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On a Certain Transcendental Integral Function in the Theory of Interpolation.

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On a Certain Transcendental Integral Function In the Theory of Interpolation,

by

KINNOSUKE OGURA, Ôsaka.

1. Consider a transcendental integral function $f(z)$ of the complex variable z $(z=re^{i\theta})$ such that it does not become infinite even at $z=\infty$ so long as z is real, and $|f(z)|$ becomes infinite to a lower order than

$$e^{\pi r|\sin\theta|}$$

when z approaches infinity. Such a function will be called the *cardinal function.*

The notion of cardinal functions is of fundamental importance in the theory of interpolation from the reason that:

I. It can be constructed analytically in a simple manner when the values

$$f(0),\ f(+1),\ f(-1),\ f(+2),\ f(-2),\ \cdots\cdots$$

are given: in fact,

$$f(z)=\frac{\sin \pi z}{\pi}\sum_{\nu=-\infty}^{+\infty}(-1)^{\nu}\frac{f(\nu)}{z-\nu} \quad (^1);$$

II. It may be represented by the Gauss formula of interpolation

$$f(z)=f_0+\frac{z}{1!}\delta f_{\frac{1}{2}}+\frac{z(z-1)}{2!}\delta^2 f_0+\frac{(z+1)z(z-1)}{3!}\delta^3 f_{\frac{1}{2}}$$

(1)

$$+\frac{(z+1)z(z-1)(z-2)}{4!}\delta^4 f_0+\cdots\cdots(^2)$$

Now we add the following remark: Prof. Whittaker stated that if

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(1) E. T. Whittaker, "On the functions which are represented by the expansions of the interpolation-theory," Proc. Roy. Soc. Edinburgh (1915), pp. 181–194; especially pp. 182–187. Although his method is very interesting and instructive for practical work, it is not free from inaccuracies. A proof of this theorem, which is simple and rigorous, can be obtained in E. Lindelöf, Calcul des résidus (1905), p. 53.

(2) Whittaker, loc. cit., pp. 191–2. We use of the following scheme:

(z) be a one-valued analytic function such that the values $\varphi(\nu)$ $(\nu=0, \pm 1, \pm 2, \cdots\cdots)$ are all finite even at infinity, then the series

$$\frac{\sin \pi z}{\pi} \sum_{\nu=-\infty}^{+\infty} (-1)^\nu \frac{\varphi(\nu)}{z-\nu}$$

represents a cardinal function $f(z)$ for which

$$f(\nu)=\varphi(\nu), \quad (\nu=0, \pm 1, \pm 2, \cdots\cdots).$$

It seems to me that this statement is *erroneous:* For, if we take any one-valued analytic function $\varphi(z)$ such that

$$\varphi(0)=0, \ \varphi(-1)=-1, \ \varphi(-2)=+1, \ \varphi(-3)=-1, \ \varphi(-4)=+1, \ \cdots\cdots,$$
$$\varphi(+1)=+1, \ \varphi(+2)=-1, \ \varphi(+3)=+1, \ \varphi(+4)=-1, \ \cdots\cdots,$$

the corresponding series diverges at $z=\dfrac{1}{2}$.

Moreover the converse of Theorem II is not true. Although a transcendental integral function $\varphi(z)$ may be expressed by the Gauss formula, the function is *not* cardinal in general. For example, if we take the function $\varphi(z)=e^z$, the corresponding Gauss formula becomes

$$1+(e-1)\frac{z}{1!}+e^{-1}(e-1)^2\frac{z(z-1)}{2!}+e^{-1}(e-1)^3\frac{(z+1)z(z-1)}{3!}$$
$$+e^{-2}(e-1)^4\frac{(z+1)z(z-1)(z-2)}{4!}$$
$$+e^{-2}(e-1)^5\frac{(z+2)(z+1)z(z-1)(z-2)}{5!}+\cdots\cdots,$$

which converges to e^z itself for every value of z ([1]). On the other

Argument	Entry					
-2	f_{-2}	....	....	....	....	....
		$\delta f_{-3/2}$	....	....	....	....
-1	f_{-1}		$\delta^2 f_{-1}$	....	....	....
		$\delta f_{-1/2}$		$\delta^3 f_{-1/2}$	....	....
0	f_0		$\rho^2 f_0$		$\delta^4 f_0$	....
		$\delta f_{1/2}$		$\delta^3 f_{1/2}$	....	....
$+1$	f_1		$\delta^2 f_1$	....	....	....
		$\delta f_{3/2}$	...	....	....	....
$+2$	f_2	....	....	....	....	....

