Chapter XXI.

§ 2. Ordinary

Dif. eq. of the second order.

92. General remarks.

$$\frac{d^2y}{dx^2}=f\left(x,y,\frac{dy}{dx}\right).$$

type I, $\frac{d^2y}{dx^2}=f(x)$.

type II. $\frac{d^2y}{dx^2}=\pm k^2y$. $\frac{d^2y}{dx^2}=f(y)$.

} コノ二ツハ諸君ノ学ブ所ナリ.

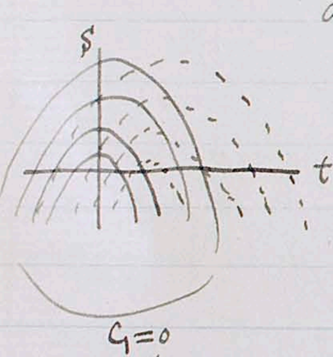
一般解ハ二ツノ const. C_1, C_2 ヲ含ム.

例.

~~$\frac{d^2y}{dx^2}=-g$. $\frac{dy}{dx}=-gx+c_1$, $y=\ \ +c_1x+c_2$.~~

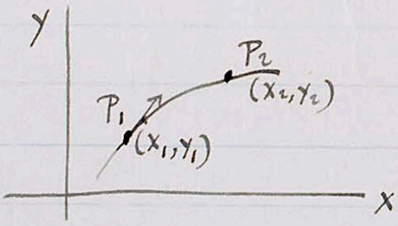
$\frac{d^2s}{dt^2}=-g$. $\frac{ds}{dt}=-gt+c_1$, $s=-\frac{1}{2}gt^2+c_1t+c_2$.

c_1, c_2 ノ意味如何.



$C_1=0$ ノ場合

二重ニ無限ノ曲線ヲ現ハス



二点ヲ通ル曲線ハ一般ニ一ツニ限ル.

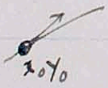
何トナレバ.

$$\begin{cases} s_1=-\frac{1}{2}gt_1^2+c_1t_1+c_2 \\ s_2=-\frac{1}{2}gt_2^2+c_1t_2+c_2 \end{cases}$$

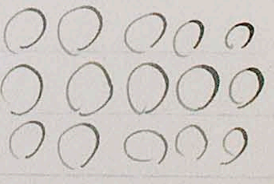
コレヨリ c_1, c_2 ガ定マル.

マタ 一点ト方向トヲ与ヘレバ曲線ハ定マル.

何トナレバ: $\left(\frac{ds}{dt}\right)$. $\begin{cases} -gt_0+c_1=v_0 \\ -\frac{1}{2}gt_0^2+c_1t_0+c_2=s_0. \end{cases}$



逆ニ 二重ニ無限ノ曲線群ノ微分方程式



B. 半径 r (一定) ナル円

$(x-a)^2+(y-b)^2=r^2$.

$x-a+(y-b)\frac{dy}{dx}=0$. $(x-a)^2=(y-b)^2\left(\frac{dy}{dx}\right)^2$. $(y-b)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$.

$1+\left(\frac{dy}{dx}\right)^2+(y-b)\frac{d^2y}{dx^2}=0$. $y-b=\pm r\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}$.

$1+\left(\frac{dy}{dx}\right)^2=\mp r\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}}\frac{d^2y}{dx^2}$.

$$\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}=\pm\frac{1}{r}.$$

93. 1 $\frac{d^2y}{dx^2}=f(x)$, 2 $\frac{d^2y}{dx^2}=f(y)$,

3 $\frac{d^2y}{dx^2}=f\left(x,\frac{dy}{dx}\right)$, 4. $\frac{d^2y}{dx^2}=f\left(y,\frac{dy}{dx}\right)$.

(3). $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$.

$\therefore \frac{dp}{dx}=f(x,p)$.

B. Catenary. $a\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

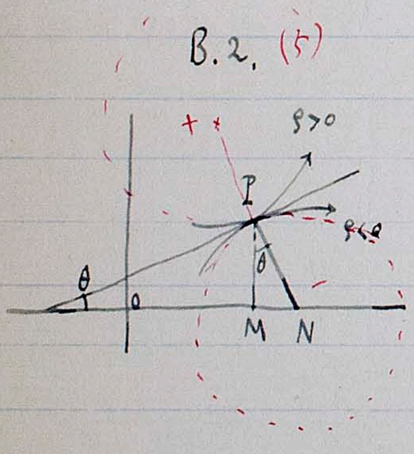
(4) $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=\frac{dp}{dy}p$

$p\frac{dp}{dy}=f(y,p)$.

