\begin{aligned}
&\sum_{d=1}^{N}\hbar g_{\bar{i}d}T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta^*_n}-\vec{e_i}}\\
&=\beta_i T^n_{\vec{\alpha_n}\vec{\beta^*_n}}+\sum_{k=1}^{N}\sum_{p=1}^{N}\frac{\hbar\left(\beta^n_k-\delta_{kp}-\delta_{ik}+1\right)\left(\beta^n_k-\delta_{kp}-\delta_{ik}+2\right)}{2}R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+2\vec{e_k}-\vec{e_i}}\\
&\quad+\sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\sum_{p=1}^{N}\hbar\left(\beta^n_k-\delta_{kp}-\delta_{ik}+1\right)\left(\beta^n_{k+l}-\delta_{(k+l),p}-\delta_{i,(k+l)}+1\right)R_{\bar{p}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\,T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+\vec{e_k}+\vec{e_{k+l}}-\vec{e_i}}.
\end{aligned}$$

Proof. *Changing the summation of Theorem 10.5,*

$$
\begin{aligned}
&\hbar\sum_{n=1}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta_n^*}}\sum_{d=1}^{N} g_{\bar i d}T^{n-1}_{\alpha_n\vec{-}\vec{e_d}\vec{\beta_n^*}-\vec{e_i}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta_n^*}-\vec{e_i}}g\right)\\
&=\sum_{n=0}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta_n^*}}\beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta_n^*}-\vec{e_i}}g\right)\\
&\qquad+\hbar\sum_{n=0}^{\infty}\sum_{k=1}^{N}\sum_{p=1}^{N}\sum_{\vec{\alpha_n}\vec{\beta_n^*}}\frac{\left(\beta_k^n-\delta_{kp}-\delta_{ik}+1\right)\left(\beta_k^n-\delta_{kp}-\delta_{ik}+2\right)}{2}R_{\bar p}{}^{\bar k\bar k}{}_{\bar i}\\
&\qquad\quad\times T^n_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_p}+2\vec{e_k}-\vec{e_i}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta_n^*}-\vec{e_i}}g\right)\\
&\qquad+\hbar\sum_{n=0}^{\infty}\sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\sum_{p=1}^{N}\sum_{\vec{\alpha_n}\vec{\beta_n^*}}\left(\beta_k^n-\delta_{kp}-\delta_{ik}+1\right)\left(\beta_{k+l}^n-\delta_{(k+l),p}-\delta_{i,(k+l)}+1\right)R_{\bar p}{}^{\overline{k+l}\bar k}{}_{\bar i}\\
&\qquad\quad\times T^n_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_p}+\vec{e_k}+e_{\vec{k+l}}-\vec{e_i}}\left(D^{\vec{\alpha_n}}f\right)\left(D^{\vec{\beta_n^*}-\vec{e_i}}g\right),
\end{aligned}
$$

and this implies the theorem. □

6.2 One and two dimensional cases

By using Theorem 10.6 we can provide explicit star products for locally symmetric Kähler manifolds. In this section, an explicit expression of a star product of a one-dimensional locally symmetric Kähler manifold is constructed as an example. A two-dimensional locally symmetric Kähler manifold is also considered.

At first, we study an explicit expression of a star product of a one-dimensional locally symmetric Kähler manifold. Formal discussions are given in [32], and star products are studied in [25]. Complex surfaces with arbitrary genus are known as a example of such manifolds when we chose proper coordinates and metrics. The Scalar curvature R is defined as

$$R = g^{i\bar j}R_{i\bar j} = R_{\bar l}{}^{\bar j\bar l}{}_{\bar j}.$$

Proposition 6.7. *Let M be a one-dimensional locally symmetric Kähler manifold $(N=1)$ and f and g be smooth functions on M. The star product with separation of variables for f and g can be described as*

