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On the Theory of Stäckel Curvature.

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On the Theory of Stäckel Curvature (1),

by

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In his interesting memoir "Die begleitenden Grenzkugeln krummer Flächen," Prof. P. Stäckel (2) has endeavoured to build up a new theory of curvature. When a surface is referred to the lines of curvature, the radius of "begleitende Kugel" is given by

$$\frac{\sigma E\, du^2 + \tau\, G\, dv^2}{\sigma L\, du^2 + \tau\, N\, dv^2},$$

where σ, τ denote the Laplace-Darboux invariants of the differential equation

$$\frac{\partial^2 \vartheta}{\partial u \partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial \vartheta}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial \vartheta}{\partial v}.$$

The main object of this paper is to find some properties of *the Stäckel curvature,* i. e.

$$\frac{1}{S} \equiv \frac{\sigma L\, du^2 + \tau\, N\, dv^2}{\sigma E\, du^2 + \tau\, G\, dv^2},$$

and to determine some remarkable curves appearing in this theory from the standpoint of Laplace transformations, where we meet the modern theory of congruence established by Prof. E. J. Wilczynski.

This paper contains, moreover, some incidental remarks concerning a geometrical characterization of an isothermal system and the vanishing directions of the "Parameterkrümmung" due to Prof. A Voss.

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(1) Read before the Physico-Mathematical Society of Japan, April, 1919.

(2) Sitzungsberichte d. Heidelberger Akademie d. Wiss., Jahrg. 1915, 3. Abh. pp. 1-34.

PART I.

Elementary Properties of the Stäckel Curvature.

The inverse curvature.

1. First of all I will introduce the notion of the inverse curvature. Let

$$f_1 \equiv E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$
$$f_2 \equiv L\,du^2 + 2\,M\,du\,dv + N\,dv^2$$

be the first and second fundamental forms of a surface

$$x = x\,(u, v), \qquad y = y\,(u, v), \qquad z = z\,(u, v),$$

and let $J(f_1, f_2)$ be the Jacobian of f_1, f_2. Then

$$f_1 = 0, \quad f_2 = 0, \quad J(f_1, f_2) = 0, \quad J(f_1, J(f_1, f_2)) = 0, \quad J(f_2, J(f_1, f_2)) = 0$$

are the equations to the minimal lines, the asymptotic lines, the lines of curvature, the lines of torsion and the characteristic lines respectively. I have already proved that the minimal lines, the lines of curvature and the lines of torsion form a cycle in the sense that the directions of any one of these families are the double rays of the involution determined by the directions of the other two; similarly the asymptotic lines, the lines of curvature and the characteristic lines form another cycle; and these five families form a closed system (1).

Now the normal curvature $\frac{1}{R}$ is given by

$$\frac{1}{R} = \frac{f_2}{f_1},$$

and the centre of normal curvature $[R]$ has the coordinates

$$x + RX, \qquad y + RY, \qquad z + RZ,$$

where X, Y, Z are the direction-cosines of the normal to the surface at (x, y, z).

(1) K. Ogura, "Some theorems concerning binary quadratic forms and their applications to the differential geometry," Science Reports of Tôhoku Imp. University, 5 (1916), p. 95; T. Hayashi, "On the usual parametric curves on a surface," Ibid., p 63; Ogura, "On the theory of representation of surfaces," Tôhoku Math. Journal, 12 (1917), p. 237.

In a similar way, an analogous quantity defined by

$$\frac{1}{I} = \frac{J(f_2, J(f_1, f_2))}{J(f_1, J(f_1, f_2))}$$

will be called the *inverse curvature* and the point

$$x + IX, \qquad y + IY, \qquad z + IZ$$

the *centre of inverse curvature* $[I]$.

If the surface be referred to the lines of curvature, we have

$$(1) \qquad \frac{1}{R} = \frac{L\,du^2 + N\,dv^2}{E\,du^2 + G\,dv^2},$$

$$(2) \qquad \frac{1}{I} = \frac{L\,du^2 - N\,dv^2}{E\,du^2 - G\,dv^2},$$

from which we obtain the relation

$$\left(R - \frac{R_1 + R_2}{2}\right)\left(I - \frac{R_1 + R_2}{2}\right) = \left(\frac{R_1 - R_2}{2}\right)^2,$$

R_1, R_2 being the principal radii of normal curvature.

Hence, *when the direction $dv : du$ varies about any given point (u, v), the centres of normal and inverse curvatures $[R]$, $[I]$ form the involution on the normal to the surface at the given point, the double points being the centres of principal curvature $[R_1]$, $[R_2]$; when the direction is given, the centre of inverse curvature $[I]$ is the inverse of the centre of normal curvature $[R]$ with respect to the sphere, the segment $\overline{[R_1]\,[R_2]}$ being a diameter.*

2. Moreover *there exists a kind of duality between the normal and inverse curvatures.* The following are some typical cases:

The normal curvature $\frac{1}{R}$.	*The inverse curvature* $\frac{1}{I}$.
$\frac{1}{R}$ becomes zero for the asymptotic lines and infinity for the minimal lines.	$\frac{1}{I}$ becomes zero for the characteristic lines and infinity for the lines of torsion.
It takes the extreme values $\frac{1}{R_1}$, $\frac{1}{R_2}$ for the lines of curvature.	It takes the extreme values $\frac{1}{I_1}\left(=\frac{1}{R_1}\right)$, $\frac{1}{I_2}\left(=\frac{1}{R_2}\right)$ for the lines of curvature.
The lines of torsion (the characteristic lines) are the curves for which R is the harmonic mean (the arithmetic mean) of R_1 and R_2.	The minimal lines(the asymptotic lines) are the curves for which I is the harmonic mean (the arithmetic mean) of I_1 and I_2.
In order that the two pairs of directions, on two surfaces Σ and $\overline{\Sigma}$, for which $R^2 = \overline{R}^2$ may be harmonic in the conformal repre-	In order that the two pairs of directions, on two surfaces Σ and $\overline{\Sigma}$, for which $I^2 = \overline{I}^2$ may be harmonic in the representation

sentation (i. e. the representation preserving the minimal lines) [or in the representation preserving the asymptotic lines], it is necessary and sufficient that the total curvatures K, $\overline{K}$ should be equal at corresponding points. The double rays of the involution determined by the two pairs of directions above mentioned form the common conjugate system [or the common orthogonal system].

