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On the Theory of Approximating Functions with Applications to Geometry, Law of Errors and Conduction of Heat.

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On the Theory of Approximating Functions with Applications to Geometry, Law of Errors and Conduction of Heat,

by

KINNOSUKE OGURA, Ôsaka.

The object of this paper is to treat the convergency of the function

$$\frac{\int_a^b f(t)\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}, \qquad \varphi_n(x,t) \geqq 0 \qquad (a \leqq x, t \leqq b),$$

when the positive variable n increases indefinitely and that of the analogous formula of interpolation, together with some special cases and certain applications.

CONTENTS.

Convergency of approximating functions.

1. I will begin with the following theorem:

Theorem I. *Let $\varphi_n(x,t)$ be limited, non-negative and integrable* (1) *with respect to t in the domain $a \leqq x, t \leqq b$* (2)*; and let both*

$$\frac{\int_a^{a_1} \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} \quad \text{and} \quad \frac{\int_{b_1}^b \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}$$

converge to zero uniformly for $a \leqq a_1 < a_2 \leqq x \leqq b_2 < b_1 \leqq b$, when the positive variable n increases indefinitely.

Again let $f(x)$ be an arbitrary function which is limited and integrable in the interval $a \leqq x \leqq b$, and M_1 be the fluctuation of $f(x)$ in $a_1 \leqq x \leqq b_1$, and δ be a positive number assigned at pleasure.

If we consider the sequence

$$(1) \qquad F_n[f(x)] = \frac{\int_a^b f(t)\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt},$$

we can find a positive number N independent of x, such that

$$|f(x) - F_n[f(x)]| < M_1 + \delta, \qquad n > N$$

in the interval $a_2 \leqq x \leqq b_2$.

If $f(x)$ be continuous at $x = x_0$ $(a < x_0 < b)$, then $F_n[f(x)]$ converges to $f(x)$ uniformly in the vicinity of that point (3).

Since

$$f(x) = \frac{\int_a^b f(x)\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt},$$

we have

$$f(x) - F_n[f(x)] = \frac{\int_a^b [f(x) - f(t)]\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}.$$

If we denote by M the fluctuation of $f(x)$ in the interval $a \leqq x \leqq b$, then

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(1) *All integrals in this paper are to be taken in Riemann's sense.*

(2) Throughout this paper a and b are supposed to be any two finite numbers such that $a < b$, unless the contrary is stated.

(3) Prof. Fujiwara kindly informed me that the latter part of this theorem can not be derived from a similar theorem due to Prof. Hobson. See Hobson, "On a general convergence theorem, etc.," Proc. London Math. Soc., (2), 6 (1908), p. 349.

$$(2) \qquad |f(x) - f(t)| \leqq M, \qquad a \leqq x, t \leqq b.$$

Hence

$$|f(x) - F_n[f(x)]| \leqq \frac{\int_a^b |f(x) - f(t)|\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} \leqq M \frac{\int_a^b \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} = M.$$

Now consider two subintervals $D_1(a_1, b_1)$ and $D_2(a_2, b_2)$ such that

$$a \leqq a_1 < a_2 < b_2 < b_1 \leqq b.$$

Then by the assumption

$$|f(x) - f(t)| \leqq M_1, \qquad a_1 \leqq x, t \leqq b_1;$$

so that

$$\left|\frac{\int_{a_1}^{b_1} [f(x) - f(t)]\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}\right| \leqq M_1 \frac{\int_{a_1}^{b_1} \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} \leqq M_1 \frac{\int_a^b \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} = M_1,$$

for x in D_1.

Also from the assumption we can find a positive number N (independent of x), corresponding to any assigned positive number δ, such that

$$\left.\begin{aligned} \left|\frac{\int_a^{a_1} [f(x) - f(t)]\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}\right| &\leqq M \frac{\int_a^{a_1} \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} < \frac{\delta}{2}, \\ \left|\frac{\int_{b_1}^b [f(x) - f(t)]\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt}\right| &\leqq M \frac{\int_{b_1}^b \varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt} < \frac{\delta}{2}, \end{aligned}\right\} \quad n > N,$$

for x in D_2.

Consequently we have

$$(3) \qquad |f(x) - F_n[f(x)]| < M_1 + \delta, \qquad n > N$$

for x in D_2, which proves the former part of the theorem.

Up to this point we have placed no restriction on $f(x)$ for continuity. We assume now that $f(x)$ is continuous at $x = x_0$ $(a < x_0 < b)$. Let ε be a positive number at pleasure; then we can take D_1, D_2 such that

$$a \leqq a_1 < a_2 < x_0 < b_2 < b_1 \leqq b \qquad \text{and} \qquad M_1 < \varepsilon.$$

Hence for $a_2 \leqq x \leqq b_2$

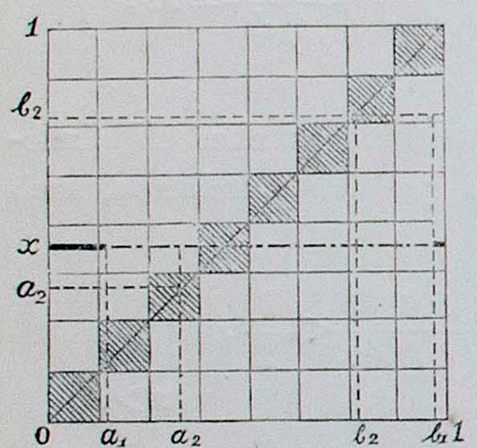
$$|f(x)-F_n[f(x)]|<\varepsilon+\delta, \qquad n>N.$$

Thus $F_n[f(x)]$ converges to $f(x)$ uniformly in the vicinity of $x=x_0$, which proves the latter part of the theorem.

Lastly let us turn to the general case. Since $f(x)$ is limited and integrable in $a\leqq x\leqq b$, it follows from Lebesgue's well known theorem that the points of discontinuity of that function form a set of measure zero in the given interval. Therefore $F_n[f(x)]$ *converges uniformly to* $f(x)$ *in the interval* $a<a_2\leqq x\leqq b_2<b$, *except a set of points of measure zero.*

2. Here I will add some simple examples:

I. Divide the square enclosed by $t=0$, $t=1$, $x=0$, $x=1$ into 2^{2n} $(n=1, 2, \cdots\cdots)$ equal squares as indicated in the figure; and define the function by $\varphi_n(x, t)=2^n$ at any point within the squares (including the edges) along the diagonal $x-t=0$ (1), $\varphi_n(x, t)=0$ at any other point within the given (large) square (including the boundary).



Then

$$\varphi_n(x, t)\geqq 0, \quad (0\leqq x, t\leqq 1)$$

and

$$\int_0^1 \varphi_n(x, t)\, dt=1 \quad \text{when} \quad x\neq\frac{k}{2^n}$$

$$=2 \quad \text{when} \quad x=\frac{k}{2^n}$$

$$(k=1, 2, \cdots\cdots, 2^{n-1}).$$

If we choose a positive integer N such that

$$a_2-a_1>\frac{1}{2^N}, \qquad b_1-b_2>\frac{1}{2^N}, \qquad (0\leqq a_1<a_2<b_2\leqq 1),$$

we have for $\qquad a_2\leqq x\leqq b_2$

$$\int_0^{a_1} \varphi_n(x, t)\, dt=0, \qquad \int_{b_1}^1 \varphi_n(x, t)\, dt=0, \qquad n>N.$$

Therefore $\varphi_n(x, t)$ satisfies the conditions imposed in Theorem I.

II. As the second example I will prove *the theorem due to Prof. A. Haar* (2):

(1) We may take the other diagonal instead of this.

(2) Haar, "Zur Theorie der orthogonalen Funktionensysteme," Math. Ann., 69 (1910), p. 361.

Let us define a system of orthogonal functions by

$$\begin{array}{ll}
\chi_0(t)=1 & 0\leqq t\leqq 1; \\
\chi_n^{(k)}(t)=0 & 0\leqq t\leqq \dfrac{2k-2}{2^n} \\
\quad =\dfrac{1}{2}\sqrt{2^{n-1}} & t=\dfrac{2k-2}{2^n} \\
\quad =\sqrt{2^{n-1}} & \dfrac{2k-2}{2^n}<t<\dfrac{2k-1}{2^n} \\
\quad =0 & t=\dfrac{2k-1}{2^n} \\
\quad =-\sqrt{2^{n-1}} & \dfrac{2k-1}{2^n}<t<\dfrac{2k}{2^n} \\
\quad =-\dfrac{1}{2}\sqrt{2^{n-1}} & t=\dfrac{2k}{2^n} \\
\quad =0 & \dfrac{2k}{2^n}<t\leqq 1,
\end{array}$$

$$(n=1, 2, \cdots\cdots;\ k=1, 2, \cdots\cdots, 2^{n-1})$$

and let us put

$$K_n^{(k)}(x, t)=\chi_0(x)\,\chi_0(t)+\cdots\cdots+\chi_n^{(1)}(x)\,\chi_n^{(1)}(t)+\cdots\cdots+\chi_n^{(k)}(x)\,\chi_n^{(k)}(t).$$

Then the sequence

$$\int_0^1 f(t)\, K_n^{(k)}(x, t)\, dt$$

converges uniformly to $f(x)$ in the interval $0<a_2\leqq x\leqq b_2<1$, except a set of points of measure zero.

Since

$$K_n^{(k)}(x, t)\geqq 0, \qquad \int_0^1 K_n^{(k)}(x, t)\, dt=1 \qquad (0\leqq x, t\leqq 1),$$

in order to prove this, it is sufficient to consider the value of the function $K_n^{(k)}(x, t)$ graphically in a similar way as in the last example. (See the paper of Prof. Haar.)

III. As the third example we may take *the well known integral of Prof. Fejér* (1):

(1) Fejér, "Untersuchungen über Fouriersche Reihen," Math. Ann. 58 (1904), p. 51.

$$\int_0^{2\pi} f(t)\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt, \qquad (n=1,2,\cdots\cdots).$$

In this case it is well known that

$$\int_0^{2\pi}\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt=1$$

and

$$\lim_{n=\infty}\int_0^{a_1}\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt=0,$$

$$\lim_{n=\infty}\int_{b_1}^{2\pi}\frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt=0,$$

uniformly for

$$0\leqq a_1<a_2\leqq x\leqq b_2<b_1\leqq 2\pi.$$

Therefore Fejér's integral converges uniformly to $f(x)$ in the interval $0<a_2\leqq x\leqq b_2<2\pi$, except a set of points of measure zero.

3. Next I will prove the converse theorem:

Theorem Ia. *Let* $\varphi_n(x,t)$ *be a function which is limited, non-negative and integrable with respect to* t *in the domain* $a\leqq x, t\leqq b$. *If*

$$F_n[f(x)]=\frac{\int_a^b f(t)\,\varphi_n(x,t)\,dt}{\int_a^b \varphi_n(x,t)\,dt},$$

corresponding to any function $f(x)$ *which is limited and integrable in the interval* $a\leqq x\leqq b$, *converge to that function uniformly in the interval* $a<a_2\leqq x\leqq b_2<b$, *then both*

$$\frac{\int_a^{a_1}\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt} \quad \text{and} \quad \frac{\int_{b_1}^b\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt},$$

$$(a\leqq a_1<a_2\leqq x\leqq b_2<b_1\leqq b)$$

should necessarily converge to zero uniformly.

In order to prove this, let us take the function $f(x)$ defined by

$$\begin{array}{ll} f(x)=0 & a\leqq x\leqq a_1 \\ \quad =\dfrac{x-a_1}{a_2-a_1} & a_1<x<a_2 \\ \quad =1 & a_2\leqq x\leqq b_2 \\ \quad =\dfrac{x-b_1}{b_2-b_1} & b_2<x<b_1 \\ \quad =0 & b_1\leqq x\leqq b. \end{array}$$

Then

$$\begin{array}{lll} \int_a^{a_1}[f(x)-f(t)]\,\varphi_n(x,t)\,dt=\int_a^{a_1}\varphi_n(x,t)\,dt, & & \int_{a_1}^{a_2}>0, \\ \int_{a_2}^{b_2} & =0, & \int_{b_2}^{b_1}>0, \\ \int_{b_1}^{b}. & =\int_{b_1}^{b}\varphi_n(x,t)\,dt. & \end{array}$$

Consequently

$$f(x)-F_n[f(x)]=\frac{1}{\int_a^b\varphi_n(x,t)\,dt}\left[\int_a^{a_1}[f(x)-f(t)]\,\varphi_n(x,t)\,dt+\int_{a_1}^{a_2}\right.$$

$$\left.+\int_{a_2}^{b_2}+\int_{b_2}^{b_1}+\int_{b_1}^{b}\right]>\frac{\int_a^{a_1}\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt}+\frac{\int_{b_1}^b\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt}.$$

But from the assumption we can find a positive number N, independent of x ($a_2\leqq x\leqq b_2$), corresponding to any positive number ε, such that

$$|f(x)-F_n[f(x)]|<\varepsilon, \qquad n>N;$$

so that we have

$$0<\frac{\int_a^{a_1}\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt}+\frac{\int_{b_1}^b\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt}<\varepsilon, \qquad n>N.$$

Since $\varphi_n(x,t)$ is non-negative in the domain $a\leqq x,t\leqq b$, the above inequalities show us that both

$$\frac{\int_a^{a_1}\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt} \quad \text{and} \quad \frac{\int_{b_1}^b\varphi_n(x,t)\,dt}{\int_a^b\varphi_n(x,t)\,dt} \qquad (a_2\leqq x\leqq b_2)$$

converge to zero uniformly.

