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On the Differential Geometry of a Line Congruence.

BY

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Reprinted from the Science Reports of the Tôhoku Imperial University, Vol. V, No. 2.

SENDAI, JAPAN

May, 1916

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Five ruled surfaces of a line congruence.

1. We may define a line congruence by means of the coordinates (x, y, z) of any point on the surface of reference in terms of two parameters u, v, and by the direction-cosines (X, Y, Z) of the line l passing through the point in terms of these parameters. If we put

$$\mathfrak{E} = \sum\left(\frac{\partial X}{\partial u}\right)^2, \quad \mathfrak{F} = \sum \frac{\partial X}{\partial u}\frac{\partial X}{\partial v}, \quad \mathfrak{G} = \sum\left(\frac{\partial X}{\partial v}\right)^2,$$

$$\mathfrak{L} = \frac{1}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}}\left(\mathfrak{F}\cdot\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} - \mathfrak{E}\cdot\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u}\right),$$

$$\mathfrak{M} + \lambda = \frac{1}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}}\left(\mathfrak{F}\cdot\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial v} - \mathfrak{E}\cdot\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial v}\right),$$

$$\mathfrak{M} - \lambda = \frac{2}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}}\left(\mathfrak{G}\cdot\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} - \mathfrak{F}\cdot\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u}\right),$$

$$\mathfrak{N} = \frac{1}{\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}}\left(\mathfrak{G}\cdot\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial v} - \mathfrak{F}\cdot\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial v}\right),$$

then the two fundamental forms of Prof. G. Sannia[1] have the expressions :

$$f_1 = d\sigma^2 = \sum dX^2 = \mathfrak{E}\, du^2 + 2\mathfrak{F}\, du dv + \mathfrak{G}\, dv^2, \tag{1}$$

$$f_2 = -\mu = \mathfrak{L}\, du^2 + 2\mathfrak{M}\, du dv + \mathfrak{N} dv^2, \tag{2}$$

where $d\sigma$ is the linear element of the spherical representation, that is,

(1) G. Sannia, "Nuova esposizione della geometria infinitesimale delle congruenze rettilinee", Annali di Mat., (3) **15** (1908), 143; Sannia, "Geometria differentiale delle congruenze rettilinee", Math. Ann., **68** (1910), 409. See also K. Zindler, Liniengeometrie mit Anwendungen, Bd. 2(1906), Abschn. II; Zindler, "Bemerkungen zum Berichte des Herrn Sannia über seine Arbeiten zur differentiellen Liniengeometrie", Math. Ann., **69** (1910) 446.

the infinitesimal angle between the two lines $l(u,v)$ and $l'(u+du,\ v+dv)$; and μ is the moment of these two lines.

The discriminants of these two forms are

$$\Delta(f_1)=\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2>0, \qquad \Delta(f_2)=\mathfrak{L}\mathfrak{N}-\mathfrak{M}^2$$

respectively and the simultaneous invariant is

$$\Theta(f_1,f_2)=\mathfrak{E}\mathfrak{N}-2\mathfrak{F}\mathfrak{M}+\mathfrak{G}\mathfrak{L}. \tag{3}$$

The congruence is said to be *elliptic*, *parabolic* or *hyperbolic* according as $\Delta(f_2)>0,\ =0$ or <0; and to be *normal* when

$$\Theta(f_1,f_2)=0. \tag{3$'$}$$

The angle ω between $l'(u+du,\ v+dv)$ and $l''(u+\delta u,\ v+\delta v)$ for $l(u,v)$ is the angle between the corresponding *central planes* for l, and hence is given by

$$\cos\omega=\frac{\mathfrak{E}\,du\delta u+\mathfrak{F}\,(du\delta v+dv\delta u)+\mathfrak{G}\,dv\delta v}{\sqrt{\mathfrak{E}\,du^2+2\mathfrak{F}\,dudv+\mathfrak{G}\,dv^2}\,\sqrt{\mathfrak{E}\,\delta u^2+2\mathfrak{F}\,\delta u\delta v+\mathfrak{G}\,\delta v^2}}.$$

The lines of congruence which pass through a curve on the surface of reference form a ruled surface. Such a curve, and consequently a ruled surface of the congruence, is determined by a relation between u and v. Hence a differential equation of the form

$$A\,(u,v)\,du+B\,(u,v)\,dv=0$$

defines a family of ruled surfaces, and the *direction of progress* of the ruled surface passing through the line $l(u,v)$ is determined by $dv:du=-A:B$. Particularly, u=const. and v=const. are called the *parametric surfaces.*

For the present note we will confine ourselves to consider the case, *excluding the isotropic congruence:*

$$\frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{M}}{\mathfrak{F}}=\frac{\mathfrak{N}}{\mathfrak{G}}$$

and the parabolic congruence:

$$\mathfrak{L}\mathfrak{N}-\mathfrak{M}^2=0.$$

2. Now the differential equation

$$f_1=0 \tag{1$'$}$$

defines the two families of *imaginary ruled surfaces, through $l(u,v)$, whose spherical representations are the minimal lines*(1).