preserving the lines of torsion [or in the representation preserving the characteristic lines], it is necessary and sufficient that the total curvatures K, $\overline{K}$ should be equal at corresponding points. The double rays of the involution determined by the two pairs of directions above mentioned form the common inverse-conjugate system [or the common inverse-orthogonal system](1).

The Stäckel curvature.

3. *When the lines of curvature are parametric* (2), the *Stäckel curvature* S^{-1} (the reciprocal of the radius of "begleitende Kugel" due to Stäckel) is given by

$$(3) \qquad \frac{1}{S}=\frac{\sigma\, L\, du^2+\tau\, N dv^2}{\sigma\, E\, du^2+\tau\, G\, dv^2},$$

where

$$\sigma=\frac{\partial^2 \log \sqrt{E}}{\partial u\, \partial v}-\frac{\partial \log \sqrt{G}}{\partial u}\cdot\frac{\partial \log \sqrt{E}}{\partial v},$$

$$\tau=\frac{\partial^2 \log \sqrt{G}}{\partial u\, \partial v}-\frac{\partial \log \sqrt{E}}{\partial v}\cdot\frac{\partial \log \sqrt{G}}{\partial u};$$

and the *centre of Stäckel curvature* $[S]$ (the centre of "begleitende Kugel") has the coordinates $x+SX$, $y+SY$, $z+SZ$.

Eliminating $dv:du$ from (1) and (2), we have

$$(\tau-\sigma)\, LN\cdot SR+(\sigma\, GL-\tau\, EN)\, S$$
$$+(\sigma\, EN-\tau\, GL)\, R+(\tau-\sigma)\, EG=0,$$

and after a short calculation we get

$$\frac{\overline{[S][R_1]}}{\overline{[R][R_1]}}:\frac{\overline{[S][R_2]}}{\overline{[R][R_2]}}=\frac{\tau}{\sigma}.$$

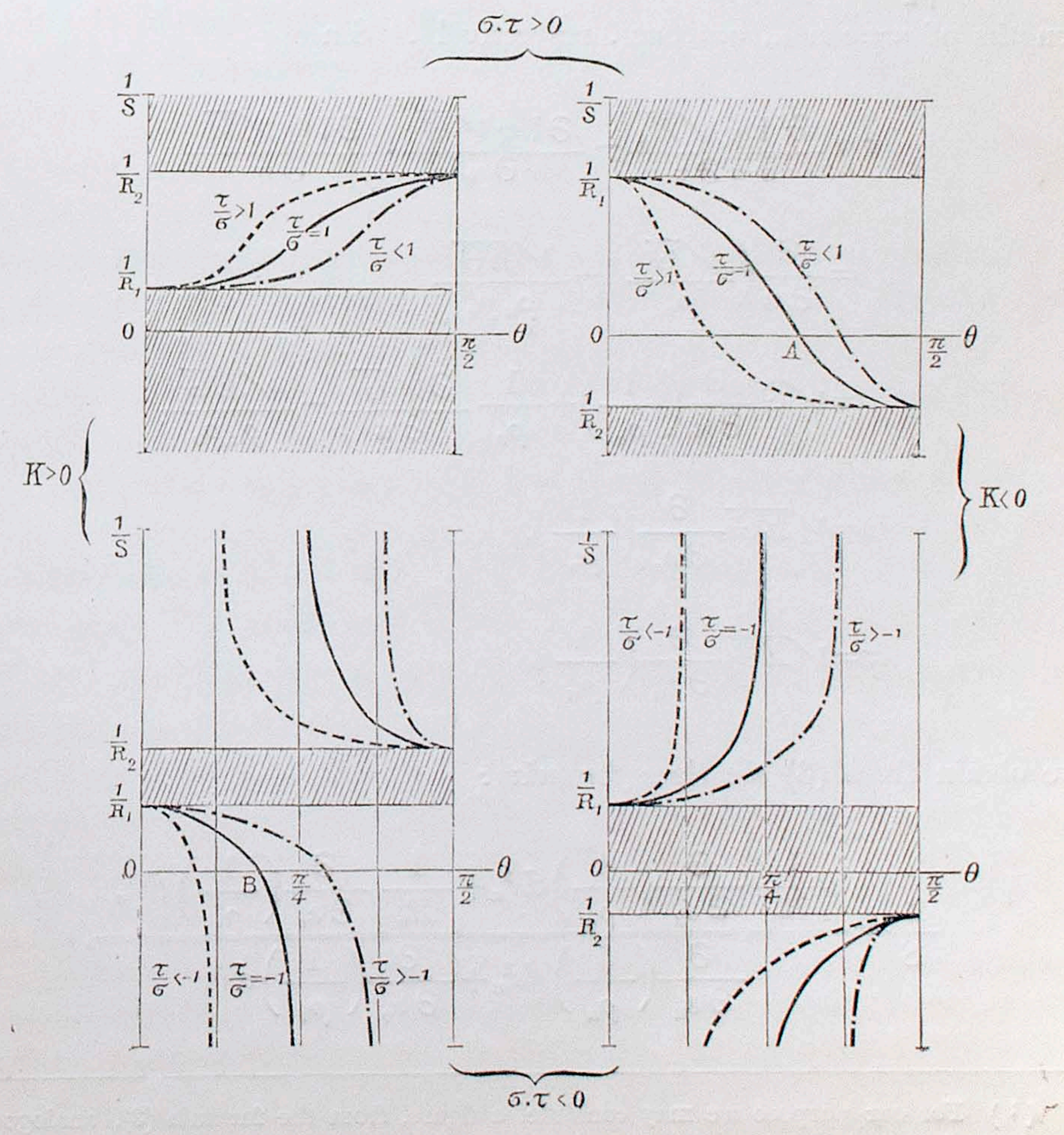
Therefore, *when the direction $dv:du$ varies about any given point on a surface, the centres of normal and Stäckel curvatures form the projective range, the self-corresponding points being the centres of principal curvature and the double ratio $([S]\,[R]\,[R_1]\,[R_2])$ being equal to $\dfrac{\tau}{\sigma}$.*

