Construction and investigation of parameterized families of operator means

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March, 2017

Acknowledgements

I would like to thank Professor Masahiro Yanagida for his guidance and advice.

I would like to thank Professor Takeaki Yamazaki from Toyo University for his helpful suggestions and comments.

Contents

1 Introduction **3**
1.1 Operator monotone functions and operator means 3
1.2 Several examples 6

2 Operator monotonicity of $\exp\{f(x)\}$ **8**
2.1 Characterization 8
2.2 Applications 10

3 A method to obtain families of operator monotone functions **15**
3.1 The method and its application 16
3.2 Interpolation of some operator means 19

4 2-parameter Stolarsky mean **22**
4.1 Range of the two parameters 23
4.1.1 Known part 23
4.1.2 Trivial part 23
4.1.3 Extension from the Stolarsky mean 23
4.1.4 Investigation of the range 24

Chapter 1

Introduction

1.1 Operator monotone functions and operator means

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. An operator A is called a positive semi-definite operator if and only if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. We denote by $\mathcal{B}(\mathcal{H})_+$ the set of all positive semi-definite operators in $\mathcal{B}(\mathcal{H})$. $A > 0$ (A is positive definite) means A is positive semi-definite and invertible. For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$, $A \leq B$ denotes that $B - A$ is positive semi-definite.

Throughout this thesis, we assume that functions are not constants. A real-valued continuous function $f(x)$ defined on an interval I in $\mathbb{R}$ is called an *operator monotone function* if for every pair $A, B \in \mathcal{B}(\mathcal{H})$ whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I, $A \leq B$ implies $f(A) \leq f(B)$. Many researchers have studied operator monotone functions and obtained many results. Here we introduce a characterization of operator monotone functions established by Löwner [6];

Theorem L ([6]). *Let $-\infty \leq a < b \leq \infty$ and f be a real-valued function on (a, b) . Then the following are equivalent:*
(1) *f is an operator monotone function on (a, b),*
(2) *f has an analytic continuation to the upper half plane $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Im z > 0\}$, and $z \in \mathbb{C}^+$ implies $f(z) \in \mathbb{C}^+$, where $\Im z$ is the imaginary part of z.*

We call a real-valued function $f(x)$ on an interval in $\mathbb{R}$ a *Pick function* if $f(x)$ has an analytic continuation to $\mathbb{C}^+$ which maps $\mathbb{C}^+$ into itself. The above theorem shows a relationship between operator monotone functions and Pick functions. When we check operator monotonicity of a function $f(x)$, we often consider the argument of $f(z)$ $(z \in \mathbb{C}^+)$.

The next theorem gives a typical example of operator monotone functions;

Theorem L-H ([3], [6]). *Let $A, B \in \mathcal{B}(\mathcal{H})$. If $0 \le A \le B$, then*

$$A^\alpha \le B^\alpha$$

holds for every $\alpha \in (0, 1]$.

This result says that the function $f_\alpha(x) := x^\alpha$ is an operator monotone function on $[0, \infty)$ for every $\alpha \in (0, 1]$, and is obtained by Theorem L as follows;

Let $f_\alpha(x) = x^\alpha$ $(0 < \alpha \le 1)$. f_α has an analytic continuation to the upper half plane $\mathbb{C}^+$. If $z \in \mathbb{C}^+$ we can suppose that $0 < \arg z < \pi$. Then

$$0 < \arg z^\alpha = \alpha \arg z \le \arg z < \pi.$$

So we have $f_\alpha(z) = z^\alpha \in \mathbb{C}^+$ and $f_\alpha(x)$ is operator monotone by Theorem L. It is also shown by Theorem L that f_α is not operator monotone for $\alpha > 1$. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = B \text{ and } A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \not\le \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = B^2.$$

Moreover, the logarithmic function $\log x$ is an operator monotone function on $(0, \infty)$, too, which is shown as follows;

Assume that $0 < A \le B$. From Theorem L-H, we have

$$\frac{A^\alpha - I}{\alpha} \le \frac{B^\alpha - I}{\alpha}$$

for every $\alpha \in (0, 1]$. By tending $\alpha \searrow 0$, both sides of the above inequality converge to $\log A$ and $\log B$ in the norm topology, respectively. So

$$\log A \le \log B$$

holds for $0 < A \le B$, namely, $\log x$ is an operator monotone function.

A map $\mathfrak{M}(\cdot, \cdot)$: $\mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ is called an *operator mean* [5] if $\mathfrak{M}(\cdot, \cdot)$ satisfies the following four conditions for $A, B, C, D \in \mathcal{B}(\mathcal{H})_+$;

(1) $A \leq C$ and $B \leq D$ imply $\mathfrak{M}(A,B) \leq \mathfrak{M}(C,D)$,
(2) $X(\mathfrak{M}(A,B))X \leq \mathfrak{M}(XAX, XBX)$ for all self-adjoint $X \in \mathcal{B}(\mathcal{H})$,
(3) $A_n \searrow A$ and $B_n \searrow B$ imply $\mathfrak{M}(A_n, B_n) \searrow \mathfrak{M}(A,B)$,
(4) $\mathfrak{M}(I,I) = I$.
($A_n \searrow A$ denotes that $A_1 \geq A_2 \geq \cdots$ and A_n converges to A in the strong operator topology.)

Trivial examples of operator means are the arithmetic mean and the harmonic mean in the following example;

Example (The arithmetic and harmonic means)**.**
(1) The map $\mathfrak{A}(\cdot,\cdot) : \mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ defined by

$$\mathfrak{A}(A,B) := \frac{A+B}{2}$$

is an operator mean, and called the arithmetic mean.
(2) The map $\mathfrak{H}(\cdot,\cdot) : \mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ defined by

$$\mathfrak{H}(A,B) := 2\big(A^{-1} + B^{-1}\big)^{-1}$$

is an operator mean, and called the harmonic mean.

We can easily show that both the arithmetic and harmonic means are operator means, but it is pretty difficult for other cases to confirm that they are operator means.

Here we introduce a quite useful theorem to study operator means. This theorem shows a relationship between operator means and operator monotone functions;

Theorem K-A (Kubo-Ando [5])**.** *For any operator mean $\mathfrak{M}(\cdot,\cdot)$, there uniquely exists an operator monotone function $f \geq 0$ on $[0,\infty)$ with $f(1) = 1$ such that*

$$f(x)I = \mathfrak{M}(I, xI), \quad x \geq 0,$$

where I means the identity operator. Then the following hold:
(1) *The map $\mathfrak{M}(\cdot,\cdot) \mapsto f$ is a one-to-one onto affine mapping from the set of all operator means to that of all non-negative operator monotone functions on $[0,\infty)$ with $f(1) = 1$. Moreover, the map preserves the order, i.e., for $\mathfrak{M}(\cdot,\cdot) \mapsto f,\ \mathfrak{N}(\cdot,\cdot) \mapsto g$,*

$$\mathfrak{M}(A,B) \leq \mathfrak{N}(A,B) \quad (\forall A, B \in \mathcal{B}(\mathcal{H})_+) \iff f(x) \leq g(x) \quad (\forall x \geq 0).$$

(2) *When $A > 0$, $\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$.*

The function $f(x)$ in Theorem K-A is called the *representing function* of $\mathfrak{M}(\cdot,\cdot)$. The representing functions of $\mathfrak{A}(\cdot,\cdot)$ and $\mathfrak{H}(\cdot,\cdot)$ are $A(x) := \frac{x+1}{2}$ and $H(x) := \frac{2x}{x+1}$, respectively. From this theorem, it is enough to consider operator monotone functions when we study operator means.

