Charlier (Lund)
Arne Fisher (和蘭)

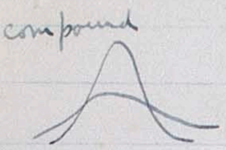
On the Frequency Curves.

Pearson (1902) ハ <u>simple</u> + <u>unimodal</u> frequency curve ヲ 七ツノ type ニ 分ケタ.

Elderton 11 個

Kelley 25 個

compound



$$y = \frac{N}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2},$$

$$y = y_0\left(1+\frac{x}{l_1}\right)^{m_1}\left(1-\frac{x}{l_2}\right)^{m_2},$$

$$y = y_0\left(1+\frac{x}{l}\right)^{p} e^{-\frac{x}{d}},$$

$$\cdots \qquad \cdots\cdots$$

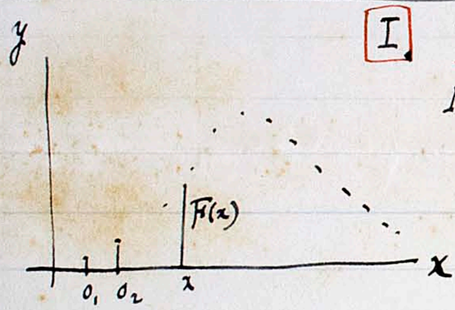
$$\frac{1}{y}\frac{dy}{dx} = \frac{x-a}{c_0 + c_1 x + c_2 x^2 + \cdots}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x-a}{c_0}.$$

$$\log y = A + \frac{(x-a)^2}{2c_0}.$$

Gauss

$\boxed{\text{I}}$. Frequency function

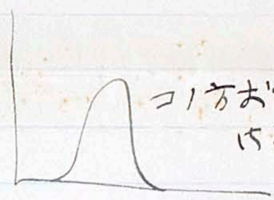


N.

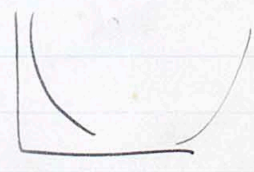
$$\sum F(a_k) = N.$$

$$\int_{-\infty}^{+\infty} F(x)\,dx = N.$$

$$F(x) = N\,\varphi(x). \qquad \int_{-\infty}^{+\infty} \varphi(x)\,dx = 1.$$



コノ方ガヨク
代表スル~



Laplace (F) general — Poisson (F)

Gauss (G) normal

series 1879

Gram (Denm)

Gauss,
又拉若

Quetelet (B) (skew) – Bravais (F) — Opperman (H.) — Thiele (Denm) 1889 — Charlier (Swed.) 1905-

Edgeworth (E)

Jørgensen 1916

Wicksell 1919

Pearson (E) 1895 — Elderton

– –

frequency function ハ

$$F(x, \lambda_1, \lambda_2, \dots, \lambda_N).$$

$\lambda_1 = \Phi_1(o_1, o_2, \cdots o_N),$ λ ハ observation ニ 特有ナ peculiar properties.

$\dots \quad \dots$ o_k ハ observation.

$\lambda_N = \Phi_N(o_1, o_2 \cdots o_N)$ Φ ハ o ノ symmetric function ナルコト λ ハ ト assumeスル.

power sum

其ノ fundamental sym. funct.

$$\begin{cases} S_0 = o_1^0 + o_2^0 + \cdots + o_N^0 = N. \\ S_1 = o_1 + o_2 + \cdots + o_N = \sum o_k \\ S_2 = o_1^2 \quad \cdots = \sum o_k^2 \\ \cancel{S_N = o_1^N + o_2^N + \cdots + o_N^N = \sum o_k^N} \end{cases}$$

イキー

1つ function トセテ 表ハセルコトハ (algebraic form)

然ラバ λ ハ s ノ如何ナル function ヲ取レバ 実際最モ便利ナルカ. ソレハ Thiele ノ semi-invariants ナルモノデアル.