(1) Ogura, loc. cit., (the second paper).

(2) The Stäckel curvature has its essential meaning when and only when the surface is referred to the lines of curvature. For the surface in which $\sigma=\tau=0$ identically, see § 7 below.

Similarly, *when the direction $dv:du$ varies about any given point on surface, the centres of inverse and Stäckel curvatures form the projective range, the self-corresponding points being the centres of principal curvature and the double ratio $([S]\,[I]\,[R_1]\,[R_2])$ being equal to $-\dfrac{\tau}{\sigma}$.*

By means of these theorems we can easily see the change of the Stäckel curvature when the direction $dv:du$ varies about a given point on a surface. If the independent variable θ be taken as the angle between the directions whose radii of Stäckel curvature are S and R_1, we obtain the following graphs of $1:S$.



Throughout these figures R_1 is assumed to be greater than R_2; and the abscissae of the points A and B are

$$\operatorname{arc\,tg}\sqrt{-\frac{R_2}{R_1}} \quad \text{(the asymptotic direction)}$$

and

$$\operatorname{arc\,tg}\sqrt{\frac{R_2}{R_1}} \quad \text{(the characteristic direction)}$$

respectively (1). (For other noteworthy directions, see §§ 12–14).

4. Let the lines of curvature be taken as the parametric curves and let $\frac{1}{\rho_u}$, $\frac{1}{\rho_v}$ and s_u, s_v be the geodesic curvatures and the arc-lengths of u=const., v=const. respectively. Since

$$\sigma=\frac{\partial^2 \log\sqrt{E}}{\partial u\,\partial v}-\frac{\partial \log\sqrt{G}}{\partial u}\cdot\frac{\partial \log\sqrt{E}}{\partial v}$$

$$=\frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\rho_v}\right)-\frac{\sqrt{EG}}{\rho_u\,\rho_v}$$

$$=\sqrt{G}\frac{\partial}{\partial u}\left(\frac{1}{\rho_v}\right)+\frac{1}{\rho_v}\frac{\partial\sqrt{G}}{\partial u}-\frac{\sqrt{EG}}{\rho_u\,\rho_v}$$

$$=\sqrt{EG}\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right),$$

$$\tau=\sqrt{EG}\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right),$$

we obtain from (3) *the three formulae:*

$$\frac{1}{S}=\frac{\frac{1}{R_u}\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)ds^2_u+\frac{1}{R_v}\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)ds^2_v}{\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)ds^2_u+\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)ds^2_v},$$

(1) For our purpose we may construct a figure, from the formula (3), analogous to Dupin's indicatrix.

$$(4)\qquad T\left(\frac{1}{S}-\frac{1}{R}\right)=\frac{\left[\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)-\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)\right]ds_u\,ds_v}{\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)ds^2_u+\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)ds^2_v},$$

$$(5)\qquad T^2\left(\frac{2}{R}-H\right)\left(\frac{1}{S}-\frac{1}{I}\right)=\frac{\left[\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)+\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)\right](ds^2_u+ds^2_v)}{\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)ds^2_u+\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)ds^2_v},$$

where $\frac{1}{T}$ and H denote the geodesic torsion and mean curvature respectively.

Isothermic and inverse-isothermic surfaces.

5. It follows from (4) that S is equal to R for all directions at any point of the surface, when and only when the quantity (1)

$$\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)-\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)$$

vanishes identically, which is nothing but the well known condition that the parametric curves should form an isothermal system. Hence we have the theorem due to Prof. Stäckel (2):

The Stäckel curvature is identically equal to the normal curvature when and only when the surface is isothermic (3).

I take this opportunity to give *a purely geometric characterization of an isothermal system.* Recently Dr. G. M. Green (4) gave two such characterizations; it should be noted that, however, one of them breaks down when the isothermal system consists of the lines of curvature, whereas the other can be applicable only when the surface is not developable.

Let u=const., v=const. be an orthogonal system and let w=const. be the isogonals which cut u=const. at a constant angle ϕ. Then

(1) For this quantity, see G. Ricci, Lezioni sulla teoria delle superficie (1898), p. 214; R. v. Lilienthal, Vorlesungen über Differentialgeometrie, 1 (1908), pp. 167–169.

(2) Stäckel, loc. cit., p. 31.

(3) Excluding the plane, the sphere, and the surface in which $\sigma=\tau=0$ identically (§ 7).