where $\delta f_{-1/2}=f_0-f_{-1}$, $\delta f_{1/2}=f_1-f_0$, $\delta^2 f_0=\delta f_{1/2}-\delta f_{-1/2}$, etc.
For the Gauss formula, see D. Gibb, A course in interpolation and numerical integration (1915), p. 26.

([1]) G. Faber, "Beitrag zur Theorie der ganzen Funktionen," Math. Ann., 70 (1911), p. 65.

hand e^z is not cardinal. Indeed it will be easily seen that

$$\varphi(z)=e^{az+b} \quad (a, b \text{ being constants})$$

is cardinal when and only when

$$a=i\beta,$$

β being real and $-\pi<\beta<\pi$; and in such a case $|\varphi(z)|$ is constant so long as z is real.

2. If the transcendental integral function

$$f(z)=\sum_{\nu=0}^{\infty} c_\nu z^\nu=\sum_{\nu=0}^{\infty}\frac{f^{(\nu)}(0)}{\nu!}z^\nu$$

be cardinal, we have from the definition

$$(2) \qquad |f(z)|<e^{\pi|z|} \quad \text{for } |z|>R,$$

R being a sufficiently large positive quantity; so that *the cardinal function is of order (ordre apparent) unity at most.*

Applying a well known theorem ([1]) to the inequality ([2]) we obtain

$$(3) \qquad \overline{\lim_{\nu=\infty}} \sqrt[\nu]{|f^{(\nu)}(0)|} \leqq \pi.$$

Now we can prove the theorem: *Any cardinal function, excluding*

$$A e^{i\beta z}$$

(A being a constant and β being real and $-\pi<\beta<\pi$), has an infinite number of zero-points.

Let us suppose that a cardinal function $f(z)$ has no root. Then $f(z)$ has the form:

$$f(z)=e^{g(z)},$$

where $g(z)$ denotes integral function. If $G(r, \theta)$ be the real part of $g(z)$,

$$|e^{g(z)}|=e^{G(r,\theta)}.$$

It is well known ([2]) that the inequality

$$G(r, \theta)<\pi r$$

(for every value of r such that $r>R$, and for every value of θ) does not hold, unless $g(z)$ has the form

$$g(z)=az+b, \quad (a, b \text{ being constants}).$$

([1]) A. Pringsheim, "Elementare Theorie der ganzen transcendenten Funktionen von endlicher Ordnung," Math. Ann., 58 (1904), p. 266.

([2]) Hadamard's theorem (cited below, p. 186). See, for example, Pringsheim, loc. cit., p. 284.

But if $g(z)=az+b$, the function $e^{g(z)}$ is cardinal when and only when

$$a=i\beta,$$

β being real and $-\pi<\beta<\pi$. Also if $g(z)$ be not of the form $az+b$, there exists a certain point (r_1, θ_1) for which

$$G(r_1, \theta_1)\geqq \pi r_1, \quad r_1>R.$$

Hence for $z_1=r_1 e^{i\theta_1}$ $(r_1>R)$ we have

$$\left|e^{g(z_1)}\right|\geqq e^{\pi r_1}, \quad r_1>R:$$

so that $e^{g(z)}$ is not cardinal.

Next suppose that a cardinal function $f(z)$ has p zero-points $b_1, b_2, \cdots, b_p$. Then $f(z)$ takes the form

$$f(z)=(z-b_1)(z-b_2)\cdots(z-b_p)\,e^{g(z)}.$$

When $g(z)$ reduces to a constant, $|f(z)|$ becomes infinity at $z=\infty$ so long as z is real, which contradicts with the supposition. When $g(z)$ is not constant, we have already shown that $e^{g(z)}$ is not cardinal, unless $g(z)$ has the form

$$g(z)=i\beta z+b,$$

where β is real and $-\pi<\beta<\pi$, and b any constant.

