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Strong instability of standing waves with negative energy for double power nonlinear Schrödinger equations

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Abstract. We study the strong instability of ground-state standing waves $e^{i\omega t}\phi_\omega(x)$ for N-dimensional nonlinear Schrödinger equations with focusing double power nonlinearity. One is L^2-subcritical, and the other is L^2-supercritical. The strong instability of standing waves with positive energy was proven by Ohta and Yamaguchi (2015). In this paper, we improve the previous result, that is, we prove that if $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, the standing wave is strongly unstable, where S_ω is the action, and $\phi_\omega^\lambda(x) := \lambda^{N/2}\phi_\omega(\lambda x)$ is the L^2-invariant scaling.

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§1. Introduction

In this paper, we consider the nonlinear Schrödinger equation with double power nonlinearity

$$\text{(NLS)} \qquad i\partial_t u = -\Delta u - a|u|^{p-1}u - b|u|^{q-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N,$$

where

$$N\in\mathbb{N}, \quad a>0, \quad b>0, \quad 1<p<1+\frac{4}{N}<q<1+\frac{4}{N-2}, \tag{1.1}$$

and $u\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ is the unknown function of $(t,x)\in\mathbb{R}\times\mathbb{R}^N$. Here, $1+4/(N-2)$ stands for ∞ if $N=1$ or 2. Eq. (NLS) appears in various regions of mathematical physics (see [1, 6, 20] and references therein).

The Cauchy problem for (NLS) is locally well-posed in the energy space $H^1(\mathbb{R}^N)$ (see, e.g., [4, 9]), that is, for each $u_0 \in H^1(\mathbb{R}^N)$, there exist the maximal lifespan $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and a unique solution u of (NLS) belonging to $C([0, T_{\max}), H^1(\mathbb{R}^N))$ with $u(0) = u_0$ such that if $T_{\max} < \infty$, then $\|\nabla u(t)\|_{L^2} \to \infty$ as $t \nearrow T_{\max}$. In the case $T_{\max} < \infty$, we say that the solution $u(t)$ *blows up in finite time.* Moreover, (NLS) satisfies the two conservation laws

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

for all $t \in [0, T_{\max})$, where E is the energy defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

Furthermore, if

$$u_0 \in \Sigma := \{v \in H^1(\mathbb{R}^N) \mid \|xv\|_{L^2} < \infty\}, \tag{1.2}$$

then the solution $u(t)$ of (NLS) with $u(0) = u_0$ belongs to $C([0, T_{\max}), \Sigma)$ and satisfies the virial identity

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8Q(u(t)) \tag{1.3}$$

for all $t \in [0, T_{\max})$ (see [4, Section 6.5]), where $v^\lambda(x) = \lambda^{N/2} v(\lambda x)$ and

$$\begin{aligned} Q(v) &= \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1} \\ &= \|\nabla v\|_{L^2}^2 - \frac{aN(p-1)}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} - \frac{bN(q-1)}{2(q+1)}\|v\|_{L^{q+1}}^{q+1}. \end{aligned} \tag{1.4}$$

Eq. (NLS) has standing wave solutions of the form $e^{i\omega t}\phi(x)$, where $\omega > 0$ and $\phi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of the stationary equation

$$-\Delta\phi + \omega\phi - a|\phi|^{p-1}\phi - b|\phi|^{q-1}\phi = 0, \quad x \in \mathbb{R}^N. \tag{1.5}$$

Eq. (1.5) can be rewritten as $S_\omega'(\phi) = 0$, where S_ω is the action defined by

$$\begin{aligned} S_\omega(v) &= E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 \\ &= \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1}. \end{aligned}$$

It is known that if $\omega > 0$, then (1.5) has *ground state* solutions, that is, the set

$$\mathcal{G}_\omega := \left\{ \phi \in H^1(\mathbb{R}^N) \;\middle|\; \begin{array}{l} \phi \neq 0,\ S_\omega'(\phi) = 0, \\ S_\omega(\phi) = \inf\{S_\omega(\psi) \mid \psi \neq 0,\ S_\omega'(\psi) = 0\} \end{array} \right\}$$

of nontrivial solutions to (1.5) with the minimal action is not empty (see, e.g., [3, 12, 19]).

The stability and instability of standing waves are defined as follows:

Definition 1.1. Let $\phi \in H^1(\mathbb{R}^N)$ be a nontrivial solution of (1.5).

