SUT Journal of Mathematics
Vol. 54, No. 2 (2018), 161–171

Edge connectivity and restricted edge connectivity of cartesian product of graphs

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(Received October 14, 2018; Revised November 26, 2018)

Abstract. An edge cutset $E \subset E(G)$ of a graph G is called a restricted edge cutset if every component of $G - E$ has order at least 2. We let $\lambda'(G)$ denote the minimum cardinality of a restricted edge cutset of G, and let $\delta'(G)$ denote the minimum of $\deg_G(x) + \deg_G(y) - 2$ as x and y range over all adjacent vertices of G. We let $\lambda(G)$ and $\delta(G)$ denote the edge connectivity and the minimum degree of G, respectively. Among other results, we show that if G_1 and G_2 are graphs such that $\lambda(G_i) = \delta(G_i) \geq 2$ and $\lambda'(G_i) = \delta'(G_i) \geq 2$ for each $i = 1, 2$, then $\lambda'(G_1 \otimes G_2) = \delta'(G_1 \otimes G_2) = \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}$, where $G_1 \otimes G_2$ denotes the cartesian product of G_1 and G_2.

AMS 2010 *Mathematics Subject Classification.* 05C40.

Key words and phrases. Edge connectivity, cartesian product.

§1. Introduction

We start by defining several invariants of a graph. We call an edge cutset $E \subset E(G)$ of a graph G a *restricted edge cutset* when every component of $G - E$ has at least 2 vertices. For a graph G, we define the values $\delta(G), \delta'(G), \lambda(G)$ and $\lambda'(G)$ by

$$\delta(G) := \min_{x \in V(G)} \deg_G(x) \quad \text{(the minimum degree of } G\text{)},$$

$$\delta'(G) := \min_{xy \in E(G)} \deg_G(x) + \deg_G(y) - 2,$$

$$\lambda(G) := \min\{|E| | E \text{ is an edge cutset of } G\} \quad \text{(the edge connectivity of } G\text{)},$$

$$\lambda'(G) := \min\{|E| | E \text{ is a restricted edge cutset of } G\}.$$

When G has no restricted edge cutset, for example when G is a star, we do not define $\lambda'(G)$. We remark that if $\lambda'(G)$ is defined, then $|V(G)| \geq 4$. In fact, for a connected graph G, $\lambda'(G)$ is defined if and only if $|V(G)| \geq 4$ and G is not a star (see Lemma 2.2). Among these invariants, we have inequalities $\delta(G) \geq \lambda(G)$ and $\lambda'(G) \geq \lambda(G)$. We also have $\delta'(G) \geq \lambda'(G)$ (see Lemma 2.2).

Next we introduce the notions of *super edge connected graphs* and *super restricted edge connected graphs*. A connected graph G of order at least 2 is called super edge connected when $G - E$ has a component of order 1 for any minimum edge cutset E. Similarly, a connected graph G for which $\lambda'(G)$ is defined is called super restricted edge connected when $G - E$ has a component of order 2 for any minimum restricted edge cutset E. When G has no restricted edge cutset, we define G not to be super restricted edge connected. For sufficient conditions for a graph to be super edge connected/super restricted edge connected, see [3], [4], [6], and [7].

In [5], Wang and Wang studied $\lambda(G)$ and $\lambda'(G)$ when G is the 2-expanded k-ary n-cube graph. Here for integers m, k, n with $m \geq 2, k \geq 2m+1$ and $n \geq 1$, the m-expanded k-ary n-cube graph (denoted by $m\text{-}Q_k^n$) is the graph defined as follows:

$$
\begin{aligned}
V(m\text{-}Q_k^n) &= \{(u_1, u_2, \ldots, u_n) | u_i \in \mathbb{Z}/k\mathbb{Z} \text{ for all } i \in \{1, 2, \ldots, n\}\},\\
E(m\text{-}Q_k^n) &= \{(u_1, u_2, \ldots, u_n)(v_1, v_2, \ldots, v_n) | \text{ there exists } j \in \{1, 2, \ldots, n\}\\
&\qquad \text{such that } u_j = v_j + g \text{ for some } g \in \{-m, \cdots - 2, -1, 1, 2, \ldots m\}\\
&\qquad \text{and } u_i = v_i \text{ for all } i \in \{1, 2, \ldots, n\} \setminus \{j\}\}.
\end{aligned}
$$

In [5], it is proved that if $k \geq 6$ and $n \geq 3$, then $\lambda(2\text{-}Q_k^n) = 4n$ and $\lambda'(2\text{-}Q_k^n) = 8n - 2$, and $2\text{-}Q_k^n$ is super edge connected and super restricted edge connected. In this paper, we generalize this results to $m\text{-}Q_k^n$ and show that the following statement holds.