Also, it is well-known that $\frac{x}{f(x)}$ is an operator monotone function on $(0,\infty)$ if $f(x)$ is a positive-valued operator monotone function on $(0,\infty)$, and the operator mean with the representing function $\frac{x}{f(x)}$ is called the *dual* of $\mathfrak{M}(\cdot,\cdot)$. We remark that $\mathfrak{A}(\cdot,\cdot)$ and $\mathfrak{H}(\cdot,\cdot)$ are mutually dual.

1.2 Several examples

In this section, we shall introduce well-known examples of operator means and their representing functions.

Example (The geometric mean)**.** The geometric mean $\mathfrak{G}(\cdot,\cdot) : \mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ is defined by

$$\mathfrak{G}(A,B) := A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}},$$

and its representing function is $G(x) := x^{\frac{1}{2}}$, which is operator monotone on $[0,\infty)$ as mentioned above in Theorem L-H. This mean is self-dual.

Example (The identric mean)**.**

$$ID(x) := \begin{cases} \frac{1}{e} & (x=0) \\ 1 & (x=1) \\ \frac{1}{e}x^{\frac{x}{x-1}} = \exp\left(\frac{x\log x}{x-1} - 1\right) & (x \neq 0,1) \end{cases}$$

is an operator monotone function on $[0,\infty)$ ([8]) and known as the representing function of the identric mean.

The exponential function $\exp(x)$ is well known as a function which is not operator monotone, in contrast with its inverse function $\log x$ is so. For example,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \leq \begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 1 \end{pmatrix} = B$$

and

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2} \end{pmatrix} \not\leq \frac{1}{5e} \begin{pmatrix} 3e^5 + 2 & \sqrt{6}(e^5 - 1) \\ \sqrt{6}(e^5 - 1) & 2e^5 + 3 \end{pmatrix} = \exp(B).$$

But there exists a function $f(x)$ such that $\exp\{f(x)\}$ is an operator monotone function, like $ID(x)$. In general, it is so difficult to check operator monotonicity of $\exp\{f(x)\}$ because $\exp\{f(x)\}$ is the composite function of the non-operator-monotone function $\exp(x)$ with $f(x)$. In Chapter 2, we will obtain a characterization of such functions by using Theorem L and Euler's formula. Thanks to this result, it has become easy to check operator monotonicity of $\exp\{f(x)\}$ by simple computation.

Example (The weighted power mean)**.** The map $\mathfrak{P}_{r,\alpha}(\cdot,\cdot) : \mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ defined as

$$\mathfrak{P}_{r,\alpha}(A,B) := A^{\frac{1}{2}} \left((1-\alpha)I + \alpha \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right)^r \right)^{\frac{1}{r}} A^{\frac{1}{2}}$$

is an operator mean for every $r \in [-1,1] \backslash \{0\}$, $\alpha \in (0,1]$, and its representing function is

$$P_{r,\alpha}(x) := \left((1-\alpha) + \alpha x^r \right)^{\frac{1}{r}}.$$

The case $r = 0$ is defined by the limit $P_{0,\alpha}(x) := \lim_{r \to 0} P_{r,\alpha}(x) = x^{\alpha}$. Moreover, for each fixed $x > 0$ and $\alpha \in (0,1]$, $P_{r,\alpha}(x)$ is increasing on $r \in [-1,1]$.

The above weighted power mean is a 2-parameter family of operator means, namely, a 2-parameter family of operator monotone functions on $(0,\infty)$, and interpolates many operator means. In fact, for $\alpha = \frac{1}{2}$, it coincides with the arithmetic, geometric and harmonic means if $r = 1$, $r = 0$ and $r = -1$, respectively. In Chapter 3, we will introduce a new way to get a family of operator monotone functions and construct another 2-parameter family of operator monotone functions

$$F_{r,s}(x) := \left(\frac{r(x^{r+s} - 1)}{(r+s)(x^r - 1)} \right)^{\frac{1}{s}}$$

by using $P_{r,\alpha}(x)$.

In Chapter 4, we will investigate a range of parameters such that $F_{r,s}(x)$ is operator monotone, and try to extend it.

Chapter 2

Operator monotonicity of $\exp\{f(x)\}$

2.1 Characterization

Here we give a characterization of a continuous function $f(x)$ on $(0, \infty)$ such that $\exp\{f(x)\}$ is an operator monotone function. It is clear that $f(x) = \log x$ satisfies this condition. The principal branch of $\log z$ is defined as

$$\mathrm{Log}z := \log r + i\theta \ (z := re^{i\theta}, -\pi < \theta \leq \pi).$$

It is an analytic continuation of the real logarithmic function to $\mathbb{C} \setminus (-\infty, 0]$. Moreover it is a Pick function, namely an operator monotone function by Theorem L, and satisfies $\Im \mathrm{Log}z = \theta$. In the following we think about the case $f(x)$ is not the logarithmic function:

Theorem 1 ([10])**.** *Let $f(x)$ be a continuous function on $(0, \infty)$. If $f(x)$ is not either a constant or $\log(\alpha x)$ $(\alpha > 0)$, then the following are equivalent:*

(1) $\exp\{f(x)\}$ *is an operator monotone function.*

(2) *There exists an analytic continuation satisfying*

$$0 < v(r, \theta) < \theta$$

where $u(r, \theta)$ and $v(r, \theta)$ are real-valued continuous functions such that $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ $(0 < r,\ 0 < \theta < \pi)$ (therefore $f(x)$ is operator monotone).

Remark 1. In [2] Hansen proved a necessary and sufficient condition for $\exp\{F(\log x)\}$ to be an operator monotone function, that is, F admits an analytic continuation to $\mathbb{S} := \{z \in \mathbb{C} \mid 0 < \Im z < \pi\}$ and $F(z)$ maps from $\mathbb{S}$ into itself. The condition of Theorem 1 is more rigid than this statement.

