On the Calculus of Generalisation.

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1. In this short note, I will give a somewhat rigorous proof of Oltramare's Calcul de Généralisation (1).

Let $\phi(x) = -\frac{a_0}{x} + \frac{a_1}{x^2} - \frac{a_2}{x^3} + \cdots\cdots\cdots$

be a convergent or an asymptotic series such that the associated series

$$f(u) = a_0 + a_1 \frac{u}{1!} + a_2 \frac{u^2}{2!} + \cdots\cdots\cdots$$

converges for certain positive values of u, and that the function $f(u)$ defined by the series is continuous from $u = 0$ to $u = \infty$.

Now we define the sum of the first series by the integral (2)

$$\int_0^\infty e^{ux} f(u)\, du, \qquad (\Re(x) < 0),$$

and assume that a fixed negative number l can be found such that

$$\lim_{u=\infty} e^{ul} f^{(n)}(u) = 0,$$

where n is any index of differentiation.

Then it is easy to see that the series $\phi(x)$ has the following properties:

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(1) Oltramare, Genève mém. inst. nat., **16** (1886); Calcul de Généralisation (1899). See also Abel, Oeuvres I, p 34 and II, p. 67.

(2) Bromwich, Infinite Series (1908), p. 339; Nielsen, Theorie der Gammafunktion (1906), p. 115. For the history, see Encyk. d. Math. Wiss., II_1, p. 781.

(1) The series represents the integral

$$\phi(x) = \int_0^\infty e^{ux} f(u)\, d u\,, \qquad (\Re(x) < 0)\,,$$

closely or asymptotically.

(2)
$$\phi'(x) = D\phi(x) = \int_0^\infty u\, e^{ux} f(u)\, d u\,,$$

$$\phi''(x) = D^2\phi(x) = \int_0^\infty u^2 e^{ux} f(u)\, d u\,,$$

....

$$\int_{-\infty}^{x} \phi(x)\, d x = D^{-1}\phi(x) = \int_0^\infty \frac{1}{u} e^{ux} f(u)\, d u\,,$$

....

(3) If $F(t)$ be a rational integral function of t, then

$$F(D)\,\phi(x) = \int_0^\infty F(u)\, e^{ux} f(u)\, d u\,.$$

Now when A is a positive constant and G is any great number, we have $|f^{(n)}(u)| < A e^{-ul}$, if $u > G$.

Hence $u^\beta e^{u(l+a)} |f^{(n)}(u)| < A u^\beta e^{au}$,

where a is a negative constant and β is either positive or negative. Thus the integral $\int_0^\infty | u^\beta e^{u(l+a)} f^{(n)}(u) | \, d u$ is convergent, and consequently the integral $\int_0^\infty | u^\beta e^{ux} f^{(n)}(u) | \, d u$ is convergent for $\Re(x) < l < 0$.

In the following articles, we shall define the generalisation of the function $F(u)$ by the integral $\int_0^\infty e^{ux} F(u) f(u)\, d u$.

2. In the first place, we shall consider generalisation of series.

I. Let $\psi(u) = \sum_{n=0}^{\infty} \psi_n(u)$ be uniformly convergent for all positive values of u. Since the integral $\int_0^\infty e^{ux} f(u)\, d u$ is absolutely convergent for $\Re(x) < l$, by the well-known theorem (1), we obtain the following theorem :

If $\sum_{n=0}^{\infty} \psi_n(u)$ be uniformly convergent for all positive values of u, then

(1) Bromwich, Infinite Series, p. 452.

$$\int_0^\infty \sum_{n=0}^{\infty} \psi_n(u)\, e^{ux} f(u)\, d u = \sum_{n=0}^{\infty} \int_0^\infty \psi_n(u)\, e^{ux} f(u)\, d u\,, \qquad (\Re(x) < l)\,.$$

For example, Dirichlet's Series

$$\mathfrak{D}(u) = \sum_{n=0}^{\infty} a_n e^{-k_n u}\,,$$

$$(k_0 < k_1 < k_2 < \cdots\cdots\,, \qquad \lim_{n=\infty} k_n = \infty),$$

is uniformly convergent for $u > \lambda$. Hence it follows that

$$\int_0^\infty \mathfrak{D}(u)\, e^{ux} f(u)\, d u = \sum_{n=0}^{\infty} \int_0^\infty a_n e^{-k_n u} e^{ux} f(u)\, d u$$

$$= \sum_{n=0}^{\infty} a_n \int_0^\infty e^{u(x-k_n)} f(u)\, d u$$

$$= \sum_{n=0}^{\infty} a_n \phi(x - k_n)\,,$$

provided that $\lambda < 0$, $\Re(x) < l$ and $\Re(x - k_n) < l$.

