Note on W-Curves.

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1. Consider two W-curves W_1 and W_2 belonging to the same system and given by the equations

$$x_1^a\, x_2^b\, x_3^c = k_1, \tag{1}$$

$$\xi_1^a\, \xi_2^b\, \xi_3^c = k_2, \tag{2}$$

where $\qquad a+b+c=0.$

If $Q\ (\xi_1, \xi_2, \xi_3)$ be one of the intersections of the curve W_2 and the tangent to the curve W_1 at a point $P\ (x_1, x_2, x_3)$, then the relation

$$\frac{a\xi_1}{x_1} + \frac{b\xi_2}{x_2} + \frac{c\xi_3}{x_3} = 0 \tag{3}$$

will be satisfied. Eliminating x_3 and ξ_3 from (1), (2) and (3), we get

$$\left(\frac{\xi_1}{\xi_2} \cdot \frac{x_2}{x_1}\right)^a = (-1)^c\, c^c \cdot \frac{k_2}{k_1} \cdot \left\{a\left(\frac{\xi_1}{\xi_2} \cdot \frac{x_2}{x_1}\right) + b\right\}^c.$$

Solving the last equation with respect to $\dfrac{\xi_1}{\xi_2} \cdot \dfrac{x_2}{x_1}$, we obtain

$$\frac{\xi_1}{\xi_2} \cdot \frac{x_2}{x_1} = e \qquad (i = 1, 2, \ldots.),$$

where e_i depends upon a, b, k_1 and k_2.

Now the equation of CP is

$$x_2\, X_1 - x_1\, X_2 = 0, \tag{4}$$

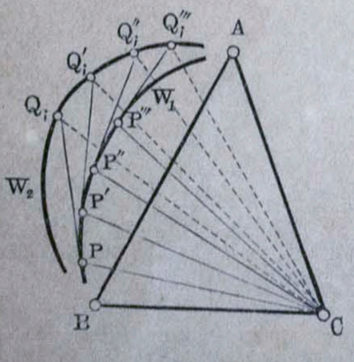
and that of CQ_i

$$\xi_2\, X_1 - \xi_1\, X_2 = 0,$$

which may be written

$$x_2\, X_1 - e_i\, x_1\, X_2 = 0. \tag{5}$$

Fig. 1.



2. Let us take two W-curves W_1, W_2 belonging to the same system, and let Q_i, Q_i', Q_i'', Q_i''' be four intersections of the curve W_2 and the four tangents to the curve W_1 at four points $P\ (x_1, x_2, x_3)$, $P'\ (x_1', x_2', x_3')$, $P''\ (x_1'', x_2'', x_3'')$ and $P'''\ (x_1''', x_2''', x_3''')$ respectively. In this case, by (4), the equations of CP, CP', CP'', CP''' are

$$x_2 X_1 - x_1 X_2 = 0, \qquad x_2' X_1 - x_1' X_2 = 0,$$
$$x_2'' X_1 - x_1'' X_2 = 0, \qquad x_2''' X_1 - x_1''' X_2 = 0$$

respectively, and by (5) those of CQ_i, CQ_i', CQ_i'', CQ_i'''

$$x_2 X_1 - e_i x_1 X_2 = 0, \qquad x_2' X_1 - e_i x_1' X_2 = 0,$$
$$x_2'' X_1 - e_i x_1'' X_2 = 0, \qquad x_2''' X_1 - e_i x_1''' X_2 = 0$$

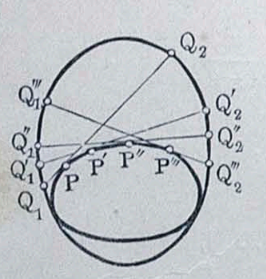
respectively. Hence we obtain

$$C(PP'P''P''') = C(Q_i Q_i' Q_i'' Q_i''') \tag{6}$$

and similar relations for A and B. Therefore we have the following theorem:

Theorem I. If two W-curves belonging to the same system, the anharmonic ratio of a pencil joining a vertex of the invariant triangle to the four points in which any four tangents to the one of the curves meet the other is equal to that of a pencil joining the vertex to the four points of contact.