- We say that the standing wave solution $e^{i\omega t}\phi$ of (NLS) is *stable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $\|u_0 - \phi\|_{H^1} < \delta$, then the solution $u(t)$ of (NLS) with $u(0) = u_0$ exists globally in time and satisfies
$$\sup_{t \geq 0} \inf_{(\theta,y) \in \mathbb{R} \times \mathbb{R}^N} \|u(t) - e^{i\theta}\phi(\cdot - y)\|_{H^1} < \varepsilon.$$
- We say that the standing wave solution $e^{i\omega t}\phi$ of (NLS) is *unstable* if it is not stable.
- We say that the standing wave solution $e^{i\omega t}\phi$ of (NLS) is *strongly unstable* if for each $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0 - \phi\|_{H^1} < \varepsilon$, and the solution $u(t)$ of (NLS) with $u(0) = u_0$ blows up in finite time.

In this paper, we study the strong instability of the standing wave solution $e^{i\omega t}\phi_\omega$ for (NLS), where $\omega > 0$, and $\phi_\omega \in \mathcal{G}_\omega$ is a ground state.

In the single-power L^2-supercritical or L^2-critical case when $a = 0$, $b > 0$, and $1 + 4/N \leq q < 1 + 4/(N-2)$, Berestycki and Cazenave [2] and Weinstein [21] proved that the standing wave is strongly unstable for any $\omega > 0$ by using variational arguments and the virial identity. On the other hand, in the L^2-subcritical case when $a > 0$, $b = 0$, and $1 < p < 1 + 4/N$, Cazenave and Lions [5] proved that the standing wave is stable for any $\omega > 0$. They show that the ground state is the unique minimizer of the action under the mass constraint $\|v\|_{L^2} = \|\phi_\omega\|_{L^2}$ up to symmetries and that the minimizing sequence in the sense that $S_\omega(v_n) \to S_\omega(\phi_\omega)$ and $\|v_n\|_{L^2} \to \|\phi_\omega\|_{L^2}$ is compact up to translation.

In the double power case when (1.1) is assumed, the argument of Ohta [14] showed the instability of standing waves for sufficiently large $\omega > 0$. In [14], it was proven that if $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then the standing wave is unstable, where $v^\lambda(x) := \lambda^{N/2} v(\lambda x)$ is the scaling, which does not change the L^2-norm. The assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} < 0$ means that $\partial_\lambda \phi_\omega^\lambda|_{\lambda=1}$ is an unstable direction, and that the ground state ϕ_ω is a saddle point of the action on the hypersurface $\{v \in H^1(\mathbb{R}^N) \mid \|v\|_{L^2} = \|\phi_\omega\|_{L^2}\}$. On the other hand, Fukuizumi [8] proved the stability of standing waves for sufficiently small $\omega > 0$ showing some coercivity of the linearized operator around the ground state. See also [13, 15] for the stability and instability in one dimensional case. The strong instability of standing waves for sufficiently large ω was proven by Ohta and Yamaguchi [17]. In [17], they proved the strong instability of standing waves with positive energy $E(\phi_\omega) > 0$ by using and modifying the idea of Zhang [22] and Le Coz [10] (see also [18] for related works).

Recently, for the nonlinear Schrödinger equation with harmonic potential

$$i\partial_t u = -\Delta u + |x|^2 u - |u|^{q-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{1.6}$$

with $1+4/N < q < 1+4/(N-2)$, Ohta [16] proved that if $\partial_\lambda^2 \tilde{S}_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, then the standing wave is strongly unstable, where $\tilde{S}_\omega$ is the corresponding action. This assumption is the same one as in Ohta [14]. More recently, Fukaya and Ohta [7] proved the strong instability of standing waves for nonlinear Schrödinger equation with an attractive inverse power potential

$$i\partial_t u = -\Delta u - \frac{\gamma}{|x|^\alpha} u - |u|^{q-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{1.7}$$