Next the differential equation

$$f_2=0 \tag{2$'$}$$

denotes the *developable surfaces* through l(2). They are real or imaginary according as the congruence is hyperbolic or elliptic.

If $J(f_1,f_2)$ be the Jacobian of f_1 and f_2, then

$$J(f_1,f_2)=\begin{vmatrix}\mathfrak{E}\,du+\mathfrak{F}\,dv & \mathfrak{F}\,du+\mathfrak{G}\,dv\\ \mathfrak{L}\,du+\mathfrak{M}dv & \mathfrak{M}du+\mathfrak{N}\,dv\end{vmatrix}; \tag{4}$$

but since the *parameter of distribution* is

$$p=-\frac{\mathfrak{L}\,du^2+2\mathfrak{M}\,dudv+\mathfrak{N}dv^2}{\mathfrak{E}\,du^2+2\mathfrak{F}\,dudv+\mathfrak{G}\,dv^2}, \tag{5}$$

the *surfaces of distribution* (after Prof. Sannia)(3) for l have the differential equation

$$J(f_1,f_2)=0. \tag{4$'$}$$

The values of p corresponding to the surfaces of distribution are called the *principal parameters of distribution* and they are given by

$$p_1=-\frac{\mathfrak{N}}{\mathfrak{G}}, \qquad p_2=-\frac{\mathfrak{L}}{\mathfrak{E}}, \tag{6}$$

where the surfaces of distribution are taken for the parametric surfaces. Also the quantities

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(1) Compare with G. Scheffers, Einführung in die Theorie der Flächen, 1. Aufl. (1902), 215.

(2) Prof. Sannia called these the *asymptotic surfaces.* But this name is inconvenient. See St. Jolles, Fortschritte d. Math., **41** (Jahrg. 1910), 733–4.

(3) These surfaces are called the *surfaces of curvature* by Prof. Zindler and also the *mean ruled surfaces* (*rigate medie*) by other mathematicians. See P. Burgatti, "Sopra alcune formole fondamentali relative alle congruenze di retti", Atti della Acc. dei Lincei, (5) 8 (1899), 515; T. Cifarelli, "Le congruenze", Ann. di Mat., (3) **2** (1899), 139; L. P. Eisenhart, A treatise on the differential geometry of curves and surfaces (1909), 422.

$$\mathfrak{K} = p_1 p_2 = \frac{\mathfrak{L}\mathfrak{N}-\mathfrak{M}^2}{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2} = \frac{\Delta(f_2)}{\Delta(f_1)}$$

and

$$\mathfrak{H} = p_1+p_2 = \frac{2\mathfrak{F}\mathfrak{M}-\mathfrak{E}\mathfrak{N}-\mathfrak{G}\mathfrak{L}}{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2} = -\frac{\theta(f_1,f_2)}{\Delta(f_1)}$$

are called the *total* and *mean parameters* respectively. Since

$$\Delta(J(f_1,f_2)) = \frac{1}{4}\Delta^2(f_1)(4\mathfrak{K}-\mathfrak{H}^2) = -\frac{1}{4}\Delta^2(f_1)(p_1-p_2)^2 < 0,$$

the surfaces of distribution are always real.

It is well known that abscissæ of the *central point* and the *middle point* for l are given by

$$r = -\frac{\Sigma\, dxdX}{\Sigma dX^2}$$

$$= \frac{[\mathfrak{F}\mathfrak{L}-\mathfrak{E}(\mathfrak{M}-\lambda)]du^2+(\mathfrak{G}\mathfrak{L}-\mathfrak{E}\mathfrak{N}+2\mathfrak{F}\lambda)dudv+[\mathfrak{G}(\mathfrak{M}+\lambda)-\mathfrak{F}\mathfrak{N}]dv^2}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}\;(\mathfrak{E}\,du^2+2\mathfrak{F}\,dudv+\mathfrak{G}\,dv^2)}$$

and

$$r_0 = \frac{\lambda}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}}$$

respectively. Hence if we denote by q the *distance between the central point and the middle point* for l, then $q=r-r_0$; and consequently

$$q = \frac{(\mathfrak{F}\mathfrak{L}-\mathfrak{E}\mathfrak{M})du^2+(\mathfrak{G}\mathfrak{L}-\mathfrak{E}\mathfrak{N})dudv+(\mathfrak{G}\mathfrak{M}-\mathfrak{F}\mathfrak{N})dv^2}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}\;(\mathfrak{E}\,du^2+2\mathfrak{F}\,dudv+\mathfrak{G}\,dv^2)}. \tag{7}$$