$$f * g = \sum_{n=0}^{\infty}\left[\left(g_{1\bar 1}\right)^n\left\{\prod_{k=1}^{n-1}\frac{2\hbar}{2k+\hbar k\left(k-1\right)R}\right\}\left\{\left(g^{1\bar 1}\frac{\partial}{\partial z}\right)^n f\right\}\left\{\left(g^{1\bar 1}\frac{\partial}{\partial \bar z}\right)^n g\right\}\right]$$

where

$$R = R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}.$$

Proof. $N = 1, i = 1$ *and*

$$D^{\vec{\alpha_n}} f = \left(g^{1\bar{1}} \frac{\partial}{\partial z} \right)^n f, \qquad D^{\vec{\beta^*_n}} g = \left(g^{1\bar{1}} \frac{\partial}{\partial \bar{z}} \right)^n g$$

are substituted in Theorem 10.5 , then we obtain

$$\hbar \sum_{n=1}^{\infty} g_{1\bar{1}} T^{n-1} \left(D^n f_1 \right) \left(D^{n-1} f_2 \right) = \sum_{n=0}^{\infty} \left\{ n + \frac{\hbar n \left(n-1 \right)}{2} R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} \right\} T^n \left(D^n f_1 \right) \left(D^{n-1} f_2 \right)$$

or equivalently, the recurrence relation of T^n *is given as*

$$T^n = g_{1\bar{1}} \left\{ \frac{2\hbar}{2n + \hbar n \left(n-1 \right) R} \right\} T^{n-1}.$$

From Proposition 10.1 the first term T^1 *is given as* $T^1 = \hbar g_{1\bar{1}}$. *Then,* T^n *is given as*

$$T^n = \left(g^{1\bar{1}} \right)^n \prod_{k=1}^{n-1} \left\{ \frac{2\hbar}{2k + \hbar k \left(k-1 \right) R} \right\}.$$

□

Next, we discuss star products on general two-dimensional locally symmetric Kähler manifolds.

According to Proposition 10.1, for a two-dimensional locally symmetric Kähler manifold M, $T^1_{\vec{\alpha_1}\vec{\beta^*_1}}$ is given as

$$\begin{pmatrix} T^1_{(1,0),(1,0)} & T^1_{(1,0),(0,1)} \\ T^1_{(0,1),(1,0)} & T^1_{(0,1),(0,1)} \end{pmatrix} = \hbar \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}.$$

Next, we estimate $T^2_{\vec{\alpha_2}\vec{\beta^*_2}}$.

Proposition 6.8. *Let* M *be a two-dimensional locally symmetric Kähler manifold and* f *and* g *be smooth functions on* M. $T^2_{\vec{\alpha_2}\vec{\beta^*_2}}$ *given in (10.3) is obtained by*

$$\begin{pmatrix} T^2_{(2,0),(2,0)} & T^2_{(2,0),(1,1)} & T^2_{(2,0),(0,2)} \\ T^2_{(1,1),(2,0)} & T^2_{(1,1),(1,1)} & T^2_{(1,1),(0,2)} \\ T^2_{(0,2),(2,0)} & T^2_{(0,2),(1,1)} & T^2_{(0,2),(0,2)} \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} \left(g_{\bar{1}1} \right)^2 & g_{\bar{1}1} g_{\bar{2}1} & \left(g_{\bar{2}1} \right)^2 \\ 2 g_{\bar{1}1} g_{\bar{1}2} & g_{\bar{2}1} g_{\bar{1}2} + g_{\bar{1}1} g_{\bar{2}2} & 2 g_{\bar{2}1} g_{\bar{2}2} \\ \left(g_{\bar{1}2} \right)^2 & g_{\bar{2}1} g_{\bar{2}2} & \left(g_{\bar{2}2} \right)^2 \end{pmatrix} \begin{pmatrix} 2 + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}} \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & 1 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}} \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & 2 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}} \end{pmatrix}^{-1}.$$

6.3 Deformation quantization for complex Grassmann manifold

In this section, recurrence relations to obtain star products on complex Grassmann manifolds are derived. Especially we calculate star products of $\mathbb{C}P^N$. Note that this star product is also equal to the ones given in [5, 12, 28], and if we put some restriction our star product is also equal to the one given in [1], as they are shown in [27, 21]. The equivalence is also discussed in [31, 32]. In addition, recurrence relations to construct star products for $G_{2,2}$ was derived. Deformation quantization of Grassmann manifolds and flag manifolds were studied in [17, 10, 11, 24].