with $\gamma > 0$, $0 < \alpha < \min\{2, N\}$, and $1+4/N < q < 1+4/(N-2)$ under the same assumption $\partial_\lambda^2 \tilde{S}_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ as in [16] by using the idea of Ohta [16] with some modifications. The assumption $\partial_\lambda^2 \tilde{S}_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ indicates that $\|\phi_\omega^\lambda\|_{L^2} = \|\phi_\omega\|_{L^2}$, $\tilde{S}_\omega(\phi_\omega^\lambda) < \tilde{S}_\omega(\phi_\omega)$, and $\tilde{Q}(\phi_\omega^\lambda) < 0$ for all $\lambda > 1$, where $\tilde{Q}$ is the functional arising in the virial identity. In general, the assumption $\partial_\lambda^2 \tilde{S}_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ is a local property around ϕ_ω. In case of (1.6) or (1.7), however, this assumption gives global information in some sense thanks to the homogeneity of the potential energy. Due to this assumption, the inequality $\tilde{Q}(\phi_\omega^\lambda) < 0$ leads to the uniform estimate $\sup_{t\in[0,T_{\max})} \tilde{Q}(u_\lambda(t)) < 0$, where $u_\lambda(t)$ is the solution with initial data ϕ_ω^λ. This uniform estimate combined with the virial identity implies the strong instability of the standing wave.

For (NLS), the strong instability of standing waves with negative energy was not known. The aim of this paper is to prove the strong instability under the same assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ as in [7, 16]. Now, we state our main result.

Theorem 1.2. *Assume* (1.1), *$\omega > 0$, and that the ground state $\phi_\omega \in \mathcal{G}_\omega$ satisfies $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, where $\phi_\omega^\lambda(x) = \lambda^{N/2}\phi_\omega(\lambda x)$. Then the standing wave solution $e^{i\omega t}\phi_\omega$ of* (NLS) *is strongly unstable.*

Remark 1.3. In the case (1.1), $E(\phi_\omega) > 0$ implies $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} < 0$. Indeed, let $\alpha = N(p-1)/2$ and $\beta = N(q-1)/2$. Then since $Q(\phi_\omega) = \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$ and $0 < \alpha < 2 < \beta$, we have

$$\begin{aligned}
\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} &= \|\nabla \phi_\omega\|_{L^2}^2 - \frac{a\alpha(\alpha-1)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} - \frac{b\beta(\beta-1)}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1} \\
&= (\alpha+1)Q(\phi_\omega) - 2\alpha E(\phi_\omega) - \frac{b(\beta-2)(\beta-\alpha)}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1} \\
&< 0.
\end{aligned}$$

Therefore, Theorem 1.2 is an improvement of the result of Ohta and Yamaguchi [17].

To prove Theorem 1.2, we introduce the set

$$\mathcal{B}_\omega := \left\{ v \in H^1(\mathbb{R}^N) \;\middle|\; \begin{array}{ll} S_\omega(v) < S_\omega(\phi_\omega), & \|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \\ K_\omega(v) < 0, & Q(v) < 0 \end{array} \right\}, \tag{1.8}$$

where

$$\begin{aligned} K_\omega(v) &:= \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} \\ &= \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - a\|v\|_{L^{p+1}}^{p+1} - b\|v\|_{L^{q+1}}^{q+1} \end{aligned} \tag{1.9}$$

is the Nehari functional. Then we obtain the following blowup result.

Theorem 1.4. *Assume* (1.1)*,* $\omega > 0$*, and that the ground state* $\phi_\omega \in \mathcal{G}_\omega$ *satisfies* $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$*. Then the set* $\mathcal{B}_\omega$ *is invariant under the flow of* (NLS)*. Moreover, if* $u_0 \in \mathcal{B}_\omega \cap \Sigma$*, then the solution* $u(t)$ *of* (NLS) *with* $u(0) = u_0$ *blows up in finite time.*

Theorem 1.2 follows from Theorem 1.4 because the scaling of the ground state ϕ_ω^λ belongs to $\mathcal{B}_\omega \cap \Sigma$ for all $\lambda > 1$ (see Section 3 below).

The proof of Theorem 1.4 is based on the variational argument in Ohta [16] and Fukaya and Ohta [7]. Firstly, we derive the key estimate $Q(v)/2 \le S_\omega(v) - S_\omega(\phi_\omega)$ for all $v \in \mathcal{B}_\omega$ (Lemma 2.1 below). Then by using the conservation laws, the variational characterization of the ground state by the Nehari functional, and the key estimate, we show the invariance of $\mathcal{B}_\omega$ under the flow of (NLS) (Lemma 2.2 below). Combining the virial identity with the key estimate, finally, we can obtain the blowup of solutions to (NLS) with initial data belonging to $\mathcal{B}_\omega \cap \Sigma$ by the classical argument as in Berestycki and Cazenave [2].