Fig. 2.



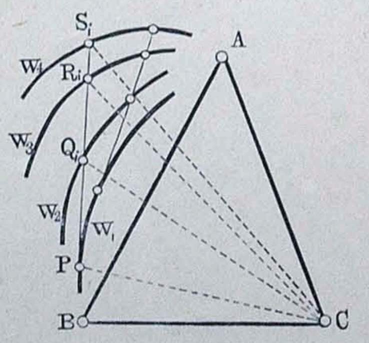
In the particular case of two conics having double contacts, we have the well-known relation

$$(P\,P'\,P''\,P''') = (Q_1\,Q_1'\,Q_1''\,Q_1''')$$
$$= (Q_2\,Q_2'\,Q_2''\,Q_2''')$$

which bears the name of Townsend or Göpel. Therefore our theorem may be considered as an extension of Townsend-Göpel's theorem.

3. Let us take four W-curves W_1, W_2, W_3, W_4 belonging to the same system, and let Q_i, R_i, S_i be three intersections of the three curves W_2, W_3, W_4 and the tangent to the curve W_1 at a point $P(x_1, x_2, x_3)$. In this case, by (4) and (5), the equations of CP, CQ_i, CR_i, CS_i are

Fig. 3.



$$x_2 X_1 - x_1 X_2 = 0, \qquad x_2 X_1 - e_i x_1 X_2 = 0,$$
$$x_2 X_1 - e_i' x_1 X_2 = 0, \qquad x_2 X_1 - e_i'' x_1 X_2 = 0$$

respectively. Hence $C(P\,Q_i\,R_i\,S_i)$ depends upon a, b, k_1, k_2, k_3, k_4, but is independent of $P(x_1, x_2, x_3)$. Thus we get the following theorem (1):

(1) Clebsch-Lindeman, Vorlesungen über Geometrie I 2, p. 997.

Theorem II. If we take four W-curves belonging to the same system, and draw the tangent to the one of the curves at a point P, then the point P and the three intersections of the tangent and the remaining three curves will form a range whose anharmonic ratio is independent of P.

When the curve W_1 reduces to an invariant point, we obtain the following corollary:

If we take three W-curves belonging to the same system and draw a line through a vertex of the invariant triangle, then the vertex and the three intersections of the line and the three curves will form a range whose anharmonic ratio is independent of the line.

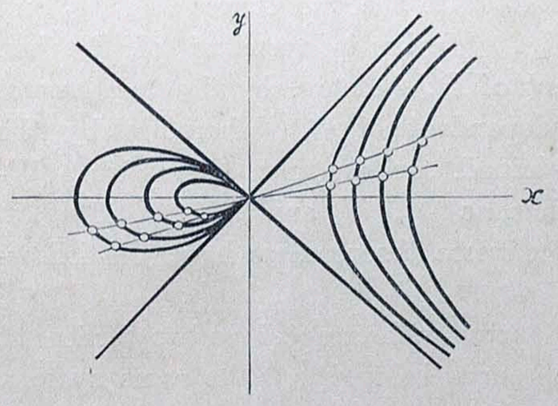
From this corollary we get the following theorem:

Theorem III. If we take four W-curves belonging to the same system, and draw a line passing through a vertex of the invariant triangle, then the four intersections of the four curves and the line will form a range whose anharmonic ratio is independent of the line.

The direct analytic or synthetic proof of this theorem is not difficult.

Corollary. If we take four inverse curves of four W-curves belonging to the same system with respect to a vertex of the invariant triangle, and draw a line through the vertex, then the four intersections of the four inverse curves and the line will form a range whose anharmonic ratio is independent of the line.

Fig. 4.



For example, four rectangular hyperbolas belonging to

$$x^2 - y^2 = \text{const.},$$

and four lemniscates belonging to

$$(x^2+y^2)^2 = \text{const.}\ (x^2-y^2)$$

will be cut in a projective range by anyline through the origin.

