On the Extreme of a Function of Several Variables.

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Peano (1) proved that the function

$$f(x, y) = (y - a^2 x^2)(y - b^2 x^2)$$

has no extreme at $(0, 0)$, yet the function $F(t) = f(t, kt)$, formed by putting $x = t$ and $y = kt$ in $f(x, y)$, has a minimum at $t = 0$.

Let $\phi(x)$ and $\Phi(x)$ be any two continuous functions at $x = 0$ such that $\phi(0) = 0$, $\Phi(0) = 0$, and

$$\lim_{x=0} \frac{1}{\phi(x)} \nless \lim_{x=0} \frac{1}{\Phi(x^2)}$$

Then the function

$$f(x, y) = \{y - a^2 \Phi(x^2)\}\{y - b^2 \Phi(x^2)\}$$

has no extreme at $(0, 0)$, but it is at a minimum at $x = 0$ along the curve $y = \phi(x)$.

Hedrick (2) applied Peano's function on the distance from a point to a surface. Here I will treat some problems of the similar nature.

Take two points P_1 and P_2 on the parabola

$$x_1 = \frac{1}{4} e^{-\frac{2}{t_1^2}}, \qquad y_1 = e^{-\frac{1}{t_1}};$$

and the straight line

$$x_2 = 1, \qquad y_2 = t_2$$

respectively. Then $\overline{P_1 P_2}^2 = f(t_1, t_2)$

$$= \left\{1 - \frac{1}{4} e^{-\frac{2}{t_1^2}}\right\}^2 + \left\{t_2 - e^{-\frac{1}{t_1^2}}\right\}^2$$

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(1) Genocchi-Peano, Calcolo differenziale e principii di calcolo integrale (1884), p XXIX; Hedrick, Ann. Math., **8** (1907), p. 172.

(2) Hedrick, Bull. Amer. Math. Soc., **14** (1908) p. 321; Saurel, ibid., **14** (1908), p. 465.

$$=\left\{t_2-\left(1-\frac{1}{\sqrt{2}}\right)e^{-\frac{1}{t_1^2}}\right\}\left\{t_2-\left(1+\frac{1}{\sqrt{2}}\right)e^{-\frac{1}{t_1^2}}\right\}+1+\frac{1}{16}e^{-\frac{4}{t_1^2}},$$

and this is not at a minimum in the position of the common normal whose feet on the two curves are $N_1(0,0)$ and $N_2(1,0)$ respectively.

But we can easily see that both the distance from N_1 to the straight line and that from N_2 to the parabola are at minima in the position N_1N_2. And if $\varphi(t_1)$ be any analytic function of t_1 that $\varphi(0)=0$, then the function $f\{t_1,\varphi(t_1)\}$ formed by putting $t_2=\varphi(t_1)$ in $f(t_1,t_2)$ is at a minimum at $t_1=0$. Hence if we take t_1 as time, it follows that whatever analytic function of time the position may be, the distance between the two corresponding points P_1 and P_2 is at a minimum in the position N_1N_2.

The similar result will be found in the case of space curves, for example, the parabola

$$x_1=e^{-\frac{1}{t_1^2}}, \quad y_1=2e^{-\frac{1}{t_1^2}}, \quad z_1=\frac{1}{2}e^{-\frac{2}{t_1^2}},$$

and the straight line

$$x_2=t_2, \quad y_2=t_2, \quad z_2=1.$$

We can also find Peano's case in a function of several variables. For example, let us take the hyperbolic paraboloid

$$x_1=e^{-\frac{1}{t_1^2}}, \quad y_1=e^{-\frac{1}{t_2^2}}, \quad z_1=\frac{1}{4}e^{-\frac{1}{t_1^2}-\frac{1}{t_2^2}},$$

and the plane

$$x_2=t_3, \quad y_2=t_4, \quad z_2=1.$$

The square of the distance of the two surfaces is given by

$$f(t_1,t_2,t_3,t_4)=\left\{t_3-e^{-\frac{1}{t_1^2}}\right\}^2+\left\{t_4-e^{-\frac{1}{t_2^2}}\right\}^2$$
$$+\left\{1-\frac{1}{4}e^{-\frac{1}{t_1^2}-\frac{1}{t_2^2}}\right\}^2,$$

and this is not at a minimum in the position of the common normal whose feet are $N_1(0,0,0)$ and $N_2(0,0,1)$ respectively.

But we see that both the distance from N_1 to the plane and that from N_2 to the paraboloid are at minima in the position N_1N_2. And if $\varphi(t_1,t_2)$, $\psi(t_1,t_2)$ be any two analytic functions of t_1,t_2 such that

$$\varphi(0,0)=0, \quad \psi(0,0)=0 \text{ and } \left[\frac{\partial(\varphi,\psi)}{\partial(t_1,t_2)}\right]_{\substack{t_1=0\\t_2=0}}\gtrless 0,$$

then the function $f\{t_1,t_2,\varphi(t_1,t_2),\psi(t_1,t_2)\}$ formed by putting $t_3=\varphi(t_1,t_2)$, $t_4=\psi(t_1,t_2)$ in $f(t_1,t_2,t_3,t_4)$ is also at a minimum in the position N_1N_2.