4. I will now prove the general theorem:

Theorem II. *If $\varphi_n(x,t)$ satisfy the conditions imposed in Theorem I, and if $f(x)$ be limited and integrable in $a \leqq x \leqq b$, then the sequence $F_n[f(x)]$ converges strongly* [1] *with respect to any positive fixed exponent to the function $f(x)$ in the interval $a \leqq x \leqq b$; that is.*

$$\lim \int_a^b |f(x) - F_n[f(x)]|^p dx = 0 \tag{4}$$

for any positive number p.

Already Prof. A. Hurwitz [2] proved this theorem for $p=2$ (and n being an integer) in the case of Fejér's integral. His method of proof may be applicable to the general case as follows:

Since $f(x)$ is integrable (in Riemann's sense) in the interval $D(a,b)$, for any pair of positive numbers ω, ε, there exists a division of D, such that the sum of the lengths of the subintervals of that division in which the fluctuation of $f(x)$ is $>\omega$, is $<\varepsilon$. If we denote these subintervals by Δ, and the remaining subintervals of D by D_1, the fluctuation of $f(x)$ is not greater than ω for each interval of D_1. Next take an interval in the inner part of each subinterval of D_1; and denote by D_2 these new subintervals and by D_3 the remaining subintervals of D_1. Lastly suppose that we have taken D_2 such that the sum of lengths of the subintervals of D_3 is smaller than ε.

Then for any point of each subinterval of D_2, from (3), we can choose a positive number N independent of x, such that

$$|f(x) - F_n[f(x)]| < \omega + \delta \qquad n > N.$$

And for any point of each subinterval of D_3 and Δ we have from (2)

$$|f(x) - F_n[f(x)]| \leqq M.$$

Now from

$$\int_a^b \left| f(x) - F_n[f(x)] \right|^p dx = \sum_{D_2} \int_{D_2} \left| f(x) - F_n[f(x)] \right|^p dx + \sum_{D_3} \int_{D_3} + \sum_{\Delta} \int_{\Delta},$$

we obtain

([1]) In the sense of Prof. F. Riesz. See F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen," Math. Ann., 69 (1910), p. 464.

([2]) Hurwitz, "Über die Fourierschen Konstanten integrierbarer Funktionen," Math. Ann., 57 (1903), p. 433.

$$\int_a^b \left| f(x) - F_n f(x) \right|^p dx < (\omega+\delta)^p \sum_{D_2} D_2 + M^p \sum_{D_3} D_3 + M^p \sum_{\Delta} \Delta \qquad n > N$$
$$< (\omega+\delta)^p (b-a) + M^p \varepsilon + M^p \varepsilon \qquad n > N.$$

Since δ, ω and ε may be taken arbitrarily small, we have

$$\lim_{n=\infty} \int_a^b \left| f(x) - F_n[f(x)] \right|^p dx = 0$$

for any positive number p, which proves the theorem.

Approximating functions of the form $\int_a^b f(t)\,\varphi_n(t-x)\,dt$.

5. Recently the convergence-problem of the functions of the form

$$\int_a^b f(t)\,\Phi_n(t-x)\,dt, \qquad \text{or} \qquad \int_a^x f(t)\,\Phi_n(t-x)\,dt$$

has been the subject of numerous researches [1]. I will now proceed to establish the theorem:

Theorem III. *Let $\varphi_n(t)$ be a function which is non-negative, limited and integrable in the interval $a-b \leqq t \leqq b-a$ and is such that*

$$\lim_{n=\infty} \int_{-c_1}^{+c_2} \varphi_n(t)\,dt = 1 \tag{5}$$

uniformly in the interval $a-b < -c_1 < 0 < c_2 < b-a$.

If $f(x)$ be limited and integrable in $a \leqq x \leqq b$, then

$$F_n[f(x)] = \int_a^b f(t)\,\varphi_n(t-x)\,dt \text{ [2]} \qquad (a \leqq x \leqq b) \tag{6}$$

converges strongly with respect to any positive exponent to the function, when n increases indefinitely.

Particularly if $f(x)$ be continuous at $x = x_0$ ($a < x_0 < b$), then the above function converges uniformly to $f(x_0)$ in the vicinity of that point.

([1]) Lebesgue, "Sur les intégrales singulières," Ann. Fac. Toulouse, (3). 1 (1909), p. 25; W. B. Ford, Studies on divergent series and summability (1916), p. 115; Ford, "On the representation of arbitrary functions by definite integrals," Amer. Jour. of Math., 38 (1916), p. 397; Y. Okada, "On measure of discontinuity and approximative representation of a function," Tôhoku Math. Journ., (1919), p. 1.

([2]) Or

$$\frac{\int_a^b f(t)\,\varphi_n(t-x)\,dt}{\int_a^b \varphi_n(t-x)\,dt}$$

The latter part of this theorem is essentially the same as one given by Prof. Lebesgue.

Suppose that x is an inner point of the interval (a, b). Then we can take a_1, a_2, b_2, b_1 such that

$$a \leqq a_1 < a_2 \leqq x \leqq b_2 < b_1 \leqq b.$$

But we have by (5)

$$\lim_{n=\infty} \int_a^b \varphi_n(t-x)\,dt = \lim_{n=\infty} \int_{a-x}^{b-x} \varphi_n(t)\,dt = 1,$$

$$\lim_{n=\infty} \int_a^{a_1} \varphi_n(t-x)\,dt = \lim_{n=\infty} \left[\int_{a-x}^{c_2} \varphi_n(t)\,dt + \int_{c_2}^{a_1-x} \varphi_n(t)\,dt \right] = 1-1=0,$$

$$\lim_{n=\infty} \int_{b_1}^{b} \varphi_n(t-x)\,dt = \lim_{n=\infty} \left[\int_{b_1-x}^{-c_1} \varphi_n(t)\,dt + \int_{-c_1}^{b-x} \varphi_n(t)\,dt \right] = -1+1=0$$

uniformly for $a_2 \leqq x \leqq b_2$. Therefore the theorem follows as an immediate consequence of Theorems I and II.

6. Next we state the converse theorem which is essentially due to Prof. Lebesgue:

Theorem IIIa. *If $\varphi_n(t)$ be a function which is non-negative, limited and integrable in the interval $a-b \leqq t \leqq b-a$, and if*

$$F_n[f(x)] = \int_a^b f(t)\,\varphi_n(t-x)\,dt \tag{6}$$

converge to $f(x)$ in the interval $a<x<b$, for any continuous function $f(x)$ $(a \leqq x \leqq b)$, then it must be

$$\lim_{n=\infty} \int_{-c_1}^{+c_2} \varphi_n(t)\,dt = 1 \quad \textit{for} \quad a-b < -c_1 < 0 < c_2 < b-a.$$

If we take $f(x)=1$, then from (6) we have

$$\lim_{n=\infty} \int_{a-x}^{b-x} \varphi_n(t)\,dt = 1, \qquad a<x<b.$$

Next let us choose a_1, a_2, b_2, b_1 such that $a \leqq a_1 < a_2 \leqq x \leqq b_2 < b_1 \leqq b$, and take the function $f(x)$ defined in § 3. Then

$$\lim_{n=\infty} \{ f(x) - F_n[f(x)] \}$$

$$= \lim_{n=\infty} \left[\int_a^{a_1} [f(x)-f(t)]\,\varphi_n(t-x)\,dt + \int_{a_1}^{a_2} + \int_{a_2}^{b_2} + \int_{b_2}^{b_1} + \int_{b_1}^{b} \right]$$

$$\geqq \lim_{n=\infty} \left[\int_a^{a_1} \varphi_n(t-x)\,dt + \int_{b_1}^{b} \varphi_n(t-x)\,dt \right];$$

so that we must have

$$\lim_{n=\infty} \int_{a-x}^{a_1-x} \varphi_n(t)\,dt = 0, \qquad \lim_{n=\infty} \int_{b_1-x}^{b-x} \varphi_n(t)\,dt = 0.$$

Case I: $0 < c_2 < (b-a)-c_1$.

Since $a+c_1<b$, we can take a_1 and b_1 such that

$$a \leqq a_1 < a+c_1 < a+c_1+c_2 = b_1 < b.$$

Then

$$\lim_{n=\infty} \int_{-c_1}^{+c_2} \varphi_n(t)\,dt = \lim_{n=\infty} \left[\int_{a-(a+c_1)}^{b-(a+c_1)} \varphi_n(t)\,dt - \int_{b_1-(a+c_1)}^{b-(a+c_1)} \varphi_n(t)\,dt \right] = 1.$$

Case II: $d \geqq (b-a)-c_1$.

In this case we can choose a_1 and b_1 such that

$$a \leqq a_1 < a+c_1 < b_1 \leqq b, \qquad a \leqq a_1 < b-c_2 < 2b-a-c_1-c_2 = b_1 \leqq b.$$

Then

$$\lim_{n=\infty} \int_{-c_1}^{+c_2} \varphi_n(t)\,dt = \lim_{n=\infty} \left[\int_{a-(a+c_1)}^{b-(a+c_1)} \varphi_n(t)\,dt + \int_{b_1-(b-c_2)}^{b-(b-c_2)} \varphi_n(t)\,dt \right] = 1.$$

7. Here I will add some illustrative examples:

I. (i) If we put

$$\varphi_n(t) = n\,|t|\,e^{-nt^2},$$

then

$$\lim_{n=\infty} \int_0^c \varphi_n(t)\,dt = \frac{1}{2} \qquad \text{for} \qquad c>0$$

uniformly. Since $\varphi_n(t)$ is even, $F_n[f(x)]$ has the form

$$n\int_a^b f(t)\,|t-x|\,e^{-n(t-x)^2}\,dt,$$

where a and b are any two fixed numbers such that $a<b$.

In similar ways we can take the following forms as approximating functions:

$$\frac{2n^2}{\pi}\int_a^b f(t)\frac{|t-x|}{1+n^4(t-x)^4}\,dt, \qquad a<b;$$

$$\frac{9n^2}{2\sqrt{3}\,\pi}\int_a^b f(t)\frac{|t-x|}{1+n^3|t-x|^3}\,dt, \qquad a<b,$$

etc. (1).

(ii) Also the function $\varphi_n(t)$, which is exceedingly simple, defined by (2)

$$\begin{aligned}\varphi_n(t)&=0 && -\infty<t<-\frac{1}{n},\\ &=\frac{n}{2} && -\frac{1}{n}\leqq t\leqq\frac{1}{n},\end{aligned}$$

(1) We can find many examples of positive functions such that

$$\lim_{n=\infty}\varphi_n(t)=0. \qquad \lim_{n=\infty}\int_0^c \varphi_n(t)\,dt=\text{a positive finite constant (independent of } c\text{)},$$

in Osgood, Bull. Amer. Math. Soc., (2), 3 (1896), p. 69; Osgood, Amer. Journ. of Math. 19 (1897), p. 155; Hayashi, Tôhoku Math. Journ, 2 (1912), p. 43.

(2) In this case if we take $a=0$, $b=1$,

$$F_n[f(t)]=\int_0^1 f(t)\,\varphi_n(t-x)\,dt=\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\varphi_n(t-x)\,dt=\frac{n}{2}\int_{-\frac{1}{n}}^{+\frac{1}{n}} f(x+t)\,dt,$$

$$\left(0<\frac{1}{n}<x<1-\frac{1}{n}\right).$$

In other words, the approximating function becomes the *integral mean* of $f(t)$ about the point $t=x$. If $f(x)$ satisfy the Lipschitz condition

$$|f(x_1)-f(x_2)|<\lambda\,|x_1-x_2|, \qquad 0<x_1,x_2<1,$$

then we can show that

$$|f(x)-F_n[f(x)]|\leqq\frac{2\lambda}{n}, \qquad \left(0<\frac{1}{n}<x<1-\frac{1}{n}\right).$$

$$=0 \qquad \frac{1}{n}<t<+\infty,$$

may be taken for our purpose; and similarly for

$$\begin{aligned}\varphi_n(t)&=0 && -\infty<t<0,\\ &=n && 0\leqq t\leqq\frac{1}{n},\\ &=0 && \frac{1}{n}<t<+\infty;\end{aligned}$$

or

$$\begin{aligned}\varphi_n(t)&=0 && -\infty<t<-\frac{1}{n},\\ &=\frac{3}{4}n(1-n^2t^2) && -\frac{1}{n}\leqq t\leqq\frac{1}{n},\\ &=0 && \frac{1}{n}<t<+\infty,\end{aligned}$$

etc..