Proposition 1.1. *Let m, k, n be integers with $m \geq 2, k \geq 2m+1$ and $n \geq 2$. Then $\lambda(m\text{-}Q_k^n) = 2mn$ and $\lambda'(m\text{-}Q_k^n) = 4mn-2$. Furthermore, $m\text{-}Q_k^n$ is super edge connected and super restricted edge connected.*

Graphs $m\text{-}Q_k^n$ have some good properties, and are useful in information theory (see [1], [2]). However, they form a rather restricted class of graphs, and it is desirable that one should obtain a more general result. In this paper, as we describe below, we actually derive Proposition 1.1 from more general results.

For integers m, k with $m \geq 2$ and $k \geq 2m+1$, let $H_{k,m}$ be the graph defined as follows:

$$
\begin{aligned}
V(H_{k,m}) &= \mathbb{Z}/k\mathbb{Z},\\
E(H_{k,m}) &= \{uv | u = v + g \text{ for some } g \in \{-m, \ldots, -2, -1, 1, 2, \ldots, m\}\}.
\end{aligned}
$$

The graph $H_{k,m}$ is called the $m-$th power of the cycle of order k. As remarked in [5], the $m-$expanded $k-$ary $n-$cube graph is the cartesian product of n copies of $H_{k,m}$. Here, for two graphs G_1 and G_2, the cartesian product $G_1 \otimes G_2$ is the graph defined as follows:

$$\begin{aligned} V(G_1 \otimes G_2) &:= \{(x, y)|x \in V(G_1), y \in V(G_2)\}, \\ E(G_1 \otimes G_2) &:= \{(x_1, y)(x_2, y)|x_1x_2 \in E(G_1), y \in V(G_2)\} \\ &\qquad \cup \{(x, y_1)(x, y_2)|x \in V(G_1), y_1y_2 \in E(G_2)\}. \end{aligned}$$

Also observe that if $m \geq 2$ and $k \geq 2m + 1$, then $\lambda(H_{k,m}) = \delta(H_{k,m}) = 2m$ and $\lambda'(H_{k,m}) = \delta'(H_{k,m}) = 4m - 2$.

Based on these observations, we prove the following two theorems, and show that Proposition 1.1 follows from them.

Theorem 1.2. *Let G_1, G_2 be graphs, and suppose that $\lambda(G_i) = \delta(G_i) \geq 1$ for each $i = 1, 2$. Then $\lambda(G_1 \otimes G_2) = \delta(G_1 \otimes G_2) = \delta(G_1) + \delta(G_2)$. Furthermore, unless either G_1 is a complete graph and $\delta(G_2) = 1$ or G_2 is a complete graph and $\delta(G_1) = 1$, $G_1 \otimes G_2$ is super edge connected.*

Theorem 1.3. *Let G_1, G_2 be graphs, and suppose that $\lambda(G_i) = \delta(G_i) \geq 2$ and $\lambda'(G_i) = \delta'(G_i) \geq 2$ for each $i = 1, 2$. Then $\lambda'(G_1 \otimes G_2) = \delta'(G_1 \otimes G_2) = \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}$. Furthermore, unless either G_1 is a complete graph and $\delta(G_2) = 2$ or G_2 is a complete graph and $\delta(G_1) = 2$, $G_1 \otimes G_2$ is super restricted edge connected.*

We prove preliminary lemmas in Section 2, and prove Theorem 1.2 and Theorem 1.3 in Section 3. In Section 4, we prove two corollaries, which immediately imply Proposition 1.1.

After submitting the first version of this paper, we become aware that Xu and Yang had already proved the following theorem.

Theorem 1.4 (Xu and Yang 2006 [8])**.** *Let G_1, G_2 be connected graphs. Then $\lambda(G_1 \otimes G_2) = \min\{\delta(G_1) + \delta(G_2), \lambda(G_1)|V(G_2)|, \lambda(G_2)|V(G_1)|\}$.*

The first assetion of Theorem 1.2 is a corollary of Theorem 1.4. However, we have decided to keep the proof of the first assertion as it was in the first version because, in our proof of the second assertion, we make use of the arguments in the proof of the first assertion.

§2. Preliminaries

In this section, we prepare some notations and lemmas. We start with two lemmas concerning the existence of a restricted edge cutset.