Complex Grassmann manifold $G_{p,q}$ is defined as a set of the whole p dimensional part vector space of $p+q$ dimensional vector space V. The local coordinate can be defined in a similar way to S. Kobayashi and K. Nomizu pp. 160-162[61].

Let U be an open subset of $G_{p,q}$. A chart (U, ϕ) is defined by

$$U := \left\{ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \in M\left(p+q, p; \mathbb{C}\right); |Y_0| \neq 0 \right\}$$

and

$$\phi : U \longrightarrow M\left(q, p; \mathbb{C}\right)$$

where

$$\phi\left(Y\right) = Y_1 Y_0^{-1}.$$

This is a holomorphic map of U onto an open subset of $p \times q$-dimensional complex space.

In this section, capital letter indices $A, B, C \cdots$ mean $aa', bb', cc' \cdots$. In the inhomogeneous coordinates $z^I := z^{ii'}$, $z^{\bar{I}} := z^{\bar{i}\bar{i}'}, (i = 1, 2, \cdots p, i' = 1, 2, \cdots q)$, the Kähler potential of $G_{p,q}$ is given as

$$\Phi = \ln\left|E_q + Z^\dagger Z\right|, \tag{6.3.1}$$

where $Z = \phi\left(Y\right) = \left(z^I\right) \in M\left(q, p; \mathbb{C}\right)$ and $E_q \in M\left(q, q; \mathbb{C}\right)$ is the unite matrix. From (12.1), the following facts are derived.

Fact 6.9. The Fubini-Study metric $(g_{I\bar{J}})$ is

$$ds^2 = 2 g_{I\bar{J}} dz^I d\bar{z}^J,$$

where

$$g_{I\bar{J}} := g_{ii'\bar{j}\bar{j}'} = \partial_I \partial_{\bar{J}} \Phi = a^{ji} b^{i'j'},\ g^{I\bar{J}} := g^{ii'\bar{j}\bar{j}'} = a_{ij} b_{j'i'}.$$

with

$$a_{ij} = \delta_{ij} + z^{ik'} \bar{z}^{jk'},\ b_{i'j'} = \delta_{i'j'} + \bar{z}^{ki'} z^{kj'}.$$

Fact 6.10. The curvature of a complex Grassmann manifold is

$$R_{\bar{A}}{}^{\bar{C}\bar{D}}{}_{\bar{B}} = g^{P\bar{C}} g^{Q\bar{D}} R_{\bar{A}PQ\bar{B}} = -\delta_{\overline{ab'}\,\overline{C}} \delta_{\overline{ba'}\,\overline{D}} - \delta_{\overline{ba'}\,\overline{C}} \delta_{\overline{ab'}\,\overline{D}}, \tag{6.3.2}$$

where

$$\delta_{\overline{ab'}\,\overline{cd'}} = \begin{cases} 1 & (a = c, b' = d') \\ 0 & (otherwise) \end{cases} .$$

From these facts, we can derive the recurrence relations to determine star products on the Grassmann manifolds.

A function similar to the determinant is defined on the matrix space.

Definition 6.11. Let $C = (C_{k,l})_{1 \leq k \leq n, 1 \leq l \leq n}$ be a $n \times n$ matrix. We define $| \cdot |^+$ as a $\mathbb{C}$-valued function on $M(n, n; \mathbb{C})$ such that

$$|C|^+ := \sum_{\sigma_n \in S_n} \prod_{k=1}^{n} C_{k,\sigma_n(k)}.$$

Example 6.12. Here we show some examples. These suggest some properties like determinant.