Proof. (1) $\Longrightarrow$ (2).
Since $\exp\{f(x)\}$ is operator monotone, it is a Pick function by Theorem L, so there exists an analytic continuation to the upper half plane $\mathbb{C}^+$ and $z \in \mathbb{C}^+$ implies $\exp\{f(z)\} \in \mathbb{C}^+$. For $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, let $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. Using Euler's formula, we obtain

$$\exp\{f(z)\} = \exp\{u(r,\theta)\}\big(\cos\{v(r,\theta)\} + i\sin\{v(r,\theta)\}\big).$$

So we have $\Im \exp\{f(z)\} = \exp\{u(r,\theta)\}\sin\{v(r,\theta)\}$, and hence $0 < \sin\{v(r,\theta)\}$. We note that $v(r,\theta)$ is continuous on its domain. From these facts, we can find that

$$2n\pi < v(r,\theta) < (2n+1)\pi$$

holds for the unique $n \in \mathbb{Z}$. Since $f(z)$ takes real values on the real axis, $\lim_{\theta\to 0} v(r,\theta) = 0$, so that $n = 0$, namely,

$$0 < v(r,\theta) < \pi.$$

On the other hand, from the operator monotonicity of $\exp\{f(x)\}$ and the assumption of Theorem 1, $x[\exp\{f(x)\}]^{-1}$ is a positive operator monotone function on $(0,\infty)$, too. So we get

$$\begin{aligned} z[\exp\{f(z)\}]^{-1} &= \exp\{\mathrm{Log} z - f(z)\} \\ &= \exp\{(\log r - u(r,\theta)) + i(\theta - v(r,\theta))\} \\ &= \exp\{\log r - u(r,\theta)\}\big(\cos\{\theta - v(r,\theta)\} + i\sin\{\theta - v(r,\theta)\}\big). \end{aligned}$$

From the above,

$$2m\pi < \theta - v(r,\theta) < (2m+1)\pi$$

holds for the unique $m \in \mathbb{Z}$. Moreover, $0 < v(r,\theta) < \pi$ and $0 < \theta < \pi$ imply

$$-\pi < \theta - v(r,\theta) < \pi.$$

From these facts, $v(r,\theta)$ must satisfy $0 < \theta - v(r,\theta) < \pi$, so we get

$$0 < v(r,\theta) < \theta.$$

(2) $\Longrightarrow$ (1).
Since the set of all holomorphic functions is closed under composition,

$\exp\{f(z)\}$ is holomorphic on the upper half plane $\mathbb{C}^+$. Let $z = re^{i\theta} \in \mathbb{C}^+$. From the assumption $0 < v(r,\theta) < \theta < \pi$,

$$0 < \sin\{v(r,\theta)\} \leq 1.$$

So we have

$$0 < \exp\{u(r,\theta)\} \sin\{v(r,\theta)\} = \Im \exp\{f(z)\}$$

and find

$$z \in \mathbb{C}^+ \Longrightarrow \exp\{f(z)\} \in \mathbb{C}^+,$$

that is, $\exp\{f(x)\}$ is a Pick function, and thus it is operator monotone by Theorem L. □

2.2 Applications

By Theorem 1, we can check numerically that $\exp\{f(x)\}$ is operator monotone or not if the imaginary part of $f(z)$ can be expressed concretely. Now we apply Theorem 1 and get some examples by "only" using simple computation.

Example 1 (The harmonic mean)**.**

$$H(x) = \frac{2x}{x+1}$$

is an operator monotone function on $[0,\infty)$, but $\exp\{H(x)\}$ is not operator monotone. Actually, by putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$H(z) = \frac{2(r^2 + r\cos\theta) + i(2r\sin\theta)}{r^2 + 1 + 2r\cos\theta}$$

and

$$v(r,\theta) := \Im H(z) = \frac{2r\sin\theta}{r^2 + 1 + 2r\cos\theta}.$$

When $r = 1, \theta = \frac{3}{4}\pi$, we get $v\left(1, \frac{3}{4}\pi\right) = \sqrt{2} + 1 > \frac{3}{4}\pi$, hence we can find $\exp\{H(x)\}$ is not an operator monotone function by Theorem 1.

Example 2 (The logarithmic mean)**.**

$$L(x) := \begin{cases} 0 & (x = 0) \\ 1 & (x = 1) \\ \dfrac{x-1}{\log x} & (x \neq 0, 1) \end{cases}$$

is an operator monotone function on $[0, \infty)$, and is the representing function of the logarithmic mean. But $\exp\{L(x)\}$ is not operator monotone. Actually, by putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$L(z) = \frac{\big\{(\log r)(r\cos\theta - 1) + r\theta\sin\theta\big\} + i\big\{(r\log r)\sin\theta - \theta(r\cos\theta - 1)\big\}}{(\log r)^2 + \theta^2}$$

and

$$v(r,\theta) := \Im L(z) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{(\log r)^2 + \theta^2}.$$

Since $\sin\theta > 0$, $\lim_{r\to\infty} v(r,\theta) = \infty$. Therefore, we find that $\exp\{L(x)\}$ is not an operator monotone function by Theorem 1. Actually, when $r = \exp\left\{\frac{\pi}{2}\right\}, \theta = \frac{\pi}{2}$, we get $v\left(\exp\left\{\frac{\pi}{2}\right\}, \frac{\pi}{2}\right) = \frac{\exp\{\frac{\pi}{2}\} + 1}{\pi} = 1.8495\cdots > 1.5707\cdots = \frac{\pi}{2}$.

Example 3 (The dual of the logarithmic mean)**.**

$$DL(x) := \begin{cases} 0 & (x = 0) \\ 1 & (x = 1) \\ \dfrac{x\log x}{x - 1} & (x \neq 0, 1) \end{cases}$$

is an operator monotone function on $[0, \infty)$ and $\exp\{DL(x)\}$ is operator monotone, too. In the following we verify that $DL(x)$ satisfies the condition of Theorem 1:
By putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$DL(z) = \frac{r\Big[\big\{(r - \cos\theta)\log r + \theta\sin\theta\big\} + i\big\{\theta(r - \cos\theta) - (\log r)\sin\theta\big\}\Big]}{r^2 + 1 - 2r\cos\theta}$$

and

$$v(r,\theta) := \Im DL(z) = \frac{r}{r^2 + 1 - 2r\cos\theta}\big\{\theta(r - \cos\theta) - (\log r)\sin\theta\big\}.$$

From operator monotonicity of $DL(x)$ we have $0 < v(r,\theta)$ by Theorem L. In the following we show $v(r,\theta) < \theta$, which is equivalent to $r\big\{\theta\cos\theta - (\log r)\sin\theta\big\} < \theta$. By using the following inequalities

$$\theta\cos\theta < \sin\theta < \theta \ (0 < \theta < \pi), \ \ r(1 - \log r) \leq 1 \ (0 < r),$$

we obtain

$$\begin{aligned} r\big\{\theta\cos\theta-(\log r)\sin\theta\big\} &< r\big\{\sin\theta-(\log r)\sin\theta\big\} \\ &= r(1-\log r)\sin\theta \\ &\le \sin\theta < \theta. \end{aligned}$$