(4) Green, "Some geometric characterizations of isothermal nets on a curved surface," Trans. Amer. Math. Soc., 18 (1917), p. 480.

$$\frac{1}{\rho_w}=\frac{\cos\phi}{\rho_u}-\frac{\sin\phi}{\rho_v}$$

and

$$\frac{\partial}{\partial s_w}\left(\frac{1}{\rho_w}\right)=\cos\phi\cdot\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_w}\right)+\sin\phi\cdot\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_w}\right)$$
$$=\cos^2\phi\cdot\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)+\cos\phi\sin\phi\left[\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_u}\right)-\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_v}\right)\right]-\sin^2\phi\cdot\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right).$$

If ϕ_1 and ϕ_2 be the angles which satisfy the equation

$$\text{(6)}\quad \cos^2\phi\cdot\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)+\cos\phi\sin\phi\left[\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_u}\right)-\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_v}\right)\right]-\sin^2\phi\cdot\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)=0,$$

we have

$$\phi_2-\phi_1=\pm\frac{\pi}{2}$$

when and only when

$$\text{(7)}\quad \frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)-\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)=0\ (^1).$$

In the above we have assumed that (6) is not an identity. If this be the case,

$$\frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)=0,\qquad \frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)=0$$

identically, so that the orthogonal system consists of curves of constant geodesic curvature; hence by a well known theorem this system must be isothermic.

(1) An analogous condition for a system of plane curves defined by the differential equation

$$\frac{dy}{dx}=\operatorname{tg}\lambda(x,y)$$

was alrea y obtained by Prof. E. Kasner. See Kasner, "The Riccati differential equations which represent isothermal systems," Bull. Amer. Math. Soc., 10 (1905), p. 342; H. W. Reddick, "Systems of plane curves whose intrinsic equations are analogous to the intrinsic equation of an isothermal system," Annals of math., II, 14 (1913), p. 179.

Consequently, for each isogonals c of a family γ of ∞^1 curves which passes through a point P, we can draw the curve g which osculate c at that point and which has constant geodesic curvature (the osculating geodesic circle of c); and *there are either*

(i) *two directions which are orthogonal,*

or (ii) ∞^1 *directions*

through each point in which the osculating geodesic circles hyperosculate their corresponding isogonals, when and only when the family γ *and its orthogonal trajectories form an isothermal system.*

6. Again, if ϕ_1 and ϕ_2 be the angles which satisfy the equation (6), we have

$$\phi_1+\phi_2=\pm\frac{\pi}{2}$$

(i. e. the two directions corresponding to ϕ_1, ϕ_2 are inverse-orthogonal with respect to u=const.), when and only when

$$\text{(8)}\quad \frac{\partial}{\partial s_u}\left(\frac{1}{\rho_u}\right)+\frac{\partial}{\partial s_v}\left(\frac{1}{\rho_v}\right)=0.$$

This gives *a geometric characterization of the condition* (8) (1).

When an orthogonal system u=const., v=const. satisfies the condition (8), it will be called *an inverse-isothermic system;* and especially a surface whose lines of curvature are inverse-isothermic *an inverse-isothermic surface.*

Using this definition and recalling the formula (5) we have:

The Stäckel curvature is identically equal to the inverse curvature when and only when the surface is inverse-isothermic (2).

If the lines of curvature be taken as the parametric curves, the Codazzi equations may be written

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial\log\sqrt{E}}{\partial v}=\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right),$$
$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial\log\sqrt{G}}{\partial u}=\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right);$$

whence

(1) Compare with Reddick, loc. cit.

(2) Excluding the plane, the sphere, and the surface in which $\sigma=\tau=0$ identically (§ 7).

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2\sigma=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{R_2}\right)$$
$$+\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right)+\frac{\partial}{\partial u}\left(\frac{1}{R_2}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right),$$
$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2\tau=\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{R_1}\right)$$
$$+\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right)+\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_1}\right).$$

Therefore we have the condition for the inverse-isothermic surface (1):

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_1}\right)$$
$$+\frac{\partial}{\partial u}\left(\frac{1}{R_2}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right)+2\frac{\partial}{\partial u}\left(\frac{1}{R_1}\right)\frac{\partial}{\partial v}\left(\frac{1}{R_2}\right).$$

7. Lastly consider the surface such that σ and τ vanish identically when the lines of curvature are parametric. Prof. Stäckel (2) remarked that the surface under consideration is *Peterson's P-surface* whose generating conical curves are the lines of curvature; and moreover K. M. Peterson (3) dealt with such a conical line of curvature at length.

On the other hand we have seen in § 5 that each of the lines of curvature has constant geodesic curvature for and only for the surface under consideration. Such a surface has been treated by Bonnet, Ribaucour and Darboux (4); and especially by Darboux's theorem we can infer:

When a surface is referred to the lines of curvature, the Stäckel curvature becomes indeterminate for the surface of revolution, the cone, the cylinder, and their transforms by inversion and for only these; and in such a case the lines of curvature are conical curves.

Transformations.

8. I will begin with *the general representation of two surfaces Σ and*

(1) A modified form of the analogous condition for the isothermic surface is found in Darboux, Théorie des surfaces, 2 (1889), p. 251.
(2) Stäckel, loc. cit.
(3) Peterson, "Sur les courbes tracées sur les surfaces," Annales de Toulouse, II, 7 (1905), pp. 56–65.
(4) Darboux, Théorie des surfaces, 3 (1894), pp. 121–122.

$\bar\Sigma$ in which the lines of curvature are preserved. If the lines of curvature be taken as the parametric curves, the representation will take the form

$$u=\bar u,\qquad v=\bar v;$$

and therefore

$$\frac{1}{S}=\frac{\sigma L\,du^2+\tau N\,dv^2}{\sigma E\,du^2+\tau G\,dv^2},\qquad \frac{1}{\bar S}=\frac{\bar\sigma \bar L\,du^2+\bar\tau \bar N\,dv^2}{\bar\sigma \bar E\,du^2+\bar\tau \bar G\,dv^2}.$$

Eliminating $dv:du$ from these, we get the bilinear relation between S and $\bar S$:

$$S\bar S\begin{vmatrix}\sigma L & \tau N\\ \bar\sigma\bar L & \bar\tau\bar N\end{vmatrix}-S\begin{vmatrix}\sigma L & \tau N\\ \bar\sigma\bar E & \bar\tau\bar G\end{vmatrix}-\bar S\begin{vmatrix}\sigma E & \tau G\\ \bar\sigma\bar L & \bar\tau\bar N\end{vmatrix}$$
$$+\begin{vmatrix}\sigma E & \tau G\\ \bar\sigma\bar E & \bar\tau\bar G\end{vmatrix}=0.$$

Hence *the straight lines joining the centres of Stäckel curvature for corresponding directions at corresponding points form a system of generators of a quadric.*

This result is also true, if the word "Stäckel curvature" be replaced by "normal curvature" (1) or "inverse curvature" respectively.