Thiele ハ

$$s_0 e^{\lambda_1 \frac{\omega}{1!} + \lambda_2 \frac{\omega^2}{2!} + \lambda_3 \frac{\omega^3}{3!} + \cdots} = s_0 + s_1 \frac{\omega}{1!} + s_2 \frac{\omega^2}{2!} + s_3 \frac{\omega^3}{3!} + \cdots$$

ナル ω ニ関スル identity ニヨリテ λ ヲ define セリ.

$$= e^{o_1\omega} + e^{o_2\omega} + e^{o_3\omega} + \cdots + e^{o_N\omega}.$$

即チ

$$\begin{cases} s_1 = \lambda_1 s_0 \\ s_2 = \lambda_1 s_1 + \lambda_2 s_0 \\ s_3 = \lambda_1 s_2 + 2\lambda_2 s_1 + \lambda_3 s_2 \\ \dots \quad \dots \end{cases} \qquad \begin{cases} \lambda_1 = \frac{s_1}{s_0}, \\ \lambda_2 = \frac{s_0 s_2 - s_1^2}{s_0^2}, \\ \lambda_3 = \frac{s_0^2 s_3 - 3 s_0 s_1 s_2 + 2 s_1^3}{s_0^3}, \\ \dots \quad \dots \end{cases}$$

何故ニ λ ヲ semi-invariant ト称スルカノ理由.

今 origin ト unit トヲ変ヘレバ, observed value ハ linear transformation

$$o_k' = a o_k + c$$

ヲ受ケル.

$$s_0 e^{\lambda_1' \frac{\omega}{1!} + \lambda_2' \frac{\omega^2}{2!} + \cdots} = e^{(ao_1+c)\omega} + e^{(ao_2+c)\omega} + \cdots$$

コレヨリ

$$\begin{cases} \lambda_1' = a\lambda_1 + c, \\ \lambda_2' = a^2 \lambda_2 \\ \lambda_3' = a^3 \lambda_3 \\ \dots \quad \dots \end{cases}$$

$\boxed{\text{II.}}$ frequency function ノ determination

$$s_0 e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \cdots} = e^{o_1\omega} + e^{o_2\omega} + \cdots + e^{o_N\omega}.$$

コノ o ハ必ズシモ相異ルヲ要セズ. o_k ガ $N\varphi(o_k)$ ダケアルトセバ,

$$N e^{\frac{\lambda_1\omega}{1!} + \frac{\lambda_2\omega^2}{2!} + \cdots} = \sum N \varphi(o_k) e^{o_k\omega}.$$

continuous variable ニ就テハ

$$= \int_{-\infty}^{+\infty} \varphi(x) e^{x\omega} dx.$$

故ニ

$$e^{\frac{\lambda_1}{1!}(i\omega) + \frac{\lambda_2(i\omega)^2}{2!} + \cdots} = \int_{-\infty}^{+\infty} \varphi(x) e^{xi\omega} dx.$$

$$\begin{cases} \Psi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x) e^{x\omega i} dx \quad \text{ナラバ} \\ \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(\omega) e^{-x\omega i} d\omega. \end{cases}$$

(Fourier ノ integral equation)

故ニ

$$e^{\frac{\lambda_1}{1!}(i\omega) + \frac{\lambda_2(i\omega)^2}{2!} + \cdots} \equiv \sqrt{2\pi}\, \Psi(\omega)$$

トオケバ,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{\frac{\lambda_1}{1!}(i\omega) + \frac{\lambda_2}{2!}(i\omega)^2 + \cdots}}{\sqrt{2\pi}} e^{-x\omega i} d\omega$$

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\frac{\lambda_1}{1!}(i\omega) + \frac{\lambda_2}{2!}(i\omega)^2 + \cdots} \cdot e^{-x\omega i}\, d\omega$$

コレ即チ semi-invariants ノ term デ表ハサレタ最モ一般ナル frequency function ノ形デアル.

