With the Author's Compliments.

KINNOSUKE OGURA,

On the Conservative Field of Force.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. **17**, Nos. **1**, **2**.
edited by TSURUICHI HAYASHI, College of Science,
Tôhoku Imperial University, Sendai, Japan,
with the collaboration of Messrs.
M. FUJIWARA, J. ISHIWARA, T. KUBOTA, S. KAKEYA, and T. KOJIMA.

February 1920

On the Conservative Field of Force,

by

KINNOSUKE OGURA, Ôsaka.

I.

1. Consider the motion of a particle which is free to move on a smooth surface, and is acted on by a conservative force. If we take an isothermal system as the parametric curves on the surface, the element of arc length may be written

$$ds^2 = \lambda(u,v)(du^2+dv^2).$$

Let the field be defined by the force-function $U(u,v)$ and the energy constant h, and put

$$\varphi = \sqrt{2(U+h)}.$$

Then the differential equation of the orbits is

$$\frac{d^2v}{du^2} = \left[\frac{\partial}{\partial v}\log(\varphi\sqrt{\lambda}) - \frac{\partial}{\partial u}\log(\varphi\sqrt{\lambda})\cdot\frac{dv}{du}\right]\left[1+\left(\frac{dv}{du}\right)^2\right].$$

For each orbit c which passes through a point P, we can draw the curve g which osculates c at that point and which has constant geodesic curvature (the osculating geodesic circle of c); and it was shown by Prof. J. Lipka that there are two directions, which are orthogonal, through each point in which the osculating geodesic circles hyperosculate their corresponding orbits. These directions are determined by

(1) $$\alpha\, du^2 - \beta\, du\, dv - \alpha\, dv^2 = 0 \ (^1),$$

where

(2) $$\alpha = 4\lambda^2\left[\frac{\partial}{\partial u}\log(\varphi\sqrt{\lambda})\cdot\frac{\partial}{\partial v}\log(\varphi\sqrt{\lambda}) - \frac{\partial^2}{\partial u\,\partial v}\log(\varphi\sqrt{\lambda})\right] + 2\lambda\frac{\partial^2\lambda}{\partial u\,\partial v} - 3\frac{\partial\lambda}{\partial u}\frac{\partial\lambda}{\partial v},$$

(1) Lipka, "Geometric characterization of isogonal trajectories on a surface," Annals of Math., Ser. II, Vol. 15 (1913), p. 71. For the case of the plane, the corresponding result was obtained by Prof. E. Kasner.

$$\beta=4\lambda^2\Big\{\frac{\partial^2}{\partial v^2}\log(\varphi\sqrt{\lambda})-\frac{\partial^2}{\partial u^2}\log(\varphi\sqrt{\lambda})$$
$$+\Big[\frac{\partial}{\partial u}\log(\varphi\sqrt{\lambda})\Big]^2-\Big[\frac{\partial}{\partial v}\log(\varphi\sqrt{\lambda})\Big]^2\Big\}$$
$$+2\lambda\Big(\frac{\partial^2\lambda}{\partial u^2}-\frac{\partial^2\lambda}{\partial v^2}\Big)-3\Big[\Big(\frac{\partial\lambda}{\partial u}\Big)^2-\Big(\frac{\partial\lambda}{\partial v}\Big)^2\Big].$$

For the sake of brevity, the curves defined by (1) will be called *the Kasner curves.*

Now I will prove the theorem: *The Kasner curves coincide with the equipotential lines and the lines of force when and only when the lines of force are geodesics, that is, the lines of force may be orbits in the field.*

The equipotential lines and the lines of force are given by

$$\Big(\frac{\partial\varphi}{\partial u}du+\frac{\partial\varphi}{\partial v}dv\Big)\Big(\frac{\partial\varphi}{\partial v}du-\frac{\partial\varphi}{\partial u}dv\Big)=0,$$

i. e.

$$(3)\qquad \frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}du^2-\Big[\Big(\frac{\partial\varphi}{\partial u}\Big)^2-\Big(\frac{\partial\varphi}{\partial v}\Big)^2\Big]du\,dv-\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}dv^2=0.$$

In order that (1) and (3) coincide, it is necessary and sufficient that

$$\frac{\alpha}{\dfrac{\partial\varphi}{\partial u}\dfrac{\partial\varphi}{\partial v}}=\frac{\beta}{\Big(\dfrac{\partial\varphi}{\partial u}\Big)^2-\Big(\dfrac{\partial\varphi}{\partial v}\Big)^2}.$$