1.
$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^+ = c_{11}c_{22} + c_{12}c_{21}$$

2.
$$\begin{aligned} & \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^+ \\ &= c_{11}c_{22}c_{33} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} \\ &= c_{11} \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix}^+ + c_{12} \begin{vmatrix} c_{11} & c_{13} \\ c_{31} & c_{33} \end{vmatrix}^+ + c_{13} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^+ \end{aligned}$$

Remark 6.13. Similar to a determinant

$$\left|{}^tC\right|^+ = |C|^+,$$

where ${}^tC$ is a transposed matrix of C.

The following is a proposition similar to cofactor expansion of a determinant.

Proposition 6.14.

$$|C|^+ = \begin{vmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{vmatrix}^+ = \sum_{j=1}^{n} c_{ij} \begin{vmatrix} c_{11} & \cdots & \hat{c_{1j}} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{c_{i1}} & \cdots & \hat{c_{ij}} & \cdots & \hat{c_{in}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & \hat{c_{nj}} & \cdots & c_{nn} \end{vmatrix}^+$$

Proof. *A proof for this function is similar to the case of determinants.* ☐

Definition 6.15. A matrix $G^{\vec{\alpha_n},\vec{\beta_n^*}}$ is defined by using the Riemannian metrics on M. Its elements are metrics on M and are located as follows. $\vec{\alpha_n}$ and $\vec{\beta_n}$ are elements of $\mathbb{Z}^N$.

$$G^{\vec{\alpha_n},\vec{\beta_n^*}} = \begin{pmatrix} \tilde{G}_{11} & \cdots & \tilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{n1} & \cdots & \tilde{G}_{nn} \end{pmatrix}$$

where

$$\tilde{G}_{pq} =: g_{p\bar{q}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M\left(\alpha_p^n, \beta_q^n; \mathbb{C}\right)$$

i. e.

$$G^{\vec{\alpha_n},\vec{\beta_n^*}} = \left(\begin{array}{ccc|c|ccc}
g_{1\bar{1}} & \cdots & g_{1\bar{1}} & & g_{1\bar{N}} & \cdots & g_{1\bar{N}} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
g_{1\bar{1}} & \cdots & g_{1\bar{1}} & & g_{1\bar{N}} & \cdots & g_{1\bar{N}} \\ \hline
 & \vdots & & \ddots & & \vdots & \\ \hline
g_{N\bar{1}} & \cdots & g_{N\bar{1}} & & g_{N\bar{N}} & \cdots & g_{N\bar{N}} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
g_{N\bar{1}} & \cdots & g_{N\bar{1}} & & g_{N\bar{N}} & \cdots & g_{N\bar{N}} \\
\multicolumn{3}{c}{\underbrace{\hspace{3cm}}_{\beta_1^n}} & \multicolumn{1}{c}{\cdots} & \multicolumn{3}{c}{\underbrace{\hspace{3cm}}_{\beta_N^n}}
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}\alpha_1^n \\
\vdots \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\}\alpha_N^n \\
\\
\end{array}.$$

For example $N = 2, \vec{\alpha_3} = (2,1), \vec{\beta_3^*} = (1,2)^*$, then $G^{\vec{\alpha_3},\vec{\beta_3^*}}$ is determined as

$$G^{\vec{\alpha_3},\vec{\beta_3^*}} = \left(\begin{array}{c|cc} g_{1\bar{1}} & g_{1\bar{2}} & g_{1\bar{2}} \\ g_{1\bar{1}} & g_{1\bar{2}} & g_{1\bar{2}} \\ \hline g_{2\bar{1}} & g_{2\bar{2}} & g_{2\bar{2}} \end{array}\right).$$

From Proposition 12.3, we obtain the following corollary.