Lemma 2.1. *Let E be an edge cutset of a connected graph G, and suppose that $G - E$ has two or more components of order at least 2. Then E contains a restricted edge cutset.*

Proof. Let $F_0 \subset E$ be an edge cutset which minimizes the number of components of order 1 among the edge cutsets F with $F \subset E$ such that $G - F$ has two or more components of order at least 2. We show that F_0 is a restricted edge cutset. Suppose that $G - F_0$ has a component C of order 1 and write $V(C) = \{v\}$. Let $vw \in F_0$. Then $F_0 \setminus \{vw\}$ is also an edge cutset having the property that $G - (F_0 \setminus \{vw\})$ has two or more components of order at least 2. On the other hand, the number of components of order 1 in $G - (F_0 \setminus \{vw\})$ is less than the number of components of order 1 in $G - F_0$. This contradicts the minimum choice of F_0. □

Lemma 2.2. *Let G be a connected graph, and suppose that $|G| \geq 4$ and G is not a star. Then $\lambda'(G)$ is defined and $\lambda'(G) \leq \delta'(G) \leq 2(|V(G)| - 2)$. Furthermore, if $\delta'(G) = 2(|V(G)| - 2)$, then G is a complete graph.*

Proof. Let uv be an edge of G with $\deg_G(u) + \deg_G(v) - 2 = \delta'(G)$. Suppose that $E(G - \{u, v\}) = \emptyset$. We can write $V(G) \setminus \{u, v\} = A \cup B$ with $A \cap B = \emptyset$ so that each vertex in A is adjacent to u and each vertex in B is adjacent to v. Since G is not a star, we can take A and B so that they further satisfy $A \neq \emptyset$ and $B \neq \emptyset$. Then $\deg_G(v) \geq |B| + 1$. Take $x \in A$. From $E(G - \{u, v\}) = \emptyset$, we get $\deg_G(x) \leq 2$. On the other hand, $\deg_G(x) \geq \deg_G(v)$ by the minimality of $\deg_G(u) + \deg_G(v)$. Hence $2 \geq \deg_G(x) \geq \deg_G(v) \geq |B| + 1$. Consequently $|B| = 1$ and $\deg_G(x) = 2$, which forces $xv \in E(G)$. This implies $\deg_G(v) \geq |B| + 2 = 3$, which contradicts the fact that $\deg_G(x) \geq \deg_G(v)$. Thus $E(G - \{u, v\}) \neq \emptyset$. Let E be the set of edges joining $\{u, v\}$ and $V(G) \setminus \{u, v\}$. Then $\delta'(G) = |E| \leq 2(|V(G)| - 2)$. Since $E(G - \{u, v\}) \neq \emptyset$, it follows from Lemma 2.1 that E contains a restricted edge cutset E'. Therefore $\lambda'(G)$ is defined, and $\lambda'(G) \leq |E'| \leq |E| = \delta'(G) \leq 2(|V(G)| - 2)$.

Now assume that $\delta'(G) = 2(|V(G)| - 2)$. Then $\deg_G(u) = \deg_G(v) = |V(G)| - 1$, which implies that each of u and v is adjacent to all vertices in $V(G) \setminus \{u, v\}$. From the minimality of $\deg_G(u) + \deg_G(v)$, we see that $\deg_G(x) = |V(G)| - 1$ for every $x \in V(G) \setminus \{u, v\}$. This means that G is a complete graph. □

Throughout the rest of this section, we let G_1, G_2 be graphs. We investigate edge cutsets of $G_1 \otimes G_2$. We define subset E_1, E_2 of $E(G_1 \otimes G_2)$ as follows:

$$E_1 := \{(x_1, y)(x_2, y) | x_1x_2 \in E(G_1), y \in V(G_2)\},$$
$$E_2 := \{(x, y_1)(x, y_2) | x \in V(G_1), y_1y_2 \in E(G_2)\}.$$

It is clear that $E_1 \cap E_2 = \emptyset$ and $E(G_1 \otimes G_2) = E_1 \cup E_2$; thus $\{E_1, E_2\}$ is a partition of $E(G_1 \otimes G_2)$. Next we define the *projections* p_1 and p_2. The mapping from $V(G_1 \otimes G_2)$ to $V(G_1)$ which associates $x \in V(G_1)$ with each $(x, y) \in V(G_1 \otimes G_2)$ is denoted by p_1; similarly, the mapping from $V(G_1 \otimes G_2)$ to $V(G_2)$ which associates $y \in V(G_2)$ with each $(x, y) \in V(G_1 \otimes G_2)$ is denoted by p_2. For $S \subset V(G_1 \otimes G_2)$, $p_1(S)$ and $p_2(S)$ denote the image of S by p_1 and p_2, respectively; thus

$$p_1(S) := \{x \in V(G_1) | (x, y) \in S \text{ for some } y \in V(G_2)\},$$
$$p_2(S) := \{y \in V(G_2) | (x, y) \in S \text{ for some } x \in V(G_1)\}.$$

We can regard $G_1 \otimes G_2$ as $|V(G_2)|$ copies of G_1 joined by edges in E_2. For $v \in V(G_2)$, G_1^v denotes the copy of G_1 corresponding to v; i.e., G_1^v is the subgraph of $G_1 \otimes G_2$ induced by $\{(x, v) | x \in V(G_1)\}$. Similarly we define G_2^v as the copy of G_2 corresponding to v for $v \in V(G_1)$.