Since (2) may be written

$$\alpha=\frac{4\lambda^2}{\varphi^2}\Big(2\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}-\varphi\frac{\partial^2\varphi}{\partial u\,\partial v}\Big)+\frac{2\lambda}{\varphi}\Big(\frac{\partial\lambda}{\partial u}\frac{\partial\varphi}{\partial v}+\frac{\partial\lambda}{\partial v}\frac{\partial\varphi}{\partial u}\Big),$$

$$\beta=\frac{4\lambda^2}{\varphi^2}\Big\{2\Big[\Big(\frac{\partial\varphi}{\partial u}\Big)^2-\Big(\frac{\partial\varphi}{\partial v}\Big)^2\Big]-\varphi\Big(\frac{\partial^2\varphi}{\partial v^2}-\frac{\partial^2\varphi}{\partial v^2}\Big)\Big\}$$
$$+\frac{4\lambda}{\varphi}\Big(\frac{\partial\lambda}{\partial u}\frac{\partial\varphi}{\partial u}-\frac{\partial\lambda}{\partial v}\frac{\partial\varphi}{\partial v}\Big),$$

the above condition becomes

$$\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\Big(\frac{\partial^2\varphi}{\partial u^2}-\frac{\partial^2\varphi}{\partial v^2}\Big)-\Big[\Big(\frac{\partial\varphi}{\partial u}\Big)^2-\Big(\frac{\partial\varphi}{\partial v}\Big)^2\Big]\frac{\partial^2\varphi}{\partial u\,\partial v}$$
$$=\frac{1}{2\lambda}\Big(\frac{\partial\lambda}{\partial u}\frac{\partial\varphi}{\partial v}-\frac{\partial\lambda}{\partial v}\frac{\partial\varphi}{\partial u}\Big)\Big[\Big(\frac{\partial\varphi}{\partial u}\Big)^2+\Big(\frac{\partial\varphi}{\partial v}\Big)^2\Big],$$

or

$$\frac{\partial}{\partial u}\left[\frac{\sqrt{\lambda}\,\dfrac{\partial\varphi}{\partial v}}{\sqrt{\Big(\dfrac{\partial\varphi}{\partial u}\Big)^2+\Big(\dfrac{\partial\varphi}{\partial v}\Big)^2}}\right]-\frac{\partial}{\partial v}\left[\frac{\sqrt{\lambda}\,\dfrac{\partial\varphi}{\partial u}}{\sqrt{\Big(\dfrac{\partial\varphi}{\partial u}\Big)^2+\Big(\dfrac{\partial\varphi}{\partial v}\Big)^2}}\right]=0.$$

This is nothing but the condition that the lines of force

$$\frac{\partial\varphi}{\partial v}du-\frac{\partial\varphi}{\partial u}dv=0$$

should be geodesics.

2. Now a necessary and sufficient condition that there exists a conservative force under which each family of an orthogonal system of curves can be orbits is that these families form an isothermal system; and if u, v be isothermal parameters and $u=$const., $v=$const. be orbits the linear element is of the form

$$ds^2=\lambda(du^2+dv^2),$$

and then we have

$$\varphi=\frac{k}{\sqrt{\lambda}},$$

k being an arbitrary constant; so that the Kasner curves take the form

$$\Big(2\lambda\frac{\partial^2\lambda}{\partial u\,\partial v}-3\frac{\partial\lambda}{\partial u}\frac{\partial\lambda}{\partial v}\Big)(du^2-dv^2)$$
$$-\Big\{2\lambda\Big(\frac{\partial^2\lambda}{\partial u^2}-\frac{\partial^2\lambda}{\partial v^2}\Big)-3\Big[\Big(\frac{\partial\lambda}{\partial u}\Big)^2-\Big(\frac{\partial\lambda}{\partial v}\Big)^2\Big]\Big\}du\,dv=0.$$

Particularly for the isothermal surface which is referred to the lines of curvature, the differential equation

$$\frac{\partial^2\vartheta}{\partial u\,\partial v}=\frac{1}{2}\frac{\partial\log\lambda}{\partial v}\frac{\partial\vartheta}{\partial u}+\frac{1}{2}\frac{\partial\log\lambda}{\partial u}\frac{\partial\vartheta}{\partial v}$$

has equal Laplace-Darboux invariants σ, τ, such that

$$4\lambda^2\sigma=4\lambda^2\tau$$
$$=2\lambda\frac{\partial^2\lambda}{\partial u\,\partial v}-3\frac{\partial\lambda}{\partial u}\frac{\partial\lambda}{\partial v}.$$

Hence if and only if

$$\sigma=\tau=0,$$

the Kasner curves coincide with the lines of curvature.