Example 4.

$$IL(x) := -L(x)^{-1} = \begin{cases} -1 & (x=1) \\ -\dfrac{\log x}{x-1} & (x\neq 1) \end{cases}$$

is a negative operator monotone function on $(0,\infty)$ and $\exp\{IL(x)\}$ is operator monotone, too. In the following we verify that $IL(x)$ satisfies the condition of Theorem 1:
By putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$IL(z) = -\frac{\{(\log r)(r\cos\theta-1)+r\theta\sin\theta\}+i\{\theta(r\cos\theta-1)-(r\log r)\sin\theta\}}{r^2+1-2r\cos\theta}$$

and

$$v(r,\theta) := \Im IL(z) = \frac{(r\log r)\sin\theta-\theta(r\cos\theta-1)}{r^2+1-2r\cos\theta}.$$

From operator monotonicity of $IL(x)$ we have $0 < v(r,\theta)$ by Theorem L. In the following we show $v(r,\theta) < \theta$, which is equivalent to $(\log r)\sin\theta + \theta\cos\theta < r\theta$. By using the following inequalities

$$\theta\cos\theta < \sin\theta < \theta \ (0<\theta<\pi), \ \log r + 1 \le r \ (0 < r),$$

we obtain

$$\begin{aligned} (\log r)\sin\theta+\theta\cos\theta &< (\log r)\sin\theta + \sin\theta \\ &= (\log r + 1)\sin\theta \\ &\le r\sin\theta < r\theta. \end{aligned}$$

Results of Example 3 and Example 4 are extended as follows;

Theorem 2 ([10])**.** *Let*

$$DL_p(x) := \begin{cases} \dfrac{1}{p} & (x=1) \\ \dfrac{x^p\log x}{x^p-1} & (x\neq 1). \end{cases}$$

Then $\exp\{DL_p(x)\}$ *is an operator monotone function on* $(0,\infty)$ *for all* $p \in [-1,1]\setminus\{0\}$.

Remark 2. Theorem 2 can be also proved from Example 3 and the well-known fact that if $f(x)$ is operator monotone on $(0,\infty)$ then so is $f(x^p)^{\frac{1}{p}}$ for $p \in [-1,1] \setminus \{0\}$ since

$$\exp\{DL_p(x)\} = \exp\left\{\frac{1}{p}DL(x^p)\right\} = \big\{\exp\{DL(x^p)\}\big\}^{\frac{1}{p}}.$$

Proof. Firstly, we show that $DL_p(x)$ satisfies the condition of Theorem 1 for the case $p \in (0,1]$:
By putting $z = re^{i\theta}$ $(0 < r, 0 < \theta < \pi)$, we have

$$DL_p(z) = \frac{r^p\Big[\big\{(r^p - \cos(p\theta))\log r + \theta\sin(p\theta)\big\} + i\big\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\big\}\Big]}{r^{2p} + 1 - 2r^p\cos(p\theta)}$$

and

$$v(r,\theta) := \Im DL_p(z) = \frac{r^p}{r^{2p}+1-2r^p\cos(p\theta)}\big\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\big\}.$$

In the following we show $0 < v(r,\theta) < \theta$.

(1) Proof of $v(r,\theta) < \theta$;
We shall show $r^p\theta\cos(p\theta) - (r^p\log r)\sin(p\theta) < \theta$ since it is equivalent to $v(r,\theta) < \theta$.

$$\begin{aligned} r^p\theta\cos(p\theta) - (r^p\log r)\sin(p\theta) &< r^p\left(\frac{1}{p}\right)\sin(p\theta) - (r^p\log r)\sin(p\theta) \\ &= \left(\frac{1}{p}\right)(r^p - r^p\log r^p)\sin(p\theta) \\ &\leq \left(\frac{1}{p}\right)\sin(p\theta) < \left(\frac{1}{p}\right)(p\theta) = \theta. \end{aligned}$$

(2) Proof of $0 < v(r,\theta)$;
This inequality follows from the fact that $DL_p(x)$ is the composite function of operator monotone functions $\dfrac{1}{p}DL(x)$ and x^p. Here we give another proof by only using simple computation from Theorem 1. It is enough to show $(\log r)\sin(p\theta) < \theta(r^p - \cos(p\theta))$.

When $r = 1$, the inequality holds clearly. When $1 < r$,

$$\begin{aligned} (\log r)\sin(p\theta) &= \left(\frac{1}{p}\right)(\log r^p)\sin(p\theta) \\ &< \left(\frac{1}{p}\right)(r^p - 1)(p\theta) < (r^p - \cos(p\theta))\theta. \end{aligned}$$

When $0 < r < 1$,

$$\begin{aligned}(\log r)\sin(p\theta) &= \left(\frac{1}{p}\right)(\log r^p)\sin(p\theta) \\ &< \left(\frac{1}{p}\right)(r^p - 1)(p\theta)\cos(p\theta) \\ &= \theta\big(r^p\cos(p\theta) - \cos(p\theta)\big) < \theta(r^p - \cos(p\theta)).\end{aligned}$$

Next, when $p \in [-1, 0)$,

$$DL_p(z) = \frac{z^p \mathrm{Log} z}{z^p - 1} = \frac{z^{-p} z^p \mathrm{Log} z}{z^{-p}(z^p - 1)} = \frac{\mathrm{Log} z}{1 - z^{|p|}}$$

and

$$\nu(r,\theta) := \Im DL_p(re^{i\theta}) = \frac{(r^{|p|}\log r)\sin(|p|\theta) - \theta(r^{|p|}\cos(|p|\theta) - 1)}{r^{2|p|} + 1 - 2r^{|p|}\cos(|p|\theta)}.$$

We can show $0 < \nu(r,\theta) < \theta$ by the same technique. □

Chapter 3

A method to obtain families of operator monotone functions

In Chapter 1, we showed the operator monotonicity of $\log x$ by that of x^α ($\alpha \in (0,1]$). Here we give a proof of operator monotonicity of $\dfrac{x-1}{\log x}$, the representing function of the logaritmic mean (see Example 2), by using x^α again;

x^α is an operator monotone function on $(0,\infty)$ for every $\alpha \in (0,1]$. Moreover, for fixed $x > 0$, x^α is continuous on α and we have

$$\int_0^1 x^\alpha d\alpha = \frac{x-1}{\log x}.$$

Since the set of all operator monotone functions on an interval (a,b) is closed under addition, $\dfrac{x-1}{\log x}$ is an operator monotone function on $(0,\infty)$.