We now prove some lemmas. We use them to obtain lower bounds of $\lambda(G_1 \otimes G_2)$ and $\lambda'(G_1 \otimes G_2)$ in Section 3.

Lemma 2.3. *Let G_1, G_2 be connected graphs of order at least 2. Let $E \subset E(G_1 \otimes G_2)$ be a minimal edge cutset of $G_1 \otimes G_2$, and let C_1, C_2 be the components of $(G_1 \otimes G_2) - E$. Then one of the following holds:*

- **(i)** *$p_i(V(C_1)) = p_i(V(C_2)) = V(G_i)$ for some i and $|E| \geq |V(G_i)|\lambda(G_{3-i})$; or*
- **(ii)** *$p_1(V(C_j)) \subsetneq V(G_1), p_2(V(C_j)) \subsetneq V(G_2)$ for some j and $|E| \geq \lambda(G_1) + \lambda(G_2)$ and, if we have $|E| = \lambda(G_1) + \lambda(G_2)$, then $|V(C_j)| = 1$.*

Proof. First assume $p_i(V(C_1)) = p_i(V(C_2)) = V(G_i)$ for some i. We may assume that $p_1(V(C_1)) = p_1(V(C_2)) = V(G_1)$ without loss of generality. Then for each $v \in V(G_1)$, $V(G_2^v) \cap V(C_1) \neq \emptyset$ and $V(G_2^v) \cap V(C_2) \neq \emptyset$. These two vertex sets are separated in G_2^v by $E \cap E(G_2^v)$, and hence $|E \cap E(G_2^v)| \geq \lambda(G_2)$. Consequently, $|E| \geq \sum_{v \in V(G_1)} |E \cap E(G_2^v)| \geq |V(G_1)|\lambda(G_2)$, which implies that (i) holds.

Thus we may assume that we have $p_1(V(C_1)) \subsetneq V(G_1)$ or $p_1(V(C_2)) \subsetneq V(G_1)$, and we also have $p_2(V(C_1)) \subsetneq V(G_2)$ or $p_2(V(C_2)) \subsetneq V(G_2)$. By the symmetry of roles of C_1 and C_2, we may assume $p_1(V(C_1)) \subsetneq V(G_1)$. Let $x \in V(G_1) \setminus p_1(V(C_1))$. Then for any $y \in V(G_2)$, (x, y) does not belong to C_1, and hence it belongs to C_2. Thus $p_2(V(C_2)) = V(G_2)$. In view of the assumption made at the beginning of this paragraph, this implies $p_2(V(C_1)) \subsetneq V(G_2)$. Consequently, for each $v \in p_1(V(C_1))$, $V(G_2^v) \cap V(C_1) \neq \emptyset$ and $V(G_2^v) \cap V(C_2) \neq \emptyset$. Arguing as in the first paragraph, we therefore obtain $|E \cap E_2| \geq |p_1(V(C_1))|\lambda(G_2)$. Similarly we obtain $|E \cap E_1| \geq |p_2(V(C_1))|\lambda(G_1)$. It now follows that

$$\begin{aligned}|E| &= |E \cap E_1| + |E \cap E_2| \\ &\geq |p_2(V(C_1))|\lambda(G_1) + |p_1(V(C_1))|\lambda(G_2) \\ &\geq \lambda(G_1) + \lambda(G_2).\end{aligned}$$

Further if $|E| = \lambda(G_1) + \lambda(G_2)$, then $|p_2(V(C_1))|\lambda(G_1) + |p_1(V(C_1))|\lambda(G_2) = \lambda(G_1) + \lambda(G_2)$, and hence $|p_1(V(C_1))| = |p_2(V(C_1))| = 1$, which implies $|V(C_1)| = 1$. Thus (ii) holds. □

Lemma 2.4. *Let G_1, G_2 be graphs, and suppose that $\lambda(G_i) \geq 2$ and $\lambda'(G_i)$ is defined for each $i = 1, 2$. Let $E \subset E(G_1 \otimes G_2)$ be a minimal restricted edge cutset of $G_1 \otimes G_2$, and let C_1, C_2 be the components of $(G_1 \otimes G_2) - E$. Then at least one of the following holds:*