We prove the key estimate $Q/2 \le S_\omega - S_\omega(\phi_\omega)$ on $\mathcal{B}_\omega$ following the proof of the same estimate for (1.7) in [7, Lemma 3.2]. The proof relies on the variational characterization of the ground state by the Nehari functional

$$S_\omega(\phi_\omega) = \inf\{S_\omega(v) \mid v \ne 0,\ K_\omega(v) = 0\}$$

and the property of the graph of the function $\lambda \mapsto S_\omega(v^\lambda)$. Note that the graph of $S_\omega(v^\lambda)$ for (NLS) has the same property as that for (1.7). In the case of (1.7), since the action $\tilde{S}_\omega$ can be expressed by use of the Nehari functional $\tilde{K}_\omega(v) := \partial_\lambda \tilde{S}_\omega(\lambda v)|_{\lambda=1}$ as

$$\tilde{S}_\omega(v) = \frac{1}{2}\tilde{K}_\omega(v) + \frac{q-1}{2(q+1)}\|v\|_{L^{q+1}}^{q+1}, \tag{1.10}$$

the above variational characterization can be written by use of L^{q+1}-norm. Therefore, in [7], not only the action but also L^{q+1}-norm was used effectively.

On the other hand, in the case of (NLS), the action S_ω cannot be expressed as (1.10) because (NLS) has double power nonlinearity. Due to this fact, we can not directly apply the proof in [7]. However, in this case, we see that the action can be expressed as

$$S_\omega(v) = \frac{1}{2}K_\omega(v) + \frac{1}{2}F(v),$$

where

$$F(v) = \frac{a(p-1)}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{b(q-1)}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

Therefore, we can use F instead of L^{q+1}-norm. By applying the argument in [7] using F, although the calculation processes differ from that in [7], we can prove the key estimate above.

At the end of this section, we remark that the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ is not a necessary condition for the instability of standing waves (see [18, Section 4] for related remarks). However, in [7, 16] and this paper, this assumption plays a very important role in the proof of the strong instability of standing waves. It is still an open problem whether the unstable standing wave is strongly unstable or not if the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ is broken.

The rest of this paper is organized as follows: In Section 2, we prove Theorem 1.4, that is, we prove that if $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, then the solution of (NLS) with $u(0) = u_0 \in \mathcal{B}_\omega \cap \Sigma$ blows up in finite time. In Section 3, we prove the strong instability of standing waves by using Theorem 1.4.

§2. Blowup

In this section, we prove Theorem 1.4. Throughout this section, we assume (1.1) and $\omega > 0$. Recall that the ground state $\phi_\omega \in \mathcal{G}_\omega$ satisfies $K_\omega(\phi_\omega) = 0$ and the variational characterization

$$S_\omega(\phi_\omega) = \inf\{S_\omega(v) \mid v \ne 0,\ K_\omega(v) = 0\} \tag{2.1}$$

(see, e.g., [11, 12]), where K_ω is the Nehari functional defined in (1.9). Note that the action S_ω is expressed as

$$S_\omega(v) = \frac{1}{2}K_\omega(v) + \frac{1}{2}F(v), \tag{2.2}$$

where

$$F(v) = \frac{a(p-1)}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{b(q-1)}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

Therefore, the characterization (2.1) is rewritten as

$$F(\phi_\omega) = \inf\left\{F(v) \mid v \ne 0,\ K_\omega(v) = 0\right\}. \tag{2.3}$$

Let

$$\alpha = \frac{N(p-1)}{2}, \quad \beta = \frac{N(q-1)}{2}.$$

Using this notation, we have

$$\begin{aligned}
S_\omega(v^\lambda) &= \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}, \\
K_\omega(v^\lambda) &= \lambda^2\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - a\lambda^\alpha\|v\|_{L^{p+1}}^{p+1} - b\lambda^\beta\|v\|_{L^{q+1}}^{q+1}, \\
\frac{N}{2}F(v^\lambda) &= \frac{a\alpha\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{b\beta\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}, \\
Q(v) &= \|\nabla v\|_{L^2}^2 - \frac{a\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}, \\
\partial_\lambda^2 S_\omega(v^\lambda)|_{\lambda=1} &= \|\nabla v\|_{L^2}^2 - \frac{a\alpha(\alpha-1)}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\beta(\beta-1)}{q+1}\|v\|_{L^{q+1}}^{q+1},
\end{aligned}$$

where $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$. Note that by $S'_\omega(\phi_\omega) = 0$, we have

$$K_\omega(\phi_\omega) = \langle S'_\omega(\phi_\omega), \phi_\omega\rangle = 0, \quad Q(\phi_\omega) = \langle S'_\omega(\phi_\omega), \partial_\lambda \phi_\omega^\lambda|_{\lambda=1}\rangle = 0.$$

Firstly, we prove the key lemma in the proof.