II. The following are some classical approximating functions belonging to our type:

(i) $$\frac{\int_a^b f(t)\,[1-(t-x)^2]^n\,dt}{2\int_0^1(1-t^2)^n\,dt}, \qquad (b-a\leqq 1).$$

When n is taken to be a positive integer, this function becomes a *polynomial* (Landau's polynomial in the case where $a=0$, $b=1$) (1).

(ii) *Trigonometrical polynomials:*

$$\frac{1}{2n\pi}\int_a^b f(t)\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt, \qquad (b-a\leqq 2\pi,\ n=1,2\cdots\cdots).$$

(Fejér's integral in the case where $a=0$, $b=2\pi$).

$$\frac{\int_a^b f(t)\left[\cos\frac{1}{2}(t-x)\right]^{2n} dt}{2\int_0^\pi\left[\cos\frac{t}{2}\right]^{2n} dt} \qquad (b-a\leqq 2\pi, \quad n=1,2,\cdots\cdots).$$

(1) Landau, "Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion," Rend. Palermo, 25 (1908), p. 337.

(Vallée Poussin's integral in the case where $a=-\pi$, $b=+\pi$) [1].

(iii) $$\frac{n}{\sqrt{\pi}}\int_a^b f(t)\, e^{-n^2(t-x)^2}\, dt, \quad (a<b)\ [2].$$

(Weierstrass' *transcendental integral function* in the case where $a=-\infty$, $b=+\infty$) [3].

8. Incidentally I will prove the following theorem whose application will be found in § 23 below.

Theorem IV. *Let $\varphi(t)$ and $\psi(t)$ be even, non-negative, limited and integrable in the interval $0 \leqq t \leqq b-a$; and let $\dfrac{\varphi(t)}{\psi(t)}$ varies monotonously when t increases from 0 to $\dfrac{b-a}{2}$. And let $f(x)$ be a function which is limited and integrable in the interval $a \leqq x \leqq b$, and*

$$f(x_1) < (or >) f(x_2) \quad when \quad \left| x_1 - \frac{a+b}{2} \right| > \left| x_2 - \frac{a+b}{2} \right|.$$

Then if $\dfrac{\varphi(t)}{\psi(t)}$ decrease,

$$\frac{\int_a^b f(t)\psi\left(t-\frac{a+b}{2}\right)dt}{\int_a^b \psi\left(t-\frac{a+b}{2}\right)dt} < (or >) \frac{\int_a^b f(t)\varphi\left(t-\frac{a+b}{2}\right)dt}{\int_a^q \varphi\left(t-\frac{a+b}{2}\right)dt} < (or >) f\left(\frac{a+b}{2}\right),$$

and if $\dfrac{\varphi(t)}{\psi(t)}$ increase,

$$\frac{\int_a^b f(t)\varphi\left(t-\frac{a+b}{2}\right)dt}{\int_a^b \varphi\left(t-\frac{a+b}{2}\right)dt} < (or >) \frac{\int_a^b f(t)\psi\left(t-\frac{a+b}{2}\right)dt}{\int_a^b \psi\left(t-\frac{a+b}{2}\right)dt} < (or >) f\left(\frac{a+b}{2}\right).$$

I will now confine myself to the case where $\dfrac{\varphi(t)}{\psi(t)}$ decreases when t

(1) Vallée Poussin, "Sur l'approximation des fonctions d'une variable réelle, etc.," Bull. Acad. Belgique, (1908), p. 193.

(2) Ford, loc. cit. (the latter), p. 401.

(3) Weierstrass, "Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen," Sitz. Akad. Berlin (1885), p. 633, p. 789. See also Maurer, "Über die Mittelwerthe der Funktion einer reellen Variabeln." Math. Ann. 47 (1896), p. 263; Lebesgue, loc. cit., p. 91.

increases from 0 to $\dfrac{b-a}{2}$ and $f(x_1)<f(x_2)$ for $\left|x_1-\dfrac{a+b}{2}\right|>\left|x_2-\dfrac{a+b}{2}\right|$.

When t increases from a to $\dfrac{a+b}{2}$, $t-\dfrac{a+b}{2}$ increases from $-\dfrac{b-a}{2}$ to 0; and since $\varphi(t)$ and $\psi(t)$ are even, $\dfrac{\varphi\left(t-\frac{a+b}{2}\right)}{\psi\left(t-\frac{a+b}{2}\right)}$ increases monotonously.

Moreover $f(t)$ increases monotonously when t increases from a to $\dfrac{a+b}{2}$. Consequently by an extension of Tchebycheff's inequality [1] we have

$$\frac{\int_a^{\frac{a+b}{2}} f(t)\psi\left(t-\frac{a+b}{2}\right)dt}{\int_a^{\frac{a+b}{2}} \psi\left(t-\frac{a+b}{2}\right)dt} < \frac{\int_a^{\frac{a+b}{2}} f(t)\varphi\left(t-\frac{a+b}{2}\right)dt}{\int_a^{\frac{a+b}{2}} \varphi\left(t-\frac{a+b}{2}\right)dt}.$$

In a similar way we can prove that

$$\frac{\int_{\frac{a+b}{2}}^b f(t)\psi\left(t-\frac{a+b}{2}\right)dt}{\int_{\frac{a+b}{2}}^b \psi\left(t-\frac{a+b}{2}\right)dt} < \frac{\int_{\frac{a+b}{2}}^b f(t)\varphi\left(t-\frac{a+b}{2}\right)dt}{\int_{\frac{a+b}{2}}^b \varphi\left(t-\frac{a+b}{2}\right)dt}.$$

Adding these inequalities and remembering

$$\int_a^{\frac{a+b}{2}} \frac{\varphi}{\psi}\left(t-\frac{a+b}{2}\right)dt = \int_{\frac{a+b}{2}}^b \frac{\varphi}{\psi}\left(t-\frac{a+b}{2}\right)dt$$
$$= \frac{1}{2}\int_a^b \frac{\varphi}{\psi}\left(t-\frac{a+b}{2}\right)dt,$$

we obtain

$$\frac{\int_a^b f(t)\psi\left(t-\frac{a+b}{2}\right)dt}{\int_a^b \psi\left(t-\frac{a+b}{2}\right)dt} < \frac{\int_a^b f(t)\varphi\left(t-\frac{a+b}{2}\right)dt}{\int_a^b \varphi\left(t-\frac{a+b}{2}\right)dt}.$$

(1) Correspondance d'Hermite et de Stieltjes, II, p. 193 (The letter of Stieltjes, dated December 2, 1891). See also Fujiwara, "Ein von Brunn vermuteter Satz, u.s.w.," Tohoku Math. Journ., 13 (1918), p. 231.

Lastly since

$$f(t)<f\left(\frac{a+b}{2}\right), \quad \text{for} \quad a\leqq t<\frac{a+b}{2}, \quad \frac{a+b}{2}<t\leqq b,$$

we have

$$\int_a^b\left[f(t)-f\left(\frac{a+b}{2}\right)\right]\varphi\left(t-\frac{a+b}{2}\right)dt<0,$$

which proves the theorem.

For example, if we put

$$\varphi(t)=\frac{(1-t^4)^n}{2\int_0^1(1-t^4)^n dt}, \quad \psi(t)=\frac{(1-t^2)^n}{2\int_0^1(1-t^2)^n dt}, \quad (n>0),$$

we get

$$\frac{\varphi(t)}{\psi(t)}=\frac{\int_0^1(1-t^2)^n dt}{\int_0^1(1-t^4)^n dt}\cdot(1+t^2)^n,$$

which increases monotonously when t increases from 0 to 1. Hence

$$\left|f\left(\frac{1}{2}\right)-\frac{\int_0^1 f(t)\,\varphi\left(t-\frac{1}{2}\right)dt}{\int_0^1\varphi\left(t-\frac{1}{2}\right)dt}\right|>\left|f\left(\frac{1}{2}\right)-\frac{\int_0^1 f(t)\,\psi\left(t-\frac{1}{2}\right)dt}{\int_0^1\psi\left(t-\frac{1}{2}\right)dt}\right|.$$

Again if we put

$$\varphi(t)=\frac{(1-t^2)^{n+1}}{2\int_0^1(1-t^2)^{n+1}dt}, \quad \psi(t)=\frac{(1-t^2)^n}{2\int_0^1(1-t^2)^n dt}, \quad (n>0),$$

we obtain

$$\frac{\varphi(t)}{\psi(t)}=\frac{\int_0^1(1-t^2)^n dt}{\int_0^1(1-t^2)^{n+1}dt}\cdot(1-t^2),$$

which decreases monotonously when t increases from 0 to 1. Therefore

$$\left|f\left(\frac{1}{2}\right)-\frac{\int_0^1 f(t)\,\varphi\left(t-\frac{1}{2}\right)dt}{\int_0^1\varphi\left(t-\frac{1}{2}\right)dt}\right|<\left|f\left(\frac{1}{2}\right)-\frac{\int_0^1 f(t)\,\psi\left(t-\frac{1}{2}\right)dt}{\int_0^1\psi\left(t-\frac{1}{2}\right)dt}\right|.$$

9. Now we pass to the differentiation of the function

$$\int_a^b f(t)\,\varphi_n(t-x)\,dt.$$

Let $\varphi_n(t)$ be a function which is limited and non-negative, and has the continuous derivative $\varphi_n'(t)$ in $a-b\leqq t\leqq b-a$ and moreover possesses the property

$$(5) \qquad \lim_{n=\infty}\int_{-c_2}^{+c_2}\varphi_n(t)\,dt=1, \qquad a-b<-c_1<0<c_2<b-a$$

uniformly. Again let $f(x)$ have the continuous derivative $f'(x)$ in $a\leqq x\leqq b$.

Then from

$$(6) \qquad F_n[f(x)]=\int_a^b f(t)\,\varphi_n(t-x)\,dt, \quad a\leqq x\leqq b,$$

we have

$$\frac{d}{dx}F_n[f(x)]=\int_a^b f(t)\frac{\partial}{\partial x}\varphi_n(t-x)\,dt.$$

Since

$$\int_a^b f'(t)\,\varphi_n(t-x)\,dt=\Big[f(t)\,\varphi_n(t-x)\Big]_{t=a}^{t=b}-\int_a^b f(t)\frac{\partial}{\partial t}\varphi_n(t-x)\,dt$$

$$=f(b)\,\varphi_n(b-x)-f(a)\,\varphi_n(a-x)+\int_a^b f(t)\frac{\partial}{\partial t}\varphi_n(t-x)\,dt,$$

we obtain

$$(7) \qquad \frac{\partial}{\partial x}F_n[f(x)]=\int_a^b f'(t)\varphi_n(t-x)dt+f(a)\varphi_n(a-x)-f(b)\varphi_n(b-x).$$

But by the assumption that $f(x)$ is continuous in the interval $a\leqq x\leqq b$, we find from Theorem III that

$$\lim_{n=\infty}\int_a^b f'(t)\,\varphi_n(t-x)\,dt=f'(x)$$

uniformly for the interval $a<x<b$; whence

(8)

$$\lim_{n=\infty}\begin{matrix}\text{sup.}\\\text{inf.}\end{matrix}\frac{\partial}{\partial x}F_n[f(x)]=f'(x)+\lim_{n=\infty}\begin{matrix}\text{sup.}\\\text{inf.}\end{matrix}[f(a)\varphi_n(a-x)-f(b)\varphi_n(b-x)].$$

From this we have the theorem immediately:

Theorem V. *Let $\varphi_n(t)$ be a function which satisfies the conditions cited above and further*

$$\lim_{n=\infty}\varphi_n(t)=0\ (^1)$$

(1) See the foot-note in § 8, I (i).

uniformly for

$$a-b<t<0 \quad \text{and} \quad 0<t<b-a.$$

If $f(x)$ have the continuous derivative $f'(x)$ in $a\leqq x\leqq b$, and put

$$F_n[f(x)]=\int_a f(t)\,\varphi_n(t-x)\,dt,$$

then

$$\lim_{n=\infty}\frac{\partial}{\partial x}F_n[f(x)]=f'(x)$$

uniformly in the interval $a<x<b$.

As an example, if we take

$$\varphi_n(t)=\frac{1}{2n\pi}\left[\frac{\sin n\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 \qquad -2\pi<t<0,\ 0<t<2\pi,$$

$$=\frac{n}{2\pi} \qquad t=0,\ 2\pi,$$

we have

$$\lim_{n=\infty}\varphi_n(t)=0, \quad -2\pi<t<0, \quad 0<t<2\pi$$

uniformly. Hence the approximating function [*the first mean of* Fourier's *series* corresponding to $f(x)$] has the properties

$$\lim_{n=\infty}F_n[f(x)]=f(x), \qquad 0<x<2\pi,$$

$$\lim_{n=\infty}\frac{\partial}{\partial x}F_n[f(x)]=f'(x), \qquad 0<x<2\pi$$

uniformly, provided that $f'(x)$ is continuous in the interval $0\leqq x\leqq 2\pi$. This is the result which was already proved by Prof. Fejér (1).

As the second example, if we take

$$\varphi_n(t)=\sqrt{\frac{n}{\pi}}(1-t^2)^n, \qquad 0\leqq t\leqq 1,$$

we have

$$\lim_{n=\infty}\varphi_n(t)=0, \qquad -1<t<0,\ 0<t<1$$

uniformly. It follows from this the following theorem at once, which was shown by Prof. Vallée Poussin (2): If $L_n(x)$ be Landau's

(1) Fejér, loc. cit., p. 61.