特ニ first approximation トシテ

λ_1, λ_2 マデヲ取レバ;

$$\varphi_0(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(\lambda_1 - x) - \frac{\lambda_2}{2}\omega^2}\, d\omega$$

$$= \frac{1}{2\pi}\left[\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{\lambda_2\omega^2}{2}} \cos[(\lambda_1 - x)\omega]\, d\omega}_{\text{Laplace integral}} + \underbrace{i\int_{-\infty}^{+\infty} e^{-\frac{\lambda_2\omega^2}{2}} \sin[\quad]\, d\omega}_{0}\right]$$

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi\lambda_2}}\, e^{-\frac{(\lambda_1 - x)^2}{2\lambda_2}}$$

$\left\{\begin{array}{ll} \lambda_1 = M & \text{(mean)} \\ \sqrt{\lambda_2} = \sigma & \text{(standard deviation)} \end{array}\right.$ トオケバ $\varphi_0(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-\frac{(x-M)^2}{2\sigma^2}}$ Pearson

特ニ M ヲ origin ニ取リ σ ヲ unit ニ選ベバ $z = \frac{x-M}{\sigma}$

$$\varphi_0(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}$$

一般ニ integrate スルコトハ困難ナル故, 他ノ方面カラ研究スル.

III. Hermite ノ polynomial.

~~√2π ヲトリ去ル~~

$\varphi_0(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ ノ derivatives ヲ順次ニ作リテ

$$\begin{aligned} & && = H_0(z)\varphi_0(z) \\ \varphi_1(z) &= -z\cdot\varphi_0(z) && = -H_1(z)\cdot\varphi_0(z) \\ \varphi_2(z) &= (z^2-1)\cdot\varphi_0(z) && = (-1)^2 H_2(z)\cdot\varphi_0(z) \\ \varphi_3(z) &= -(z^3-3z)\cdot\varphi_0(z) && = (-1)^3 H_3(z)\cdot\varphi_0(z), \\ &\cdots\cdots && \cdots\cdots \end{aligned}$$

$H_n(z)$ Hermite polynomial

$$\int_{-\infty}^{+\infty} \varphi_n(z) H_m(z)\, dz = 0 \qquad m \neq n$$

$$= (-1)^n n! \qquad m = n$$

$$\varphi(z) = c_0\varphi_0(z) + c_1\varphi_1(z) + c_2\varphi_2(z) + \cdots$$

$$c_0 = 1, \qquad c_n = \frac{(-1)^n}{n!}\int_{-\infty}^{+\infty} \varphi(z) H_n(z)\, dz.$$

$n = 1, 2, \cdots$

$$F(z) = N\,\varphi(z) = N \sum c_n \varphi_n(z). \qquad c_0 = 1,$$

$$\cancel{F(z) = N \cdot \sum c_n H_n(z)\, e^{-\frac{z^2}{2}}}.$$

$$c_n = \frac{(-1)^n}{n!\cdot N}\int_{-\infty}^{+\infty} F(z)\, H_n(z)\, dz$$

$$c_n = [illegible] \int_{-\infty}^{+\infty} H_n(z)\cdot F(z)\, dz.$$

$H_n(z)$ [illegible] polynomial [illegible]

c_n [illegible] $\quad s_r = \int_{-\infty}^{+\infty} z^r F(z)\, dz \quad$ rth moment (Pearson)

1 function [illegible]. $\quad N\varphi(o_1)o_1^r + N\varphi(o_2)\,o_2^r + \cdots \quad (= s_r).$

$\boxed{\text{IV.}}$ Standard form.