It is noteworthy that p and q are the intrinsic quantities of the congruence; while r and r_0 depend upon the surface of reference.

Now since

$$J(f_1, J(f_1,f_2)) = \begin{vmatrix} dv^2 & \mathfrak{E} & \mathfrak{E}\mathfrak{M}-\mathfrak{F}\mathfrak{L} \\ -dudv & \mathfrak{F} & \frac{1}{2}(\mathfrak{E}\mathfrak{N}-\mathfrak{G}\mathfrak{L}) \\ du^2 & \mathfrak{G} & \mathfrak{F}\mathfrak{N}-\mathfrak{G}\mathfrak{M} \end{vmatrix}, \tag{8}$$

$$\Delta(J(f_1,J(f_1,f_2))) = \frac{1}{4}\Delta^3(f_1)(4\mathfrak{K}-\mathfrak{H}^2) < 0,$$

the *principal surfaces* of the congruence for l are determined by

$$J(f_1, J(f_1,f_2)) = 0 \tag{8$'$}$$

and they are always real.

3. Similarly, since

$$J(f_2, J(f_1,f_2)) = \begin{vmatrix} dv^2 & \mathfrak{L} & \mathfrak{E}\mathfrak{M}-\mathfrak{F}\mathfrak{L} \\ -dudv & \mathfrak{M} & \frac{1}{2}(\mathfrak{E}\mathfrak{N}-\mathfrak{G}\mathfrak{L}) \\ du^2 & \mathfrak{N} & \mathfrak{F}\mathfrak{N}-\mathfrak{G}\mathfrak{M} \end{vmatrix}, \tag{9}$$

$$\Delta(J(f_2, J(f_1,f_2))) = \frac{1}{4}\Delta^2(f_1)\,\Delta(f_2)(4\mathfrak{K}-\mathfrak{H}^2),$$

the ruled surfaces

$$J(f_2, J(f_1,f_2)) = 0 \tag{9$'$}$$

are real or imaginary according as the congruence is elliptic or hyperbolic. Now we proceed to prove the theorem:

For an elliptic congruence there exists a unique system of conjugate ruled surfaces S for which the angle between the directions of progress for any line l is the minimum angle between conjugate directions of progress for the line l; it is the only conjugate system whose directions of progress are symmetric with respect to those of the surfaces of distribution, and the equation to S is (9)$'$.

In order to prove this, we note that the two ruled surfaces corresponding to the directions $dv:du$ and $\delta v:\delta u$ are *conjugate* when and only when the relation

$$\mathfrak{L}\,du\delta u+\mathfrak{M}\,(du\delta v+dv\delta u)+\mathfrak{N}\,dv\delta v = 0 \tag{10}$$

is satisfied.

Now let us take the surfaces of distribution for the parametric surfaces. Then equation (10) becomes

$$\mathfrak{L}\,du\delta u+\mathfrak{N}\,dv\delta v = 0. \tag{11}$$

If the angles which the central planes corresponding to $dv:du$ and $\delta v:\delta u$ make with the central plane corresponding to $v=$const. for l are denoted by θ and θ', we have

$$\operatorname{tg}\theta=\sqrt{\frac{\mathfrak{G}}{\mathfrak{E}}\frac{dv}{du}},\quad \operatorname{tg}\theta'=\sqrt{\frac{\mathfrak{G}}{\mathfrak{E}}\frac{\delta v}{\delta u}};$$

so that, by the aid of (6), equation (11) may be written

$$\operatorname{tg}\theta\operatorname{tg}\theta'=-\frac{p_2}{p_1}, \tag{12}$$

or

$$\operatorname{tg}(\theta-\theta')=\frac{p_2\cot\theta+p_1\operatorname{tg}\theta}{p_2-p_1}.$$

If we differentiate the right hand member of this equation with respect to θ and equate the result to zero, we obtain

$$\operatorname{tg}\theta=\pm\sqrt{\frac{p_2}{p_1}}. \tag{13}$$

Then from (12) we have

$$\operatorname{tg}\theta'=\mp\sqrt{\frac{p_2}{p_1}},$$

so that

$$\theta'=-\theta.$$

Conversely, when $\theta'=-\theta$, equation (12) becomes (13).