(2) Vallée Poussin, Cours d'analyse infinitésimale, t 2, 2. ed. (1912), p. 129.

polynomial, then

$$\lim_{n=\infty}L_n(x)=f(x), \qquad 0<x<1,$$

$$\lim_{n=\infty}\frac{d}{dx}L_n(x)=f'(x), \qquad 0<x<1$$

uniformly, provided that $f'(x)$ is continuous in the interval $0\leqq x\leqq 1$.

10. I. *Let $\varphi_n(t)$ be a function which is limited and non-negative, and has the continuous derivative in $a-b\leqq t\leqq b-a$, and moreover possesses the property*

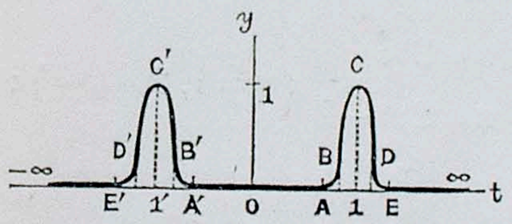
$$\lim_{n=\infty}\int_{-c_1}^{+c_2}\varphi_n(t)\,dt=\frac{1}{2}, \quad a-b<-c_1<0<c_2<b-a$$

uniformly. Let $f(x)$ have the continuous derivative in $a\leqq x\leqq b$, and moreover let $\lim\limits_{n=\infty}\frac{d}{dx}F_n[f(x)]$ exist. But $f'(x)$ is not equal to $\lim\limits_{n=\infty}\frac{d}{dx}F_n[f(x)]$ throughout the interval $a<x<b$ in general.

In order to show this, take the upper half of the $(t,y)-$ plane, and consider the parabola passing through

$$\left(1-\frac{1}{n},0\right),\ (1,1),\ \left(1+\frac{1}{n},0\right)$$

and having $(1,1)$ as the vertex. Draw the two circular arcs which touch the parabola and also touch the $t-$axis at $\left(1-\frac{2}{n},0\right)$ and $\left(1+\frac{2}{n},0\right)$ respectively; then construct the curve $0ABCDE\infty$ as indicated in the figure.



This curve combined with its image $-\infty E'D'C'B'A'$ with respect to the $y-$axis is taken for the graph of the function

$$y=\psi_n(t), \qquad -\infty<t<+\infty.$$

Then $\psi_n(t)$ is limited, even and non-negative, and its derivative is continuous, and

$$\lim_{n=\infty}\psi_n(t)=1 \qquad |t|=1$$

$$=0 \qquad |t|\neq 1;$$

and

$$\lim_{n=\infty}\int_0^c\psi_n(t)\,dt=0 \qquad c\neq 0$$

uniformly [1].

Now define the function $\varphi_n(t)$ by

$$\varphi_n(t)=n\,|\,t\,|\,e^{-nt^2}+\psi_n(t).$$

Then $\varphi_n(t)$ is limited, even and non-negative, and its derivative is continuous, and

$$\lim_{n=\infty}\varphi_n(t)=1 \qquad |\,t\,|=1$$
$$=0 \qquad |\,t\,|\neq 1;$$

and moreover

$$\lim_{n=\infty}\int_0^c \varphi_n(t)\,dt=\frac{1}{2}, \qquad c>0$$

uniformly. Consequently (8) becomes

$$\lim_{n=\infty}\frac{d}{dx}F_n[f(x)]=f'(x) \qquad \text{when } |\,x\,|\neq 1$$
$$=f'(x)+f(0)-f(1) \qquad \text{when } |\,x\,|=1.$$

II. *Let $\varphi_n(t)$ be a function which satisfies the conditions imposed in* I; *and let $f(x)$ have the continuous derivative in $a\leqq x\leqq b$. Then the $\frac{d}{dx}F_n[f(x)]$ $(a<x<b)$ does not converge in general.*

To show this, consider the function $\chi_n(t)$ $(n=1, 2, \cdots\cdots)$ such that

$$\chi_n(t)=\psi_{2m}(t) \quad \text{defined in I. above} \quad \text{when } n=2m$$
$$=0 \quad \text{when } n=2m-1 \qquad (m=1, 2, \cdots\cdots);$$

and put

$$\varphi_n(t)=n\,|\,t\,|\,e^{-nt^2}+\chi_n(t), \quad (n=1, 2, \cdots\cdots).$$

Then $\varphi_n(t)$ is limited, even and non-negative, and its derivative is continuous, and moreover

$$\lim_{n=\infty}\int_0^c \varphi_n(t)\,dt=\frac{1}{2}, \qquad c>0$$

(1) We may take

$$\psi_n(t)=\frac{1}{2}\left\{1+\cos\left[\frac{n}{2}\Big(|\,t\,|-1\Big)\right]\right\} \quad \text{for } 1-\frac{2\pi}{n}\leqq|\,t\,|\leqq 1+\frac{2\pi}{n} \qquad (n=1, 2, \ldots.),$$
$$=0 \qquad \text{for } 0\leqq|\,t\,|\leqq 1-\frac{2\pi}{n}, \quad 1+\frac{2\pi}{n}<|\,t\,|<+\infty,$$

instead of the function defined in the text.

uniformly; and

$$\lim_{n=\infty}\varphi_n(t)=0 \qquad \text{when } |\,t\,|\neq 1,$$

but the sequence $\varphi_n(t)$ oscillates between 0 and 1 when $|\,t\,|=1$.

Consequently when $|\,x\,|\neq 1$, $\frac{d}{dx}F_n[f(x)]$ converges to $f'(x)$; on the contrary when $|\,x\,|=1$, this sequence oscillates between $f'(x)$ and $f'(x)+f(0)-f(1)$.

11. On the other hand I can state the theorem:

Theorem VI. *Let $\varphi_n(t)$ be a function which satisfies the conditions imposed in* 1 *of* § 10. *If $f(x)$ have the continuous derivative in the interval $a\leqq x\leqq b$, then $\frac{d}{dx}F_n[f(x)]$ converges strongly with respect to exponent* 1 *to the function $f'(x)$ in the interval $a<a_2\leqq x\leqq b_2<b$; that is,*

$$\lim_{n=\infty}\int_{a_2}^{b_2}\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|dx=0.$$

Since $f'(x)$ is continuous in the interval $a\leqq x\leqq b$, it follows from Theorem I that, if we put

$$\int_a^b f'(t)\,\varphi_n(t-x)\,dt=f'(x)+\eta_n(x),$$

we can find a positive number N independent of x $(a<a_2\leqq x\leqq b_2<b)$, corresponding to any assigned positive number ε, such that

$$|\,\eta_n(x)\,|<\varepsilon, \qquad n>N.$$

Hence we have from (7)

$$(9)\quad \left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|<\varepsilon+|f(a)|\cdot\varphi_n(a-x)+|f(b)|\cdot\varphi_n(b-x),$$
$$a<a_2\leqq x\leqq b_2<b, \qquad n>N.$$

Now from the supposition that

$$\lim_{n=\infty}\int_{-c_1}^{+c_2}\varphi_n(t)\,dt=1, \quad a-b<-c_1<0<c_2<b-a,$$

we have

$$\lim_{n=\infty}\int_{a_2}^{b_2}\varphi_n(a-x)\,dx=-\lim_{n=\infty}\left[\int_{a-a_2}^{c_2}\varphi_n(t)\,dt+\int_{c_2}^{a-b_2}\varphi_n(t)\,dt\right]$$
$$=-1+1=0;$$

and similarly

$$\lim_{n=\infty}\int_{a_2}^{b_2}\varphi_n(b-x)\,dx=0.$$

Hence

$$\lim_{n=\infty}\int_{a_2}^{b_2}\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|dx\leqq\varepsilon(b_2-a_2).$$

Since ε may be taken arbitrarily small, we obtain

$$\lim_{n=\infty}\int_{a_2}^{b_2}\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|dx=0,$$

which proves the theorem.

Further we have the theorem:

Theorem VII. *Let $\varphi_n(t)$ be a function which satisfies the conditions imposed in* I. *of* § 10, *and moreover let*

$$|\varphi_n(t)|<K \quad for \quad a-b<t<0,\ 0<t<b-a,$$

where K is a finite constant independent of n and t. If $f(x)$ have the continuous derivative in $a\leqq x\leqq b$, then $\dfrac{d}{dx}F_n[f(x)]$ converges strongly with respect to exponent 2 *to the function $f'(x)$ in the interval*

$$a<a_2\leqq x\leqq b_2<b.$$

And there exists a sequence of positive integers

$$n_1<n_2<\cdots\cdots<n_k<\cdots\cdots,$$

for which $\dfrac{d}{dx}F_{n_k}[f(x)]$ converges to $f'(x)$ essential-uniformly in the interval $a<a_2\leqq x\leqq b_2<b$.

From the inequalities (9) and

$$|\varphi_n(t)|<K \quad \text{for} \quad a-b<t<0,\ 0<t<b-a,$$

we obtain

$$\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|^2$$
$$<\varepsilon^2+2\varepsilon K[|f(a)|+|f(b)|]+\{|f(a)|\varphi_n(a-x)+|f(b)|\varphi_n(b-x)\}^2$$
$$<\varepsilon^2+2\varepsilon K[|f(a)|+|f(b)|]+K|f(a)|^2\varphi_n(a-x)$$
$$+2K|f(a)|\cdot|f(b)|\cdot\varphi_n(a-x)+K|f(b)|^2\varphi_n(b-x),$$
$$a<a_2\leqq x\leqq b_2<b,\quad n>N.$$

But since

$$\lim_{n=\infty}\int_{a_2}^{b_2}\varphi_n(a-x)\,dx=\lim_{n=\infty}\int_{a_2}^{b_2}\varphi_n(b-x)\,dx=0,$$

we have

$$\lim_{n=\infty}\int_{a_2}^{b_2}\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|^2dx\leqq\left\{\varepsilon^2+2\varepsilon K[|f(a)|+|f(b)|]\right\}(b_2-a_2);$$

i.e.

$$\lim_{n=\infty}\int_{a_2}^{b_2}\left|\frac{d}{dx}F_n[f(x)]-f'(x)\right|^2dx=0,$$

which proves the first part of the theorem. In other words, $\dfrac{\partial}{\partial x}F_n[f(x)]$ *converges on the average* (in the sense of Prof. E. Fischer) *to $f'(x)$ in the interval $a<a_2\leqq x\leqq b_2<b$.*

The second part of the theorem follows from this immediately by a theorem due to Prof. Weyl[1].

12. I pass now to the integration of approximating functions.

Theorem VIII. *Let $\varphi_n(t)$ be a function which satisfies the conditions imposed in* I. *of* § 10. *If $f(x)$ be limited and integrable in the interval $a\leqq x\leqq b$, then $\int_a^x F_n[f(x)]\,dx$ converges uniformly to $\int_a^x f(x)\,dx$ in the interval $a<a$, $x<b$.*

If we put

$$\int_a^t f(t)\,dt=\psi(t),$$
$$a<a\leqq x<b,$$
$$\int_a^x\varphi_n(t-x)\,dx=\Phi_n(t,x)$$

$f(t)$ is limited and integrable in $a\leqq t\leqq b$, and hence $\psi(t)$ is continuous in the same interval; so that

$$\int_a^b f(t)\,\Phi_n(t,x)\,dt=\Big[\psi(t)\,\Phi_n(t,x)\Big]_{t=a}^{t=b}-\int_a^b\psi(t)\frac{\partial}{\partial t}\Phi_n(t,x)\,dt.$$

But since

$$\frac{\partial}{\partial t}\Phi_n(t,x)=\int_a^x\frac{\partial}{\partial t}\varphi_n(t-x)\,dx=-\int_a^x\frac{\partial}{\partial x}\varphi_n(t-x)\,dx$$
$$=\varphi_n(t-a)-\varphi_n(t-x),$$

(1) Weyl, "Über die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten," Math. Ann., 67 (1909), p. 243.

the above equation becomes

$$\int_a^b \psi(t)\,\varphi_n(t-x)\,dt = \int_a^b f(t)\,\Phi_n(t,x)\,dt + \int_a^b \psi(t)\,\varphi_n(t-a)\,dt$$
$$+\psi(a)\,\Phi_n(a,x) - \psi(b)\,\Phi_n(b,x).$$

Now

$$\lim_{n=\infty} \Phi_n(a,x) = \lim_{n=\infty} \int_a^x \varphi_n(a-x)\,dx$$
$$= -\lim_{n=\infty} \left[\int_{a-a}^{c_2} \varphi_n(t)\,dt + \int_{c_2}^{a-x} \varphi_n(t)\,dt \right]$$
$$= -1+1=0\,;$$

similarly

$$\lim_{n=\infty} \Phi_n(b,x) = 0.$$

Since $\psi(t)$ is continuous in the interval $a \leqq t \leqq b$,

$$\int_a^b \psi(t)\,\varphi_n(t-x)\,dt$$

converges uniformly to $\psi(x)$ $(a<x<b)$; consequently

$$\lim_{n=\infty} \int_a^b \psi(t)\,\varphi_n(t-a)\,dt = \psi(a) = 0.$$

Therefore

$$\lim_{n=\infty} \int_a^b f(t)\,\Phi_n(t,x)\,dt = \psi(x)$$

uniformly for $a<x<b$, which proves the theorem.