Corollary 6.16. *For a matrix* $G^{\vec{\alpha_n},\vec{\beta^*_n}}$,

$$\left|G^{\vec{\alpha_n},\vec{\beta^*_n}}\right|^+ = \sum_{J=1}^{N} \beta^n_J g_{\bar{J}I} \left|G^{\vec{\alpha_n}-\vec{e_I},\vec{\beta^*_n}-\vec{e_J}}\right|^+ = \sum_{K=1}^{N} \alpha^n_K g_{\bar{I}K} \left|G^{\vec{\alpha_n}-\vec{e_K},\vec{\beta^*_n}-\vec{e_I}}\right|^+ .$$

6.4 Deformation quantization for a complex projective space

In this subsection, we obtain concrete expression of star products on $\mathbb{C}P^N$. A complex projective space $\mathbb{C}P^N$ is a Grassmann manifold $G_{1,N}$ by definition.

Proposition 6.17. *Let M be a complex projective space and f and g be smooth functions on M. The recurrence relation of $T^n_{\vec{\alpha_n}\vec{\beta^*_n}}$ given in (10.3) is*

$$T^n_{\vec{\alpha_n}\vec{\beta^*_n}} = \sum_{d=1}^{N} \frac{\hbar g_{\bar{i}d}}{(1+\hbar-\hbar n)\,\beta^n_i} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta^*_n}-\vec{e_i}}. \tag{6.4.1}$$

Proof. *The curvature (12.2) is substituted for Theorem 10.6, and the following is proved.*

$$\begin{aligned}
&\sum_{d=1}^{N} \hbar g_{\bar{i}d} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta^*_n}-\vec{e_i}} \\
&= \beta_i T^n_{\vec{\alpha_n}\vec{\beta^*_n}} + \sum_{k=1}^{N}\sum_{p=1}^{N} \frac{\hbar\left(\beta^n_k-\delta_{kp}-\delta_{ik}+1\right)\left(\beta^n_k-\delta_{kp}-\delta_{ik}+2\right)}{2} R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+2\vec{e_k}-\vec{e_i}} \\
&\quad + \sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\sum_{p=1}^{N} \hbar\left(\beta^n_k-\delta_{kp}-\delta_{ik}+1\right)\left(\beta^n_{k+l}-\delta_{(k+l),p}-\delta_{i,(k+l)}+1\right) R_{\bar{p}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+\vec{e_k}+e_{\vec{k+l}}-\vec{e_i}}.
\end{aligned}$$

We also use

$$R_{\bar{p}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}} = -\delta_{\bar{p},\overline{k+l}}\delta_{\bar{i},\bar{k}} - \delta_{\bar{i},\overline{k+l}}\delta_{\bar{p},\bar{k}},\, R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}} = -\delta_{\bar{p},\bar{k}}\delta_{\bar{i},\bar{k}} - \delta_{\bar{i},\bar{k}}\delta_{\bar{p},\bar{k}},$$

then the above is rewritten as

$$
\begin{aligned}
&\sum_{d=1}^{N} \hbar g_{\bar{i}d} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta^*_n}-\vec{e_i}} \\
&= \beta_i T^n_{\vec{\alpha_n}\vec{\beta^*_n}} - \sum_{k=1}^{N}\sum_{p=1}^{N} \frac{\hbar\left(\beta^n_k - \delta_{kp} - \delta_{ik} + 1\right)\left(\beta^n_k - \delta_{kp} - \delta_{ik} + 2\right)\left(\delta_{\overline{p},\bar{k}}\delta_{\bar{i},\bar{k}} + \delta_{\bar{i},\bar{k}}\delta_{\overline{p},\bar{k}}\right)}{2} T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+2\vec{e_k}-\vec{e_i}} \\
&\quad - \sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\sum_{p=1}^{N} \hbar\left(\beta^n_k - \delta_{kp} - \delta_{ik} + 1\right)\left(\beta^n_{k+l} - \delta_{(k+l),p} - \delta_{i,(k+l)} + 1\right) \\
&\qquad \times \left(\delta_{\overline{p},\overline{k+l}}\delta_{\bar{i},\bar{k}} + \delta_{\bar{i},\overline{k+l}}\delta_{\overline{p},\bar{k}}\right) T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_p}+\vec{e_k}+e_{\vec{k+l}}-\vec{e_i}}.
\end{aligned}
$$