We can apply the similar generalisation to the factorial series

$$\Omega(u) = \sum_{n=0}^{\infty} \frac{n!\; a_n}{u(u+1)\cdots\cdots(u+n)}\,.$$

II. If $\psi(u) = \sum_{n=0}^{\infty} \psi_n(u)$ be uniformly convergent in any fixed positive interval which may be arbitrarily great, but do not converge uniformly in an infinite interval, we must take another method. As an example, here we shall derive Taylor's series by the generalisation of the exponential series.

Now
$$\phi(x + a) = \int_0^\infty e^{u(x+a)} f(u)\, d u$$

$$= \int_0^\infty e^{au} . e^{ux} f(u)\, d u$$

$$= \sum_{k=0}^{n-1} \frac{a^k}{k!} \int_0^\infty u^k e^{ux} f(u)\, d u + \frac{a^n}{n!} \int_0^\infty u^n e^{u(x+\theta a)} f(u)\, du$$

$$= \sum_{k=0}^{n-1} \frac{a^k}{k!} \phi^{(k)}(x) + \frac{a^n}{n!} \int_0^\infty u^n e^{u(x+\theta a)} f(u)\, d u\,,$$

$$\Re(x) < l, \qquad \Re(x + a) < l, \qquad 0 < \theta < 1.$$

But the integral $\int_0^\infty u^n | e^{u(x+\theta a)} f(u) | \, d u$ is convergent for $\Re(x) < l < 0$ or $\Re(x + a) < l < 0$, and in this case

$$\lim_{n=\infty}\int_0^\infty \frac{(a\,u)^n}{n\,!}\,e^{u(x+\theta a)}\,f(u)\,d\,u = 0\,.$$

Hence we find

$$\phi\,(x+a) = \sum_{n=0}^{\infty}\frac{a^n}{n\,!}\,\phi^{(n)}\,(x)\,, \qquad (\Re\,(x) < l\,,\ \Re\,(x+a) < l)\,,$$

which shows that the function $\phi\,(x)$ can be expressed as Taylor's series in the above mentioned half plane.

3. We shall now consider Euler's summation formula.

From the identity

$$\frac{1}{e^u-1} = e^{-u} + e^{-2u} + \cdots\cdots\cdots + e^{-nu} + \frac{e^{-(n+1)u}}{1-e^{-u}}\,,$$

we have

$$\int_0^\infty \frac{e^{ux}}{e^u-1}f(u)\,d\,u = \int_0^\infty e^{(x-1)u}f(u)\,d\,u + \cdots\cdots\cdots$$

$$+\int_0^\infty e^{(x-n)u}f(u)\,d\,u + \int_0^\infty \frac{e^{-(n+1)u}\,.\,e^{xu}}{1-e^{-u}}f(u)\,d\,u$$

where x is real and smaller than l.

If we denote $\displaystyle\int_0^\infty \frac{e^{-(n+1)u}\,.\,e^{xu}}{1-e^{-u}}f(u)\,d\,u$ by R_n,

then

$$|\,R_n\,| = \int_0^\infty \frac{e^{-(n+1)u}\,.\,e^{xu}}{1-e^{-u}}\,|f(u)\,|\,d\,u$$

$$= \int_0^\infty \frac{u}{e^u-1}\,\frac{e^{(x-n)u}}{u}\,|f(u)\,|\,d\,u$$

$$< \int_0^\infty \frac{e^{(x-n)u}}{u}\,|\,f(u)\,|\,d\,u$$

$$< \int_0^\infty \frac{|\,f(u)\,|}{u}\,d\,u\,.$$

But, since $\displaystyle\int_G^\infty e^{(x-n)u}\,|f(u)\,|\,d\,u$ is convergent, $\displaystyle\int_G^\infty \frac{e^{(x-n)u}}{u}\,|\,f(u)\,|du$ is also convergent; and $\displaystyle\int_0^\epsilon \frac{|\,f(u)\,|}{u}\,d\,u$ is convergent, provided that we can satisfy one of the conditions

$$|f(u)\,| < A\,u^\alpha\left(\log\frac{1}{u}\right)^\beta \quad \text{for small values of } u\,,$$

where either (i) $\alpha > 0$ or (ii) $\alpha = 0$, $\beta < -1$.