Now the necessary and sufficient condition that the central planes corresponding to $dv:du$ and $\delta v:\delta u$ should be symmetric with respect to the planes of distribution is

$$(\mathfrak{F}\mathfrak{L}-\mathfrak{E}\mathfrak{M})du\delta u+\frac{1}{2}(\mathfrak{G}\mathfrak{L}-\mathfrak{E}\mathfrak{N})(du\delta v+dv\delta u)+(\mathfrak{G}\mathfrak{M}-\mathfrak{F}\mathfrak{N})dv\delta v=0. \tag{14}$$

Eliminating δu and δv between (10) and (14), we obtain the equation

$$\begin{vmatrix} \mathfrak{L}\,du+\mathfrak{M}\,dv & 2(\mathfrak{F}\mathfrak{L}-\mathfrak{E}\mathfrak{M})du+(\mathfrak{G}\mathfrak{L}-\mathfrak{E}\mathfrak{N})dv \\ \mathfrak{M}\,du+\mathfrak{N}\,dv & (\mathfrak{G}\mathfrak{L}-\mathfrak{E}\mathfrak{N})du+2(\mathfrak{G}\mathfrak{M}-\mathfrak{F}\mathfrak{N})dv \end{vmatrix}=0,$$

which proves our theorem. The ruled surfaces (9)′ will be called the *characteristic surfaces* for $l(u,v)$.

Lastly we add the two formulæ:

$$\begin{aligned} J(J(f_1,f_2),J(f_1,J(f_1,f_2)))&=\varDelta(J(f_1,f_2))\cdot f_1 \\ &=-\frac{1}{4}(\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2)^2(4\mathfrak{K}-\mathfrak{H}^2)(\mathfrak{E}\,du^2+2\mathfrak{F}\,dudv+\mathfrak{G}\,dv^2), \end{aligned} \tag{15}$$

$$\begin{aligned} J(J(f_1,f_2),J(f_2,J(f_1,f_2)))&=\varDelta(J(f_1,f_2))\cdot f_2 \\ &=-\frac{1}{4}(\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2)^2(4\mathfrak{K}-\mathfrak{H}^2)(\mathfrak{L}\,du^2+2\mathfrak{M}\,dudv+\mathfrak{N}\,dv^2). \end{aligned} \tag{16}$$

In the particular case where the congruence is normal, we have

$$\Theta(f_1,f_2)=0; \tag{3$'$}$$

whence it is seen that

$$J(f_1,J(f_1,f_2))=0 \quad\text{and}\quad J(f_2,J(f_1,f_2))=0$$

are equivalent to

$$f_2=0 \quad\text{and}\quad f_1=0$$

respectively.

Consequently, *for the normal congruence, the principal surfaces and the characteristic surfaces coincide with the developable surfaces and the ruled surfaces whose spherical representations are the minimal lines respectively.*

4. Thus we have from § 2 and § 3

$$\frac{f_2}{f_1}=p,\quad \frac{J(f_1,f_2)}{f_1}=-\sqrt{\varDelta(f_1)}\cdot q,\quad \frac{J(f_1,f_2)}{f_2}=\sqrt{\varDelta(f_1)}\cdot\frac{q}{p}; \tag{17}$$

also we can show that

$$\begin{aligned} \frac{J(f_1,f_2)}{J(f_1,J(f_1,f_2))}&=\frac{J(f_1,f_2)}{\frac{1}{2}\Theta(f_1,f_2)\cdot f_1-\varDelta(f_1)\cdot f_1}=\frac{q}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}\left(\frac{\mathfrak{H}}{2}-p\right)}, \\ \frac{J(f_1,f_2)}{J(f_2,J(f_1,f_2))}&=\frac{J(f_1,f_2)}{\varDelta(f_2)\cdot f_1-\frac{1}{2}\Theta(f_1,f_2)\cdot f_2}=\frac{q}{\sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}\left(\frac{\mathfrak{H}}{2}p-\mathfrak{K}\right)}. \end{aligned} \tag{18}$$