The theorem follows from this. □

Theorem 6.18. *Let f and g be smooth functions on a projective space $\mathbb{C}P^N$. A star product with separation of variables on a projective space $\mathbb{C}P^N$ is given as*

$$
f * g = f \cdot g + \sum_{n=1}^{\infty}\sum_{\vec{\alpha_n}\vec{\beta^*_n}} \left|G^{\vec{\alpha_n},\vec{\beta^*_n}}\right|^{+} \left\{\prod_{k=0}^{n} \frac{\hbar}{\left(1+\hbar-\hbar k\right)\alpha^n_k!\beta^n_k!}\right\} \left(D^{\vec{\alpha_n}} f\right)\left(D^{\vec{\beta^*_n}} g\right). \tag{6.4.2}
$$

Proof. *We show that*

$$
T^n_{\vec{\alpha_n}\vec{\beta^*_n}} = \left|G^{\vec{\alpha_n},\vec{\beta^*_n}}\right|^{+} \left\{\prod_{k=0}^{n} \frac{\hbar}{\left(1+\hbar-\hbar k\right)\alpha^n_k!\beta^n_k!}\right\}
$$

satisfies (12.3). The R. H. S of (12.3) for this case is given as

$$
\sum_{d=1}^{N} \frac{\hbar g_{\bar{i}d}}{\left(1+\hbar-\hbar n\right)\beta^n_i} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta^*_n}-\vec{e_i}} = \sum_{d=1}^{N} g_{\bar{i}d}\alpha^n_d \left|G^{\vec{\alpha_n}-\vec{e_d},\vec{\beta^*_n}-\vec{e_i}}\right|^{+} \frac{\hbar}{\left(1+\hbar-\hbar n\right)} \prod_{k=0}^{n-1} \frac{\hbar}{\left(1+\hbar-\hbar k\right)\alpha^n_k!\beta^n_k!}.
$$

Using Corollary 12.4, R. H. S. of the above is rewritten as

$$
\left|G^{\vec{\alpha_n},\vec{\beta^*_n}}\right|^{+} \prod_{k=0}^{n} \frac{\hbar}{\left(1+\hbar-\hbar k\right)\alpha^n_k!\beta^n_k!}.
$$

This shows the given $T^n_{\vec{\alpha_n}\vec{\beta^*_n}}$ satisfies the recurrence relation (12.3). □

Fact 6.19. Let f and g be smooth functions on a projective space $\mathbb{C}P^N$. A star product on a projective space $\mathbb{C}P^N$ is given in [28] as

$$\begin{aligned}
f\tilde{*}g &= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar)g^{\bar{j}_1 k_1}\cdots g^{\bar{j}_n k_n}}{n!\Gamma(1+1/\hbar)} \left(\nabla_{\bar{j}_1}\cdots\nabla_{\bar{j}_n} f\right)\left(\nabla_{k_1}\cdots\nabla_{k_n} g\right) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar)}{n!\Gamma(1+1/\hbar)} \left(D^{k_1}\cdots D^{k_n} f\right)\left(\nabla_{k_1}\cdots\nabla_{k_n} g\right) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar)g_{\bar{m}_1 k_1}\cdots g_{\bar{m}_n k_n}}{n!\Gamma(1+1/\hbar)} \left(D^{k_1}\cdots D^{k_n} f\right)\left(D^{\bar{m}_1}\cdots D^{\bar{m}_n} g\right). \qquad (6.4.3)
\end{aligned}$$

As mentioned in Section 2, the star product with separation of variables is uniquely determined. This fact means (12.4) coincides with (12.5). This coincidence is easily checked from Definition 60.