Hence when the above conditions are satisfied, the remainder R_n exists; and

$$\lim_{n\,\infty} R_n = \int_0^\infty \lim_{n=\infty} e^{-(n+1)u}\,.\,\frac{e^{xu}}{1-e^{-u}}f(u)\,d\,u$$

$$= 0\,,$$

which leads to the equation

$$\int_0^\infty \frac{e^{ux}}{e^u-1}f(u)\,du = \sum_{n=1}^{\infty}\phi\,(x-n)\,.$$

Now introducing Legendre's formula

$$\frac{1}{e^u-1} = \frac{1}{u} - \frac{1}{2} + 2\int_0^\infty \frac{\sin u\,t}{e^{2\pi t}-1}\,d\,t\,,$$

we get

$$\int_0^\infty \frac{e^{ux}}{e^u-1}f(u)\,d\,u = \int_0^\infty \frac{e^{ux}}{u}f(u)\,d\,u - \frac{1}{2}\int_0^\infty e^{ux}f(u)\,d\,u$$

$$+\,2\int_0^\infty e^{ux}f(u)\,d\,u\int_0^\infty \frac{\sin u\,t}{e^{2\pi t}-1}\,d\,t\,.$$

But

$$\int_0^\infty \frac{|\sin u\,t\,|}{e^{2\pi t}-1}\,d\,t < \int_0^\infty \frac{u\,t}{e^{2\pi t}-1}\,d\,t = \frac{u}{24}\,,$$

$$\int_0^\infty e^{ux}\,|\,f(u)\sin u\,t\,|\,d\,u < \int_0^\infty e^{ux}\,|\,f(u)\,|\,d\,u\,,$$

$$\int_0^\infty e^{ux}\,|\,f(u)\,|\,d\,u\int_0^\infty \frac{|\sin u\,t\,|}{e^{2\pi t}-1}\,d\,t < \frac{1}{24}\int_0^\infty u\,e^{ux}\,|\,f(u)\,|\,d\,u\,,$$

and accordingly the three integrals on the left hand side are all convergent.

Thus by the well-known theorem (1) we find

$$\int_0^\infty e^{ux}f(u)\,d\,u\int_0^\infty \frac{\sin u\,t}{e^{2\pi t}-1}\,d\,t = \int_0^\infty \frac{d\,t}{e^{2\pi t}-1}\int_0^\infty e^{ux}f(u)\sin u\,t\;d\,u$$

$$= \int_0^\infty \frac{d\,t}{e^{2\pi}-1}\int_0^\infty e^{ux}\,.\,\frac{e^{uti}-e^{-uti}}{2\,i}\,f(u)\,d\,u$$

$$= \frac{1}{2\,i}\int_0^\infty \frac{d\,t}{e^{2\pi t}-1}\left\{\int_0^\infty e^{u(x+ti)}f(u)\,d\,u - \int_0^\infty e^{u(x-ti)}f(u)\,d\,u\right\}$$

$$= \frac{1}{2\,i}\int_0^\infty \frac{\phi\,(x+t\,i) - \phi\,(x-t\,i)}{e^{2\pi t}-1}\,d\,t\,,$$

and in the domain $\Re\,(x) < l$, $\Re\,(x \pm t\,i) < l$, that is $x < l$, this integral is equal to

(1) Ibid, p. 457.

$$\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots\cdots\right\}$$
$$=\int_0^b \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots\cdots\right\}$$
$$+\int_b^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots\cdots\right\}.$$

Now if we can choose a positive number b such that either the integral

$$\int_0^b \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\,|\,\phi'(x)\,|+\frac{t^3}{3!}\,|\,\phi'''(x)\,|+\cdots\cdots\cdots\right\}$$

or the series

$$\frac{|\,\phi'(x)\,|}{1!}\int_0^b \frac{t\,dt}{e^{2\pi t}-1}+\frac{|\,\phi'''(x)\,|}{3!}\int_0^b \frac{t^3\,dt}{e^{2\pi t}-1}+\cdots\cdots\cdots$$

is convergent, then by Hardy's theorem (¹)

$$\int_0^b \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots\cdots\right\}$$
$$=\frac{\phi'(x)}{1!}\int_0^b \frac{t\,dt}{e^{2\pi t}-1}-\frac{\phi'''(x)}{3!}\int_0^b \frac{t^3\,dt}{e^{2\pi t}-1}+\cdots\cdots\cdots.$$