In the above proof, we used an integral with respect to $\alpha \in [0,1]$. By the same technique, we can obtain a 1-parameter family of operator monotone functions as follows;

$$P_{r,\alpha}(x) = \left((1-\alpha) + \alpha x^r\right)^{\frac{1}{r}}$$

is operator monotone on $(0,\infty)$ for every $r \in [-1,1] \setminus \{0\}, \alpha \in (0,1]$ (the weighted power mean), and we have

$$\int_0^1 P_{r,\alpha}(x)d\alpha = \frac{r(x^{r+1}-1)}{(r+1)(x^r-1)}.$$

Hence $\left\{ \frac{r(x^{r+1}-1)}{(r+1)(x^r-1)} \right\}_{r\in[-1,1]\setminus\{0\}}$ is a family of operator monotone functions on $(0,\infty)$. We note that these functions are the representing functions of a well-known operator mean as the power difference mean (see Example 5).

In this chapter, we will develop this technique and obtain a new method to get families of operator monotone functions. Moreover, we will construct a 2-parameter family of operator monotone functions by applying it.

3.1 The method and its application

For a natural number k, let $u(x)$ be a positive function on $[0,\infty)$ defined by

$$u(x) := r\prod_{i=1}^{k}(x+a_i)^{p_i} \quad (0 \le a = a_1 < a_2 < \cdots < a_k = b, 1 \le p_1, 0 < p_i, 0 < r).$$

We remark that $u(x)$ is a strictly increasing function on $[-a,\infty)$, and it has the inverse function $u^{-1}(x)$. M. Uchiyama has obtained the following result in [9].

Lemma U ([9 Theorem 2.1])**.** *The inverse function u^{-1} of u is an operator monotone function on $[0,\infty)$.*

Theorem 3 ([11])**.** *Let μ be a probability measure on $[0,1]$ and $\{f(\alpha;x) \mid \alpha \in [0,1]\}$ be a family of positive-valued operator monotone functions of $x \ge 0$. Assume for each $x \ge 0$, the map $\alpha \mapsto f(\alpha;x)$ is continuous. Then*

$$F(x) := u\left(\int_0^1 u^{-1}\left(f(\alpha;x)\right)d\mu(\alpha) + b - a\right)$$

is an operator monotone function.

Proof. It is enough by Theorem L to show that $u\left(\sum_j \beta_j u^{-1}(f(\alpha_j;x)) + b - a\right)$ is a Pick function for any $\alpha_1,\cdots,\alpha_m \in [0,1]$ and positive numbers $\beta_1,\ldots,\beta_m$ which satisfy $\sum_{j=1}^m \beta_j = 1$. From the assumption, $u^{-1}(f(\alpha_j;x))$ is operator monotone by Lemma U, and hence a Pick function by Theorem L. Thus for a complex number $z \in \mathbb{C}^+$, we have

$$\begin{aligned}
0 < \arg &\left(u \left(\sum_j \beta_j u^{-1}(f(\alpha_j; z)) + b - a \right) \right) \\
&= \sum_i p_i \arg \left(\sum_j \beta_j u^{-1}(f(\alpha_j; z)) + b - a + a_i \right) \\
&\leq \sum_i p_i \arg \left(\sum_j \beta_j u^{-1}(f(\alpha_j; z)) + b \right) \\
&\leq \sum_i p_i \max_j \left\{ \arg \left(u^{-1}(f(\alpha_j; z)) + b \right) \right\} \\
&\leq \max_j \left\{ \sum_i p_i \arg \left(u^{-1}(f(\alpha_j; z)) + a_i \right) \right\} \\
&= \max_j \left\{ \arg(u(u^{-1}(f(\alpha_j; z))) \right\} = \max_j \left\{ \arg f(\alpha_j; z) \right\} < \pi,
\end{aligned}$$

where the last inequality follows from that $f(\alpha_j; x)$ is a Pick function. Hence $u\left(\sum_j \beta_j u^{-1}(f(\alpha_j; x)) + b - a\right)$ is a Pick function. Therefore the proof is completed. □

Corollary 1 ([11]). *Under the same assumptions of Theorem 3,*

$$\Phi(x) := \left[u \left(\int_0^1 u^{-1}(f(\alpha; x)^{-1}) d\mu(\alpha) + b - a \right) \right]^{-1}$$

is an operator monotone function.

Proof. Since $f(\alpha; x^{-1})^{-1}$ is operator monotone,

$$\Psi(x) := u \left(\int_0^1 u^{-1} \left(f(\alpha; x^{-1})^{-1} \right) d\mu(\alpha) + b - a \right)$$

is operator monotone by Theorem 3. So we have

$$\Phi(x) = \Psi(x^{-1})^{-1} = \left[u \left(\int_0^1 u^{-1}(f(\alpha; x)^{-1}) d\mu(\alpha) + b - a \right) \right]^{-1}$$

is operator monotone. □

As easy consequences of Theorem 3 and Corollary 1, we obtain the following family of operator monotone functions.

Corollary 2 ([11])**.** *Under the same assumptions of Theorem 3, for each* $p \in [-1,1] \setminus \{0\}$,

$$F_p(x) := \left(\int_0^1 f(\alpha; x)^p d\mu(\alpha) \right)^{\frac{1}{p}}$$

is an operator monotone function of $x \geq 0$*. Moreover, for each fixed* $x \geq 0$, $F_p(x)$ *is increasing on* $p \in [-1,1] \setminus \{0\}$.

Proof. Operator monotonicity of $F_p(x)$ can be obtained by putting $u(x) = x^{\frac{1}{p}}$ ($p \in (0,1]$) in Theorem 3 and Corollary 1. So we have only to prove the monotonicity of $F_p(x)$ on $p \in [-1,1] \setminus \{0\}$. Since x^α is a concave function for $\alpha \in [0,1]$, then for $0 < p < q \leq 1$,

$$\int_0^1 f(\alpha; x)^p d\mu(\alpha) = \int_0^1 f(\alpha; x)^{q \cdot \frac{p}{q}} d\mu(\alpha) \leq \left(\int_0^1 f(\alpha; x)^q d\mu(\alpha) \right)^{\frac{p}{q}}.$$

Hence we have $F_p(x) \leq F_q(x)$.

We can prove the case $-1 \leq q < p < 0$ by the same way. Next, we shall show $F_{-p}(x) \leq F_p(x)$ for $0 < p \leq 1$. Since x^{-1} is a convex function we have

$$\int_0^1 f(\alpha; x)^{-p} d\mu(\alpha) \geq \left(\int_0^1 f(\alpha; x)^p d\mu(\alpha) \right)^{-1}.$$

Taking the power $\frac{-1}{p}$ on both sides, we have $F_{-p}(x) \leq F_p(x)$. □

By using l'Hospital's theorem we can obtain Corollary 3 easily.

Corollary 3 ([11])**.** *Under the same assumptions of Theorem 3,*

$$F_0(x) := \lim_{p \to 0} F_p(x) = \exp \left(\int_0^1 \log f(\alpha; x) d\mu(\alpha) \right)$$

is operator monotone, where $F_p(x)$ *is defined in Corollary 2. Moreover,* $F_p(x) \leq F_0(x) \leq F_q(x)$ *hold for all* $p \in [-1,0)$ *and* $q \in (0,1]$.