9. As a particular case I consider the isometric representation

$$u=\bar u,\qquad v=\bar v,$$

in which the lines of curvature $u=$const., $v=$const. correspond to the lines of curvature $\bar u=$const., $\bar v=$const., respectively. Then we have

$$E=\bar E,\qquad F=\bar F=0,\qquad G=\bar G,$$
$$M=\bar M=0,\qquad \sigma=\bar\sigma,\qquad \tau=\bar\tau.$$

Also since the total curvature is absolutely invariant,

$$LN=\bar L\bar N.$$

The directions for which

$$S^2=\bar S^2$$

are given by

$$\sigma(L\pm\bar L)\,du^2+\tau(N\pm\bar N)\,dv^2=0.$$

(1) Ogura, loc. cit., (the second paper), in which it will be found a general discussion for some problems of similar nature.

In consequence of

$$\sigma\tau(L-\bar{L})(N+\bar{N})+\sigma\tau(L+\bar{L})(N-\bar{N})$$
$$=2\sigma\tau(LN-\bar{L}\bar{N})=0,$$

we obtain the theorem:

In the isometric representation preserving the lines of curvature the two pairs of directions corresponding to

$$S^2=\bar{S}^2$$

are separated harmonically.

10. Now we pass to the transformation by reciprocal radii:

(9) $$\bar{x}=\frac{-x}{x^2+y^2+z^2},\qquad \bar{y}=\frac{-y}{x^2+y^2+z^2},\qquad \bar{z}=\frac{-z}{x^2+y^2+z^2}.$$

Let the lines of curvature be taken as the parametric curves of a surface Σ; then the transformed surface $\bar{\Sigma}$ has the fundamental quantities:

$$\bar{E}=\frac{E}{r^4},\qquad \bar{F}=\frac{F}{r^4}=0,\qquad \bar{G}=\frac{G}{r^4},$$

$$\bar{L}=\frac{1}{r^2}\left(L+2\frac{W}{r^2}E\right),\qquad \bar{M}=\frac{1}{r^2}\left(M+2\frac{W}{r^2}F\right)=0,$$

$$\bar{N}=\frac{1}{r^2}\left(N+2\frac{W}{r^2}G\right),$$

where we have

$$r^2\equiv x^2+y^2+z^2,\qquad W\equiv xX+yY+zZ,$$

X, Y, Z being the direction-cosines of the normals to the surface Σ at (x, y, z). Moreover, as Prof. P. Calapso has remarked (1), σ and τ are absolute invariants for the inversion. Therefore we get

(10) $$\frac{1}{\bar{S}}=\frac{r^2}{S}+2W.$$

If $[S]\ (\xi,\eta,\zeta)$ and $[\bar{S}]\ (\bar{\xi},\bar{\eta},\bar{\zeta})$ be the centres of Stäckel curvature of Σ and $\bar{\Sigma}$ for corresponding directions at the corresponding points (x, y, z) and $(\bar{x},\bar{y},\bar{z})$ respectively, then

$$\xi=x+XS,\qquad \eta=y+YS,\qquad \zeta=z+ZS;$$
$$\bar{\xi}=\bar{x}+\bar{X}\bar{S},\qquad \bar{\eta}=\bar{y}+\bar{Y}\bar{S},\qquad \bar{\zeta}=\bar{z}+\bar{Z}\bar{S}.$$

(1) Calapso, "Sugl'invarianti del gruppo delle trasformazioni conformi dello spazio," Rend. Palermo, 22 (1906), p. 211-212.

where

$$\bar{X}=\frac{2}{r^2}xW-X,\qquad \bar{Y}=\frac{2}{r^2}yW-Y,\qquad \bar{Z}=\frac{2}{r^2}zW-Z;$$

from which we obtain

$$\frac{\bar{\xi}}{\xi}=\frac{\bar{\eta}}{\eta}=\frac{\bar{\zeta}}{\zeta}=-\frac{1}{r^2+2WS}.$$

Consequently, *for inverse surfaces, the centres of Stäckel curvature for corresponding directions at corresponding points lie on a straight line with the centre of inversion.*

11. It is easily seen that the above theorem holds good, if the word "Stäckel curvature" be replaced by "normal curvature" (1) or "inverse curvature" respectively. Now I will consider the converse problem:

To find the general quantity $l\left(u, v, \frac{dv}{du}\right)$ *such that any two corresponding points*

$$\xi=x+Xl,\qquad \eta=y+Yl,\qquad \zeta=z+Zl$$

and

$$\bar{\xi}=\bar{x}+\bar{X}\bar{l},\qquad \bar{\eta}=\bar{y}+\bar{Y}\bar{l},\qquad \bar{\zeta}=\bar{z}+\bar{Z}\bar{l},$$

$x, X, \ldots$ and $\bar{x}, \bar{X}, \ldots$ having the same meaning as above, lie on a straight line with the centre of inversion (9).