- **(i)** *$p_i(V(C_1)) = p_i(V(C_2)) = V(G_i)$ for some i, and $|E| \geq |V(G_i)|\lambda(G_{3-i})$;*
- **(ii)** *$p_1(V(C_j)) \subsetneq V(G_1)$ and $p_2(V(C_j)) \subsetneq V(G_2)$ for some j, $E \cap E_1 \neq \emptyset$, and $|E| \geq \lambda'(G_1) + 2\lambda(G_2)$ and, if we have $|E| = \lambda'(G_1) + 2\lambda(G_2)$, then $|V(C_j)| = 2$; or*
- **(iii)** *$p_1(V(C_j)) \subsetneq V(G_1)$ and $p_2(V(C_j)) \subsetneq V(G_2)$ for some j, $E \cap E_2 \neq \emptyset$, and $|E| \geq 2\lambda(G_1) + \lambda'(G_2)$ and, if we have $|E| = 2\lambda(G_1) + \lambda'(G_2)$, then $|V(C_j)| = 2$.*

Proof. Arguing as in the first paragraph of the proof of Lemma 2.3, we see that if $p_i(V(C_1)) = p_i(V(C_2)) = V(G_i)$ for some i, then (i) holds. Thus arguing as in the first half of the second paragraph of the proof of Lemma 2.3, we may assume $p_1(V(C_1)) \subsetneq V(G_1)$ and $p_2(V(C_1)) \subsetneq V(G_2)$. Since E is a restricted edge cutset, C_1 contains an edge. Let $e \in E(C_1)$ be an edge. Assume first that $e \in E_1$. We show that (ii) holds. For each $v \in p_1(V(C_1))$, we have $V(G_2^v) \cap V(C_1) \neq \emptyset$ and $V(G_2^v) \cap V(C_2) \neq \emptyset$, and hence $|E \cap E_2| \geq |p_1(V(C_1))|\lambda(G_2)$. Let $v \in V(G_2)$ be the vertex such that $e \in E(G_1^v)$. We focus on G_1^v. We have $V(G_1^v) \cap V(C_1) \neq \emptyset$ and $V(G_1^v) \cap V(C_2) \neq \emptyset$. Now we distinguish two cases.

Case 1: $E(G_1^v) \cap E(C_2) \neq \emptyset$.

In this case, $|E \cap E(G_1^v)| \geq \lambda'(G_1)$ by Lemma 2.1. Hence

$$\begin{aligned}|E| &= |E \cap E(G_1^v)| + |E \cap E_2| + |E \cap (E_1 \setminus E(G_1^v))| \\ &\geq \lambda'(G_1) + |p_1(V(C_1))|\lambda(G_2) + |E \cap (E_1 \setminus E(G_1^v))| \\ &\geq \lambda'(G_1) + 2\lambda(G_2) + |E \cap (E_1 \setminus E(G_1^v))|.\end{aligned}$$

Thus $|E| \geq \lambda'(G_1) + 2\lambda(G_2) + |E \cap (E_1 \setminus E(G_1^v))|$.

Now if $|E| = \lambda'(G_1)+2\lambda(G_2)$, then $|p_1(V(C_1))| = 2$ and $|E\cap(E_1\setminus E(G_1^v))| = 0$ by the above inequality. From $|E\cap(E_1\setminus E(G_1^v))| = 0$, we get $|p_2(V(C_1))| = 1$, which implies $|V(C_1)| = 2$.

Case 2: $E(G_1^v) \cap E(C_2) = \emptyset$.

There are at least $|V(G_1)| - |p_1(V(C_1))|$ vertices in $V(G_1^v) \cap V(C_2)$ and they are isolated in $G_1^v - C_1$. Hence every edge of G_1^v incident with a vertex in $V(G_1^v) \cap V(C_2)$ is contained in $E \cap E(G_1^v)$. Since $\delta(G_i) \geq \lambda(G_i) \geq 2$ for each i, it follows from Lemma 2.2 that

$$\begin{aligned}
|E| &= |E \cap E(G_1^v)| + |E \cap E_2| + |E \cap (E_1 \setminus E(G_1^v))| \\
&\geq (|V(G_1)| - |p_1(V(C_1))|)\delta(G_1) + |p_1(V(C_1))|\lambda(G_2) \\
&\geq 2(|V(G_1)| - |p_1(V(C_1))|) + (|p_1(V(C_1))| - 2)\lambda(G_2) + 2\lambda(G_2) \\
&\geq 2(|V(G_1)| - |p_1(V(C_1))|) + 2(|p_1(V(C_1))| - 2) + 2\lambda(G_2) \\
&= 2(|V(G_1)| - 2) + 2\lambda(G_2) \\
&\geq \lambda'(G_1) + 2\lambda(G_2).
\end{aligned}$$