Therefore $F_p(x)$ is increasing of $p \in [-1,1]$ and operator monotone on $x \geq 0$. By putting $f(\alpha; x) = P_{r,\alpha}(x)$ (the representing function of the weighted power mean) and $\mu(\alpha) = \alpha$, we now get a 2-parameter operator monotone function in the following;

Theorem 4 ([11]). *For $r \in [-1,1]$ and $s \in [-1,1]$,*

$$F_{r,s}(x) := \begin{cases} \left(\int_0^1 \left((1-\alpha)+\alpha x^r\right)^{\frac{s}{r}} d\alpha\right)^{\frac{1}{s}} & = \left(\frac{r(x^{r+s}-1)}{(r+s)(x^r-1)}\right)^{\frac{1}{s}} & (r,s\neq 0, r\neq -s) \\ \left(\int_0^1 \left((1-\alpha)+\alpha x^r\right)^{-1} d\alpha\right)^{\frac{1}{-r}} & = \left(\frac{x^r-1}{r\log x}\right)^{\frac{1}{r}} & (r,s\neq 0, r=-s) \\ \left(\int_0^1 x^{\alpha s} d\alpha\right)^{\frac{1}{s}} & = \left(\frac{x^s-1}{s\log x}\right)^{\frac{1}{s}} & (r=0, s\neq 0) \\ \exp\left\{\int_0^1 \log\left((1-\alpha)+\alpha x^r\right)^{\frac{1}{r}} d\alpha\right\} & = \exp\left\{\frac{1}{r}\left(\frac{x^r\log x^r}{x^r-1}-1\right)\right\} & (r\neq 0, s=0) \\ \exp\left\{\int_0^1 \log x^{\alpha} d\alpha\right\} & = x^{\frac{1}{2}} & (r,s=0) \end{cases}$$

is an operator monotone function of $x \geq 0$. Moreover, for each fixed $x \geq 0$, $F_{r,s}(x)$ is increasing on each $r \in [-1,1]$ and $s \in [-1,1]$.

Since $F_{r,s}(1) = 1$, $F_{r,s}(x)$ is the representing function of an operator mean for $r,s \in [-1,1]$. This family interpolates some famous 1-parameter family of operator monotone functions.

3.2 Interpolation of some operator means

Example 5 (The power difference mean [1], [4]). From operator monotonicity of $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$, we have that

$$F_{r,-1}(x) = \frac{(r-1)(x^r-1)}{r(x^{r-1}-1)} \quad (-1 \leq r < 1, r \neq 0)$$

is operator monotone. Also we have that

$$F_{r,1}(x) = \frac{r(x^{r+1}-1)}{(r+1)(x^r-1)} = \frac{(q-1)(x^q-1)}{q(x^{q-1}-1)} \quad (0 < q \leq 2, q \neq 1)$$

is operator monotone, too. So we obtain a 1-parameter family $\{PD_r(x)\}_{r\in[-1,2]}$ of operator monotone functions such that

$$PD_r(x) := \frac{(r-1)(x^r-1)}{r(x^{r-1}-1)} \quad (-1 \leq r \leq 2, r \neq 0, 1).$$

This family is called the power difference mean and the optimality of its range of the parameter $-1 \leq r \leq 2$ is well known (e.g., see [4 Proposition 4.2]).

Example 6 (The power mean)**.** If $r = s$, then

$$F_{r,r}(x) = \left(\frac{x^r + 1}{2}\right)^{\frac{1}{r}} \quad (-1 \leq r \leq 1, r \neq 0).$$

This function is the representing function of the power mean $P_r(x)$, and is the case of $\alpha = \dfrac{1}{2}$ of the weighted power mean. Namely, $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ interpolates the power mean.

Example 7 (The Stolarsky mean [8])**.** If $r = 1$ and $s = p - 1$, then $F_{r,s}(x)$ coincides with

$$S_p(x) := \left(\frac{p(x-1)}{x^p - 1}\right)^{\frac{1}{1-p}} \quad (p \neq 0, 1),$$

and by Theorem 4 we obtain the fact that $S_p(x)$ is operator monotone for $0 \leq p \leq 2$. $S_p(x)$ is the representing function of the Stolarsky mean, and the range $-2 \leq p \leq 2$ is optimal for which $S_p(x)$ is operator monotone ([8]). Namely, $S_p(x)$ is not operator monotone if $p \in (-\infty, -2) \cup (2, \infty)$.

Remark 3. We cannot prove operator monotonicity of $S_p(x)$ for $-2 \leq p < 0$ by Theorem 4, because $s = p-1 \in [-1, 1]$. So we think that the range of the parameters $(r, s) \in [-1, 1] \times [-1, 1]$ of $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ such that $F_{r,s}(x)$ is operator monotone is not optimal, and try to extend it in the next chapter.

Example 8 (Order among means obtained from $F_{r,s}(x)$)**.** Since $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ has monotonicity for its parameters r and s, we can observe the following inequalities;

$$\frac{2x}{x+1} \leq \frac{x \log x}{x-1} \leq x^{\frac{1}{2}} \leq \frac{x-1}{\log x} \leq \exp\left\{\frac{x \log x}{x-1} - 1\right\} \leq \frac{x+1}{2}. \quad (3.1)$$

These six functions are the representing functions of the harmonic, dual of logarithmic, geometric, logarithmic, identric, and arithmetic means, respectively, so we have obtained order among these operartor means.

We can summarize concrete examples of $F_{r,s}(x)$ in the following table, where $\alpha \in (-1, 0)$ and $\beta \in (0, 1)$.

$s\backslash r$	-1	α	0	β	1
-1	$H(x) = A(x^{-1})^{-1}$	$PD_\alpha(x)$	$L(x^{-1})^{-1}$	$PD_\beta(x)$	$L(x)$
α	$S_{1-\alpha}(x^{-1})^{-1}$	$P_\alpha(x)$	$L(x^\alpha)^{\frac{1}{\alpha}}$	$F_{\beta,\alpha}(x)$	$S_{1+\alpha}(x)$
0	$ID(x^{-1})^{-1}$	$ID(x^\alpha)^{\frac{1}{\alpha}}$	$G(x)$	$ID(x^\beta)^{\frac{1}{\beta}}$	$ID(x)$
β	$S_{1-\beta}(x^{-1})^{-1}$	$F_{\alpha,\beta}(x)$	$L(x^\beta)^{\frac{1}{\beta}}$	$P_\beta(x)$	$S_{1+\beta}(x)$
1	$L(x^{-1})^{-1}$	$PD_{1+\alpha}(x)$	$L(x)$	$PD_{1+\beta}(x)$	$A(x)$