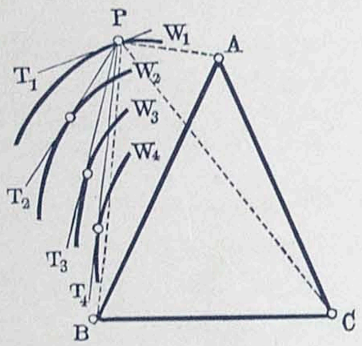
4. Now reciprocating Theorem II with respect to

$$x_1^2 + x_2^2 + x_3^2 = 0,$$

we get the following theorem:

Theorem II'. If we take four W-curves W_1, W_2, W_3, W_4 belonging to the same system, and draw three tangents to the three curves W_2, W_3, W_4 from a point P on the curve W_1, then the three tangents

Fig. 5.



and the tangent at P to the curve W_1 will form a pencil whose anharmonic ratio is independent of P.

Corollary. If we take two W-curves W_1, W_2 belonging to the same system, and draw the tangent at a point P to the curve W_1, and a tangent from P to the curve W_2, then these two tangents and the two lines PA, PB will form a pencil whose anharmonic ratio is independent of P.

Now I will prove the converse of this corollary.

Let us take two W-curves having the same invariant triangle ABC, and draw the tangent PT at a point P to the curve W, and the tangent PT' to another curve W' from P. If $P(ABTT')$ be independent of P, then $P(ABCT')$ will be independent of P; because $P(ABCT)$ is independent of P. Therefore, by the well-known theorem (1), W' is a W-curve belonging to the same system as W.

Next I will prove the converse of theorem II'.

Take W_1, W_2, W_3 be three W-curves belonging to the same system. Let us draw a tangent PT_1 at a point P to the curve W_1, and three tangents PT_2, PT_3, PT' from P to the two curves W_2, W_3 and another curve W'. If $P(T_1\, T_2\, T_3\, T')$ be independent of P, then $P(ABT_1\, T')$ will be independent of P; because, by the corollary of Theorem II', $P(ABT_2\, T_3)$ is independent of P. Therefore by the converse of the corollary just obtained, we see that W' is also a W-curve belonging to the same system as W_1, W_2 and W_3.

As a particular case of this converse theorem, we have the following corollary:

The quasi-evolute (2) of a W-curve with respect to two vertices of the invariant triangle is also a W-curve belonging to the same system.

Reciprocating this converse, we see that the converse of Theorem II is also true.

5. In the last place, I will give another proof of the theorem due to Mr. Kubota (3).

(1) Lie-Scheffers, Vorlesungen über continuirliche Gruppen, p. 76.
(2) Salmon-Fiedler, Analytische Geometrie der höheren ebenen Kurven (1882), p. 114.
(3) T. Kubota, Tokyô Su. Buts. Kw. K. [2], **5**, p. 61, "Theorem I."

and
$$\mu.\ \xi = \frac{p_{14}-4p_{32}}{4i}, \qquad \mu.\ R = p_{34},$$
$$\mu.\ \eta = \frac{p_{14}+4p_{32}}{4}, \qquad \mu = 2p_{24}.$$
$$\mu.\ p = \frac{p_{31}}{2},$$

~~Hence we see that a circle-congruence (1) of the nth order will be transformed into a line-congruence of the same order.~~

1 2. If M_{ik} be the moment of two null-lines l_i and l_k, and T_{ik} the common tangent of the two corresponding circles C_i and C_k, then

$$M_{ik} = \frac{(\rho_i-\rho_k)(s_i-s_k)-(\sigma_i-\sigma_k)(r_i-r_k)}{\sqrt{r_i^2+s_i^2+1}\,\sqrt{r_k^2+s_k^2+1}}$$

and
$$T_{ik}^2 = (\xi_i-\xi_k)^2+(\eta_i-\eta_k)^2-(R_i-R_k)^2$$
$$= -\frac{(\rho_i-\rho_k)(s_i-s_k)-(\sigma_i-\sigma_k)(r_i-r_k)}{4s_is_k\sqrt{r_i^2+s_i^2+1}\,\sqrt{r_k^2+s_k^2+1}}.$$