Consequently, *there exists a conservative field of force in which the* *lines of curvature are orbits and the Kasner curves simultaneously, when and only when the surface belongs to Peterson's P-surface having conical lines of curvature, that is, the surface of revolution, the cone, the cylinder, or their transforms by inversion*(1).

II.

3. A particle is free to move on a surface under a conservative force. Now I will establish *a connection among any 2 ∞^1 orbits, the total curvature of the surface and the force function.*

Let $u=$const., $v=$const. be any two sets of the orbits under the force derivable from a force-function $U(u, v)$, the total energy being h; and let us put

$$\varphi=\sqrt{2(U+h)}.$$

Then we have (2)

$$\frac{\partial \log \varphi}{\partial u}=\frac{F}{E}\begin{Bmatrix}1 & 1\\ & 2\end{Bmatrix}+\frac{E}{G}\begin{Bmatrix}2 & 2\\ & 1\end{Bmatrix},$$

$$\frac{\partial \log \varphi}{\partial v}=\frac{G}{E}\begin{Bmatrix}1 & 1\\ & 2\end{Bmatrix}+\frac{F}{G}\begin{Bmatrix}2 & 2\\ & 1\end{Bmatrix},$$

where E, F, G are the fundamental quantities of the first order of the surface and $\begin{Bmatrix}1 & 1\\ & 2\end{Bmatrix}$, $\begin{Bmatrix}2 & 2\\ & 1\end{Bmatrix}$ the Christoffel symbols as usual.

By use of Liouville's formula for the total curvature K, we obtain

$$K=\frac{1}{\sqrt{EG-F^2}}\frac{\partial^2\omega}{\partial u\,\partial v}+\frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\frac{\sqrt{EG-F^2}}{G}\begin{Bmatrix}2 & 2\\ & 1\end{Bmatrix}\right]+\frac{\partial}{\partial v}\left[\frac{\sqrt{EG-F^2}}{E}\begin{Bmatrix}1 & 1\\ & 2\end{Bmatrix}\right]\right\}$$

$$=\frac{1}{\sqrt{EG-F^2}}\frac{\partial^2\omega}{\partial u\,\partial v}+\frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\frac{G\dfrac{\partial \log\varphi}{\partial u}-F\dfrac{\partial \log\varphi}{\partial v}}{\sqrt{EG-F^2}}+\frac{\partial}{\partial v}\frac{E\dfrac{\partial \log\varphi}{\partial v}-F\dfrac{\partial \log\varphi}{\partial u}}{\sqrt{EG-F^2}}\right\},$$

(1) See K. Ogura, "On the theory of Stäckel curvature," Tôhoku Math. Journal, Vol. 16 (1919), p. 270 § 7.

(2) Ogura, Trajectories in the conservative field of force," Tôhoku Math. Journal, Vol. 7 (1915), p. 179

ω standing for the angle between the parametric curves.

Consequently we find *the relation*:

$$\Delta_2(\log\varphi)=K-\frac{1}{\sqrt{EG-F^2}}\frac{\partial^2\omega}{\partial u\,\partial v},$$

the left hand being the differential parameter of the second order of $\log\varphi$.

When and only when the angle ω is of the form $\omega=f(u)+g(v)$, the above formula becomes

$$\Delta_2(\log\varphi)=K.$$

It follows from this the following result immediately: *Let two sets of ∞^1 curves on a surface, which form an isogonal system, be orbits in a certain conservative field of force. In order that $\log\varphi$ be the isothermal parameter of the isothermic system (consisting of the equipotential lines and the lines of force), it is necessary and sufficient that the surface should be developable.*

4. Lastly I will add some remarks concerning the applicability of two surfaces. Let $u=$const., $v=$const. and $\bar{u}=$const., $\bar{v}=$const. be orbits on two surfaces S and $\bar{S}$ respectively corresponding to the functions $\varphi(u, v)$ and $\bar{\varphi}(\bar{u}, \bar{v})$; and let

(A) $$K(u, v)=\bar{K}(\bar{u}, \bar{v})$$

and

(B) $$\varphi(u, v)=c\,\bar{\varphi}(\bar{u}, \bar{v}),$$

c being a constant, for the representation

$$u=\bar{u},\quad v=\bar{v}.$$

These two conditions $(A), (B)$ are *necessary*, but *not sufficient* to secure that these two surfaces should be isometric in this representation.