Chapter 4

2-parameter Stolarsky mean

In this chapter we treat $\{F_{r,s}(x)\}$ in the following form;

$$S_{p,\alpha}(x) := F_{p,\alpha-p}(x) = \begin{cases} \left(\dfrac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha-p}} & (0 \neq p \neq \alpha \neq 0) \\ \left(\dfrac{x^p - 1}{p \log x}\right)^{\frac{1}{p}} & (p \neq \alpha = 0) \\ \left(\dfrac{x^\alpha - 1}{\alpha \log x}\right)^{\frac{1}{\alpha}} & (0 = p \neq \alpha) \\ \exp\left\{\dfrac{1}{p}\left(\dfrac{x^p \log x^p}{x^p - 1} - 1\right)\right\} & (p = \alpha \neq 0) \\ x^{\frac{1}{2}} & (p = \alpha = 0). \end{cases}$$

We call it *2-parameter Stolarsky mean.* Nagisa and Wada [7] have obtained an equivalent condition of (p, α) such that $S_{p,\alpha}(x)$ is operator monotone. Their characterization, however, has not given any explicit condition in terms of p and α, therefore, we have not known the concrete form of the optimal range of (p, α) such that $S_{p,\alpha}(x)$ is operator monotone, yet. On the other hand, we have obtained a part of this range in Chapter 3. In this chapter, we shall extend and investigate it from the results in the previous chapters.

4.1 Range of the two parameters

4.1.1 Known part

We have known so far that $S_{p,\alpha}(x)$ is operator monotone if

$$p - 1 \leq \alpha \leq p + 1,\ p \in [-1, 1]$$

from the operator monotonicity of $F_{r,s}(x)$ shown in Chapter 3. On the other hand, it was shown in [7] that the following function

$$h_{p,\alpha}(x) := \frac{\alpha(x^p - 1)}{p(x^\alpha - 1)}$$

is operator monotone if and only if $(p, \alpha) \in \left\{(p, \alpha) \in \mathbb{R}^2 \,\middle|\, 0 < p - \alpha \leq 1, p \geq -1 \text{ and } \alpha \leq 1\right\} \cup ([0, 1] \times [-1, 0]) \backslash \{(0, 0)\}$. Also, if $\alpha \leq p - 1$, then $\frac{1}{p-\alpha} \in (0, 1]$. From these results and Theorem L-H, we can find that $S_{p,\alpha}(x) = h_{p,\alpha}(x)^{\frac{1}{p-\alpha}}$ is operator monotone if $(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 | \alpha \leq p - 1, p \leq 1 \text{ and } -1 \leq \alpha\}$. Therefore, we obtain the fact that $S_{p,\alpha}(x)$ is operator monotone if

$$(p, \alpha) \in \left\{(p, \alpha) \in \mathbb{R}^2 \,\middle|\, p - 1 \leq \alpha \leq p + 1,\ p \in [-1, 1]\right\} \cup ([0, 1] \times [-1, 0]).$$

4.1.2 Trivial part

There is a domain where $S_{p,\alpha}(x)$ is trivially operator monotone. If $\alpha = -p \neq 0$, then

$$S_{p,-p}(x) = \left(\frac{p(x^{-p} - 1)}{(-p)(x^p - 1)}\right)^{\frac{1}{-2p}} = \left(\frac{1}{x^p}\right)^{\frac{1}{-2p}} = x^{\frac{1}{2}}.$$

Hence, we find that operator monotonicity of $S_{p,\alpha}(x)$ always holds if $\alpha = -p$.

4.1.3 Extension from the Stolarsky mean

From Theorem L and operator monotonicity of the 1-parameter family $\{S_p(x)\}_{p \in [-2,2]}$, $z \in \mathbb{C}^+$ implies $S_p(z) \in \mathbb{C}^+$ for all $p \in [-2, 2]$, namely, we can assume that the argument of $S_p(z)$ has the following property

$$0 < \arg\left(\frac{p(z - 1)}{z^p - 1}\right)^{\frac{1}{1-p}} \left(= \frac{1}{1 - p} \arg\left(\frac{p(z - 1)}{z^p - 1}\right)\right) < \pi$$

$(z \in \mathbb{C}^+,\ -2 \leq p \leq 2)$. So we get

$$0 < \arg\left(\frac{p(z - 1)}{z^p - 1}\right) < (1 - p)\pi \quad (-2 \leq p < 1),$$

$$0 < \arg\left(\frac{z^p - 1}{p(z-1)}\right) < (p-1)\pi \quad (1 < p \le 2).$$

By these inequalities we obtain

$$\begin{aligned} 0 &< \arg\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)}\right)^{\frac{1}{\alpha-p}} \\ &= \frac{1}{\alpha - p}\left\{\arg\left(\frac{z^\alpha - 1}{\alpha(z-1)}\right) + \arg\left(\frac{p(z-1)}{z^p - 1}\right)\right\} \\ &< \frac{1}{\alpha - p}\big\{(\alpha - 1)\pi + (1-p)\pi\big\} = \pi, \end{aligned}$$

that is, $S_{p,\alpha}(z) \in \mathbb{C}^+$ for the case $-2 \le p < 1 < \alpha \le 2$. Hence we have $S_{p,\alpha}(x)$ is operator monotone by Theorem L. On the other hand,

$$\begin{aligned} S_{-p}(x^{-1})^{-1} &= \left(\frac{(-p)(x^{-1} - 1)}{(x^{-1})^{-p} - 1}\right)^{\frac{-1}{1-(-p)}} \\ &= \left(\frac{(-p)(1-x)}{x(x^p - 1)}\right)^{\frac{-1}{1+p}} = \left(\frac{x(x^p - 1)}{p(x-1)}\right)^{\frac{1}{1+p}} \end{aligned}$$

is operator monotone for $-2 \le p \le 2$, too. So we have

$$0 < \frac{1}{1+p}\arg\left(\frac{z(z^p - 1)}{p(z-1)}\right) < \pi \ (z \in \mathbb{C}^+,\ -2 \le p \le 2)$$

by Theorem L and get the following relation similarly for the case $-2 \le \alpha < -1 < p \le 2$;

$$\begin{aligned} 0 &< \arg\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)}\right)^{\frac{1}{\alpha-p}} \\ &= \frac{1}{p - \alpha}\left\{\arg\left(\frac{z(z^p - 1)}{p(z-1)}\right) + \arg\left(\frac{\alpha(z-1)}{z(z^\alpha - 1)}\right)\right\} \\ &< \frac{1}{p - \alpha}\big\{(1+p)\pi - (1+\alpha)\pi\big\} = \pi, \end{aligned}$$

that is, $S_{p,\alpha}(z) \in \mathbb{C}^+$, and we have $S_{p,\alpha}(x)$ is operator monotone by Theorem L.