Suppose that $|E| = \lambda'(G_1) + 2\lambda(G_2)$. Then $\delta(G_1) = 2$ and $\lambda'(G_1) = 2(|V(G_1)|-2)$. By Lemma 2.2, G_1 is a complete graph. Hence from $\delta(G_1) = 2$, we see that $|V(G_1)| = 3$, which contradicts the assumption that $\lambda'(G_1)$ is defined. Thus $|E| > \lambda'(G_1) + 2\lambda(G_2)$.

We have shown that if $e \in E_1$, then (ii) holds. Similarly, if $e \in E_2$, then (iii) holds. This completes the proof of the lemma. □

§3. Proof of Main Theorems

First we prove Theorem 1.2.

Proof. First we verify that $\lambda(G_1 \otimes G_2) \leq \delta(G_1 \otimes G_2) \leq \delta(G_1) + \delta(G_2)$. Let $x \in V(G_1), y \in V(G_2)$ be vertices which attain the minimum degree of G_1 and G_2, respectively; namely, $\deg_{G_1}(x) = \delta(G_1)$ and $\deg_{G_2}(y) = \delta(G_2)$. Then $\deg_{G_1\otimes G_2}((x,y)) = \deg_{G_1}(x) + \deg_{G_2}(y) = \delta(G_1) + \delta(G_2)$ by the definition of cartesian product. Since we clearly have $\lambda(G_1 \otimes G_2) \leq \delta(G_1 \otimes G_2)$, we get $\lambda(G_1 \otimes G_2) \leq \delta(G_1 \otimes G_2) \leq \delta(G_1) + \delta(G_2)$.

Next we prove $\lambda(G_1 \otimes G_2) \geq \delta(G_1) + \delta(G_2)$. Let E be an edge cutset of $G_1 \otimes G_2$. We may assume that E is a minimal edge cutset. By Lemma 2.3,

$$|E| \geq \min\{|V(G_1)|\lambda(G_2), |V(G_2)|\lambda(G_1), \lambda(G_1) + \lambda(G_2)\}.$$

By easy calculations, we get

$$|V(G_1)|\lambda(G_2) \geq |V(G_1)| + \lambda(G_2) - 1 \geq \lambda(G_1) + \lambda(G_2).$$

Similarly $|V(G_2)|\lambda(G_1) \geq \lambda(G_1) + \lambda(G_2)$. Since $\lambda(G_i) = \delta(G_i)$ for each i by assumption, it follows that $|E| \geq \lambda(G_1) + \lambda(G_2) = \delta(G_1) + \delta(G_2)$. Since E is an arbitrary edge cutset, we get $\lambda(G_1 \otimes G_2) \geq \delta(G_1) + \delta(G_2)$. Combining this with the inequality proved in the first paragraph, we obtain $\lambda(G_1 \otimes G_2) = \delta(G_1 \otimes G_2) = \delta(G_1) + \delta(G_2)$

We now prove the last assertion of the theorem. Suppose that $G_1 \otimes G_2$ is not super edge connected, and let E be an edge cutset with $|E| = \delta(G_1) + \delta(G_2)$ such that $(G_1 \otimes G_2) - E$ has no component of order 1. From Lemma 2.3, we see that (i) of Lemma 2.3 holds. By the calculations in the preceding paragraph, we have either $|V(G_1)|\lambda(G_2) = |V(G_1)| + \lambda(G_2) - 1 = \lambda(G_1) + \lambda(G_2)$ or $|V(G_2)|\lambda(G_1) = |V(G_2)| + \lambda(G_1) - 1 = \lambda(G_2) + \lambda(G_1)$. If $|V(G_1)|\lambda(G_2) = |V(G_1)| + \lambda(G_2) - 1 = \lambda(G_1) + \lambda(G_2)$, then it follows from the first equality that $\lambda(G_2) = 1$, and it follows from the second equality that G_1 is a complete graph. Similarly, if $|V(G_2)|\lambda(G_1) = |V(G_2)| + \lambda(G_1) - 1 = \lambda(G_2) + \lambda(G_1)$, then $\lambda(G_1) = 1$ and G_2 is a complete graph. This completes the proof of Theorem 1.2. □

Next we prove Theorem 1.3. The outline of the proof is the same as the proof of Theorem 1.2, though some calculations are somewhat complicated.