Convergency of interpolating functions.

13. Take four fixed numbers a_1, a_2, b_2, b_1 in a given finite interval (a, b) such that

$$a \leqq a_1 < a_2 < b_2 < b_1 \leqq b\,;$$

and let us put

$$t_i = a + \frac{b-a}{n} i, \quad i=0, 1, 2, \cdots, n\,;$$

and determine positive integers $\mu_1, \mu_2, \nu_2, \nu_1$ such that

$$t_{\mu_1} \leqq a_1 < t_{\mu_1+1}, \qquad t_{\mu_2} \leqq a_1 < t_{\mu_1+1},$$
$$t_{\nu_2} < b_2 \leqq t_{\nu_2+1}, \qquad t_{\nu_1} < b_1 \leqq t_{\nu_1+1},$$

these integers being functions of n. Lastly let γ_0, γ_1 be two fixed non-negative integers independent of n (1).

Now I can state the following theorem after the analogy of Theorem I and II:

Theorem IX. *Let $\varphi_n(x, t)$ be limited and non-negative in the domain $a \leqq x,\ t \leqq b$ and let both*

$$\frac{\sum_{i=\gamma_0}^{\mu_1} \varphi_n(x, t_i)}{\sum_{i=\gamma_0}^{n-\gamma_1} \varphi_n(x, t_i)} \quad \text{and} \quad \frac{\sum_{i=\nu_1+1}^{n-\gamma_1} \varphi_n(x, t_i)}{\sum_{i=\gamma_0}^{n-\gamma_1} \varphi_n(x, t_i)}$$

converge uniformly to zero for $a_2 \leqq x \leqq b_2$, when the positive integer n increases indefinitely. Again let $f(x)$ be an arbitrary function which is limited in the interval $a \leqq x \leqq b$, and M_1 be the fluctuation of $f(x)$ in the interval $a_1 \leqq x \leqq b_1$, and δ be a positive number assigned at pleasure.

If we take the sequence of interpolating functions

$$\Psi_n[f(x)] = \frac{\sum_{i=\gamma_0}^{n-\gamma_1} f(t_i)\,\varphi_n(x, t_i)}{\sum_{i=\gamma_0}^{n-\gamma_1} \varphi_n(x, t_i)} \tag{10}$$

we can find a positive integer N, independent of x, such that

$$|f(x) - \Psi_n[f(x)]| < M_1 + \delta, \quad n > N$$

in the interval

$$a_2 \leqq x \leqq b_2.$$

If $f(x)$ be continuous at $x = x_0$ $(a < x_0 < b)$, then $\Psi_n[f(x)]$ converges to $f(x)$ uniformly in the vicinity of that point.

Since

$$f(x) = \frac{\sum_{i=\gamma_0}^{n-\gamma_1} f(x)\,\varphi_n(x, t_i)}{\sum_{i=\gamma_0}^{n-\gamma_1} \varphi_n(x, t_i)},$$

we have

(1) We may assume that

$$\gamma_0 \leqq \mu_1, \qquad \nu_1 + 1 \leqq n - \gamma_1.$$

$$f(x)-\Psi_n[f(x)]=\frac{\sum\limits_{i=\gamma_0}^{n-\gamma_1}[f(x)-f(t_i)]\,\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}.$$

Let M be the fluctuation of $f(x)$ in the interval $a\leqq x\leqq b$; then

$$|f(x)-f(t_i)|\leqq M, \quad a\leqq x\leqq b;$$

so that

$$|f(x)-\Psi_n[f(x)]|\leqq M, \quad a\leqq x\leqq b.$$

Also since

$$|f(x)-f(t_i)|\leqq M_1, \quad a_1\leqq x\leqq b_1, \quad \mu_1+1\leqq i\leqq \nu_1,$$

we get

$$\left|\frac{\sum\limits_{i=\mu_1+1}^{\nu_1}[f(x)-f(t_i)]\,\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}\right|\leqq M_1\frac{\sum\limits_{i=\mu_1+1}^{\nu_1}\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}\leqq M_1$$

for $a_1\leqq x\leqq b_1$.

But by the assumption we can find a positive integer N independent of x, corresponding to any assigned positive number δ such that

$$\left|\frac{\sum\limits_{i=\gamma_0}^{\mu_1}[f(x)-f(t_i)]\,\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}\right|\leqq M\frac{\sum\limits_{i=\gamma_0}^{\mu_1}\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}<\frac{\delta}{2}, \quad n>N,$$

$$\left|\frac{\sum\limits_{i=\nu_1+1}^{n-\gamma_1}[f(x)-f(t_i)]\,\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}\right|\leqq M\frac{\sum\limits_{i=\nu_1+1}^{n-\gamma_1}\varphi_n(x,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(x,t_i)}<\frac{\delta}{2}, \quad n>N$$

in the interval $a_2\leqq x\leqq b_2$.

Consequently

$$|f(x)-\Psi_n[f(x)]|<M_1+\delta, \quad n>N$$

for $a_2\leqq x\leqq b_2$.

The latter part of the theorem can be proved in the same way as in § 1.

Theorem X. *If $\varphi_n(x,t)$ satisfy the conditions imposed in Theorem IX, and if $f(x)$ be limited and integrable in the interval $a\leqq x\leqq b$, then the sequence $\Psi_n[f(x)]$ converges strongly to the function $f(x)$ with respect to any positive exponent in the interval $a\leqq x\leqq b$. Also the sequence converges uniformly to $f(x)$ in the interval $a<a_2\leqq x\leqq b_2<b$, except a set of points of measure zero.*

As an example, consider the function

$$\varphi_n(x,t)=\left[\frac{\sin\frac{n}{2}(t-x)}{n\sin\frac{1}{2}(t-x)}\right]^2 \quad \text{when} \quad x\neq t,$$
$$=1 \quad \text{when} \quad x=t;$$

and put

$$a=0, \quad b=2\pi, \quad t_i=\frac{2\pi i}{n}, \quad \gamma_0=1, \quad \gamma_1=0.$$

Then it is easily seen that

$$\sum_{i=1}^{n}\varphi_n(x,t_i)=1.$$

Next for $\mu_1(\nu_1)$ and x such that

$$t_{\mu_1}\leqq a_1<a_2\leqq x\leqq b_2<b_1\leqq t_{\nu_1+1}$$

we can find a fixed positive quantity ω for which

$$\left|\sin\frac{1}{2}(t_i-x)\right|>\omega.$$

Hence

$$\frac{1}{n^2}\sum_{i=1}^{\mu_1}\left[\frac{\sin\frac{n}{2}(t_i-x)}{\sin\frac{1}{2}(t_i-x)}\right]^2<\frac{1}{\omega^2n^2}\sum_{i=1}^{\mu_1}\left[\sin\frac{n}{2}(t_i-x)\right]^2$$
$$<\frac{\mu_1}{\omega^2n^2}<\frac{1}{\omega^2 n};$$

so that

$$\lim_{n=\infty}\sum_{i=1}^{\mu_1}\varphi_n(x,t_i)=0, \quad a_2\leqq x\leqq b_2$$

uniformly. Similarly

$$\lim_{n=\infty}\sum_{i=\nu_1+1}^{n}\varphi_n(x,t_i)=0,\qquad a_2\leqq x\leqq b_2$$

uniformly.

Hence we have *the formula of trigonometric interpolation* :

$$\Psi_n[f(x)]=\sum_{i=1}^{n}f\left(\frac{2\pi i}{n}\right)\left[\frac{\sin\frac{n}{2}\left(\frac{2\pi i}{n}-x\right)}{n\sin\frac{1}{2}\left(\frac{2\pi i}{n}-x\right)}\right]^2,$$

which is due to Prof. R. D. Jackson (1).

14. As an application of Theorems IX and X, I will establish the following theorem which is analogous to Theorem III :

Theorem XI. *Let $\varphi_n(x)$ be non-negative and limited in the interval $a-b\leqq x\leqq b-a$, and let h, k be any positive integers which satisfy*

$$a+\gamma\leqq t_h,\ t_k\leqq b-\delta,$$

γ, δ being any assigned small positive numbers. Moreover assume that there exists a positive integer N independent of h, k, corresponding to a given arbitrary positive quantity η, such that

$$\text{(11)}\qquad \left|1-\sum_{i=-h}^{+k}\varphi_n\left(v+\frac{b-a}{n}i\right)\right|<\eta,\quad n>N$$

for

$$\frac{a-b}{n}<v\leqq 0\quad\left(or\quad 0\leqq v<\frac{b-a}{n}\right).$$

If $f(x)$ be limited and integrable in the interval $a\leqq x\leqq b$, then the sequence

$$\text{(12)}\qquad \Psi_n[f(x)]=\sum_{i=r_0}^{n-r_1}f(t_i)\,\varphi_n(t_i-x),\quad a\leqq x\leqq b$$

converges strongly with respect to any positive exponent to $f(x)$.

If $f(x)$ be continuous at $x=x_0$ $(a<x_0<b)$, then the above sequence converges uniformly to $f(x_0)$ in the vicinity of that point (2).

Suppose that x is an inner point of the interval (a, b). We can

(1) R. D. Jackson, "A formula of trigonometric interpolation," Rend. Palermo, 37 (1914), p. 371.

(2) Compare with Jackson, "Note on trigonometric interpolation," Rend. Palermo. 39 (1915), p. 230.

take four fixed numbers a_1, a_2, b_2, b_1 such that $a<a_1<a_2<x<b_2<b_1<b$, and then $\mu_1, \mu_2, \nu_2, \nu_1$ as before (§ 13). Now let

$$t_\lambda\leqq x<t_{\lambda+1};$$

then

$$\frac{a-b}{n}<t_\lambda-x\leqq 0\quad\text{and}\quad a_2\leqq t_\lambda<b_2.$$

If we choose γ and δ such that

$$a_1-a<2\lambda, 2\delta;\ a_2-a_1>\gamma, \delta;\ b-b_2>2\gamma, 2\delta;\ b_1-b_2>\gamma, \delta;$$

and take a positive integer n_0 for which

$$t_{\lambda-1}>a_1+\gamma,\ t_{\lambda-r_0}\geqq a_2-\gamma,\ t_{n-r_1}>b_2+\gamma,\quad n>n_0,$$

then

$$a+\gamma<a_2-\gamma\leqq t_{\lambda-r_0}<b_2<b-\delta,$$

$$a+\gamma<a+t_{n-r_1}-b_2\leqq a+t_{n-r_1}-t_\lambda$$

$$=t_{n-r_1-\lambda}=t_{n-r_1}-\frac{b-a}{n}\lambda\leqq b-(a_2-a)<b-\delta,$$

$$a+\gamma<a+t_{\lambda-1}-a_1\leqq a+t_{\lambda-1}-t_{\mu_1}$$

$$=t_{\lambda-\mu_1}-1=t_\lambda-\frac{b-a}{n}(\mu_1+1)\leqq b-(a_1-a)<b-\delta,$$

$$n>n_0;$$

that is,

$$a+\gamma<t_{\lambda-r_0},\ t_{n-r_1-\lambda},\ t_{\lambda-\mu_1-1}<b-\delta,\quad n>n_0.$$

Consequently it follows from the assumption that there exists a positive integer N independent of λ (that is, independent of x), such that

$$\left|1-\sum_{i=-(\lambda-r_0)}^{n-r_1-\lambda}\varphi_n\left[(t_\lambda-x)+\frac{b-a}{n}i\right]\right|$$

$$=\left|1-\sum_{i=r_0}^{n-r_1}\varphi_n(t_i-x)\right|<\eta,\quad n>N(>n_0);$$

so that

$$\lim_{n=\infty}\sum_{i=r_0}^{n-r_1}\varphi_n(t_i-x)=1$$

uniformly for $a_2<x<b_2$.

Again

$$\left|1-\sum_{i=-(\lambda-\mu_1-1)}^{n+r_1-\lambda}\varphi_n\left[(t_i-x)+\frac{b-a}{n}i\right]\right.$$

$$=\left|1-\sum_{i=\mu_1+1}^{n-r_1}\varphi_n(t_i-x)\right|<\eta,\quad n>N(>n_0);$$

whence

$$\lim_{n=\infty}\sum_{i=\mu_1+1}^{n-r_1}\varphi_n(t_i-x)=1.\quad a_2<x<b_2$$

uniformly. Therefore

$$\lim_{n=\infty}\sum_{i=r_0}^{m_1}\varphi_n(t_i-x)=\lim_{n=\infty}\sum_{i=r_0}^{n-r_1}\varphi_n(t_i-x)-\lim_{n=\infty}\sum_{i=\mu_1+1}^{n-r_1}\varphi_n(t_i-x)=1-1=0$$

uniformly for $a_2<x<b_2$.