B. 1. $\frac{d^2s}{dt^2}=F(s)$. $\frac{ds}{dt}=p=v$

$v\frac{dv}{ds}=F(s)$. $\frac{1}{2}v^2=\int F(s)ds+C$.

B. 2. (5) $PN=\rho$ (± ヲ取ル)



$PN=\frac{y}{\cos\theta}=y\sqrt{1+tg^2\theta}=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

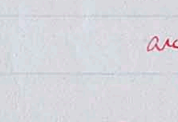
$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\pm\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$.

$\pm y\frac{d^2y}{dx^2}=1+\left(\frac{dy}{dx}\right)^2$

(+)ノ場合. $yp\frac{dp}{dy}=1+p^2$ $\frac{p}{1+p^2}dp=\frac{dy}{y}$.

$\frac{1}{2}\log(1+p^2)=\log y+\log c$ $\sqrt{1+p^2}=cy$. $p=\sqrt{c^2y^2-1}\ \left(=\frac{dy}{dx}\right)$

$\int\frac{dy}{\sqrt{c^2y^2-1}}=x+c'$, $\log\left(cy+\sqrt{c^2y^2-1}\right)=c(x+c')$.



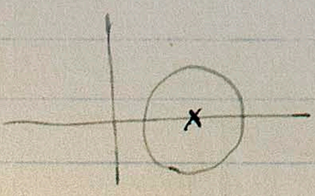
ナル故 Catenary.

(−)ノ場合

$-py\frac{dp}{dy}=1+p^2$. $p=\sqrt{\frac{c}{y^2}-1}$ $\int\frac{y\,dy}{\sqrt{c-y^2}}=x+c'$

$-(c-y^2)^{\frac{1}{2}}=x+c'$ $c-y^2=(x+c')^2$

$x^2+y^2+Ax+B=0$.



円

演習

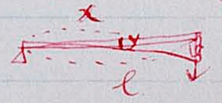
1. $s=A\sin kt+B\cos kt$ ヨリ A, B ヲ消去セヨ.

2. $\alpha=v^2+1$ ナル運動ノ法ヲ定

$\mathrm{arc\,tg}\,v=t+c$, $v=tg(t+c)$, $s=-\log\cos(t+c)+c'$.

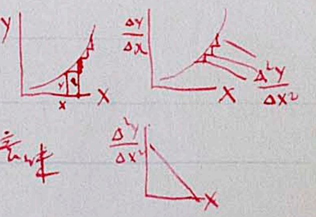
3. $\frac{d^2y}{dx^2}=e^{2y}$.

4.



$\frac{d^2y}{dx^2}=K(l-x)$. (K is const)

実験上ノ意味



94. Homogeneous linear equation

$$A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=0.$$

$(A, B, C$ ハ const.$)$

先ヅ $y=e^{mx}$ トオク.

$(Am^2+Bm+C)e^{mx}=0$ 故ニ

$$Am^2+Bm+C=0.$$

(I). $\underline{\Delta\equiv B^2-4AC>0}$ 二次方程式ハ二ツノ実根ヲ有ス.

ナラバ,

$$y=C_1e^{m_1x}+C_2e^{m_2x}.$$

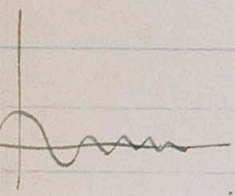
(II), $\Delta\equiv B^2-4AC\leqq 0$ ナル場合.

$y=e^{rx}.v$ トオケバ $\qquad \frac{dy}{dx}=re^{rx}.v+e^{rx}\frac{dv}{dx}=e^{rx}\left(rv+\frac{dv}{dx}\right)$

$$A\frac{d^2v}{dx^2}+(2rA+B)\frac{dv}{dx}+(Ar^2+Br+c)v=0.$$

$2rA+B=0$ 即チ $r=-\frac{B}{2A}$ トオケバ

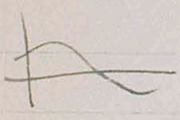
$$\frac{d^2v}{dx^2}=\frac{B^2-4AC}{4A^2}v=-k^2v. \qquad k=\frac{\sqrt{4AC-B^2}}{2A}=\frac{\sqrt{-\Delta}}{2A} \quad \text{実数}$$



$\underline{\Delta<0}$ ナル場合.