Already I proved the following theorem[1]: If

$$\varDelta(f_1)\neq0,\quad \varDelta(f_2)\neq0,\quad 4\varDelta(f_1)\varDelta(f_2)-\Theta^2(f_1,f_2)\neq0,$$

then the values of t, for which

(1) K. Ogura, "Some theorems concerning binary quadratic forms and their applications to the differential geometry," Science Reports of the Tôhoku Imperial University, Series I, **5** (1916), 95.

$$\frac{f_2}{f_1}\Bigg|_{\frac{dv}{du}=t}, \qquad \frac{J(f_1,f_2)}{f_1}\Bigg|_{\frac{dv}{du}=t}, \qquad \frac{J(f_1,f_2)}{f_2}\Bigg|_{\frac{dv}{du}=t},$$

$$\frac{J(f_1,f_2)}{J(f_1,J(f_1,f_2))}\Bigg|_{\frac{dv}{du}=t}, \qquad \frac{J(f_1,f_2)}{J(f_2,J(f_1,f_2))}\Bigg|_{\frac{dv}{du}=t}$$

are extremes respectively, are given by

$$J(f_1,f_2)\Big|_{\frac{dv}{du}=t}=0, \quad J(f_1,J(f_1,f_2))\Big|_{\frac{dv}{du}=t}=0, \quad J(f_2,J(f_1,f_2))\Big|_{\frac{dv}{du}=t}=0,$$

$$J(J(f_1,J(f_1,f_2)),J(f_1,f_2))\Big|_{\frac{dv}{du}=t}=0 \quad \text{or} \quad f_1\Big|_{\frac{dv}{du}=t}=0,$$

$$J(J(f_2,J(f_1,f_2)),J(f_1,f_2))\Big|_{\frac{dv}{du}=t}=0 \quad \text{or} \quad f_2\Big|_{\frac{dv}{du}=t}=0$$

respectively.

Therefore we can find *the five families of ruled surfaces of the congruence corresponding to the extremes of*

$$p, \quad q, \quad \frac{q}{p}, \quad \frac{q}{\frac{\mathfrak{H}}{2}-p}, \quad \frac{q}{\frac{\mathfrak{H}}{2}p-\mathfrak{K}}$$

respectively. In the following table we will also give the ruled surfaces corresponding to the infinities of the above quantities.

	Extremals		Infinities
p	$J(f_1,f_2)=0$	Surfaces of distribution(1) (always real)	Ruled surfaces whose spherical representations are the minimal lines (always imaginary)
q	$J(f_1,J(f_1,f_2))=0$	Principal surfaces (always real)	Ruled surfaces whose spherical representations are the minimal lines (always imaginary)
$\frac{q}{p}$	$J(f_2,J(f_1,f_2))=0$	Characteristic surfaces ($\mathfrak{K}>0$, real; $\mathfrak{K}<0$, imaginary)	Developable surfaces ($\mathfrak{K}>0$, imaginary; $\mathfrak{K}<0$, real)
$\frac{q}{\frac{\mathfrak{H}}{2}-p}$	$J(J(f_1,J(f_1,f_2)),J(f_1,f_2))=0$ (or $f_1=0$)	Ruled surfaces whose spherical representations are the minimal lines (always imaginary)	Principal surfaces (always real)
$\frac{q}{\frac{\mathfrak{H}}{2}p-\mathfrak{K}}$	$J(J(f_2,J(f_1,f_2)),J(f_1,f_2))=0$ (or $f_2=0$)	Developable surfaces ($\mathfrak{K}>0$, imaginary; $\mathfrak{K}<0$, rea	Characteristic surfaces ($\mathfrak{K}<0$, real; $\mathfrak{K}>0$, imaginary)

(1) Zindler, Liniengeometrie, Bd. 2, 85.

The normal congruence.

	Extremals		Infinities
p	$J(f_1,f_2)=0$	Surfaces of distribution (always real)	Ruled surfaces whose spherical representations are the minimal lines (always imaginary)
q	$J(f_1,J(f_1,f_2))=0$ (or $f_2=0$)	Principal surfaces = Developable surfaces (always real)	Characteristic surfaces = Ruled surfaces whose spherical representations are the minimal lines (always imaginary)
$\frac{q}{p}$	$J(f_2,J(f_1,f_2))=0$ (or $f_1=0$)	Characteristic surfaces = Ruled surfaces whose spherical representations are the minimal lines (always imaginary)	Principal surfaces = Developable surfaces (always real)