And since the series $\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots$ is uniformly convergent in any fixed interval $b\leqq t\leqq c$, where c is arbitrary, we have

$$\int_b^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\cdots\cdots\right\}=\frac{\phi'(x)}{1!}\int_b^\infty \frac{t\,dt}{e^{2\pi t}-1}-\cdots\cdots,$$

provided that either

$$\int_b^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\,|\,\phi'(x)\,|+\cdots\cdots\right\}\quad\text{or}\quad\frac{|\,\phi'(x)\,|}{1!}\int_0^\infty \frac{t\,dt}{e^{2\pi t}-1}+\cdots\cdots$$

is convergent.

Hence if either the integral $\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\,|\,\phi'(x)\,|+\cdots\cdots\right\}$ or the series $\frac{|\,\phi'(x)\,|}{1!}\int_0^\infty \frac{t\,dt}{e^{2\pi t}-1}+\cdots\cdots$ is convergent, then

$$\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left\{\frac{t}{1!}\phi'(x)-\frac{t^3}{3!}\phi'''(x)+\cdots\cdots\cdots\right\}$$
$$=\frac{\phi'(x)}{1!}\int_0^\infty \frac{t\,dt}{e^{2\pi t}-1}-\frac{\phi'''(x)}{3!}\int_0^\infty \frac{t^3\,dt}{e^{2\pi t}-1}+\cdots\cdots\cdots$$

(¹) Ibid, p. 450 (Theorem C).

Thus we arrive at Euler's formula of summation

$$\sum_{n=1}^\infty \phi(x-n)=\int_{-\infty}^x \phi(x)dx-\frac{1}{2}\phi(x)+\frac{1}{2!}B_1\phi'(x)-\frac{1}{4!}B_2\phi'''(x)+\cdots,$$

where B_1, B_2, ⋯⋯ denote Bernoulli's numbers.

4. Here we shall introduce the integral

$$\Phi(x)=\int_c^\infty e^{ux}f(u)\,du,\qquad(\Re(x)<0),$$

where $f(u)$ is the same function as before and c is a positive constant.

In this case, we have also

$$\Phi'(x)=\int_c^\infty u\,e^{ux}f(u)\,du,$$
$$\cdots\cdots\cdots\quad\cdots\cdots\cdots\cdots\cdots\cdots.$$

Let us consider Legendre's series

$$e^{au}=1+\frac{a}{1!}u\,e^{bu}+\frac{a(a-2b)}{2!}u^2e^{2bu}+\frac{a(a-3b)^2}{3!}u^3e^{3bu}+\cdots\cdots,$$

which is convergent for $|\,u\,e^{bu}\,|<\frac{1}{|\,b\,|\,e}$, and let us assume that b s negative. Then we can easily see that the series is uniformly convergent for $|\,u\,|>-\frac{1}{b}$.

Now $\int_c^\infty |\,e^{ux}f(u)\,|\,du$ is convergent for $c>-\frac{1}{b}$ and Legendre's series is uniformly conveigent for all values of u greater than c, and consequently we get

$$\int_c^\infty e^{u(x+a)}f(u)\,du=\int_c^\infty e^{ux}f(u)\,du\left\{1+\frac{a}{1!}u\,e^{bu}\right.$$
$$\left.+\frac{a(a-2b)}{2!}u^2e^{2bu}+\cdots\cdots\right\}$$
$$=\int_c^\infty e^{ux}f(u)\,du+\frac{a}{1!}\int_c^\infty u\,e^{(x+b)u}f(u)\,du$$
$$+\frac{a(a-2b)}{2!}\int_c^\infty u^2e^{(x+2b)u}f(u)\,du+\cdots$$
$$(\Re(x)<l,\ \Re(x+a)<l).$$

Thus we find

$$\Phi(x+a)=\Phi(x)+\frac{a}{1!}\Phi'(x+b)+\frac{a(a-2b)}{2!}\Phi''(x+2b)+\cdots\cdots\cdots$$

which is nothing but the well-known formula due to Abel (¹).

(¹) Abel, Oeuvre II, p. 73.