$c_0 = 1. \qquad s_0 = N.$

$$c_1 = -\frac{1}{N}\int_{-\infty}^{+\infty} z\, F(z)\, dz = -\frac{s_1}{s_0}$$

$c_1 = 0$ [illegible], mean value $= 0$. $\quad M = 0.$

$$N\varphi(o_1)\cdot o_1 + N\varphi(o_2)\cdot o_2 + \cdots = 0.$$

$s_1 = 0. \qquad \lambda_1 = \frac{s_1}{s_0} = 0 \quad \therefore \lambda_1 = 0.$

$$c_2 = \frac{1}{2N}\int_{-\infty}^{+\infty} (z^2 - 1)\, F(z)\, dz = \frac{1}{2N}\left[\int_{-\infty}^{+\infty} z^2 F(z)\, dz - \int_{-\infty}^{+\infty} F(z)\, dz\right]$$

$c_2 = 0$ [illegible] $\quad = \frac{s_2 - s_0}{2s_0} = 0. \quad \therefore s_2 = s_0.$

$$\lambda_2 = \frac{s_2 s_0 - s_1^2}{s_0^2} = \frac{s_0^2}{s_0^2} = 1. \qquad \therefore \sigma = 1.$$

[illegible] transformation [illegible]

$$z = \frac{x}{\sigma} - \frac{M}{\sigma}.$$

$$c_3 = -\frac{1}{3!}\left(\frac{s_3 - 3s_1}{s_0}\right) = -\frac{1}{3!}\,\frac{s_3}{s_0} = \frac{-1}{3!}\lambda_3'$$

$$c_4 = \frac{1}{4!}\left(\frac{s_4}{s_0} - 3\right) = \frac{1}{4!}\lambda_4'$$

$\ldots$

$$\left\{\begin{array}{l} \lambda_3' = \dfrac{\lambda_3}{\sigma^3}. \\ \lambda_4' = \dfrac{\lambda_4}{\sigma^4}. \\ \ldots \end{array}\right.$$

$$F(z) = N\left[\varphi_0(z) + c_3\varphi_3(z) + c_4\varphi_4(z) + \cdots\right].$$

$3c_3 = S \qquad$ Schiefheit

$3c_4 = E \qquad$ Exzess

American public utility corporation

Age	x	pensioned functionaries $F(x)$	cal	$x\,F(x)$	$x^2 F(x)$	$x^3 F(x)$	$x^4 F(x)$
35-39	-6	1	2	-6	36	-216	1296
40-44	-5	6	5	-30	150	-750	3750
45-49	-4	17	17				
50-54	-3	48	48				
55-59	-2	118	118				
60-64	-1	224	219				
65-69	0	286	291				
	Sum	700		-708	1586	-4518	15418
70-74	+1	248	241	248	248	248	248
75-79	+2	128	126	128	256	1024	2048
80-84	+3	38	44				
85-89	+4	13	15				
90-94	+5	2	3				
95 over	+6	1	1				
	Sum	430		686	1396	3596	11248
		1130		-22	2982	-922	41892
		$= S_0 = N$		$= S_1$	$= S_2$	$= S_3$	$= S_4$

$\lambda_1 = \frac{S_1}{S_0} = -0.0195, = M$

$\lambda_2 = \frac{S_0 S_2 - S_1^2}{S_0^2} = 2.6374$

$\sigma = \sqrt{\lambda_2} = 1.6240$

$$\varphi_0(x) = \frac{1}{1.6240\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+0.0195}{1.6240}\right)^2}$$

$\lambda_3 = \frac{S_0^2 S_3 - 3 S_0 S_1 S_2 + 2 S_1^3}{S_0^3} = -0.6617$

$\lambda_3' = \frac{\lambda_3}{\sigma^3} = -0.1545$

$c_3 = -\frac{\lambda_3'}{3!} = 0.0258$

$\lambda_4 = \quad = 2.6450$

$\lambda_4' = \frac{\lambda_4}{\sigma^4} = 0.3803$

$c_4 = \frac{\lambda_4'}{4!} = 0.0158$

$$F(x) = 1130\left[\varphi_0(z) + 0.0258\,\varphi_3(z) + 0.0158\,\varphi_4(z)\right].$$