With the Author's Compliments.

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On the summability of series of Sturm-Liouville's functions.

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On the Summability of Series of Sturm-Liouville's Functions,

BY

K. OGURA in Sendai.

In the series $\sum_{n=0}^{\infty} u_n$, if we put

$$\overset{(r)}{S_n}(u_n) = u_n + \binom{r+1}{1} u_{n-1} + \binom{r+2}{2} u_{n-2} + \dots + \binom{r+n}{n} u_0$$

$$= s_n + \binom{r}{1} s_{n+1} + \binom{r+1}{2} s_{n-2} + \dots + \binom{r+n-1}{n} s_0,$$

r being a number not negatively integral, and

$$\overset{(1)}{S_n}(u_n) = s_n = u_0 + u_1 + \dots + u_n,$$

$$\overset{(r)}{A_n} = \binom{r+n}{n} = \frac{(r+1)(r+2)\dots(r+n)}{n!},$$

then we have

$$\overset{(r-1)}{S_n}\left(\overset{(1)}{S_n}(u_n)\right) = \overset{(r-1)}{S_n}(s_n)$$

$$= s_n + \binom{r}{1} s_{n-1} + \binom{r+1}{2} s_{n-2} + \dots + \binom{r+n-1}{n} s_0$$

$$= \overset{(r)}{S_n}(u_n).$$

Let us consider the Fourier series

$$\sum_{n=0}^{\infty} u_n = v_0(x)\int_0^{\pi} f(x)v_0(x)dx + v_1(z)\int_0^{\pi} f(x)v_1(x)dx + v_2(z)\int_0^{\pi} f(x)v_{\_}(x)dx + \dots$$

with respect to the orthogonal functions $v_0(z)$, $v_1(z)$, $v_2(z)$, ..., where

$$\int_0^{\pi} v_p(x)\, v_q(x)\, dx = 0 \quad (p \neq q), \quad \int_0^{\pi} \{v_p(x)\}^2 dx = 1.$$

If we put

$$K_n(z, t) = \sum_{p=0}^{\infty} v_p(z)\, v_p(t) = \overset{(1)}{S_n}\{v_n(z)\, v_n(t)\} \qquad (p = 0, 1, 2, \dots),$$

$$\overset{*(r)}{K_n}(z, t) = \frac{1}{A_n^{(r)}} \overset{(r-1)}{S_n}\{K_n(z, t)\} = \frac{1}{A_n^{(r)}} \overset{(r)}{S_n}\{v_n(z)\, v_n(t)\},$$

$$[f(z)]_n = \int_0^\pi K_n(z,t) f(t)\,dt = \overset{(1)}{S_n}\left\{v_n(z)\int_0^\pi f(t)\,v_n(t)\,dt\right\} = \overset{(1)}{S_n}(u_n),$$

$$[f^{*(r)}(z)]_n = \frac{1}{A_n^{(r)}}\overset{(r-1)}{S_n}\{[f(z)]_n\} = \frac{1}{A_n^{(r)}}\overset{(r)}{S_n}(u_n),$$

then we get the relation

$$[f^{*(r)}(z)]_n = \int_0^\pi \overset{*(r)}{K_n}(z,t) f(t)\,dt.$$

But from the equation

$$[v_n^{*(r)}(z)]_{n+p} = v_n(z)\left\{1+\binom{r}{1}+\binom{r+1}{2}+\dots+\binom{r+p-1}{p}\right\} : \binom{r+n+p}{n+p}$$

$$= v_n(z)\binom{r+p}{p} : \binom{r+n+p}{n+p} \qquad (p=0,1,2,\dots),$$

which is obtained by short calculations, we see that

$$\lim_{p=\infty}[v_n^{*(r)}(z)]_{n+p} = v_n(z).$$

Hence by a lemma due to Haar,(1) we have the theorem:

Fourier series with respect to the orthogonal functions, corresponding to a function which lies in the domain of the orthogonal functions, is summable (C_r) for any value of r, provided that

$$\int_0^\pi \left|\overset{*(r)}{K_n}(z,t)\right| dt$$

is smaller than a finite number independent of n and z.