Since $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ may be written

$$\bar{\xi}=\frac{1}{r^2}[-x+(2xW-Xr^2)\bar{l}],\ \ldots$$

we have

$$\frac{\bar{\xi}}{\xi}=\frac{(2W\bar{l}-1)x-r^2\bar{l}X}{r^2(x+lX)},\ \ldots$$

In order that (ξ,η,ζ) and $(\bar{\xi},\bar{\eta},\bar{\zeta})$ lie on a straight line passing through $(0, 0, 0)$, it is necessary and sufficient that

$$\frac{\bar{\xi}}{\xi}=\frac{\bar{\eta}}{\eta}=\frac{\bar{\zeta}}{\zeta},$$

or

(1) For example, see Salmon, Analytic geometry of three dimensions, 5. ed., 2 (1915), p. 157.

$$\frac{(2W\bar{l}-1)\frac{x}{X}-r^2\bar{l}}{r^2\left(\frac{x}{X}+l\right)}=\frac{(2W\bar{l}-1)\frac{y}{Y}-r^2\bar{l}}{r^2\left(\frac{y}{Y}+l\right)}$$

$$=\frac{(2W\bar{l}-1)\frac{z}{Z}-r^2\bar{l}}{r^2\left(\frac{z}{Z}+l\right)}(=\lambda,\ \text{say});$$

whence

$$\frac{X}{x}=\frac{Y}{y}=\frac{Z}{z}=\frac{2W\bar{l}-1-\lambda r^2}{r^2(l\lambda+\bar{l})}$$

which is impossible in general, unless

$$l\lambda+\bar{l}=0, \qquad 2W\bar{l}-1-\lambda r^2=0;$$

or

$$\frac{2W\bar{l}-1}{r^2}=-\frac{\bar{l}}{l},$$

i. e.

(10)′ $$\frac{1}{l}=\frac{r^2}{l}+2W.$$

It follows from this that *the expression*

$$\left(\frac{1}{l}-\frac{1}{R}\right):\left(\frac{1}{I}-\frac{1}{R}\right)$$

should be absolutely invariant for the inversion (9); in other words, *the quantity* l *must take the form*

$$\frac{1}{l}=\frac{1}{R}+\left(\frac{1}{I}-\frac{1}{R}\right)\Psi\left(u,v,\frac{dv}{du}\right),$$

where Ψ *denotes an absolute invariant for the inversion* (9). This gives the general solution of our problem.

For example, if we put

$$\Psi\equiv\frac{\sigma-\tau}{2}\;\frac{E\,du^2-G\,dv^2}{\sigma E\,du^2+\tau G\,dv^2},$$

$\frac{1}{l}$ becomes the Stäckel curvature $\frac{1}{S}$.

PART II.

Remarkable Curves corresponding to Certain Special Values of the Stäckel Curvature.

A closed system.

12. Let the lines of curvature be taken as the parametric curves of a surface Σ (1), and let us put

$$\varphi_1\equiv\sigma E\,du^2+\tau G\,dv^2,$$
$$\varphi_2\equiv\sigma L\,du^2+\tau N\,dv^2.$$

Then the five families of curves

$$\varphi_1=0, \qquad \varphi_2=0,$$

$J(\varphi_1,\varphi_2)=0$ (i. e. the lines of curvature),

$J(\varphi_1,J(\varphi_1,\varphi_2))=0$ (i. e. $\sigma E\,du^2-\tau G\,dv^2=0$),

$J(\varphi_2,J(\varphi_1,\varphi_2))=0$ (i. e. $\sigma L\,du^2-\tau N\,dv^2=0$)

are important in developing the theory of the Stäckel curvature, just as the five families stated in § 1 play the fundamental rôle in the classical theory of curvature.

Since

$$\frac{1}{S}=\frac{\varphi_2}{\varphi_1},$$

$\varphi_1=0$ *or* $\varphi_2=0$ *gives the directions for which the Stäckel curvature becomes infinity or zero respectively;* and $J(\varphi_1,\varphi_2)=0$ *corresponds the extreme values of that curvature (which are equal to the radii of principal curvature).* Lastly $J(\varphi_1,J(\varphi_1,\varphi_2))=0$ *and* $J(\varphi_2,J(\varphi_1,\varphi_2))=0$ *are the curves whose radii of Stäckel curvature are the harmonic and arithmetic means of the principal radii of curvature respectively.*

Moreover

$$\varphi_1=0, \qquad J(\varphi_1,\varphi_2)=0, \qquad J(\varphi_1,J(\varphi_1,\varphi_2))=0\ (2)$$

form a cycle in the sense that the directions of any one of these families are the double rays of the involution determined by the directions of the other two; similarly

(1) Excluding the plane, the sphere, and the surface mentioned in § 7.

(2) *Each of these three families is preserved by the transformation of reciprocal radii.*

$$\varphi_2=0, \quad J(\varphi_1, \varphi_2)=0, \quad J(\varphi_2, J(\varphi_1, \varphi_2))=0$$

form another cycle; and these five families form a closed system.

In the following paragraphs I will deal with the two families $\varphi_1=0$ and $\varphi_2=0$ from the standpoint of Laplace transforms.

The curves $\varphi_1=0$.