6.5 Deformation quantization for a $G_{2,2}$

In this subsection, we derive the recurrence relation to obtain concrete expression of star products on a Grassmann manifold $G_{2,2}$. The inhomogeneous coordinates are $z^{11'}, z^{12'}, z^{21'}$ and $z^{22'}$. To decide the order of coordinates is useful in order to calculate the finite sum. We set the order:$11' < 12' < 21' < 22'$. In this subsection, j is used as "Not i". That means that if $i = 1$ then $j = 2$ and if $i = 2$ then $j = 1$. For example, if $I = ii' = 11'$, then $ij' = 12', ji' = 21', J = 22'$. If $I = ii' = 12'$, then $ij' = 11', ji' = 22', J = 21'$. A finite sum is defined as

$$\sum_{D=1}^{4} a_D := a_{11'} + a_{12'} + a_{21'} + a_{22'}.$$

Theorem 6.20. *Let f and g be smooth functions on $G_{2,2}$. The recurrence relation of $T^n_{\vec{\alpha_n}\vec{\beta^*_n}}$ given in (10.3) is*

$$\begin{aligned}
&\beta_I\left(1+\hbar-\hbar\beta^n_I-\hbar\beta^n_{ji'}-\hbar\beta^n_{ij'}\right)\ T^n_{\vec{\alpha_n}\vec{\beta^*_n}} - \hbar\left(\beta^n_{ij'}+1\right)\left(\beta^n_{ji'}+1\right)\ T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_J}+\vec{e_{ij'}}+\vec{e_{ji'}}-\vec{e_I}} \\
&= \hbar g_{\bar{I}I}T^{n-1}_{\vec{\alpha_n}-\vec{e_I}\vec{\beta^*_n}-\vec{e_I}} + \hbar g_{\bar{I}ij'}T^{n-1}_{\vec{\alpha_n}-\vec{e_{ij'}}\vec{\beta^*_n}-\vec{e_I}} + \hbar g_{\bar{I}ji'}T^{n-1}_{\vec{\alpha_n}-\vec{e_{ji'}}\vec{\beta^*_n}-\vec{e_I}} + \hbar g_{\bar{I}J}T^{n-1}_{\vec{\alpha_n}-\vec{e_J}\vec{\beta^*_n}-\vec{e_I}}. \qquad (6.5.1)
\end{aligned}$$

for each I.

Proof. *The curvature (12.2) is substituted into Theorem 10.6, and the following is obtained.*

$$\begin{aligned}
&\sum_{D=1}^{4} \hbar g_{\bar{I}D} T^{n-1}_{\vec{\alpha_n}-\vec{e_D}\vec{\beta_n^*}-\vec{e_I}} \\
&= \beta_I T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \\
&- \sum_{K=1}^{4}\sum_{P=1}^{4} \frac{\hbar\left(\beta^n_K - \delta_{KP} - \delta_{IK} + 1\right)\left(\beta^n_K - \delta_{KP} - \delta_{IK} + 2\right)\left(\delta_{\overline{pi'},\bar{K}}\delta_{\overline{ip'},\bar{K}} + \delta_{\overline{ip'},\bar{K}}\delta_{\overline{pi'},\bar{K}}\right)}{2} T^n_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_P}+2\vec{e_K}-\vec{e_I}} \\
&\quad - \sum_{K=1}^{4-1}\sum_{L=1}^{4-K}\sum_{P=1}^{4} \hbar\left(\beta^n_K - \delta_{KP} - \delta_{IK} + 1\right)\left(\beta^n_{K+L} - \delta_{(K+L),P} - \delta_{I,(K+L)} + 1\right) \\
&\qquad \times \left(\delta_{\overline{pi'},\overline{K+L}}\delta_{\overline{ip'},\bar{K}} + \delta_{\overline{ip'},\overline{K+L}}\delta_{\overline{pi'},\bar{K}}\right) T^n_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_P}+\vec{e_K}+\vec{e_{K+L}}-\vec{e_I}} \\
&= \beta_I \left\{1 + \hbar - \hbar\beta^n_I - \hbar\beta^n_{ji'} - \hbar\beta^n_{ij'}\right\} T^n_{\vec{\alpha_n}\vec{\beta_n^*}} - \hbar\left(\beta^n_{ij'} + 1\right)\left(\beta^n_{ji'} + 1\right) T^n_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_J}+\vec{e_{ij'}}+\vec{e_{ji'}}-\vec{e_I}}
\end{aligned}$$