By applying this theorem I will prove the following theorem which may be considered as *an extension of two theorems of Haar's* (2) *and M. Riesz-Chapman's:* (3)

Any Fourier series with respect to Sturm-Liouville's functions, corresponding to a function which lies in the domain of Sturm-Liouville's functions is always summable (C_r) for any value of $r > 0$. Particularly, any Fourier series with respect to Sturm-Liouville's functions, corresponding to a continuous function $f(x)$ for $0 < x < \pi$, is always summable (C_r) for any value of $r > 0$; and the sum is equal to $f(x)$ for $0 < x < \pi$.

Now in the case of Sturm-Liouville's function $v_n(x)$, it is wellknown that

(1) Haar, Math. Ann., 69 (1910), p. 357.

(2) Haar, lot. cit. p. 358.

(3) M. Riesz, Comptes Rendus, November 22, 1909; Chapman, Proc. London Math. Soc., II 9 (1910-11), p. 390; Chapman, Quarterly Journal of Math., 43 (1911), p. 1.

$$K_n(a,t) = \frac{2}{\pi}\sum_{p=0}^{n}\cos pa\cdot\cos pt + \Phi_n(a,t)$$

$$= \frac{1}{\pi}\left\{\frac{\sin(2n+1)\frac{t-a}{2}}{2\sin\frac{t-a}{2}} + \frac{\sin(2n+1)\frac{t+a}{2}}{2\sin\frac{t+a}{2}} + 1\right\} + \Phi_n(a,t)$$

where $\quad |\Phi_n(a,t)| \quad$ and $\quad \left|\dfrac{\sin(2n+1)\frac{t+a}{2}}{2\sin\frac{t+a}{2}}\right|$

are smaller than a finite number G independent of n, a, t, assuming that $\quad 0 < a < \pi,\ 0 \leqq t \leqq \pi.$

But since $\qquad \overset{(r-1)}{S_n}(G) : \overset{(r)}{A_n} = G,$

in order to prove our theorem, we are only to shew that

$$\int_0^\pi \left|\frac{1}{A_n^{(r)}}\overset{(r-1)}{S_n}\left\{\frac{\sin(2n+1)\frac{t-a}{2}}{2\sin\frac{t-a}{2}}\right\}\right| dt$$

is smaller than a finite number independent of n and a.

Now

$$\int_0^\pi \left|\frac{1}{A_n^{(r)}}\overset{(r-1)}{S_n}\left\{\frac{\sin(2n+1)\frac{t-a}{2}}{2\sin\frac{t-a}{2}}\right\}\right| dt < \int_{-\frac{a}{2}}^{\frac{\pi-a}{2}} \Psi(n,r,\vartheta)\,d\vartheta$$

$$= \int_{-\frac{a}{2}}^{-\epsilon}\Psi\,d\vartheta + \int_{-}^{0}\Psi\,d\vartheta + \int_0^{\epsilon}\Psi\,d\vartheta + \int_{\epsilon}^{\frac{\pi-a}{2}}\Psi\,d\vartheta,$$

where ε is an arbitrary small positive number and

$$\Psi(n,r,\vartheta) = \left|\frac{1}{A_n^{(r)}}\overset{(r-1)}{S_n}\left\{\frac{\sin(2n+1)\vartheta}{\sin\vartheta}\right\}\right|.$$

But it is easily seen that

$$\int_{-\frac{a}{2}}^{-\epsilon}\Psi\,d\vartheta < K, \quad \int_{2\epsilon}^{\frac{\pi-a}{2}}\Psi\,d\vartheta < K', \quad (r>0);$$

and Chapman (1) has shewn that

(1) Chapman, loc. cit.

$$\int_{-\epsilon}^{0} \Psi \, d\vartheta = \int_{0}^{\epsilon} \Psi \, d\vartheta < K'' \qquad (0 < r < 1),$$

where K, K' and K'' are independent of n and a.

Hence our theorem is proved.

We can get similar theorems for the summability of the two series treated in my previous note.(1)

Sendai, November 1911.

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(1) This Journal, 1 (1911-12), p. 121-133.

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