Lemma 2.1. *Assume that $\phi_\omega \in \mathcal{G}_\omega$ satisfies $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$. Let $v \in H^1(\mathbb{R}^N)$ satisfy*

$$v \neq 0, \quad \|v\|_{L^2}^2 \le \|\phi_\omega\|_{L^2}^2, \quad K_\omega(v) \le 0, \quad Q(v) \le 0.$$

Then

$$\frac{Q(v)}{2} \le S_\omega(v) - S_\omega(\phi_\omega).$$

Proof. Since $\lim_{\lambda\searrow 0} K_\omega(v^\lambda) = \omega\|v\|_{L^2}^2 > 0$ and $K_\omega(v) \le 0$, there exists $\lambda_0 \in (0,1]$ such that $K_\omega(v^{\lambda_0}) = 0$. By the definition of the scaling v^λ and (2.3), we have

$$\|v^{\lambda_0}\|_{L^2} = \|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \tag{2.4}$$

$$\frac{N}{2}F(\phi_\omega) \le \frac{N}{2}F(v^{\lambda_0}) = \frac{a\alpha\lambda_0^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} + \frac{b\beta\lambda_0^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}. \tag{2.5}$$

Now, we define

$$\begin{aligned}
f(\lambda) &= S_\omega(v^\lambda) - \frac{\lambda^2}{2}Q(v) \\
&= \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a}{p+1}\left(\lambda^\alpha - \frac{\alpha\lambda^2}{2}\right)\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\left(\lambda^\beta - \frac{\beta\lambda^2}{2}\right)\|v\|_{L^{q+1}}^{q+1}.
\end{aligned}$$

for $\lambda \in (0,1]$. If we have $f(\lambda_0) \le f(1)$, then by (2.1) and $Q(v) \le 0$, we obtain

$$S_\omega(\phi_\omega) \le S_\omega(v^{\lambda_0}) \le S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2} Q(v) \le S_\omega(v) - \frac{Q(v)}{2}. \tag{2.6}$$

This is the desired inequality.

In what follows, we prove the inequality $f(\lambda_0) \le f(1)$. This is equivalent to

$$\frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} \le \frac{b}{q+1} \cdot \frac{2\lambda_0^\beta - \beta\lambda_0^2 - 2 + \beta}{\alpha\lambda_0^2 - 2\lambda_0^\alpha - \alpha + 2}\|v\|_{L^{q+1}}^{q+1}. \tag{2.7}$$

Since

$$\frac{p+1}{\alpha} + \frac{2}{\beta} = \frac{2}{N} + \frac{2}{\beta} + \frac{2}{\alpha} = \frac{q+1}{\beta} + \frac{2}{\alpha}, \tag{2.8}$$

we have

$$\begin{aligned}
&K_\omega(\phi_\omega) + \frac{2}{\alpha\beta}\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} - \left(1 + \frac{2}{\alpha\beta}\right) Q(\phi_\omega) \\
&= \omega\|\phi_\omega\|_{L^2}^2 - \frac{a\alpha}{p+1}\left(\frac{p+1}{\alpha} + \frac{2}{\beta} - 1 - \frac{4}{\alpha\beta}\right)\|\phi_\omega\|_{L^{p+1}}^{p+1} \\
&\quad - \frac{b\beta}{q+1}\left(\frac{q+1}{\beta} + \frac{2}{\alpha} - 1 - \frac{4}{\alpha\beta}\right)\|\phi_\omega\|_{L^{q+1}}^{q+1} \\
&= \omega\|\phi_\omega\|_{L^2}^2 - \left(\frac{q+1}{\beta} + \frac{2}{\alpha} - 1 - \frac{4}{\alpha\beta}\right)\frac{N}{2}F(\phi_\omega).
\end{aligned}$$

Therefore, by $K_\omega(\phi_\omega) = Q(\phi_\omega) = 0$ and the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, we obtain

$$\omega\|\phi_\omega\|_{L^2}^2 \le \left(\frac{q+1}{\beta} + \frac{2}{\alpha} - 1 - \frac{4}{\alpha\beta}\right)\frac{N}{2}F(\phi_\omega).$$