$$v=C_1\sin kv+C_2\cos kv$$

$$\therefore\ y=e^{rx}\left[C_1\sin kv+C_2\cos kv\right]. \qquad r=-\frac{B}{2A}, \quad k=\frac{\sqrt{4AC-B^2}}{2A}.$$



$$r\pm ik=\frac{-B\pm\overset{i}{\sqrt{-(B^2-4AC)}}}{2A}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$\underline{\Delta=0}$ ナル場合 $\qquad k=0 \qquad \frac{d^2v}{dx^2}=0. \qquad v=C_1x+C_2.$

$$y=e^{rx}(C_1x+C_2). \qquad r=-\frac{B}{2A}.$$

B.1. $3\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=0.$

$3m^2-4m+1=0. \qquad m=1, \frac{1}{3} \qquad y=C_1e^{x}+C_2e^{\frac{x}{3}}.$

B.2. $3\frac{d^2y}{dx^2}-4\frac{dy}{dx}+\frac{4}{3}y=0$

$3m^2-4m+\frac{4}{3}=0 \qquad m=\frac{2}{3}, \frac{2}{3}. \qquad y=e^{\frac{2}{3}x}(C_1x+C_2)$

$r=-\frac{B}{2A}=\frac{2}{3}.$

B.3. $3\frac{d^2y}{dx^2}-4\frac{dy}{dx}+2y=0$

$3m^2-4m+2=0. \qquad m=\frac{1}{3}(\cancel{2\pm\sqrt{-2}})=\frac{2}{3}\pm\frac{\sqrt{-2}}{3}.$

$$y=e^{\frac{2}{3}x}\left(C_1\cos\frac{\sqrt{2}}{3}x+C_1\sin\frac{\sqrt{2}}{3}x\right)$$

Remark.

$\Delta < 0$

~~$y = C_1 e^{\frac{-B}{2A}+\frac{\sqrt{-(B^2-4AC)}}{2A}}$~~

$m_0 = r \pm ik.$

$$y = C_1 e^{(r+ik)x} + C_2 e^{(r-ik)x}$$
$$= C_1 e^{rx} e^{ikx} + C_2 e^{rx} \cdot e^{-ikx}$$
$$= e^{rx}\left[C_1 e^{ikx} + C_2 e^{-ikx} \right]$$

$e^{ix} = \cos x + i \sin x.$

(Euler)

$$= e^{rx}\left[C_1 (\cos kx + i \sin kx) + C_2 (\cos kx - i \sin kx) \right]$$
$$= e^{rx}\left[(C_1 + C_2)\cos kx + (C_1 i - C_2 i)\sin kx \right]$$
$$= e^{rx}(A_1 \cos kx + A_2 \sin kx).$$

Exercises

$$\frac{d^2y}{dx^2} \pm k^2 y = 0$$
$$\frac{d^2y}{dx^2} + k\frac{dy}{dx} = 0$$
$$\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 14y = 0$$
$$2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$$
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$$

95. ~~Damping oscillation~~ (damped vibration)

$$\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + k^2 x = 0.$$

振動ノ條件 $f^2 - k^2 < 0.$ $m^2 + 2fm + k^2 = 0.$

$\sqrt{k^2 - f^2} = p$ トオケバ. $m = -f \pm p$

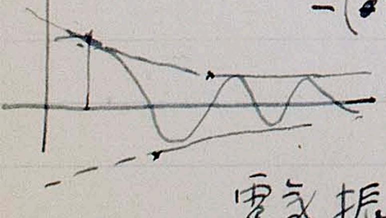
$$x = e^{-ft}(A \sin pt + B \cos pt).$$

Period $T = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{k^2 - f^2}}$ (T ハ [illegible]).

$$\frac{dx}{dt} = -f e^{-ft}(A \sin pt + B \cos pt) + e^{-ft}(Ap \cos pt - pB \sin pt) = 0.$$

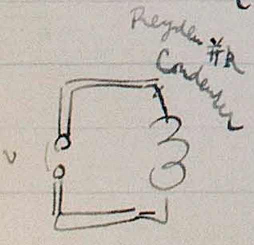
$$-f(A \sin pt + B \cos pt) + p(A \cos pt - B \sin pt) = 0.$$

$$-(fA + pB)\sin pt + (pA - fB)\cos pt = 0 \qquad \operatorname{tg} pt = \frac{fA + pB}{pA - fB}.$$



$$t = \frac{1}{p}\operatorname{arctg}\frac{fA + pB}{pA - fB}.$$

電気振動ノ場合



$$lk\frac{d^2v}{dt^2} + rk\frac{dv}{dt} + v = 0.$$

t ハ 時, v ハ potential ノ差.

l ハ ~~induction~~ inductance (感應率)

k capacity (容量) condenser ノ

r 抵抗 (輪道ノ)