13. Consider the first and minus first Laplace transforms Σ_1 and Σ_{-1} of a surface Σ, the curves of reference being the lines of curvature u=const., v=const. of Σ. If we put

$$a\equiv\begin{Bmatrix}11\\1\end{Bmatrix}=\frac{1}{2E}\frac{\partial E}{\partial u}, \qquad b\equiv\begin{Bmatrix}11\\2\end{Bmatrix}=-\frac{1}{2G}\frac{\partial E}{\partial v},$$

$$a'\equiv\begin{Bmatrix}12\\1\end{Bmatrix}=\frac{1}{2E}\frac{\partial E}{\partial v}, \qquad b'\equiv\begin{Bmatrix}12\\2\end{Bmatrix}=\frac{1}{2G}\frac{\partial G}{\partial u},$$

$$a''\equiv\begin{Bmatrix}22\\1\end{Bmatrix}=-\frac{1}{2E}\frac{\partial G}{\partial u}, \qquad b''\equiv\begin{Bmatrix}22\\2\end{Bmatrix}=\frac{1}{2G}\frac{\partial G}{\partial v},$$

the corresponding points $P_1(x_1, y_1, z_1)$ and $P_{-1}(x_{-1}, y_{-1}, z_{-1})$ of $P(x, y, z)$ are

(11) $$x_1=x-\frac{1}{a'}\frac{\partial x}{\partial v}, \qquad y_1=y-\frac{1}{a'}\frac{\partial y}{\partial v},$$
$$z_1=z-\frac{1}{a'}\frac{\partial z}{\partial v},$$

(12) $$x_{-1}=x-\frac{1}{b'}\frac{\partial x}{\partial u}, \qquad y_{-1}=y-\frac{1}{b'}\frac{\partial y}{\partial u},$$
$$z_{-1}=z-\frac{1}{b'}\frac{\partial z}{\partial u}.$$

By use of the fundamental equations

$$\frac{\partial^2 x}{\partial u^2}=a\frac{\partial x}{\partial u}+b\frac{\partial x}{\partial v}+LX,$$

$$\frac{\partial^2 x}{\partial u\,\partial v}=a'\frac{\partial x}{\partial u}+b'\frac{\partial x}{\partial v},$$

$$\frac{\partial^2 x}{\partial v^2}=a''\frac{\partial x}{\partial u}+b''\frac{\partial x}{\partial v}+NX,$$

....,

we get

$$dx_1=\left\{\left(1-\frac{b''}{a'}+\frac{1}{a'^2}\frac{\partial a'}{\partial v}\right)dv-\frac{1}{a'^2}\left(a'b'-\frac{\partial a'}{\partial u}\right)du\right\}\frac{\partial x}{\partial v}$$
$$-\frac{1}{a'}\left(a''\frac{\partial x}{\partial u}+NX\right)dv,$$

$$dx_{-1}=\left\{\left(1-\frac{a}{b'}+\frac{1}{b'^2}\frac{\partial b'}{\partial u}\right)du-\frac{1}{b'^2}\left(a'b'-\frac{\partial b'}{\partial v}\right)dv\right\}\frac{\partial x}{\partial u}$$
$$-\frac{1}{b'}\left(b\frac{\partial x}{\partial v}+LX\right)du,$$

....

After a short calculation we obtain

(13) $$dx_1\,dx_{-1}+dy_1\,dy_{-1}+dz_1dz_{-1}=A\,du^2+2\,B\,du\,dv+C\,dv^2,$$

where

(14) $$\begin{cases} A\equiv\dfrac{b}{b'\,a'^2}\left(a'b'-\dfrac{\partial a'}{\partial u}\right)G=\dfrac{1}{a'\,b'}E\,\sigma, \\ 2\,B\equiv\dfrac{LN}{a'b'}-\dfrac{a''}{a'}\left(1-\dfrac{a}{b'}+\dfrac{1}{b'^2}\dfrac{\partial b'}{\partial u}\right)E \\ \qquad -\dfrac{b}{b'}\left(1-\dfrac{b''}{a'}+\dfrac{1}{a'^2}\dfrac{\partial a'}{\partial v}\right)G, \\ C\equiv\dfrac{a''}{a'\,b'^2}\left(a'b'-\dfrac{\partial b'}{\partial v}\right)E=\dfrac{1}{a'b'}G\,\tau. \end{cases}$$

Now since u=const., v=const. are the lines of curvature, the Gauss equation becomes

$$LN=-\frac{1}{2}\frac{\partial^2 E}{\partial v^2}-\frac{1}{2}\frac{\partial^2 G}{\partial u^2}+\frac{1}{4E}\left[\left(\frac{\partial E}{\partial v}\right)^2+\frac{\partial E}{\partial u}\frac{\partial G}{\partial u}\right]$$
$$+\frac{1}{4G}\left[\left(\frac{\partial G}{\partial u}\right)^2+\frac{\partial E}{\partial v}\frac{\partial G}{\partial v}\right].$$

On the other hand

$$E\,a'=-b\,G$$

and

$$a'(a'-b'')+\frac{\partial a'}{\partial v}=-\frac{1}{E}\left[-\frac{1}{2}\frac{\partial^2 E}{\partial v^2}+\frac{1}{4E}\frac{\partial^2 E}{\partial v^2}\right.$$

$$+\frac{1}{4G}\frac{\partial E}{\partial v}\frac{\partial G}{\partial v}\Big];$$

whence

$$\frac{b}{a'}G\Big[a'(a'-b'')+\frac{\partial a'}{\partial v}\Big]=-\frac{1}{2}\frac{\partial^2 E}{\partial v^2}+\frac{1}{4E}\Big(\frac{\partial E}{\partial v}\Big)^2$$
$$+\frac{1}{4G}\frac{\partial E}{\partial v}\frac{\partial G}{\partial v};$$

and similarly

$$\frac{a''}{b'}E\Big[b'(b'-a)+\frac{\partial b'}{\partial u}\Big]=-\frac{1}{2}\frac{\partial^2 G}{\partial u^2}+\frac{1}{4G}\Big(\frac{\partial G}{\partial u}\Big)^2$$
$$+\frac{1}{4E}\frac{\partial E}{\partial u}\frac{\partial G}{\partial u}.$$

Hence the Gauss equation takes the form

$$LN=\frac{b}{a'}G\Big[a'(a'-b'')+\frac{\partial a'}{\partial v}\Big]+\frac{a''}{b'}E\Big[b'(b'-a)+\frac{\partial b'}{\partial u}\Big].$$

Consequently it follows from (14) that the coefficient B vanishes identically.