But when $e^{g(z)}$ is not cardinal we can immediately see that $f(z)$ can not be cardinal. On the other hand, when $e^{g(z)}$ is cardinal, $|f(z)|$ becomes infinity at $z=\infty$ so long as z is real. (See the last remark in § 1).

3. In this paragraph I will confine myself to the case where cardinal function has an infinite number of zero-points.

Let $b_1, b_2, b_3, \cdots\cdots$ be the zero-points (other than zero) of $f(z)$ such that

$$|b_1|\leqq|b_2|\leqq|b_3|\leqq\cdots\cdots.$$

Then there exists the well known inequality (1):

$$\overline{\lim}\sqrt[\nu]{\left|b_1 b_2\cdots b_{\nu-1}\cdot\frac{f^{(\nu)}(0)}{\nu!}\right|}>1.$$

Since

(1) J. Hadamard, "Études sur les propriétés des fonctions entières, etc.," Journ de math. (4) 9 (1893), p. 171.

$$\overline{\lim}\sqrt[\nu]{\left|b_1 b_2\cdots b_{\nu-1}\cdot\frac{f^{(\nu)}(0)}{\nu!}\right|}\cdot\overline{\lim}\sqrt[\nu]{\left|\frac{\nu!}{b_1 b_2\cdots b_{\nu-1}}\right|}$$
$$=\overline{\lim}\sqrt[\nu]{|f^{(\nu)}(0)|}$$
$$\leqq\pi,$$

we obtain

$$\overline{\lim}\sqrt[\nu]{\frac{\nu!}{|b_1 b_2\cdots b_{\nu-1}|}}<\pi,$$

or by means of Stirling's formula

$$\nu!=\sqrt{2\pi}\cdot\nu^{\nu+\frac{1}{2}}\cdot e^{-\nu+\frac{\rho_\nu}{12\nu}} \qquad (0<\rho_\nu<1),$$

we infer: *If $b_1, b_2, \cdots\cdots$ be the zero-points $(0<|b_\nu|\leqq|b_{\nu+1}|)$ of a cardinal function, then*

$$(4) \qquad \underline{\lim}\frac{\sqrt[\nu]{|b_1\cdots b_{\nu-1}|}}{\nu}>\frac{1}{e\pi}$$

and especially

$$(4)' \qquad \lim\frac{|b_\nu|}{\nu}>\frac{1}{\pi},$$

provided the limit of the left hand side exists.

On the other hand, (2) leads us to *the inequality:*

$$(4)'' \qquad \underline{\lim}\frac{|b_\nu|}{\nu}\geqq\frac{1}{(1+e)\pi} \quad (1).$$

The class (genre, Höhe) and "Grenzexponent" of the cardinal function are unity at most respectively.

4. Prof. G. Faber proved the following theorems (2):

I. In order that the series

$$(5) \qquad e_0+e_1(z-a_0)+e_2\frac{(z-a_0)(z-a_1)}{a_1-a_0}+\cdots\cdots$$
$$+e_\nu\frac{(z-a_0)(z-a_1)\cdots(z-a_{\nu-1})}{(a_1-a_0)(a_2-a_0)\cdots(a_{\nu-2}-a_0)}+\cdots\cdots,$$

where

$$|a_0|\leqq|a_1|\leqq|a_2|\leqq\cdots\cdots \text{ and } \lim|a_\nu|=\infty,$$

(1) E. Schou, "Sur la théorie des fonctions entières," Compt. rend. Paris, 125 (1897), p. 763; Pringsheim, loc. cit., p. 295.