Hence we obtain the relation
$$\frac{T_{ik}^2}{R_i R_k} = -\frac{M_{ik}}{\cos(l_i z)\cos(l_k z)}.$$

Or, if ω_{ik} be the angle between the two circles C_i and C_k, and if A_{ik} the quantity $-\dfrac{M_{ik}}{\cos(l_i z)\cos(l_k z)}$ related to the two null-lines l_i and l_k, then
$$4\sin^2\frac{\omega_{ik}}{2} = A_{ik}.$$

Therefore, when two circles have a line-element in common, the two corresponding null-lines will intersect at a point.

Next, when we consider four null-lines and their four corresponding circles, we have
$$\frac{T_{12}^2}{T_{23}^2} : \frac{T_{14}^2}{T_{43}^2} = \frac{M_{12}}{M_{23}} : \frac{M_{14}}{M_{43}}.$$

If we call the quantity $\dfrac{T_{12}}{T_{23}} : \dfrac{T_{14}}{T_{43}}$ the anharmonic ratio of the four circles, then we obtain the following theorem:

Grassmann's anharmonic ratio of four null-lines is equal to the square of the anharmonic ratio of the four corresponding circles.

Lastly, if we transform Casey's condition
$$T_{23}\,T_{14}+T_{31}\,T_{24}+T_{12}\,T_{34}=0$$

~~(1) In the sense of Lie, but not of Reye.~~

that four circles should be touched by a fifth, then we get the condition

$$\sqrt{M_{23}\,M_{14}}+\sqrt{M_{31}\,M_{24}}+\sqrt{M_{12}\,M_{34}}=0$$

that four null-lines should be cut by a fifth. This expresion is the same as Cayley's condition for the equilibrium of four forces ([1]).

3. When a system of ∞^1 circles cut two given circles C' and C'' at the given angle ω, the corresponding null-lines l' and l'' will form a system of generators of a ruled surface of the second order determined by the equations

$$p_{32}'\,p_{14}+p_{14}'\,p_{32}+p_{13}'\,p_{24}+p_{24}'\,p_{13}-2\,p_{34}'\,p_{31}+A'\,\cdot p_{34}'\,p_{34}=0,$$
$$p_{32}''\,p_{14}+p_{14}''\,p_{32}+p_{13}''\,p_{24}+p_{24}''\,p_{13}-2\,p_{34}''\,p_{31}+A''\cdot p_{34}''\,p_{34}=0,$$
$$p_{21}+p_{34}=0.$$

We shall call this system the ω-system with respect to the two null-lines l' and l''.

Particularly, a system of coaxal circles cutting C' and C'' orthogonally will be transformed into a $\frac{\pi}{2}$-system with respect to l' and l''; and a system of colunar circles touching C' and C'' into a 0-system cutting l' and l''.

Many elementary theorems in the plane geometry of circles are transformable into higher theorems in the geometry of the ruled surface of the second order. Here I will give some examples.

i) Null-lines in an ω-system with respect to two null-lines in a $\frac{\pi}{2}$-system will form a 0-system cutting other two null-lines in the $\frac{\pi}{2}$-system.

ii) If B be a $\frac{\pi}{2}$-system with respect to any two null-lines in another $\frac{\pi}{2}$-system A with respect to any two null-lines l' and l'', then B will contain the two null-lines l' and l'', and A is a $\frac{\pi}{2}$-system with respect to any two null-lines in B.

iii) Null-lines belonging to a $\frac{\pi}{2}$-system with respect to any two null-lines in a pencil of null-lines on a plane will form a pencil on another plane.

([1]) See, for example, Timerding. Geometrie der Kräfte, p. 129.