Therefore the quantity

$$dx_1\,dx_{-1}+dy_1\,dy_{-1}+dz_1\,dz_{-1}$$

vanishes when and only when $dv:du$ satisfies the relation

$$\sigma E\,du^2+\tau G\,dv^2=0,$$

which is nothing but

$$\varphi_1=0.$$

Thus we have arrived at the theorem:

The tangents to the curves, on the first and minus first Laplace transforms (the curves of reference being the lines of curvature) of a surface, corresponding to the curves

$$(15)\qquad \sigma E\,du^2+\tau G\,dv^2=0$$

on that surface are perpendicular to one another at corresponding points; this property is characteristic of the curves (15).

The curves (15) form an orthogonal system when and only when

$$\sigma+\tau=0;$$

so *a necessary and sufficient condition that a surface be inverse-isothermic is that the curves* (15) *should form an orthogonal system.*

Similarly, *a necessary and sufficient condition that a surface be isothermic is that the curves* (15) *form an inverse-orthogonal system.*

The curves $\varphi_2=0$.

14. In his famous paper entitled "The general theory of congruences," Prof. Wilczynski[1] treated the *ray-congruence*, that is, the totality of the straight lines joining the corresponding points of the first and minus first Laplace transforms Σ_1 and Σ_{-1} of a surface Σ. The developable surfaces of the ray-congruence correspond to the *ray-curves* on the surface Σ.

In the particular case where the Laplace transforms are performed with reference to the lines of curvature u=const., v=const. on the surface Σ, the differential equation of the ray-curves is of the form

$$(16)\qquad A'\,du^2+2B'\,du\,dv+C'\,dv^2=0,$$

where

$$(17)\begin{cases} A'\equiv -L\Big(a'b'-\dfrac{\partial a'}{\partial u}\Big)=L\sigma,\\ 2B'\equiv L\Big[a'(a'-b'')+\dfrac{\partial a'}{\partial v}-a''b'\Big]-N\Big[a'(b'-a)+\dfrac{\partial b'}{\partial u}-a'b\Big],\\ C'\equiv N\Big(a'b'-\dfrac{\partial b'}{\partial v}\Big)=-N\tau\ (^2). \end{cases}$$

The double lines of the involution determined by the ray-curves and the lines of curvature have the directions given by

$$\sigma L\,du^2+\tau N\,dv^2=0,$$

which coincides with

$$\varphi_2=0.$$

Thus we infer the theorem:

When the Laplace transforms are performed with reference to the

(1) Trans. Amer. Math. Soc, 16 (1915), p. 311.

(2) G. M. Green, loc. cit., equation (9). [There are some misprints in equations (8), (9), (14), (15), (16), etc. of this paper]. It is inconvenient, for our purpose, to use Wilczynski's original form.

lines of curvature $u=\text{const.}$, $v=\text{const.}$ on a surface, the tangents to the curves

$$\sigma L\,du^2+\tau N\,dv^2=0 \tag{18}$$

on that surface are the double lines of involution determined by the ray-curves and the lines of curvature; this is a property characteristic for the curves (18).

We add the following two theorems which can be easily proved:

A necessary and sufficient condition that a non-developable surface be inverse-isothermic is that the curves (18) *form a conjugate system, in other words, the ray-curves* (16) *form an inverse-conjugate system.*

A necessary and sufficient condition that a non-developable surface be isothermic is that the curves (18) *form an inverse-conjugate system, in other words, the ray-curves* (16) *form a conjugate system* (1).

15. Let the Laplace transforms be performed with reference to *any* conjugate system $u=\text{const.}$, $v=\text{const.}$ on a surface. Then the equation of the ray-curves takes the same form as (16) also, if we take

$$a=\begin{Bmatrix}11\\1\end{Bmatrix},\qquad a'=\begin{Bmatrix}12\\1\end{Bmatrix},\qquad a''=\begin{Bmatrix}22\\1\end{Bmatrix}$$

$$b=\begin{Bmatrix}11\\2\end{Bmatrix},\qquad b'=\begin{Bmatrix}12\\2\end{Bmatrix},\qquad b''=\begin{Bmatrix}22\\2\end{Bmatrix}$$

and

$$\sigma=\frac{\partial a'}{\partial u}-a'b',\qquad \tau=\frac{\partial b'}{\partial v}-a'b'.$$

Therefore if a surface be referred to a conjugate system and if σ, τ be the Laplace-Darboux invariants of the equation

$$\frac{\partial^2\vartheta}{\partial u\,\partial v}=\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\vartheta}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial\vartheta}{\partial v},$$

the tangents to the curves

$$\sigma L\,du^2+\tau N\,dv^2=0 \tag{18$'$}$$

are the double lines of involution determined by the conjugate system and the ray-curves; this property is characteristic of the curves (18)$'$.

On the other hand, the quantity

(1) The latter part of this theorem is contained in Wilczynski's theorem : A conjugate system on a non-developable surface has equal Laplace-Darboux invariants, if and only if its ray-curves also form a conjugate system.

$$\frac{1}{\sqrt{EG-F^2}}\,\frac{\sigma L\,du^2+\tau N\,dv^2}{E\,du^2+2F\,du\,dv+G\,dv^2}$$

has been treated by Prof. A. Voss in detail and called by him *the parameter curvature* ("Parameterkrümmung") (1). He said "....Ebenso existieren im Allgemeinen zwei Richtungen, für welche die Parameterkrümmung Null ist; sie sind den asymptotischen Richtungen zu vergleichen....". Now I can state the theorem:

When a surface is referred to a conjugate system, the parameter curvature vanishes for and only for the two directions which form the double lines of involution determined by the conjugate system and the ray-curves.

Ikeda near Ôsaka, February 1919.

(1) Voss, "Zur Theorie der Krümmung der Flächen," Math. Ann., 39 (1891), p. 179.

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