5. It is easily seen that the two pairs of central planes for $l(u, v)$ corresponding to the two ruled surfaces

$$\varphi_1 = A_1\,du^2 + 2B_1\,dudv + C_1\,dv^2 = 0,$$
$$\varphi_2 = A_2\,du^2 + 2B_2\,dudv + C_2\,dv^2 = 0$$

determine an involution of the pencil of planes through l; and the double planes of the involution are the pair of central planes for l corresponding to the ruled surfaces

$$J(\varphi_1, \varphi_2) = 0.$$

But I proved in the previous paper(1) that every Jacobian of the third order and any higher orders (beginning from f_1 and f_2) is equal to one of the 0^{th}, first and second orders, that is,

$$f_1, f_2; \quad J(f_1,f_2); \quad J(f_1,J(f_1,f_2)), \; J(f_2,J(f_1,f_2)),$$

up to some constant multiples; in other words, these five iterated Jacobians form the complete system. This system can be divided into the two cycles:

$$f_1, \; J(f_1,f_2), \; J(f_1,J(f_1,f_2));$$

$$f_2, \; J(f_1,f_2), \; J(f_2,J(f_1,f_2)),$$

$J(f_1,f_2)$ being common to the both.

(1) Ogura, l. c.

Therefore we arrive at the following theorem:

The ruled surfaces whose spherical representations are the minimal lines, the surfaces of distribution and the principal surfaces form a cycle in the sense that the central planes (for any line of the congruence) corresponding to any one of these three surfaces are the double planes of the involution determined by the corresponding central planes of the other two; the similar result holds good for the developable surfaces, the surfaces of distribution and the characteristic surfaces. And these five families of ruled surfaces form the complete system.

Particularly, *for the normal congruence, the characteristic surfaces (the ruled surfaces whose spherical representations are the minimal lines), the principal surfaces (the developable surfaces) and the surfaces of distribution form the complete system.*

Comparison of Kummer's and Sannia's theories.

6. In Kummer's theory of a line congruence we have the following fundamental forms[1]:

$$f_1 = \mathfrak{E}\, du^2 + 2\mathfrak{F}\, du dv + \mathfrak{G}\, dv^2, \tag{1}$$

$$f_2' = e\, du^2 + (f+f')du dv + g\, dv^2, \tag{19}$$

where

$$e = \sum \frac{\partial X}{\partial u}\frac{\partial x}{\partial u}, \quad f = \sum \frac{\partial X}{\partial u}\frac{\partial x}{\partial v}, \quad f' = \sum \frac{\partial X}{\partial v}\frac{\partial x}{\partial u}, \quad g = \sum \frac{\partial X}{\partial v}\frac{\partial x}{\partial v}.$$

Since

$$f_2' = [\mathfrak{E}(\mathfrak{M}-\lambda) - \mathfrak{F}\mathfrak{L}]du^2 + (\mathfrak{E}\mathfrak{N} - \mathfrak{G}\mathfrak{L} - 2\mathfrak{F}\lambda)du dv + [\mathfrak{F}\mathfrak{N} - \mathfrak{G}(\mathfrak{M}+\lambda)]dv^2,$$

we have

$$f_2' = J(f_1, f_2) - \lambda f_1. \tag{20}$$

It follows from this relation that

(1) E. Kummer, "Allgemeine Theorie der geradlinigen Strahlensysteme," Journal f. Math., **57** (1860), 189; K. Hensel, "Theorie der unendlich dünnen Strahlenbündel," Journal f. Math., **102** (1888), 273. See also J. Knoblauch, "Strahlensysteme und Differentialformen," Sitzungsberichte d. Berliner Math. Ges., (1915), 79.

$$J(f_1, f_2') = 0 \qquad \text{and} \qquad J(f_1, J(f_1, f_2')) = 0$$

are equivalent to

$$J(f_1, J(f_1, f_2)) = 0 \qquad \text{and} \qquad J(f_1, f_2) = 0$$

respectively. This result is easily seen geometrically by introducing the idea of the *Jacobian point*[1]; for the point $P[f_2']$ lies on the straight line joining $P[f_1]$ and $P[J(f_1, f_2)]$.