Proof. By Lemma 2.2, $\lambda'(G_1 \otimes G_2)$ is defined. First, we verify that $\lambda'(G_1 \otimes G_2) \leq \delta'(G_1 \otimes G_2) \leq \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}$. Let $x_1x_2 \in E(G_1), y \in V(G_2)$ be an edge and a vertex such that $\deg_{G_1}(x_1) + \deg_{G_1}(x_2) - 2 = \delta'(G_1)$ and $\deg_{G_2}(y) = \delta(G_2)$. Then $\deg_{G_1 \otimes G_2}(x_1, y) + \deg_{G_1 \otimes G_2}(x_2, y) - 2 = \delta'(G_1) + 2\delta(G_2)$ by the definition of cartesian product. Since $\lambda'(G_1 \otimes G_2) \leq \delta'(G_1 \otimes G_2)$ by Lemma 2.2, we get $\lambda'(G_1 \otimes G_2) \leq \delta'(G_1 \otimes G_2) \leq \delta'(G_1) + 2\delta(G_2)$. By swapping the roles of G_1 and G_2 in the above argument, we also get $\lambda'(G_1 \otimes G_2) \leq \delta'(G_1 \otimes G_2) \leq \delta'(G_2) + 2\delta(G_1)$.

Next we prove $\lambda'(G_1 \otimes G_2) \geq \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}$. Note that $|V(G_i)| \geq 4$ for each i because $\lambda'(G_i)$ is defined. Let E be a restricted edge cutset of $G_1 \otimes G_2$. We may assume that E is a minimal edge cutset. By Lemma 2.4,

$$|E| \geq \min\{|V(G_1)|\lambda(G_2), |V(G_2)|\lambda(G_1), \lambda'(G_1) + 2\lambda(G_2), \lambda'(G_2) + 2\lambda(G_1)\}$$

On the other hand, since $|V(G_1)| > 2$ and $\lambda(G_2) \geq 2$, it follows from Lemma 2.2 that,

$$|V(G_1)|\lambda(G_2) \geq 2|V(G_1)| + 2\lambda(G_2) - 4 \geq \lambda'(G_1) + 2\lambda(G_2).$$

Similarly $|V(G_2)|\lambda(G_1) \geq \lambda'(G_2) + 2\lambda(G_1)$. Since $\lambda(G_i) = \delta(G_i)$ and $\lambda'(G_i) = \delta'(G_i)$ for each i by assumption, it follows that

$$\begin{aligned}|E| &\geq \min\{\lambda'(G_1) + 2\lambda(G_2), \lambda'(G_2) + 2\lambda(G_1)\}\\ &= \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}.\end{aligned}$$

Since E is an arbitrary restricted edge cutset, we get

$$\lambda'(G_1 \otimes G_2) \geq \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}.$$

Combining this with the inequalities proved in the first paragraph, we obtain

$$\lambda'(G_1 \otimes G_2) = \delta'(G_1 \otimes G_2) = \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}.$$

We now prove the last assertion of the theorem. Suppose that $G_1 \otimes G_2$ is not super restricted edge connected. and let E be a restricted edge cutset with $|E| = \min\{\delta'(G_1) + 2\delta(G_2), \delta'(G_2) + 2\delta(G_1)\}$ such that $(G_1 \otimes G_2) - E$ has no component of order 2. From Lemma 2.4, we see that (i) of Lemma 2.4 holds. By the calculations in the preceding paragraph, we have either $|V(G_1)|\lambda(G_2) = 2|V(G_1)| + 2\lambda(G_2) - 4 = \lambda'(G_1) + 2\lambda(G_2)$ or $|V(G_2)|\lambda(G_1) = 2|V(G_2)| + 2\lambda(G_1) - 4 = \lambda'(G_2) + 2\lambda(G_1)$. If $|V(G_1)|\lambda(G_2) = 2|V(G_1)| + 2\lambda(G_2) - 4 = \lambda'(G_1) + 2\lambda(G_2)$, then since $|V(G_1)| > 2$, it follows from the first equality that $\lambda(G_2) = 2$, and it follows from the second equality and Lemma 2.2 that G_1 is a complete graph. Similarly if $|V(G_2)|\lambda(G_1) = 2|V(G_2)| + 2\lambda(G_1) - 4 = \lambda'(G_2) + 2\lambda(G_1)$, then $\lambda(G_1) = 2$ and G_2 is a complete graph. This proves the last assertion, and completes the proof of Theorem 1.3. $\square$

§4. Corollaries

In this section, we prove corollaries of Theorem 1.2 and Theorem 1.3 (note that Proposition 1.1 follows immediately from these corollaries).