5. In this article we shall apply the calculus of generalisation to functional equations.

Let us take the rational integral function of u

$$F(u) = (u-\omega_1)(u-\omega_2)\cdots\cdots(u-\omega_n),$$

and let us consider the equation

$$\int_0^\infty F(u)\, e^{ux} f(u)\, d\, u = \int_0^\infty e^{vx} f_1(v)\, d\, v,$$

where x is real and $f_1(u)$ is a function which is the same kind as $f(u)$.

By Art. **1**, we have

$$F(D)\int_0^\infty e^{ux} f(u)\, d\, u = \int_0^\infty e^{vx} f_1(v)\, d\, v,$$

or, if we put $y = \int_0^\infty e^{ux} f(u)\, d\, u$, we obtain the linear differential equation with constant coefficients

$$(D-\omega_1)(D-\omega_2)\cdots\cdots(D-\omega_n)\, y = \int_0^\infty e^{vx} f_1(v)\, d\, v.$$

Assume that $\omega_1, \omega_2, \cdots\cdots \omega_n$ are all different and their real parts are all negative. Then, since $\int_{-\infty}^x d\, x \int_0^\infty e^{vx} f_1(v)\, d\, v$ is convergent for $x < 0$, $\int_{-\infty}^x e^{-\omega_k x}\, d\, x \int_0^\infty e^{vx} f_1(v)\, dv$ is also convergent for $x < 0$. Hence a particular solution of the differential equation will be given by

$$y = \sum_{k=1}^n \frac{e^{\omega_k x}}{F'(\omega_k)} \int_{-\infty}^x e^{-\omega_k x}\, d\, x \int_0^\infty x f_1(v)\, d\, v.$$

Now $\int_0^\infty e^{vx} |f_1(v)|\, dv$ is convergent; and $\int_{-\infty}^x |e^{(v-\omega_k)x}|\, dx$ is equal to $\dfrac{e^{[v-\Re(\omega_k)]x}}{v-\Re(\omega_k)}$, since $v - \Re(\omega_k) > 0$; and

$$\int_0^\infty |f_1(v)|\, d\, v \int_{-\infty}^x |e^{(v-\omega_k)x}|\, d\, x = e^{-\Re(\omega_k)x} \int_0^\infty \frac{e^{vx}}{v-\Re(\omega_k)} |f_1(v)|\, d\, v,$$

which is convergent.

Hence
$$y = \sum_{k=1}^n \frac{e^{\omega_k x}}{F'(\omega_k)} \cdot e^{-\omega_k x} \int_0^\infty \frac{e^{vx}}{v-\omega_k} f_1(v)\, d\, v$$
$$= \sum_{k=1}^n \frac{1}{F'(\omega_k)} \int_0^\infty \frac{e^{vx}}{v-\omega_k} f_1(v)\, d\, v.$$

Or, we can express symbolically

$$\int_0^\infty e^{ux} f(u)\, d\, u = \frac{1}{F(D)} \int_0^\infty e^{vx} f_1(v)\, d\, v$$
$$= \sum_{k=1}^n \frac{1}{F'(\omega_k)} \cdot \frac{1}{D-\omega_k} \int_0^\infty e^{vx} f_1(v)\, d\, v$$
$$= \sum_{k=1}^n \frac{1}{F'(\omega_k)} \int_0^\infty \frac{e^{vx}}{v-\omega_k} f_1(v)\, d\, v$$
$$= \int_0^\infty \frac{1}{F(v)} e^{vx} f_1(v)\, d\, v.$$

When $\omega_1, \omega_2, \cdots\cdots \omega_n$ are not all di erent, by the use of the well-known formula

$$\frac{1}{(D-\omega_k)^{r+1}} \int_0^\infty e^{vx} f_1(v)\, d\, v = e^{\omega_k x} \int_{-\infty}^{x(r+1)} e^{-\omega_k x}\, d\, x^{r+1} \int_0^\infty e^{vx} f_1(v)\, d\, v$$
$$= \frac{e^{\omega_k x}}{r!} \sum_{m=0}^r (-1)^m \binom{r}{m} x^{r-m} \int_{-\infty}^x x^m e^{-\omega_k x}\, d\, x \int_0^\infty e^{xv} f_1(v)\, d\, v,$$

we arrive at the same result

Conversely, from the equation

$$\int_0^\infty \frac{1}{F(u)} e^{ux} f(u)\, d\, u = \int_0^\infty e^{vx} f_1(v)\, d\, v,$$

we get
$$\frac{1}{F(D)} \int_0^\infty e^{ux} f(u)\, d\, u = \int_0^\infty e^{vx} f_1(v)\, d\, v.$$