The theorem follows from this. □

Star products on a noncommutative $G_{2,2}$ are determined by this formula recursively. For general $G_{p,q}$, the recurrence relations are determined in a similar way.

Chapter 7

Summary

In this paper, we considered noncommutative geometry of Kähler manifolds often from the viewpoint of deformation quantization in detail when the Kähler manifold is locally symmetric. We also discussed the relationship between a gauge theory defined on noncommutative $\mathbb{R}^4$, one of the noncommutative Kähler manifolds, and the Ricci flat metric on a commutative Hermitian manifold.

From various methods for constructing noncommutative geometry, we chose a deformation quantization which replace products of functions by noncommutative products called star products. In particular for the noncommutative deformation of Kähler manifolds, we used the deformation quantization with separation of variables introduced by Karabegov.

At first we focused on the gauge theory defined on the noncommutative $\mathbb{R}^4$, which is the simplest example of a noncommutative Kähler manifold. The relationship between $U(1)$ gauge theory in the noncommutative $\mathbb{R}^4$ and the Ricci flat metric of a Hermitian manifold was discussed. Yang et al. had already discussed the relationship between the instanton of the noncommutative $U(1)$ gauge theory and the Eguchi-Hanson metric/Kähler metric. The relation is based on the idea of the Seiberg-Witten map which is the map from noncommutative gauge fields to commutative gauge fields. Yang et al. gave a new interpretation to the correspondence of gauge theories on a noncommutative space and a commutative space. We extended the researches, and we showed that noncommutative $U(1)$ instantons constitute Ricci flat metrics of Hermitian manifolds using noncommutative $U(1)$ gauge connection and metric correspondence by Yang. We have also constructed a number of actual examples. Under the asymptotically flat conditions, it was shown that this Ricci flat metric leads to an instanton solution.

Next we focused on noncommutative deformation of non-flat Kähler manifolds. It is known that the star products which are products of deformation quantization can be constructed in all

Kähler manifolds. However, even if it is known that a star product exists in a manifold, it does not mean that the star product that can be actually calculated is obtained. The explicit star products of $\mathbb{C}^n$, projective spaces, and hyperbolic spaces were obtained by the method of the deformation quantization with separation of variables, but no other examples have been known. The authors newly construct star products that holds for any Riemann surfaces, and recurrence formulas that allows star products to be constructed in algebraic ways for Grassmann manifolds.

The following are future issues. What is obtained in the deformation quantization of the local symmetric Kähler manifold is a recurrence formula that constitutes a noncommutative product. In the case of a compact Riemann surface or complex projective space, this recurrence formula can be solved, but in the case of a complex two-dimensional case or a complex Grassmann manifold, the solutions are not obvious. If we try to obtain deformation quantization of other Kähler manifolds, we need to solve the problem.

Instanton solution and Hermitian-Einstein metric are very important objects in physics, but the physical conditions are slightly different. Instanton solution is one of the solutions of gauge theory, but not all solutions. In other words, the set of gauge theory solutions is a little wider. Einstein metrics that appear in astronomy are called pseudo-Riemannian metric and are not positive. For real physics we need to study a little more.

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