4.1.4 Investigation of the range

Moreover, since $S_{p,\alpha}(x)$ is symmetric with respect to p and α, i.e., $S_{p,\alpha}(x) = S_{\alpha,p}(x)$, we can extend symmetrically the range of parameters obtained so far. Namely, we have

$$\begin{aligned}
&\bigl\{(p,\alpha)\in\mathbb{R}^2 \bigm| p-1\le\alpha\le p+1,\ p\in[-1,1]\bigr\}\cup([0,1]\times[-1,0])\\
&\longrightarrow \bigl\{(p,\alpha)\in\mathbb{R}^2 \bigm| \alpha-1\le p\le \alpha+1,\ \alpha\in[-1,1]\bigr\}\cup([-1,0]\times[0,1]),
\end{aligned}$$

$$\begin{aligned}
[-2,1)\times(1,2] &\longrightarrow (1,2]\times[-2,1),\\
(-1,2]\times[-2,-1) &\longrightarrow [-2,-1)\times(-1,2].
\end{aligned}$$

We can summarize the above-mentioned results on monotonicity of $S_{p,\alpha}(x)$ in the following form.

Theorem 5 ([10])**.** *Let*

$$S_{p,\alpha}(x) := \begin{cases}
\left(\dfrac{p(x^\alpha-1)}{\alpha(x^p-1)}\right)^{\frac{1}{\alpha-p}} & (0\ne p\ne\alpha\ne 0)\\
\left(\dfrac{x^p-1}{p\log x}\right)^{\frac{1}{p}} & (p\ne\alpha=0)\\
\left(\dfrac{x^\alpha-1}{\alpha\log x}\right)^{\frac{1}{\alpha}} & (0=p\ne\alpha)\\
\exp\left\{\dfrac{1}{p}\left(\dfrac{x^p\log x^p}{x^p-1}-1\right)\right\} & (p=\alpha\ne 0)\\
x^{\frac{1}{2}} & (p=\alpha=0).
\end{cases}$$

Then $S_{p,\alpha}(x)$ is operator monotone on $(0,\infty)$ if $(p,\alpha)\in\mathcal{A}\subset\mathbb{R}^2$, where

$$\mathcal{A} := \bigl([-2,1]\times[-1,2]\bigr)\cup\bigl([-1,2]\times[-2,1]\bigr)\cup\bigl\{(p,\alpha)\in\mathbb{R}^2 \bigm| \alpha=-p\bigr\}.$$

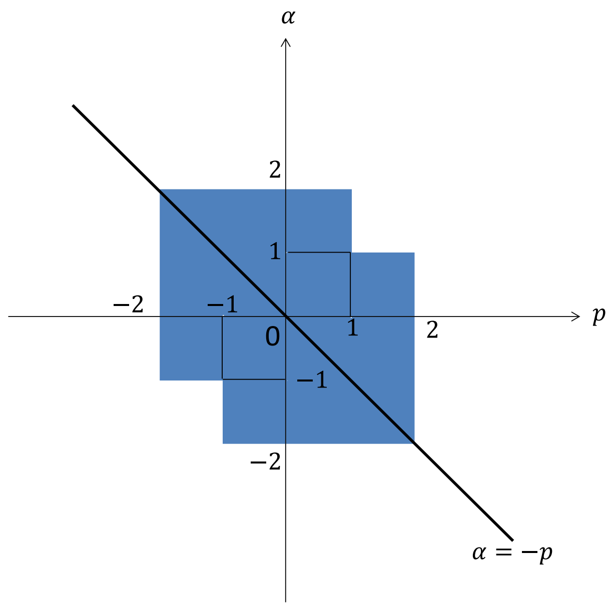

Figure of the range $\mathcal{A}$

Remark 4. $S_{p,\alpha}(x)$ is not operator monotone if $1 < |p|$ and $\alpha = p$. $S_{p,p}(x)$ coincides with $\exp\{DL_p(x) - \frac{1}{p}\}$, where $DL_p(x)$ is the one-parameter family of functions defined in Theorem 2. If $p = 2$, for example, then

$$DL_2(x) = \frac{1}{2} \times \frac{2x}{x+1} \times \frac{x \log x}{x-1}.$$

From the inequality (3.1) in Chapter 3, we have

$$\frac{2x}{x+1} \times \frac{x \log x}{x-1} \leq x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x,$$

and $DL_2(x)$ is not operator monotone from the well-known fact that a positive operator monotone function on $(0, \infty)$ with $f(1) = 1$ must satisfy $x < f(x)$ if $0 < x < 1$. And hence, $\exp\{DL_2(x)\}$ is not operator monotone by Theorem 1. Next we will see general case $1 < |p|$. From the proof of Theorem 2, we have

$$v(r, \theta) := \Im DL_p(z) = \frac{r^p}{r^{2p} + 1 - 2r^p \cos(p\theta)} \big\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\big\}.$$

By simple computation,

$$v(r, \theta) < \theta \Longleftrightarrow \big(l(p, r, \theta) =\big) r^p \left(\cos(p\theta) - (\log r)\frac{\sin(p\theta)}{\theta}\right) < 1.$$

Take θ as $\dfrac{\pi}{p} < \theta < \min\left\{\pi, \dfrac{2\pi}{p}\right\}$, then $\sin(p\theta) < 0$ and

$$\lim_{r\to\infty} l(p, r, \theta) = \infty.$$

Therefore $\exp\{DL_p(x)\} = S_{p,p}(x)$ is not operator monotone if $1 < p$ from Theorem 1. We can also show the case $p < -1$ similarly.

Bibliography

[1] J.-I. Fujii and Y. Seo, *On parametrized operator means dominated by power ones*, Sci. Math., **1** (1998), 301–306.

[2] F. Hansen, *Selfadjoint means and operator monotone functions*, Math. Ann., **256** (1981), 29–35.

[3] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.

[4] F. Hiai and H. Kosaki, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J., **48** (1999), 899–936.

[5] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1979/80), 205–224.

[6] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z., **38** (1934), 177–216.

[7] M. Nagisa and S. Wada, *Operator monotonicity of some functions*, Linear Algebra Appl., **486** (2015), 389–408.

[8] Y. Nakamura, *Classes of operator monotone functions and Stieltjes functions*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), 395–404, Oper. Theory Adv. Appl., 41, Birkhäuser, Basel, 1989.

[9] M. Uchiyama, *Operator monotone functions which are defined implicitly and operator inequalities*, J. Funct. Anal., **175** (2000), 330–347.

[10] Y. Udagawa, *Operator monotonicity of a 2-parameter family of functions and* $\exp\{f(x)\}$ *related to the Stolarsky mean*, to appear in Oper. Matrices.

[11] Y. Udagawa, S. Wada, T. Yamazaki and M. Yanagida, *On a family of operator means involving the power difference means*, Linear Algebra Appl., **485** (2015), 124–131.