Hence we have the differential equation

$$\int_0^\infty e^{ux} f(u)\, d\, u = F(D) \int_0^\infty e^{vx} f_1(v)\, d\, v,$$

or, by Art. **1**,

$$\int_0^\infty e^{ux} f(u)\, d\, u = \int_0^\infty F(v)\, e^{vx} f_1(v)\, d\, v.$$

Thus we obtain the following theorem: If $F(u)$ be a rational integral function of u or its reciprocal, and the real parts of all the roots of the rational integral function are all negative, then from the equation

$$\int_0^\infty F(u)\, e^{ux} f(u)\, d\, u = \int_0^\infty e^{vx} f_1(v)\, d\, v,$$

we have
$$\int_0^\infty e^{ux} f(u)\, d\, u = \int_0^\infty \frac{1}{F(v)} e^{vx} f_1(v)\, d\, v.$$

As an example, we shall solve the differential equation

$$\frac{dy}{dx} = \frac{a}{x} + by \qquad (b < 0).$$

Assume that the solution may be obtained in the form

$$y = \int_0^\infty e^{ux} f(u)\, du \qquad (x < 0).$$

Then $$\frac{dy}{dx} = \int_0^\infty u\, e^{ux} f(u)\, du \qquad (x < 0).$$

And from $$\frac{a}{x} = \int_0^\infty e^{vx} f_1(v)\, dv \qquad (x < 0),$$

we get $f_1(v) = -a$.

Hence from the given equation we have

$$\int_0^\infty (u - b)\, e^{ux} f(u)\, du = -a \int_0^\infty e^{vx}\, dv.$$

By the theorem, $$y = \int_0^\infty e^{ux} f(u)\, du$$

$$= -a \int_0^\infty \frac{e^{vx}}{v - b}\, dv \qquad (x < 0),$$

or, putting $t = -vx$, we have the solution

$$y = -a \int_0^\infty \frac{e^{-t}}{t + bx}\, dt \qquad (x < 0),$$

which is the form obtained by Borel by means of an asymptotic series.

6. In the last place, we shall solve the integral equation

$$\int_0^\infty e^{-pt} \varphi(x - qt)\, dt = F(x),$$

where $x < 0$, $p > 0$, $q > 0$, and $F(x)$ is a given function expressible in the form

$$F(x) = \int_0^\infty e^{xv} f_1(v)\, dv.$$

To find the function $\varphi(x)$, assume that

$$\varphi(x) = \int_0^\infty e^{xu} f(u)\, du.$$

Then $$\varphi(x - qt) = \int_0^\infty e^{(x-qt)u} f(u)\, du,$$

this integral being convergent.

From the given equation, we get

$$\int_0^\infty e^{-pt}\, dt \int_0^\infty e^{(x-qt)u} f(u)\, du = \int_0^\infty e^{xv} f_1(v)\, dv.$$

Now $$\int_0^\infty e^{-(p+qu)t}\, dt = \frac{1}{p+qu};$$

$$\int_0^\infty e^{(x-qt)u} |f(u)|\, du \text{ is convergent;}$$

and $$\int_0^\infty e^{xu} |f(u)|\, du \int_0^\infty e^{-(p+qu)t}\, dt = \int_0^\infty \frac{1}{p+qu} e^{xu} |f(u)|\, du,$$

which is also convergent. Hence

$$\int_0^\infty e^{xv} f_1(v)\, dv = \int_0^\infty e^{xu} f(u)\, du \int_0^\infty e^{-(p+qu)t}\, dt$$

$$= \int_0^\infty \frac{1}{p+qu} e^{xu} f(u)\, du.$$

Therefore, by Art. **5**, we have

$$\int_0^\infty e^{xu} f(u)\, du = \int_0^\infty (p + qv)\, e^{xv} f_1(v)\, dv$$

$$= p \int_0^\infty e^{vx} f_1(v)\, dv + q \int_0^\infty v\, e^{vx} f_1(v)\, dv,$$

which leads to the solution

$$\varphi(x) = p\, F(x) + q\, F'(x).$$

In the similar way, we can treat the more general equation

$$\int_\alpha^\beta \varphi(x - T_1) \, .\, T\, dt = F(x),$$

where α, β are constants, and T, T_1 are given functions of t.