Now in Sannia's theory we have the two self-conjugate triangles whose vertices are

$$P[f_1],\ P[J(f_1, f_2)],\ P[J(f_1, J(f_1, f_2))];$$

$$P[f_2],\ P[J(f_1, f_2)],\ P[J(f_2, J(f_1, f_2))]$$

respectively. On the other hand, the two self-conjugate triangles for Kummer's theory have the vertices

$$P[f_1], \quad P[J(f_1, f_2')] \ (\text{i.e. } P[J(f_1, J(f_1, f_2))]),$$

$$P[J(f_1, J(f_1, f_2'))] \ (\text{i.e. } P[J(f_1, f_2)]);$$

$$P[f_2'], \quad P[J(f_1, f_2')] \ (\text{i.e. } P[J(f_1, J(f_1 \cdot f_2))]),$$

$$P[J(f_2', J(f_1, f_2'))]$$

respectively; and *the two Jacobian points $P[f_2']$ and $P[J(f_2', J(f_1, f_2'))]$ lie on the straight line joining $P[f_1]$ and $P[J(f_1, f_2)]$.*

In fact, we can show that

$$\begin{aligned} J(f_2', J(f_1, f_2')) &= \frac{1}{4}[4(\mathfrak{E}\mathfrak{M} - \mathfrak{F}\mathfrak{L})(\mathfrak{F}\mathfrak{N} - \mathfrak{G}\mathfrak{M}) - (\mathfrak{E}\mathfrak{N} - \mathfrak{G}\mathfrak{L})^2] \cdot f_1 \\ &\quad + \lambda(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2) \cdot J(f_1, f_2) \end{aligned} \tag{21}$$

$$= -\frac{1}{4}(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2)^2 (p_1 - p_2)^2 \cdot f_1 + \lambda(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2) \cdot J(f_1, f_2);$$

and then

$$\frac{f_2'}{f_1} = -\sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2} \cdot (q + r_0), \tag{22}$$

$$\frac{J(f_2', J(f_1, f_2'))}{f_1} = -(\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2)^2 \cdot \left[\left(\frac{p_1 - p_2}{2}\right) + q r_0\right]. \tag{23}$$

(1) Ogura, l. c.

Therefore we may conclude that *in Sannia's theory the five ruled surfaces*

$$f_1=0,\quad f_2=0,\quad J(f_1,f_2)=0,\quad J(f_1,J(f_1,f_2))=0,\quad J(f_2,J(f_1,f_2))=0$$

depend upon the congruence itself, but independent of the surface of reference. On the contrary, in Kummer's theory the two ruled surfaces

$$f_2'=0,\quad J(f_2',J(f_1,f_2'))=0$$

depend not only upon the congruence itself, but upon the surface of reference.

Special classes of ruled surfaces of a line congruence.

7. Let us consider a system of $2\infty^1$ ruled surfaces of a line congruence which may be taken for the parametric surfaces $u=$const., $v=$const. .

I. *If the relation*

$$q_u=-q_v^{(1)} \tag{24}$$

be satisfied for any line l of the congruence, we have either

$$\mathfrak{F}=0 \tag{25}$$

or

$$\frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{N}}{\mathfrak{G}}\quad (\text{i.e. } p_u=p_v); \tag{26}$$

hence in this case the $2\infty^1$ ruled surfaces form an *orthogonal system*[2] or an *isoclinal system* (Sannia)[3].

Now it is well known that for the principal surfaces

$$\frac{\mathfrak{L}\,du^2+2\mathfrak{M}\,du\,dv+\mathfrak{N}\,dv^2}{\mathfrak{E}\,du^2+2\mathfrak{F}\,du\,dv+\mathfrak{G}\,dv^2}=\frac{\mathfrak{E}\mathfrak{N}-2\mathfrak{F}\mathfrak{M}+\mathfrak{G}\mathfrak{L}}{2(\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2)},$$

we have

$$p=\frac{1}{2}(p_1+p_2)^{(4)}.$$

(1) For the analogous case in the theory of surface-curves, see K. Ogura, "On the T-system on a surface", Tôhoku Math. Journal, **9** (1916), 88.

(2) Prof. Zindler called this *quasi-orthogonal system*.

(3) Prof. Zindler called this the *symmetrical system*.

(4) Zindler, Liniengeometrie, Bd. 2, 109.

Next, if we put

$$\frac{1}{p}=\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right),$$

we obtain

$$\frac{\mathfrak{L}\,du^2+2\mathfrak{M}\,du\,dv+\mathfrak{N}\,dv^2}{\mathfrak{E}\,du^2+2\mathfrak{F}\,du\,dv+\mathfrak{G}\,dv^2}=\frac{2(\mathfrak{L}\mathfrak{N}-\mathfrak{M}^2)}{\mathfrak{E}\mathfrak{N}-2\mathfrak{F}\mathfrak{M}+\mathfrak{G}\mathfrak{L}},$$

which denotes the characteristic surfaces.