Combining (2.4) and (2.5) with this inequality and using (2.8) again, we have

$$\begin{aligned}
\omega\|v\|_{L^2}^2 \le &\left(a + \frac{a}{p+1}\cdot\frac{1}{\beta}(2\alpha - \alpha\beta - 4)\right)\lambda_0^\alpha\|v\|_{L^{p+1}}^{p+1} \\
&+ \left(b + \frac{b}{q+1}\cdot\frac{1}{\alpha}(2\beta - \alpha\beta - 4)\right)\lambda_0^\beta\|v\|_{L^{q+1}}^{q+1}.
\end{aligned} \tag{2.9}$$

Moreover, it follows from $K_\omega(v^{\lambda_0})=0$, $Q(v)\le 0$, and (2.9) that

$$\begin{aligned}
a\|v\|_{L^{p+1}}^{p+1} &= \lambda_0^{2-\alpha}\|\nabla v\|_{L^2}^2+\lambda_0^{-\alpha}\omega\|v\|_{L^2}^2-b\lambda_0^{\beta-\alpha}\|v\|_{L^{q+1}}^{q+1}\\
&\le \lambda_0^{2-\alpha}\left(\frac{a\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}+\frac{b\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}\right)\\
&\quad+\left(a+\frac{a}{p+1}\cdot\frac{1}{\beta}\left(2\alpha-\alpha\beta-4\right)\right)\|v\|_{L^{p+1}}^{p+1}\\
&\quad+\left(b+\frac{b}{q+1}\cdot\frac{1}{\alpha}\left(2\beta-\alpha\beta-4\right)\right)\lambda_0^{\beta-\alpha}\|v\|_{L^{q+1}}^{q+1}-b\lambda_0^{\beta-\alpha}\|v\|_{L^{q+1}}^{q+1}\\
&=\left(a+\frac{a}{p+1}\cdot\frac{1}{\beta}\left(2\alpha-\alpha\beta-4+\alpha\beta\lambda_0^{2-\alpha}\right)\right)\|v\|_{L^{p+1}}^{p+1}\\
&\quad+\frac{b}{q+1}\cdot\frac{1}{\alpha}\left(\left(2\beta-\alpha\beta-4\right)\lambda_0^{\beta-\alpha}+\alpha\beta\lambda_0^{2-\alpha}\right)\|v\|_{L^{q+1}}^{q+1},
\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{a}{p+1}\cdot\frac{1}{\beta}\left(\alpha\beta+4-2\alpha-\alpha\beta\lambda_0^{2-\alpha}\right)\|v\|_{L^{p+1}}^{p+1}\\
&\quad\le\frac{b}{q+1}\cdot\frac{1}{\alpha}\left(\left(2\beta-\alpha\beta-4\right)\lambda_0^{\beta-\alpha}+\alpha\beta\lambda_0^{2-\alpha}\right)\|v\|_{L^{q+1}}^{q+1}.
\end{aligned}$$

Since $\alpha\beta+4-2\alpha-\alpha\beta\lambda_0^{2-\alpha}\ge 4-2\alpha>0$, this is rewritten as

$$\frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1}\le\frac{b}{q+1}\cdot\frac{\beta(2\beta-\alpha\beta-4)\lambda_0^{\beta-\alpha}+\alpha\beta^2\lambda_0^{2-\alpha}}{\alpha(\alpha\beta+4-2\alpha-\alpha\beta\lambda_0^{2-\alpha})}\|v\|_{L^{q+1}}^{q+1}. \tag{2.10}$$

In view of (2.7) and (2.10), it suffices to show that

$$\frac{\beta(2\beta-\alpha\beta-4)\lambda_0^{\beta-\alpha}+\alpha\beta^2\lambda_0^{2-\alpha}}{\alpha(\alpha\beta+4-2\alpha-\alpha\beta\lambda_0^{2-\alpha})}\le\frac{2\lambda_0^\beta-\beta\lambda_0^2-2+\beta}{\alpha\lambda_0^2-2\lambda_0^\alpha-\alpha+2}.$$