96. Empirical formula.

$$y = a e^{cx} + b e^{dx}.$$

$$\begin{array}{c|cccccc} x & x_1 \overset{\Delta x}{\frown} x_2 \overset{\Delta x}{\frown} \cdots & x_k & x_{k+1} & x_{k+2} & x_n \\ \hline y & y_1 \quad y_2 & y_k & y_{k+1} & y_{k+2} & y_n \end{array}$$

$$\left\{\begin{array}{l} y_k = a e^{cx_k} + b e^{dx_k} \\ y_{k+1} = a e^{cx_{k+1}} + b e^{dx_{k+1}} \\ y_{k+2} = a e^{cx_{k+2}} + b e^{dx_{k+2}} \end{array}\right.$$

コノ間カラ a, b ヲ消去シテ

$$\frac{y_{k+2}}{y_k} = \left(e^{c\cdot\Delta x} + e^{d\cdot\Delta x}\right)\frac{y_{k+1}}{y_k} - e^{(c+d)\Delta x}$$

$\xi = \frac{y_{k+1}}{y_k}$, $\eta = \frac{y_{k+2}}{y_k}$ トオケバ 直線トナル.

$\eta = A\xi + B.$ $\left\{\begin{array}{l} A = e^{c\cdot\Delta x} + e^{d\cdot\Delta x} \\ B = -e^{c\cdot\Delta x}\cdot e^{d\cdot\Delta x} \end{array}\right.$ コレヨリ $e^{c\cdot\Delta x}$ 及ビ $e^{d\cdot\Delta x}$ ヲ求ム. 故ニ c, d ガ定マル.

次ニ $y_k = a e^{cx_k} + b e^{dx_k}$ ヨリ $y_k e^{-dx_k} = a e^{(c-d)x_k} + b$

$X = e^{(c-d)x_k}$, $Y = y_k e^{-dx_k}$ トオケバ 直線 $Y = aX + b$

トナル. ソレヨリ a, b ヲ定メル.

Ex. β線ヲ出シテ変リツヽ Ra ノ 放射能 A ト 時 t(分) トノ関係

t	0	10	20	30	40	50	60	70	80	90	100
A	100	97.0	88.5	77.5	67.5	57.0	48.2	42.5	33.5	29.0	22.5

$$y = e^{ax}(c\cos bx + d\sin bx).$$

y_k, y_{k+1}, y_{k+2} ヲ作リ c, d ヲ消去スレバ:

$$\frac{y_{k+2}}{y_k} = 2\cos(b\cdot\Delta x)e^{a\cdot\Delta x}\ \frac{y_{k+1}}{y_k} - e^{2a\cdot\Delta x}$$

$\xi = \frac{y_{k+1}}{y_k}$, $\eta = \frac{y_{k+2}}{y_k}$ トオケバ. $\eta = A\xi + B$

$$\left\{\begin{array}{l} A = 2\cos(b\cdot\Delta x)e^{a\cdot\Delta x}, \\ B = -e^{2a\cdot\Delta x} \end{array}\right.$$

B ヨリ a ヲ求メル. A ヨリ b ヲ求メル.

次ニ $y_k = e^{ax_k}(c\cos bx_k + d\sin bx_k)$ ヨリ $y_k e^{-ax_k}\sec bx_k = c + d\,\mathrm{tg}\, bx_k$.

ニ於テ $X = \mathrm{tg}\, bx_k,\quad Y = y_k e^{-ax_k}\sec bx_k$ トオケバ $Y = c + dX$

コレヨリ c, d ヲ求メル.

Ex. 液体中ニアル振子ノ運動 t ヲ時(秒), x ヲ偏位(粍)

t	x
0	1.50
1	0.89
2	0.28
3	−0.21
4	−0.48
5	−0.55
6	−0.47
7	−0.30
8	−0.12
9	+0.03
10	0.12
11	0.17
12	0.15

97. graphical solution.

Exercises.

1. $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$ $\quad m^3 - m = 0 \quad 0,\ 1,\ -1 \quad y = c_1 + c_2 e^{x} + c_3 e^{-x}.$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$

2. ~~$\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 12\frac{dy}{dx}$~~ $\frac{d^3y}{dx^3} + 8y = 0$ $\quad m^3 + 8 = 0 \quad (m+2)\left[m^2 - 2\underset{m}{\cancel{x}} + 4\right] = 0$

$m = -2,\ 1 \pm \sqrt{3}i$ $\quad m = 1 \pm \sqrt{1-4}$

$$y = c_1 e^{-2x} + e^{x}\left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right].$$