In a similar way we can show that

$$\lim_{n=\infty}\sum_{i=\nu_1+1}^{n-r_1}\varphi_n(t_i-x)=0$$

uniformly in the same interval.

Thus the theorem follows from Theorems IX and X immediately.

Lastly we remark that *when $\varphi_n(x)$ is an even function,* (11) *may be replaced by*

$$\left|\frac{1}{2}-\sum_{i=0}^{k}\varphi_n\left(v+\frac{b-a}{n}i\right)\right|<\eta,\quad n>N,\quad 0\leqq v<\frac{b-a}{n}.$$

15. Here I will add some illustrative examples:

I. Take the function defined by

$$\varphi_n(x)=\frac{1}{2}\qquad 0\leqq x<\frac{1}{n},$$

$$=0\qquad \frac{1}{n}\leqq x\leqq 1,$$

$$\varphi_n(-x)=\varphi_n(x);$$

and put $a=0$, $b=1$.

Then for $0\leqq v<\frac{1}{n}$ and $\frac{1}{n}<\gamma,\delta$ we have

$$\sum_{i=0}^{k}\varphi_n\left(v+\frac{i}{n}\right)=\varphi_n(v)=\frac{1}{2},$$

where $0<\gamma\leqq\frac{k}{n}\leqq 1-\delta<1$; so that there exists a positive integer N independent of k, such that

$$\sum_{i=0}^{k}\varphi_n\left(v+\frac{i}{n}\right)=\frac{1}{2},\quad n>N.$$

The interpolating function is

$$\Psi_n[f(x)]=\sum_{i=r_0}^{n-r_1}f\left(\frac{i}{n}\right)\varphi_n\left(\frac{i}{n}-x\right),\quad 0\leqq x\leqq 1\ (^1).$$

II. Next if we put

$$\varphi_n(x)=\frac{1}{4}\left[1+\cos\frac{n}{2}\left(x-\frac{2\pi}{n}\right)\right]$$

$$\text{for } 0\leqq x\leqq\frac{4\pi}{n},\quad 2\pi-\frac{4\pi}{n}\leqq x\leqq 2\pi,$$

$$=0\qquad\text{for }\frac{4\pi}{n}<x<2\pi-\frac{4\pi}{n},$$

$$\varphi_n(-x)=\varphi_n(x);$$

and put $a=0$, $b=2\pi$, then for $0\leqq v<\frac{4\pi}{n}<\gamma,\delta$ we have

$$\sum_{i=0}^{k}\varphi_n\left(v+\frac{2\pi i}{n}\right)=\varphi_n(v)+\varphi_n\left(v+\frac{2\pi}{n}\right)=\frac{1}{2},$$

where

$$0<\gamma\leqq\frac{2\pi k}{n}\leqq 2\pi-\delta<2\pi;$$

whence there exists a positive integer N independent of k such that

$$\sum_{i=0}^{k}\varphi_n\left(v+\frac{2\pi i}{n}\right)=\frac{1}{2},\qquad n>N.$$

Consequently the interpolation formula becomes

$$\Psi_n[f(x)]=\sum_{i=r_0}^{n-r_1}f\left(\frac{2\pi i}{n}\right)\varphi_n\left(\frac{2\pi i}{n}-x\right),\quad 0\leqq x\leqq 2\pi,$$

---

(1) If $\frac{\lambda}{n}<x_0<\frac{\lambda+1}{n}$, then $\Psi_n[f(x_0)]=\frac{1}{2}\left[f\left(\frac{\lambda}{n}\right)+f\left(\frac{\lambda+1}{n}\right)\right]$.

which is essentially the same as one used by Prof. G. Faber [1].

III. Lastly let us take the even function

$$\varphi_n(x)=\frac{(1-x^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n};$$

and let

$$a=0,\quad b=1,\quad t_i=\frac{i}{n},\quad r_0=0,\quad r_1=0.$$

Since $0\leqq v<\frac{1}{n}$ and then $t_i\leqq v+\frac{i}{n}<t_{i+1}$, we have

$$\sum_{i=1}^{k+1}\varphi_n(t_i)<\sum_{i=0}^{k}\varphi_n\left(v+\frac{i}{n}\right)<\sum_{i=0}^{k}\varphi_n(t_i).$$

Now

$$\sum_{i=0}^{k}\varphi_n(t_i)=\frac{\sum_{i=0}^{n}(1-t_i^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n}-\frac{\sum_{i=k+1}^{n}(1-t_i^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n}.$$

If we put

$$0<\varepsilon<t_{k+1}<1-\delta<1,$$

then

$$\sum_{i=k+1}^{n}(1-t_i^2)^n<\sum_{i=k+1}^{n}(1-\varepsilon^2)^n<n(1-\varepsilon^2)^n.$$

On the other hand

$$\sum_{i=0}^{n}(1-t_i^2)^n=\sum_{i=0}^{n}\left(1-\frac{i^2}{n^2}\right)^n>\sum_{i=0}^{[\sqrt{n}]}\left(1-\frac{n}{n^2}\right)^n$$
$$>[\sqrt{n}]\left(1-\frac{1}{n}\right)^n,$$

where $[\sqrt{n}\,]$ denotes the integral part of $\sqrt{n}$. It follows therefore that

$$\left|\frac{1}{2}-\sum_{i=0}^{k}\varphi_n(t_i)\right|=\frac{\sum_{i=k+1}^{n}(1-t_i^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n}<\frac{n(1-\varepsilon^2)^n}{2[\sqrt{n}\,]}\left(1-\frac{1}{n}\right)^{-n};$$

so that there exists a positive integer N independent of k, corresponding to any assigned positive quantity η, such that

[1] Faber, "Über stets konvergente Interpolationsformeln," Jahresb. Deuts. Math. Ver., 19 (1910), p. 142; Jackson, loc. cit. (the second paper).

$$\left|\frac{1}{2}-\sum_{i=0}^{k}\varphi_n(t_i)\right|<\eta,\qquad n>N.$$

The same result is true for $\sum_{i=1}^{k+1}\varphi_n(t_i)$, since

$$\sum_{i=1}^{k+1}\varphi_n(t_i)=\sum_{i=0}^{k}\varphi_n(t_i)+\varphi_n(t_{k+1})-\varphi_n(t_0).$$

Consequently the same conclusion holds for

$$\sum_{i=0}^{k}\varphi_n\left(v+\frac{i}{n}\right),\qquad \left(0\leqq v<\frac{1}{n}\right).$$

Thus we have arrived at *the formula of polynomial interpolation* due to Mr. W. G. Simon [1]:

$$\Psi_n[f(x)]=\frac{\sum_{i=0}^{n}f\left(\frac{i}{n}\right)\left[1-\left(\frac{i}{n}-x\right)^2\right]^n}{2\sum_{i=0}^{n}\left[1-\left(\frac{i}{n}\right)^2\right]^n},\qquad (0\leqq x\leqq 1).$$

It is evident that the mode of convergence of the above interpolation formula is equivalent to that of

$$\frac{\sum_{i=0}^{n}f\left(\frac{i}{n}\right)\left[1-\left(\frac{i}{n}-x\right)^2\right]^n}{\sum_{i=0}^{n}\left[1-\left(\frac{i}{n}-x\right)^2\right]^n},\qquad (0<x<1).$$

Approximating curves and surfaces.

16. Consider the plane curve C defined by

(13) $$x=x(\theta),\qquad y=y(\theta)$$

where $x(\theta)$ and $y(\theta)$ are *limited* and *integrable* in the interval $a\leqq\theta\leqq b$; and denote by D *the closed least convex domain which contains the curve* C. Let $\varphi_n(\theta,t)$ $(n=1,2,\cdots\cdots)$ be limited, non-negative and integrable with respect to t in the domain $a\leqq\theta,t\leqq b$; and let us put

$$F_n[x(\theta)]=\frac{\int_a^b x(t)\,\varphi_n(\theta,t)\,dt}{\int_a^b \varphi_n(\theta,t)\,dt},\quad F_n[y(\theta)]=\frac{\int_a^b y(t)\,\varphi_n(\theta,t)\,dt}{\int_a^b \varphi_n(\theta,t)\,dt}.$$

Then it follows by Weierstrass' theorem that all *the approximating curves* C_n:

(14) $$x=F_n[x(\theta)],\quad y=F_n[y(\theta)],\quad (n=1,2,\cdots\cdots)$$

[1] W. G. Simon "A formula of polynomial interpolation," Annals of Math., (2) 19 (1918), p. 242.

are contained in the domain D for $a \leqq \theta \leqq b$.

Let us now adopt the definition: *The distance $\delta(K_1, K_2)$ of two curves*

$$K_1: \qquad x = x_1(\theta), \qquad y = y_1(\theta), \qquad (a \leqq \theta \leqq b)$$

and

$$K_2: \qquad x = x_2(\theta), \qquad y = y_2(\theta), \qquad (a \leqq \theta \leqq b)$$

with respect to the parameter θ is measured by the integral

$$\int_a^b \sqrt{[x_1(\theta) - x_2(\theta)]^2 + [y_1(\theta) - y_2(\theta)]^2}\, d\theta \ (^1).$$

Then the distance of C and C_n with respect to θ is

$$\delta(C, C_n) = \int_a^b \sqrt{\{x(\theta) - F_n[x(\theta)]\}^2 + \{y(\theta) - F_n[y(\theta)]\}^2}\, d\theta.$$

Since we have the inequalities

$$0 \leqq \sqrt{\{x(\theta) - F_n[x(\theta)]\}^2 + \{y(\theta) - F_n[y(\theta)]\}^2}$$
$$\leqq |x(\theta) - F_n[x(\theta)]| + |y(\theta) - F_n[y(\theta)]|,$$

if

$$\lim_{n=\infty} \frac{\int_a^{a_1} \varphi_n(\theta, t)\, dt}{\int_a^b \varphi_n(\theta, t)\, dt} = 0, \qquad \lim_{n=\infty} \frac{\int_{b_1}^b \varphi_n(\theta, t)\, dt}{\int_a^b \varphi_n(\theta, t)\, dt} = 0$$

uniformly for $a \leqq a_1 < a_2 \leqq \theta \leqq b_2 < b_1 \leqq b$, then by Theorem II

$$\lim_{n=\infty} \int_a^b |x(\theta) - F_n[x(\theta)]|\, d\theta = 0,$$
$$\lim_{n=\infty} \int_a^b |y(\theta) - F_n[y(\theta)]|\, d\theta = 0;$$

so that

$$\lim_{n=\infty} \delta(C, C_n) = 0.$$

Thus we get the theorem:

Theorem XII. *Let $\varphi_n(\theta, t)$ be limited, non-negative and integrable with respect to t in the domain $a \leqq \theta, t \leqq b$, and moreover let*

$$\lim_{n=\infty} \frac{\int_a^{a_1} \varphi_p(\theta, t)\, dt}{\int_a^b \varphi_n(\theta, t)\, dt} = 0, \qquad \lim_{n=\infty} \frac{\int_{b_1}^b \varphi_n(\theta, t)\, dt}{\int_a^b \varphi_n(\theta, t)\, dt} = 0$$

(1) The distance δ thus defined satisfies the following properties:

1. $\delta(K_1, K_2)$ becomes zero when and only when

$$x_1(\theta) = x_2(\theta), \qquad y_1(\theta) = y_2(\theta), \qquad (a \leqq \theta \leqq b),$$

except a set of points of measure zero.

2. $\delta(K_1, K_2) = \delta(K_2, K_1)$.
3. $\delta(K_1, K_2) + \delta(K_2, K_3) \geqq \delta(K_1, K_3)$.

uniformly for $a \leqq a_1 < a_2 \leqq \theta \leqq b_2 < b_1 \leqq b$. Then the approximating curve C_n lies in the domain D; and the distance $\delta(C, C_n)$ becomes zero when n increases infinitely.

Further we will adopt the definition of Prof. Fréchet concerning *the limit of a sequence of curves* (1). Then from Theorem I and Fréchet's theorem (2) we can infer the following at once:

Theorem XIII. *If $\varphi_n(\theta, t)$ satisfy the conditions imposed in Theorem XII, and if $x(\theta)$ and $y(\theta)$ be continuous throughout the interval $a \leqq \theta \leqq b$, the approximating curve C_n tends to C when n increases indefinitely for $a + \varepsilon \leqq \theta \leqq b - \varepsilon$, ε being an arbitrary small positive quantity.*

Some parts of Theorem XII and XIII were already obtained by Prof. Fejér (1) for the approximating curve

$$x = \frac{1}{2n\pi} \int_0^{2\pi} x(t) \left[\frac{\sin \frac{n}{2}(t - \theta)}{\sin \frac{1}{2}(t - \theta)} \right]^2 dt,$$
$$(n = 1, 2, \cdots\cdots).$$
$$y = \frac{1}{2n\pi} \int_0^{2\pi} y(t) \left[\frac{\sin \frac{n}{2}(t - \theta)}{\sin \frac{1}{2}(t - \theta)} \right]^2 dt,$$

17. If we apply Theorem XII and XIII to *the rational curve*

$$(15) \qquad x = \frac{\int_0^1 x(t)[1 - (\theta - t)^2]^n\, dt}{\int_0^1 [1 - (\theta - t)^2]^n\, dt}, \qquad y = \frac{\int_0^1 y(t)[1 - (\theta - t)^2]^n\, dt}{\int_0^1 [1 - (\theta - t)^2]^n\, dt}, \qquad (n = 1, 2, \cdots\cdots).$$

(§ 7, II), we obtain the theorem:

Theorem XIV. *Any plane curve C defined by*

$$x = x(\theta), \qquad y = y(\theta), \qquad (0 \leqq \theta \leqq 1)$$

can be approximated by rational plane curves (15) *which are contained in the domain D.*

(1), (2) Fréchet, "Sur quelques points du calcul fonctionnel," Rend. Palermo, 22 (1906), p. 55.