This inequality follows if we have

$$\begin{aligned}
g_1(\lambda)&:=\frac{\alpha(2\lambda^\beta-\beta\lambda^2-2+\beta)(\alpha\beta+4-2\alpha-\alpha\beta\lambda^{2-\alpha})}{(\alpha\lambda^2-2\lambda^\alpha-\alpha+2)\lambda^{\beta-\alpha}}\\
&\qquad-\beta(2\beta-\alpha\beta-4)-\frac{\alpha\beta^2}{\lambda^{\beta-2}}\\
&\ge 0
\end{aligned}$$

for all $\lambda\in(0,1)$. Since $\lim_{\lambda\nearrow 1}g_1(\lambda)=0$, it is enough to show that $g_1'(\lambda)\le 0$ for all $\lambda\in(0,1)$. A direct calculation shows

$$\begin{aligned}
g_1'(\lambda)&=\frac{\alpha\lambda^{\alpha-\beta+1}}{(\alpha\lambda^2-2\lambda^\alpha-\alpha+2)^2}\\
&\quad\cdot\big((2-\alpha)(\beta-2)-2\beta\lambda^{-\alpha}+(\alpha\beta-2\alpha+4)\lambda^{-2}\big)\\
&\quad\cdot\big(2\alpha(2-\alpha)\lambda^\beta-\alpha\beta(\beta-\alpha)\lambda^2+2\beta(\beta-2)\lambda^\alpha-(2-\alpha)(\beta-2)(\beta-\alpha)\big).
\end{aligned}$$

Now, we put

$$h(\lambda) = (2-\alpha)(\beta-2) - 2\beta\lambda^{-\alpha} + (\alpha\beta - 2\alpha + 4)\lambda^{-2}.$$

Since $h(1) = 0$ and for $\lambda \in (0,1)$

$$h'(\lambda) = -2\alpha\beta(\lambda^{-3} - \lambda^{-\alpha-1}) - 4(2-\alpha)\lambda^{-3} \le 0,$$

we have $h(\lambda) \ge 0$. Thus, we only have to show that

$$g_2(\lambda) := 2\alpha(2-\alpha)\lambda^{\beta} - \alpha\beta(\beta-\alpha)\lambda^2 + 2\beta(\beta-2)\lambda^{\alpha} - (2-\alpha)(\beta-2)(\beta-\alpha) \le 0$$

for all $\lambda \in (0,1)$. Since $g_2(1) = 0$, it suffices to show that

$$g_2'(\lambda) = 2\alpha\beta\lambda^{\alpha-1}\Big((2-\alpha)\lambda^{\beta-\alpha} - (\beta-\alpha)\lambda^{2-\alpha} + \beta - 2\Big) \ge 0$$

for all $\lambda \in (0,1)$. This is equivalent to

$$g_3(\lambda) := (2-\alpha)\lambda^{\beta-\alpha} - (\beta-\alpha)\lambda^{2-\alpha} + \beta - 2 \ge 0.$$

Since $g_3(1) = 0$, and

$$g_3'(\lambda) = -(\beta-\alpha)(2-\alpha)\lambda^{1-\alpha}(1-\lambda^{\beta-2}) \le 0$$

for all $\lambda \in (0,1)$, we obtain $g_3(\lambda) \ge 0$ for all $\lambda \in (0,1)$. This implies $f(\lambda_0) \le f(1)$. Thus, the inequality (2.6) follows. This completes the proof. □

Next, we show that the set $\mathcal{B}_\omega$ given in (1.8) is invariant under the flow of (NLS).

Lemma 2.2. *Let $u_0 \in \mathcal{B}_\omega$. Then the solution $u(t)$ of* (NLS) *with $u(0) = u_0$ belongs to $\mathcal{B}_\omega$ for all $t \in [0, T_{\max})$.*

Proof. Firstly, since S_ω and $\|\cdot\|_{L^2}$ are the conserved quantities of (NLS), we have $S_\omega(u(t)) = S_\omega(u_0) < S_\omega(\phi_\omega)$ and $\|u(t)\|_{L^2} = \|u_0\|_{L^2} \le \|\phi_\omega\|_{L^2}$ for all $t \in [0, T_{\max})$. Then by (2.1), we have $K_\omega(u(t)) \ne 0$ for all $t \in [0, T_{\max})$. Moreover, $K_\omega(u_0) < 0$ and the continuity of the solution $u(t)$ imply $K_\omega(u(t)) < 0$ for all $t \in [0, T_{\max})$.