Corollary 4.1. *Let $n \geq 2$. Let $G_1, G_2, \ldots, G_n$ be graphs, and suppose that $\lambda(G_i) = \delta(G_i) \geq 1$ for each $1 \leq i \leq n$. Then*

$$\lambda(G_1 \otimes \cdots \otimes G_n) = \delta(G_1 \otimes \cdots \otimes G_n) = \sum_{1 \leq i \leq n} \delta(G_i).$$

Furthermore, unless $n = 2$ and either G_1 is a complete graph and $\delta(G_2) = 1$ or G_2 is a complete graph and $\delta(G_1) = 1$, $G_1 \otimes \cdots \otimes G_n$ is super edge connected.

Proof. We proceed by induction on n. If $n = 2$, then the desired conclusion follows from Theorem 1.2. Thus let $n \geq 3$ and assume that the proposition holds for $n-1$. Then $\lambda(G_1 \otimes \cdots \otimes G_{n-1}) = \delta(G_1 \otimes \cdots \otimes G_{n-1}) = \sum_{1 \leq i \leq n-1} \delta(G_i)$.

Hence

$$\begin{aligned}
\lambda(G_1 \otimes \cdots \otimes G_n) &= \delta(G_1 \otimes \cdots \otimes G_n) \\
&= \sum_{1 \leq i \leq n-1} \delta(G_i) + \delta(G_n) \\
&= \sum_{1 \leq i \leq n} \delta(G_i)
\end{aligned}$$

by Theorem 1.2. Further $\delta(G_1 \otimes \cdots \otimes G_{n-1}) = \sum_{1 \le i \le n-1} \delta(G_i) > 1$ and $G_1 \otimes \cdots \otimes G_{n-1}$ is not a complete graph. Consequently $G_1 \otimes \cdots \otimes G_n$ is super edge connected by Theorem 1.2. $\square$

Corollary 4.2. *Let $n \ge 2$. Let $G_1, G_2, \ldots, G_n$ be graphs and suppose that $\lambda(G_i) = \delta(G_i) \ge 2$ and $\lambda'(G_i) = \delta'(G_i) \ge 2$ for each $1 \le i \le n$. Then*

$$\lambda'(G_1 \otimes \cdots \otimes G_n) = \delta'(G_1 \otimes \cdots \otimes G_n) = \min_{1 \le j \le n} (\delta'(G_j) + 2 \sum_{\substack{1 \le i \le n \\ i \ne j}} \delta(G_i)).$$

Furthermore, unless $n = 2$ and either G_1 is a complete graph and $\delta(G_2) = 2$ or G_2 is a complete graph and $\delta(G_1) = 2$, $G_1 \otimes \cdots \otimes G_n$ is super restricted edge connected.

Proof. We proceed by induction on n. If $n = 2$, then the desired conclusion follows from Theorem 1.3. Thus let $n \ge 3$ and assume that the proposition holds for $n - 1$. Then

$$\begin{aligned} \lambda(G_1 \otimes \cdots \otimes G_{n-1}) &= \delta(G_1 \otimes \cdots \otimes G_{n-1}) \\ &= \min_{1 \le j \le n-1} (\delta'(G_j) + 2 \sum_{\substack{1 \le i \le n-1 \\ i \ne j}} \delta(G_i)) \end{aligned}$$

Hence

$$\begin{aligned} \lambda'(G_1 \otimes \cdots \otimes G_n) &= \delta'(G_1 \otimes \cdots \otimes G_n) \\ &= \min\{\delta'(G_1 \otimes \cdots \otimes G_{n-1}) + 2\delta(G_n), \\ &\quad \delta'(G_n) + 2\delta(G_1 \otimes \cdots \otimes G_{n-1})\} \\ &= \min\{\min_{1 \le j \le n-1} (\delta'(G_j) + 2 \sum_{\substack{1 \le i \le n-1 \\ i \ne j}} \delta(G_i)) + 2\delta(G_n), \\ &\quad \delta'(G_n) + 2 \sum_{1 \le i \le n-1} \delta(G_i)\} \\ &= \min_{1 \le j \le n} (\delta'(G_j) + 2 \sum_{\substack{1 \le i \le n \\ i \ne j}} \delta(G_i)). \end{aligned}$$

by Theorem 1.3 and Corollary 4.1. Further

$$\delta(G_1 \otimes \cdots \otimes G_{n-1}) = \sum_{1 \le i \le n-1} \delta(G_i) > 2$$

and $G_1 \otimes \cdots \otimes G_{n-1}$ is not a complete graph. Consequently $G_1 \otimes \cdots \otimes G_n$ is super restricted edge connected by Theorem 1.3. This proves Corollary 4.2. $\square$

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