(1) Fejér, "Über gewisse durch die Fouriersche und Laplacesche Reihe definierten Mittelkurven und Mittelflächen," Rend. Palermo, 38 (1914), p. 79. See also Ogura, "On certain mean curves defined by the series of orthogonal functions," Tôhoku Math. Journal, 15 (1919), p. 172.

This theorem holds good for the space curve:

$$x=x(\theta),\qquad y=y(\theta),\qquad z=z(\theta),\qquad (0\leqq\theta\leqq 1),$$

if we choose the approximating rational curves

$$x=\frac{\int_0^1 x(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt},\quad y=\frac{\int_0^1 y(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt},$$

$$z=\frac{\int_0^1 z(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt}.$$

Moreover the above theorem can be generalised to a surface. Consider the surface S defined by

$$(16)\qquad x=x(u,v),\qquad y=y(u,v),\qquad z=z(u,v),$$

where $x(u,v)$, $y(u,v)$, $z(u,v)$ are *limited* and *integrable* in the domain $0\leqq u, v\leqq 1$; and let $\overline{D}$ be *the closed least convex domain which contains the surface* S $(0\leqq u, v\leqq 1)$. If we put

$$(17)\qquad F_n[x(u,v)]=\frac{\int_0^1\int_0^1 x(t,\tau)\,[1-(u-t)^2]^n\,[1-(v-\tau)^2]^n\,dt\,d\tau}{\int_0^1\int_0^1[1-(u-t)^2]^n\,[1-(v-\tau)^2]^n\,dt\,d\tau},$$

....[1],

the *rational surface* S_n:

$$(18)\qquad x=F_n[x(u,v)],\qquad y=F_n[y(u,v)],\qquad z=F_n[z(u,v)],$$
$$(0\leqq u, v\leqq 1)$$

is contained in the domain $\overline{D}$. Further in a similar way as in § 16, we can prove that whenever $x(u,v)$, $y(u,v)$, $z(u,v)$ are continuous, the rational surface S_n tends to S for $\varepsilon\leqq u, v\leqq 1-\varepsilon$, ε being an arbitrary small quantity. Thus we have the theorem:

Theorem XV. *Any surface S (16) can be approximated by rational surface S_n (18) which are contained in the domain $\overline{D}$.*

18. For an example, let us take the surface of translation S:

$$(19)\qquad \begin{cases} x(u,v)=f_1(u)+\varphi_1(v),\\ y(u,v)=f_2(u)+\varphi_2(v),\\ z(u,v)=f_3(u)+\varphi_3(v),\end{cases}\qquad 0\leqq u, v\leqq 1,$$

in which the generating curves

$$C_1:\begin{cases} x_1(u)=f_1(u),\\ y_1(u)=f_2(u),\\ z_1(u)=f_3(u),\end{cases}\qquad C_2:\begin{cases} x_2(v)=\varphi_1(v),\\ y_2(v)=\varphi_2(v),\\ z_2(v)=\varphi_3(v)\end{cases}$$

(1) See Vallée Poussin, Analyse, Vol. 2, p. 133.

form a conjugate system.

In this case (17) becomes

$$F_n[x(u,v)]=\frac{\int_0^1\int_0^1[f_1(u)+\varphi_1(v)]\,[1-(u-t)^2]^n\,[1-(v-\tau)^2]^n\,dt\,d\tau}{\int_0^1\int_0^1[1-(u-t)^2]^n\,[1-(v-\tau)^2]^n\,dt\,d\tau}$$
$$=\frac{\int_0^1 f_1(u)\,[1-(u-t)^2]^n\,dt}{\int_0^1[1-(u-t)^2]^n\,dt}+\frac{\int_0^1\varphi_1(v)\,[1-(v-\tau)^2]^n\,d\tau}{\int_0^1[1-(v-\tau)^2]^n\,dt}$$
$$=F_n[x_1(u)]+F_n[x_2(v)];$$

so that the approximating surface has the form

$$(20)\qquad \begin{aligned} x&=F_n[x_1(u)]+F_n[x_2(v)],\\ y&=F_n[y_1(u)]+F_n[y_2(v)],\\ z&=F_n[z_1(u)]+F_n[z_2(v)].\end{aligned}$$

Therefore *any surface of translation S (19) can be approximated by rational surfaces of translation S_n (20) which are contained in the domain $\overline{D}$ (1). The generating curve C_1 (C_2) of the given surface S is approximated by the rational curves C_{1n} (C_{2n}) lying in the least convex domain D which contains the given curve C_1 (C_2), and moreover C_{1n}, C_{2n} are the generating curves of the surface S_n (20); the conjugate system consisting of the generating curves is preserved in this approximation.*

19. I will now consider *the approximating curves defined by* Stieltjes-Landau's polynomials:

$$x=F_n[x(\theta)]=\frac{\int_0^1 x(t)\,[1-(\theta-t)^2]^n\,dt}{2\int_0^1(1-t^2)^n\,dt},$$
$$y=F_n[y(\theta)]=\frac{\int_0^1 y(t)\,[1-(\theta-t)^2]^n\,dt}{2\int_0^1(1-t^2)^n\,dt}.\qquad (0\leqq\theta\leqq 1),$$

Since

$$F_n[x(\theta)]=\frac{\int_0^1 x(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt}\cdot\frac{\int_0^1[1-(\theta-t)^2]^n\,dt}{2\int_0^1(1-t^2)^n\,dt}$$

and

$$\int_0^1[1-(\theta-t)^2]^n\,dt=\int_{-\theta}^{1-\theta}(1-t^2)^n\,dt$$
$$=\int_0^\theta(1-t^2)^n\,dt+\int_0^{1-\theta}(1-t^2)^n\,dt$$
$$<2\int_0^1(1-t^2)^n\,dt,$$

we have

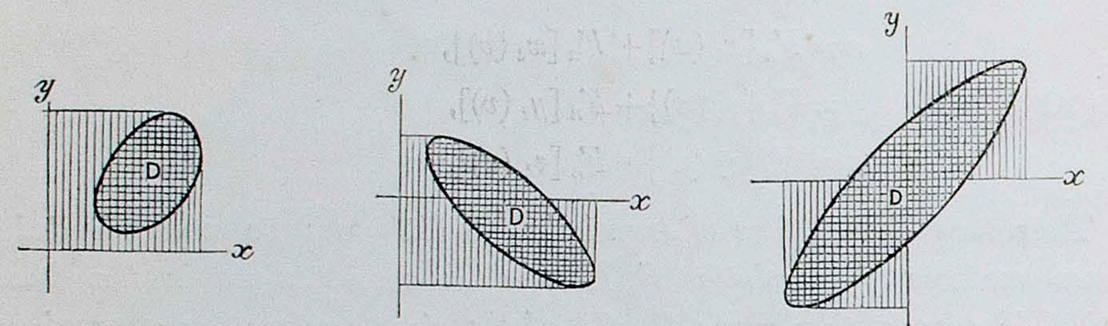
(1) In a similar way we can state the theorem: *Any ruled surface can be approximated by rational ruled surfaces which are contained in the domain D.*

$$F_n[x(\theta)]=\lambda(\theta)\cdot\frac{\int_0^1 x(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt},\qquad (0<\lambda(\theta)<1).$$

Similarly

$$F_n[y(\theta)]=\lambda(\theta)\cdot\frac{\int_0^1 y(t)\,[1-(\theta-t)^2]^n\,dt}{\int_0^1[1-(\theta-t)^2]^n\,dt}.$$

Consequently *all the approximating curves C_n lie in a domain D' determined geometrically from the shape of the least convex domain D containing the given curve C*: Some typical cases are indicated in the figures.



It follows from this theorem that *if L and l be the upper and lower limits of $x(\theta)$ $(0\leqq\theta\leqq 1)$ respectively, and if $L\geqq 0$ and $l\leqq 0$, then*

$$l\leqq F_n[x(\theta)]\leqq L\ ^{(1)}.$$

A similar result is also true for $F_n[y(\theta)]$.

20. I pass now to the approximating curves of interpolations.

Consider the plane curve C defined by

$$(21)\qquad x=x(\theta),\qquad y=y(\theta),$$

where $x(\theta)$ and $y(\theta)$ are *limited* in the interval $a\leqq\theta\leqq b$, and denote by D the closed least convex domain which contains the curve C. Let $\varphi_n(\theta,t)$ $(n=1,2,\cdots\cdots)$ be *limited* and *non-negative* in the domain $a\leqq\theta$, $t\leqq b$; and put

$$(22)\qquad \Psi_n[x(\theta)]=\frac{\sum\limits_{i=\gamma_0}^{n-\gamma_1} x(t_i)\,\varphi_n(\theta,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(\theta,t_i)},$$

$$\Psi_n[y(\theta)]=\frac{\sum\limits_{i=\gamma_0}^{n-\gamma_1} y(t_i)\,\varphi_n(\theta,t_i)}{\sum\limits_{i=\gamma_0}^{n-\gamma_1}\varphi_n(\theta,t_i)},$$

(1) Incidentally we will add a remark: Prof. F. Riesz stated *incorrectly* that these inequalities hold for $L<0$ or $l>0$ also, in his paper "Über die Approximation einer Funktion durch Polynome," Jahresb. Deuts. Math. Ver., 17 (1908), p. 208.

where γ_0, γ_1 denote two fixed non-negative integers and

$$t_i=a+\frac{b-a}{n}i.$$

Now by Weierstrass' theorem the approximating curve

$$(23)\qquad x=\Psi_n[x(\theta)],\qquad y=\Psi_n[y(\theta)],\qquad (a\leqq\theta\leqq b)$$

lies in the least convex polygon D_n which contains the $n-\gamma-\gamma_1+1$ points

$$x=x(t_i),\qquad y=y(t_i),$$

$$(i=\gamma_0,\ \gamma_0+1,\ \cdots\cdots,\ n-\gamma_1-1,\ n-\gamma_1).$$

But since these points are on the given curve C, the polygon D_n lies in the domain D. Thus we have the theorem:

Theorem XVI. *All the approximating curves* (23) *defined by the interpolation formulae* (22) *lie in the domain D.*

Further it is almost evident that the theorems analogous to Theorem XII and XIII are also true.

The approximating curves defined by Jackson's trigonometric interpolation (1)

(1) On the contrary, *the approximating curves defined by the ordinary trigonometrical interpolation:*

$$x=\sum_{i=0}^{n-1}x\Big(\frac{2\pi i}{n}\Big)\frac{\sin\frac{n}{2}\Big(\frac{2\pi i}{n}-\theta\Big)}{n\sin\frac{1}{2}\Big(\frac{2\pi i}{n}-\theta\Big)},\qquad y=\sum_{i=0}^{n-1}y\Big(\frac{2\pi i}{n}\Big)\frac{\sin\frac{n}{2}\Big(\frac{2\pi i}{n}-\theta\Big)}{n\sin\frac{1}{2}\Big(\frac{2\pi i}{n}-\theta\Big)},$$

$$(0\leqq\theta\leqq 2\pi)$$

do not lie necessarily in the domain D. For, if we take the functions

$$x(\theta)=\theta,\qquad 0\leqq\theta\leqq 2\pi;\qquad y(0)=y(2\pi)=1,$$
$$y(\theta)=0,\qquad 0<\theta<2\pi,$$

the domain D is the rectangle bounded by $x=0$, $x=2\pi$, $y=0$, $y=1$. But we have

$$\Psi_n[y(\theta)]=\frac{\sin\frac{n}{2}\theta}{n\sin\frac{\theta}{2}};\quad\text{so that}\quad \Psi_n\Big[y\Big(\frac{3\pi}{n}\Big)\Big]=-\frac{1}{n\sin\frac{3\pi}{2\pi}}<0\quad\text{for}\quad n>2.$$

The approximating curve shows us the Gibbs *phenomenon:*

$$\lim_{n=\infty}\Psi_n\Big[y\Big(\frac{3\pi}{n}\Big)\Big]=-\frac{2}{3\pi}.$$

(See Faber, "Über stetige Funktionen," Math. Ann., 69 (1910), p. 417).