Again, if we take

$$p^2=p_1p_2,$$

we obtain

$$\frac{\mathfrak{L}\,du^2+2\mathfrak{M}\,du\,dv+\mathfrak{N}\,dv^2}{\mathfrak{E}\,du^2+2\mathfrak{F}\,du\,dv+\mathfrak{G}\,dv^2}=-\sqrt{\frac{\mathfrak{L}\mathfrak{N}-\mathfrak{M}^2}{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}},$$

which may be called the *surfaces of Occhipinti*, from the similarity in the theory of surface-curves[1].

In the following table five particular families (each taken for the parametric surfaces respectively) belonging to the class $q_u=-q_v$ are mentioned.

<table>
<tr><td rowspan="3">Orthogonal system</td><td colspan="3">Surfaces of distribution</td><td rowspan="2">Principal surfaces</td></tr>
<tr><td colspan="3">$\mathfrak{F}=0,\quad \mathfrak{M}=0.$</td></tr>
<tr><td colspan="3">$p_u\neq p_v$ (in general)</td><td rowspan="2">$\mathfrak{F}=0,\ \frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{N}}{\mathfrak{G}}.$</td></tr>
<tr><td rowspan="3">Isoclinal system</td><td>Developable surfaces</td><td>Characteristic surfaces</td><td>Surfaces of Occhipinti</td></tr>
<tr><td>$\mathfrak{L}=0,\ \mathfrak{M}=0.$</td><td>$\mathfrak{M}=0,\ \frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{N}}{\mathfrak{G}}.$</td><td>$\frac{\mathfrak{L}}{\mathfrak{E}}=-\frac{\mathfrak{M}}{\mathfrak{F}}=\frac{\mathfrak{N}}{\mathfrak{G}}.$</td><td rowspan="2">$p_u=p_v=\frac{\mathfrak{H}}{2}.$</td></tr>
<tr><td>$p_u=p_v=0.$</td><td>$p_u=p_v=\frac{2\mathfrak{K}}{\mathfrak{H}}.$</td><td>$p_w=p_v=-\sqrt{\mathfrak{K}}.$</td></tr>
</table>

II. *If the relation*

$$q_u=q_v \tag{27}$$

be satisfied for any line l of the congruence, we have

(1) R. Occhipinti, "Sur un double système de lignes d'une surface", L'enseignement math., (1914), 38; T. Hayashi, "On the lines of torsion", Science Reports of the Tôhoku Imperial University, Series I, **3** (1914), 217.

$$\frac{\mathfrak{M}}{\mathfrak{F}}=\frac{1}{2}\left(\frac{\mathfrak{L}}{\mathfrak{E}}+\frac{\mathfrak{N}}{\mathfrak{G}}\right). \tag{28}$$

This system consists of the *complementary surfaces*[1].

8. I. *If the relation*

$$\left(\frac{q}{p}\right)_u=-\left(\frac{q}{p}\right)_v \tag{29}$$

be satisfied for any line l of the congruence, we have either

$$\mathfrak{M}=0 \tag{30}$$

or

$$\frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{N}}{\mathfrak{G}} \quad (\text{i.e. } p_u=p_v);$$

hence in this case the $2\infty^1$ ruled surfaces form a *conjugate system* or an *isoclinal system*.

Conjugate system	Surfaces of distribution			Characteristic surfaces
Isoclinal system	Developable sur aces	Principal surfaces	Surfaces of Occhipinti	

II. Lastly, *if the relation*

$$\left(\frac{q}{p}\right)_u=\left(\frac{q}{p}\right)_v \tag{31}$$

be satisfied for any line l of the congruence, we have

$$\frac{\mathfrak{F}}{\mathfrak{M}}=\frac{1}{2}\left(\frac{\mathfrak{E}}{\mathfrak{L}}+\frac{\mathfrak{G}}{\mathfrak{N}}\right). \tag{32}$$

This may be called the *inverse-conjugate system*, from the similarity in the theory of surface-curves[2].

January 1916.

(1) Zindler, Liniengeometrie, **2**, 138.

(2) A. Voss, "Über diejenigen Flächen, auf denen zwei Schaaren geodätischer Linien ein conjugiertes System bilden", Münchener Sitzungsberichte, **18** (1888), 97; R. D. Beetle, "A formula in the theory of surfaces", Annals of math., (2) **15** (1913–14), 179; T. Hayashi, "On the usual parametric curves on a surface", Science Reports of the Tôhoku Imperial University, Series I, **5** (1916), 63.