For example, if we take the two spiral surfaces having the linear elements (1)

$$ds^2=e^{2v}[du^2+a^2(1+u^2)^2dv^2],\quad d\bar{s}^2=e^{2\bar{v}}[d\bar{u}^2+\bar{a}^2(1+\bar{u}^2)^2d\bar{v}^2],\quad (a^2\neq\bar{a}^2)$$

respectively, we have

(1) K. Ogura, "Two surfaces having equal measures of curvature but not deformable into each other," Proc. Tôkyô Math.-Phys. Soc., Ser. II, Vol. 14 (1908), p. 338; T. Hayashi, "On the applicability of two surfaces having the same total curvature at corresponding points," Tôhoku Math. Journal, Vol. 5 (1914), p. 197. It should be noted that $u=$const., $v=$const. form an isothermal system, but u is no isothermal parameter (compare with the last theorem of this note).

$$K = -\frac{2}{(1+u^2)e^{2v}}, \quad \bar{K} = -\frac{2}{(1+\bar{u}^2)e^{2\bar{v}}};$$

$$\varphi = \frac{k\,e^{2v}}{(1+u^2)^2}, \quad \bar{\varphi} = \frac{\bar{k}\,e^{2\bar{v}}}{(1+\bar{u}^2)^2},$$

k, $\bar{k}$ being constants. But these surfaces are not isometric in the representation $u=\bar{u}$, $v=\bar{v}$ (1).

On the contrary, we can state:

If u, v and $\bar{u}, \bar{v}$ be isothermal parameters (2) of two surfaces S and $\bar{S}$ respectively, and if

(B) $$\varphi(u, v) = c\,\bar{\varphi}(\bar{u}, \bar{v}),$$

c being a constant, for the representation $u=\bar{u}$, $v=\bar{v}$, these two surfaces are isometric in the representation combined with a similitude.

If u, v and $\bar{u}, \bar{v}$ be isothermal parameters of two surfaces S and $\bar{S}$ respectively, and if

(A) $$K(u, v) = \bar{K}(\bar{u}, \bar{v})$$

and

(B) $$\varphi(u, v) = c\,\bar{\varphi}(\bar{u}, \bar{v}),$$

c being a constant, for the representation $u=\bar{u}$, $v=\bar{v}$, these two surfaces are isometric in this representation (3).

Ikeda near Ôsaka, February 1919.

---

(1) On the other hand, if we take

$$ds^2 = \left(1 + \frac{1}{u^2}\right) du^2 + u^2\, dv^2, \qquad d\bar{s}^2 = d\bar{u}^2 + (1+\bar{u}^2)\, d\bar{v}^2,$$

then

$$K = -\frac{1}{(1+u^2)^2}, \quad \bar{K} = -\frac{1}{(1+\bar{u}^2)^2}; \quad \varphi = \frac{k}{u}, \quad \bar{\varphi} = \frac{\bar{k}}{\sqrt{1+\bar{u}^2}},$$

in which $\frac{\bar{\varphi}(\bar{u}, \bar{v})}{\varphi(u, v)}$ is not constant for $u=\bar{u}$, $v=\bar{v}$. See Stäckel and Wangerin, "Zur Theorie des Gauss'schen Krümmungsmasses," Leipziger Berichte (1893), p. 163, p. 170; Scheffers, Einführung in die Theorie der Flächen (1902), p. 276.

(2) Then there exists a function φ for which $u=$const., $v=$const. may be orbits.

(3) If the condition (B) be dropped off, this result does not hold in general. To show this it will be sufficient to consider the two surfaces having the linear elements.

$$ds^2 = du^2 + dv^2, \qquad d\bar{s}^2 = e^{\bar{u}}(d\bar{u}^2 + d\bar{v}^2),$$

for which

$$K = \bar{K} = 0.$$

THE TÔHOKU MATHEMATICAL JOURNAL.

The Editor of the Journal, T. HAYASHI, College of Science, Tôhoku Imperial University, Sendai, Japan, accepts contributions from any person.

Contributions should be written legibly in English, French, German, Italian or Japanese and diagrams should be given in separate slips and in proper sizes. The author has the sole and entire scientific responsibility for his work. Every author is entitled to receive gratis 30 separate copies of his memoir; and for more copies to pay actual expenses.

All communications intended for the Journal should be addressed to the Editor.

Subscriptions to the Journal and orders for back numbers should be addressed directly to the Editor T. HAYASHI, or to the bookseller Y. ÔKURA, No. 19, Tôri-itchôme, Nihonbashi, Tôkyô, Japan, or MARUZEN COMPANY, LTD., Branch Office, Kokubunchô, Sendai, Japan.

Price per volume (consisting of four numbers) payable in advance: 3 yen = 6 shillings = 6 Mark = 7.50 francs = 1.50 dollars. Postage inclusive.