(2) Faber, loc. cit., p. 52, and p. 55.

be convergent for every value of z, it is necessary that

$$\overline{\lim}\sqrt[\nu]{|e_\nu|} \leqq 1,$$

and sufficient that

$$\overline{\lim}\sqrt[\nu]{|e_\nu|} < 1.$$

II. If a transcendental integral function $f(z)$ satisfy the condition

$$\lim\sqrt[\nu]{|a_1\, a_2 \cdots a_{\nu-1}\, c_\nu|} = 0,$$

then the function may be represented by the series (5) having the limitation

$$\lim\sqrt[\nu]{|e_\nu|} = 0.$$

In the case under consideration we have from (1) that

$$a_{2n} = -n, \quad a_{2n+1} = n+1, \quad (n=0,\ 1,\ 2, \cdots\cdots),$$

$$|e_{2n}| = \frac{n\,[(n-1)\,!]^2}{(2\,n)\,!}\,\delta^{(2\,n)} f_0,$$

$$|e_{2n+1}| = \frac{(n\,!)^2}{(2\,n+1)\,!}\,\delta^{(2\,n+1)} f_{\frac{1}{2}}\,;$$

whence by use of Stirling's formula we get

$$\lim \sqrt[\nu]{\frac{|a_1\, a_2 \cdots a_{\nu-1}|}{\nu\,!}} = \frac{1}{2},$$

$$\overline{\lim}\sqrt[2n]{|e_{2n}|} = \frac{1}{2}\ \overline{\lim}\sqrt[2n]{|\delta^{(2\,n)} f_0|}\,,$$

$$\overline{\lim}\sqrt[2n+1]{|e_{2n+1}|} = \frac{1}{2}\ \overline{\lim}\sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|}\,.$$

Therefore from Theorem I above mentioned we have:

In order that the Gauss formula of interpolation (1) for a one-valued function $f(z)$, such that the values $f(\nu)$ $(\nu=0,\ \pm1,\ \pm2, \cdots)$ are all finite be convergent for every value of z, it is necessary that

$$\overline{\lim}\sqrt[2n]{|\delta^{(2n)} f_0|} \leqq 2, \quad \overline{\lim}\sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} \leqq 2,$$

and sufficient that

$$(6) \qquad \overline{\lim}\sqrt[2n]{|\delta^{(2n)} f_0|} < 2, \quad \overline{\lim}\sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} < 2.$$

Especially consider the case where the condition (6) is satisfied; then

$$(7) \qquad \overline{\lim}\sqrt[2n]{|\delta^{(2n)} f_0|} \leqq \lambda, \quad \overline{\lim}\sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} \leqq \lambda,$$

λ being a positive constant smaller than 2, and hence

$$|e_\nu| \leqq \left(\frac{\lambda}{2}\right)^\nu, \quad \nu > N.$$

If the function $f(z)$ be represented by the Gauss formula (1), we have

$$|c_\nu| < \frac{1}{|a_1\, a_2 \cdots a_{\nu-1}|}\left[|e_\nu| + \frac{\nu}{1\,!}|e_{\nu+1}| + \frac{\nu\,(\nu+1)}{2\,!}|e_{\nu+2}| + \cdots\right]^{(1)}$$

$$< \frac{1}{|a_1\, a_2 \cdots a_{\nu-1}|}\left(\frac{\lambda}{2}\right)^\nu \left[1 + \frac{\nu}{1\,!}\left(\frac{\lambda}{2}\right) + \frac{\nu\,(\nu+1)}{2\,!}\left(\frac{\lambda}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{|a_1\, a_2 \cdots a_{\nu-1}|}\left(\frac{\lambda}{2}\right)^\nu \left(1 - \frac{\lambda}{2}\right)^{-\nu}$$

$$= \frac{1}{|a_1\, a_2 \cdots a_{\nu-1}|}\left(\frac{\lambda}{2-\lambda}\right)^\nu,$$

so that

$$\overline{\lim}\sqrt[\nu]{|f^{(\nu)}(0)|} \leqq \frac{2\,\lambda}{2-\lambda}\,.$$

But since

$$\frac{2\,\lambda}{2-\lambda} \leqq \text{ or } \geqq \pi$$

according as

$$\lambda \leqq \text{ or } \geqq \frac{2\,\pi}{2+\pi},$$

we obtain the result: *A necessary (but not sufficient) condition that a transcendental integral function $f(z)$ satisfying*

$$(7) \qquad \overline{\lim}\sqrt[2n]{|\delta^{(2n)} f_0|} \leqq \lambda, \quad \overline{\lim}\sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} \leqq \lambda, \ (0 < \lambda < 2).$$

may be cardinal, is that

$$(8) \qquad \overline{\lim}\sqrt[\nu]{|f^{(\nu)}(0)|} \leqq \frac{2\,\lambda}{2-\lambda} \text{ or } \leqq \pi$$

according as

$$0 < \lambda < \frac{2\,\pi}{2+\pi} \quad \text{or} \quad \frac{2\,\pi}{2+\lambda} \leqq \lambda < 2.$$

(1) **Faber**, loc. cit., p. 53.