Finally, we show that $Q(u(t)) < 0$ for all $t \in [0, T_{\max})$. If not, there exists $t_0 \in (0, T_{\max})$ such that $Q(u(t_0)) = 0$. Then by Lemma 2.1 and $S_\omega(u(t_0)) < S_\omega(\phi_\omega)$, we have $Q(u(t_0)) < 0$. This is a contradiction. This completes the proof. □

Finally, we prove the blowup result.

Proof of Theorem 1.4. By the virial identity (1.3), Lemmas 2.1 and 2.2, and the conservation of S_ω, we have

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8Q(u(t)) \\ &\le 16\big(S_\omega(u(t)) - S_\omega(\phi_\omega)\big) = 16\big(S_\omega(u_0) - S_\omega(\phi_\omega)\big) < 0\end{aligned}$$

for all $t \in [0, T_{\max})$. This implies $T_{\max} < \infty$. This completes the proof. □

§3. Strong instability

In this section, we prove Theorem 1.2 using Theorem 1.4. Throughout this section, we impose the assumption of Theorem 1.2.

We remark that

$$\begin{aligned} S_\omega(v^\lambda) &= \frac{1}{2}K_\omega(v^\lambda) + \frac{1}{2}F(v^\lambda) \\ &= \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}, \\ Q(v^\lambda) &= \lambda\partial_\lambda S_\omega(v^\lambda), \\ Q(\phi_\omega) &= \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0, \quad \partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0. \end{aligned}$$

Lemma 3.1. *Assume that $\phi_\omega \in \mathcal{G}_\omega$ satisfies $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$. Then $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda > 1$.*

Proof. First, by the definition of the scaling v^λ, we have $\|\phi_\omega^\lambda\|_{L^2} = \|\phi_\omega\|_{L^2}$ for all $\lambda > 1$.

Next, we show $S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega)$ and $Q(\phi_\omega^\lambda) < 0$ for all $\lambda > 1$. Note that the function $S_\omega(\phi_\omega^\lambda)$ of λ has the form $S_\omega(\phi_\omega^\lambda) = A\lambda^2 + B - C\lambda^\alpha - D\lambda^\beta$ with some positive coefficients A, B, C, and D. By $\partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$, the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$ can be rewritten as $-\beta(\beta-2)D \le -\alpha(2-\alpha)C$. Using this, we have

$$\begin{aligned} \partial_\lambda^3 S_\omega(\phi_\omega^\lambda) &= \alpha(\alpha-1)(2-\alpha)C\lambda^{\alpha-3} - \beta(\beta-1)(\beta-2)D\lambda^{\beta-3} \\ &\le -\alpha(2-\alpha)\lambda^{\alpha-3}\Big((\beta-1)\lambda^{\beta-\alpha} - (\alpha-1)\Big)C < 0 \end{aligned}$$

for all $\lambda \ge 1$. Therefore, it follows that $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda) < 0$, $\partial_\lambda S_\omega(\phi_\omega^\lambda) < 0$, and thus $S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega)$ for all $\lambda > 1$. Moreover, we have $\partial_\lambda Q(\phi_\omega^\lambda) = \partial_\lambda S_\omega(\phi_\omega^\lambda) + \lambda\partial_\lambda^2 S_\omega(\phi_\omega^\lambda) < 0$ for all $\lambda > 1$, which implies $Q(\phi_\omega^\lambda) < 0$.

Finally, we obtain

$$K_\omega(\phi_\omega^\lambda) = 2S_\omega(\phi_\omega^\lambda) - F(\phi_\omega^\lambda) < 2S_\omega(\phi_\omega) - F(\phi_\omega) = 0$$

for all $\lambda > 1$. This completes the proof. □

Now, we prove our main theorem.

Proof of Theorem 1.2. By an analogous argument in the proof of [4, Theorem 8.1.1], we see that ϕ_ω decays exponentially. This implies $\phi_\omega \in \Sigma$, where Σ is the weighted space defined in (1.2). Therefore, combining this with Lemma 3.1, we have $\phi_\omega^\lambda \in \mathcal{B}_\omega \cap \Sigma$ for all $\lambda > 1$. Thus, Theorem 1.4 implies that for any $\lambda > 1$, the solution $u(t)$ of (NLS) with $u(0) = \phi_\omega^\lambda$ blows up in finite time. Moreover, we obtain $\phi_\omega^\lambda \to \phi_\omega$ in $H^1(\mathbb{R}^N)$ as $\lambda \searrow 1$. Hence, the standing wave solution $e^{i\omega t}\phi_\omega$ of (NLS) is strongly unstable. □

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