It should be noticed that *if a transcendental integral function* $f(z)$ *satisfy* (7) *and*

$$\pi < \overline{\lim} \sqrt[\nu]{|f^{(\nu)}(0)|} \leqq \frac{2\lambda}{2-\lambda}, \quad \left(\frac{2\pi}{2+\pi} \leqq \lambda < 2\right),$$

the Gauss formula converges, but it represents no cardinal function (1).

5. The method of proof used in the last paragraph leads us to the following: In order that a transcendental integral function $f(z)$ may be represented by the Gauss formula with the limitations

$$(9) \qquad \lim \sqrt[2n]{|\delta^{(2n)} f|} = 0, \quad \lim \sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} = 0,$$

it is necessary that

$$(10) \qquad \lim \sqrt[\nu]{|f^{(\nu)}(0)|} = 0.$$

On the other hand we can state the theorem: *If a transcendental integral function* $f(z)$ *remain finite for* $z=\infty$ *so long as* z *is real, and if*

$$(10) \qquad \lim \sqrt[\nu]{|f^{(\nu)}(0)|} = 0,$$

then the function is cardinal.

Let θ be a real quantity which is arbitrarily small in the absolute value. Then the condition (10) shows us that there exists a positive quantity R such that

$$\left| \sum_{\nu=0}^{\infty} c_\nu z^\nu \right| < e^{|z| \cdot |\sin\theta|} \qquad \text{for all } |z| > R,$$

(1) As an example, let us consider the function

$$f(z) = e^{\frac{3}{2}z} + \sin 2\pi z.$$

Then

$$\overline{\lim} \sqrt[2n]{|\delta^{(2n)} f_0|} = \overline{\lim} \sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} = e^{\frac{3}{4}} - e^{-\frac{3}{4}} = 1.64\ldots,$$

so that we may take

$$\lambda = 1.65 \left(> \frac{2\pi}{2+\pi} = 1.2\cdots \right)$$

and hence

$$\frac{2\lambda}{2-\lambda} = 9.4\cdots.$$

Since

$$\overline{\lim} \sqrt[\nu]{|f^{(\nu)}(0)|} = \lim \sqrt[2n+1]{(2\pi)^{2n+1} + (\tfrac{3}{2})^{2n+1}} = 2\pi,$$

we have

$$\pi < \overline{\lim} \sqrt[\nu]{|f^{(\nu)}(0)|} < \frac{2\lambda}{2-\lambda}.$$

In fact, the Gauss formula represents the function $e^{\frac{3}{2}z}$ (but not $f(z)$ itself), which is not cardinal.

however small $|\theta|$ may be (1). Consequently by the definition $f(z)$ is a cardinal function.

But when a transcendental integral function $f(z)$ satisfies

$$(10) \qquad \lim \sqrt[\nu]{|f^{(\nu)}(0)|} = 0,$$

we have, from Faber's theorem II mentioned in § 4,

$$(9) \qquad \lim \sqrt[2n]{|\delta^{(2n)} f_0|} = 0, \quad \lim \sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} = 0.$$

Thus we have arrived at the theorem:

Let a transcendental integral function $f(z)$ *remain finite for* $z=\infty$ *so long as* z *is real. In order that this function may be represented by the Gauss formula of interpolation such that*

$$\lim \sqrt[2n]{|\delta^{(2n)} f_0|} = 0, \quad \lim \sqrt[2n+1]{|\delta^{(2n+1)} f_{\frac{1}{2}}|} = 0$$

for every value of z, *it is necessary and sufficient that the function should be a cardinal function which satisfies*

$$\lim \sqrt[\nu]{|f^{(\nu)}(0)|} = 0.$$

For such a function the interpolation by the Gauss formula is very effective.

Ikeda near Ôsaka, March 1919.

(1) Pringsheim, loc. cit,, p. 339.

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