On the other hand *this phenomenon disappears for the approximating curve defined by* Jackson's *interpolation* with respect to the functions defined above.

$$x=\Psi_n[x(\theta)]=\sum_{i=1}^{n}x\left(\frac{2\pi i}{n}\right)\left[\frac{\sin\frac{n}{2}\left(\frac{2\pi i}{n}-\theta\right)}{n\sin\frac{1}{2}\left(\frac{2\pi i}{n}-\theta\right)}\right]^2,$$

$$(0\leqq\theta\leqq2\pi)$$

$$y=\Psi_n[y(\theta)]=\sum_{i=1}^{n}y\left(\frac{2\pi i}{n}\right)\left[\frac{\sin\frac{n}{2}\left(\frac{2\pi i}{n}-\theta\right)}{n\sin\frac{1}{2}\left(\frac{2\pi i}{n}-\theta\right)}\right]^2,$$

and also those defined by

$$x=\Psi_n[x(\theta)]=\frac{\sum_{i=0}^{n}x\left(\frac{i}{n}\right)\left[1-\left(\frac{i}{n}-\theta\right)^2\right]^n}{\sum_{i=0}^{n}\left[1-\left(\frac{i}{n}-\theta\right)^2\right]^n},$$

$$(0\leqq\theta\leqq1),$$

$$y=\Psi_n[y(\theta)]=\frac{\sum_{i=0}^{n}y\left(\frac{i}{n}\right)\left[1-\left(\frac{i}{n}-\theta\right)^2\right]^n}{\sum_{i=0}^{n}\left[1-\left(\frac{i}{n}-\theta\right)^2\right]^n},$$

belong to this class. In addition to this, we can establish the theorems analogous to those in § 17 for the latter curves.

Lastly the approximating curves defined by Simon's polynomial interpolation can be treated in a similar way as in § 19 (1).

On the law of errors.

21. In the theory of errors Gauss' law is usually adopted. "As a matter of fact, however, the cases are quite exceptional in which the

(1) On the contrary, *in the ordinary equidistant polynomial interpolation, the approximating curves*

$$x=\Psi_n[x(\theta)]=\sum_{i=0}^{n-1}f(i_i)\frac{(\theta-t_0)\cdot\cdot(\theta-t_{i-1})(\theta-t_{i+1})\cdot\cdot(\theta-t_{n-1})}{(t_i-t_0)\cdot\cdot(t_i-t_{i-1})(t_i-t_{i+1})\cdot\cdot(t_i-t_{n-1})},\qquad y=\cdot\cdot,$$

$$t_i=\frac{i}{n},\qquad 0\leqq\theta\leqq1$$

do not lie necessarily in the domain D' (§ 19). (See Borel, Leçons sur les fonctions de variables réelles (1905), p. 75).

errors are found to follow really the law" (1). I will now consider a general type of the law of errors containing that of Gauss as a particular case.

When $\varphi_h(\varepsilon)$ satisfies the following five conditions, this function will be called the law of errors, ε and h being the error of observation and the measure of precision respectively (2):

(i) $\varphi_h(\varepsilon)$ is a function of ε, containing a positive parameter h, which is even, non-negative, limited and integrable in the interval $-\infty\leqq\varepsilon\leqq+\infty$.

(ii) $\varphi_h(\varepsilon)$ decreases monotonously when ε increases from 0 to ∞, (h being fixed).

(iii) $$\int_0^\infty\varphi_h(\varepsilon)\,d\varepsilon=\frac{1}{2}.$$

(iv) Let $a_{h,h'}$ be the quantity such that

$$\varphi_h(\varepsilon)>0 \qquad \text{for} \quad 0\leqq\varepsilon<a_{h,h'},$$
$$\varphi_h(\varepsilon)=\varphi_{h'}(\varepsilon)=0 \quad (h'>h) \quad \text{for} \quad a_{h,h'}\leqq\varepsilon\leqq\infty.$$

Then $\dfrac{\varphi_{h'}(\varepsilon)}{\varphi_h(\varepsilon)}$ $(h'>h)$ decreases monotonously when ε increases from 0 to $a_{h,h'}$.

(v) $$\lim_{h=\infty}\int_0^c\varphi_h(\varepsilon)\,d\varepsilon=\frac{1}{2},$$

uniformly where c is any positive finite constant independent of h.

Some illustrative examples are as follows:

Ex. I. Gauss' *law:*

$$\varphi_h(\varepsilon)=\frac{h}{\sqrt{\pi}}e^{-h^2\varepsilon^2} \quad (3), \tag{24}$$

in which

$$a_{h,h'}=\infty.$$

(1) Newcomb, "A generalized theory of the combination of observations, etc.," Amer. Journ. of Math., 8 (1886), p. 343.

(2) The former three conditions are found in common text books, for example, Helmert, Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate, 2. Aufl. (1907), pp. 7–9. But the reason why h may be taken as the measure of precision can not be interpreted by use of these three conditions only.

(3) See § 7, II (iii).

Ex. II. Helmert's *first law* (1):

(25) $$\varphi_h(\varepsilon)=\frac{h}{2} \qquad 0\leqq\varepsilon\leqq\frac{1}{h},$$
$$=0 \qquad \frac{1}{h}<\varepsilon<\infty,$$

in which

$$a_{h,h'}=\frac{1}{h} \qquad (h'>h).$$

Ex. III. Helmert's *second law* (2):

(26) $$\varphi_h(\varepsilon)=\frac{3}{4}h(1-h^2\varepsilon^2) \qquad 0\leqq\varepsilon\leqq\frac{1}{h},$$
$$=0 \qquad \frac{1}{h}<\varepsilon<\infty,$$

in which

$$a_{h,h'}=\frac{1}{h} \qquad (h'>h).$$

Ex. IV. Lastly we can take

(27) $$\varphi_h(\varepsilon)=\frac{(1-\varepsilon^2)^h}{2\int_0^1(1-\varepsilon^2)^h\,d\varepsilon} \qquad 0\leqq\varepsilon\leqq 1\ (^3),$$
$$=0 \qquad 1<\varepsilon<\infty,$$

in which

$$a_{h,h'}=1.$$

22. Now I will prove the theorem which gives the reason why h may be taken as *the measure of precision.*

Theorem XVII. *The probability $P_h(\delta)$ which an error lies between $-\delta$ and $+\delta$ increases when h increases.*

It is easily seen from the conditions (i), (ii), (iii) and (iv) that there exists one and only one value of ε ($\varepsilon=c_{h,h'}$ say) in the interval $0\leqq\varepsilon\leqq a_{h,h'}$, such that

$$\varphi_{h'}(\varepsilon)\gtreqless\varphi_h(\varepsilon) \qquad (h'>h) \qquad \text{according as} \qquad \begin{array}{l} 0\leqq\varepsilon<c_{h,h'} \\ \varepsilon=c_{h,h'} \\ c_{h,h'}<\varepsilon\leqq a_{h,h'} \end{array}$$

(1) (2) Helmert, loc. cit., p. 13. See §7, I (ii).
(3) See §7, II (i).

Hence if $0<\delta\leqq c_{h,h'}$, then

$$\int_0^\delta[\varphi_{h'}(\varepsilon)-\varphi_h(\varepsilon)]\,d\varepsilon>0.$$

Next if $\delta\geqq c_{h,h'}$, then

$$\int_0^\delta[\varphi_{h'}(\varepsilon)-\varphi_h(\varepsilon)]\,d\varepsilon$$
$$=\int_0^{c_{h,h'}}[\varphi_{h'}(\varepsilon)-\varphi_h(\varepsilon)]\,d\varepsilon-\int_{c_{h,h'}}^\delta[\varphi_h(\varepsilon)-\varphi_{h'}(\varepsilon)]\,d\varepsilon$$
$$>\int_0^{c_{h,h'}}[\varphi_{h'}(\varepsilon)-\varphi_h(\varepsilon)]\,d\varepsilon-\int_{c_{h,h'}}^\infty[\varphi_h(\varepsilon)-\varphi_{h'}(\varepsilon)]\,d\varepsilon$$
$$=\int_0^\infty[\varphi_{h'}(\varepsilon)-\varphi_h(\varepsilon)]\,d\varepsilon$$
$$=\frac{1}{2}-\frac{1}{2}=0.$$

Therefore we have the inequality in general

$$\int_0^\delta\varphi_{h'}(\varepsilon)\,d\varepsilon>\int_0^\delta\varphi_h(\varepsilon)\,d\varepsilon, \qquad h'>h.$$

But in virtue of the definition

$$P_h(\delta)=\int_{-\delta}^{+\delta}\varphi_h(\varepsilon)\,d\varepsilon=2\int_0^\delta\varphi_h(\varepsilon)\,d\varepsilon,$$

we obtain

$$P_{h'}(\delta)>P_h(\delta) \qquad \text{for} \qquad h'>h.$$

23. Next we consider the measure of exactitude.

Let $f(\varepsilon)$ be a function such that

$$f(\varepsilon)=f(-\varepsilon)$$

and

$$f(\varepsilon_1)>f(\varepsilon_2) \qquad \text{when} \qquad |\varepsilon_1|>|\varepsilon_2|;$$

and let the integral

$$R_h\equiv\int_{-\infty}^{+\infty}f(\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon$$

exist. Then the integral denotes *the risk of errors* (Fehlerrisiko) (1).

(1) Czuber, Wahrscheinlichkeitsrechnung, I (2. Aufl., 1908), p. 267.

Now I proceed to prove the theorem which gives the reason why the risk of errors may be taken as *the measure of exactitude*.

Theorem XVIII. *The risk of errors decreases when the measure of precision increases.*

Putting,

$$\varphi=\varphi_{h'}(\varepsilon),\qquad \psi=\varphi_h(\varepsilon),\qquad a=-\alpha_{h,h'},\qquad b=+\alpha_{h,h'}$$

and remembering the condtion (iv) and the definition of $f(\varepsilon)$, Theorem IV (§ 8) gives

$$\frac{\int_{-\alpha_{h,h'}}^{+\alpha_{h,h'}} f(\varepsilon)\,\varphi_{h'}(\varepsilon)\,d\varepsilon}{\int_{-\alpha_{h,h'}}^{+\alpha_{h,h'}} \varphi_{h'}(\varepsilon)\,d\varepsilon} < \frac{\int_{-\alpha_{h,h'}}^{+\alpha_{h,h'}} f(\varepsilon)\,\varphi_{h'}(\varepsilon)\,d\varepsilon}{\int_{-\alpha_{h,h'}}^{+\alpha_{h,h'}} \varphi_h(\varepsilon)\,d\varepsilon},\qquad (h'>h).$$

Since

$$\varphi_h(\varepsilon)=\varphi_{h'}(\varepsilon)=0\qquad \text{for}\qquad \alpha_{h,h'}\leqq\varepsilon\leqq\infty,$$

the above inequality becomes

$$\frac{\int_{-\infty}^{+\infty} f(\varepsilon)\,\varphi_{h'}(\varepsilon)\,d\varepsilon}{\int_{-\infty}^{+\infty} \varphi_{h'}(\varepsilon)\,d\varepsilon} < \frac{\int_{-\infty}^{+\infty} f(\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon}{\int_{-\infty}^{+\infty} \varphi_h(\varepsilon)\,d\varepsilon},\qquad (h'>h).$$

But we have from the condition (iii)

$$\int_{-\infty}^{+\infty}\varphi_h(\varepsilon)\,d\varepsilon=\int_{-\infty}^{+\infty}\varphi_{h'}(\varepsilon)\,d\varepsilon=1\,;$$

so that we arrive at

$$\int_{-\infty}^{+\infty} f(\varepsilon)\,\varphi_{h'}(\varepsilon)\,d\varepsilon<\int_{-\infty}^{+\infty} f(\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon,$$

i. e.

$$R_{h'}<R_h\qquad \text{for}\qquad h'>h.$$

It is evident that if $\alpha_{h,h'}$ be finite for any positive value of h, h', this theorem holds good when $f(\varepsilon)$ is limited and integrable in the interval $-\infty<\varepsilon<+\infty$.

Particularly, the risk of errors becomes *the average error* (Durchschnittsfehler) ϑ in the case where $f(\varepsilon)=|\varepsilon|$; and the square of *the mean error* μ in the case where $f(\varepsilon)=\varepsilon^2$: i. e.

$$\vartheta_h=\int_{-\infty}^{+\infty}|\varepsilon|\,\varphi_h(\varepsilon)\,d\varepsilon,$$

$$\mu_h^2=\int_{-\infty}^{+\infty}\varepsilon^2\,\varphi_h(\varepsilon)\,d\varepsilon.$$

For Ex. I (Gauss' law) in § 21, we have

$$\vartheta_h=\frac{1}{\sqrt{\pi}\,h},\qquad \mu_h=\frac{1}{\sqrt{2}\,h}\,;$$

for Ex. II (Helmert's first law)

$$\vartheta_h=\frac{1}{2h},\qquad \mu_h=\frac{1}{\sqrt{3}\,h}\ (^1)\,;$$

for Ex. III (Helmert's second law)

$$\vartheta_h=\frac{3}{8h},\qquad \mu_h=\frac{1}{\sqrt{5}\,h}\ (^2)\,;$$

and lastly for Ex. IV.

$$\vartheta_h=\frac{\Gamma\left(\frac{3}{2}+h\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(2+h)},\qquad \mu_h=\frac{1}{\sqrt{3+2h}}.$$

24. Let $f(x)$ be a function which is limited in $-\infty<x<+\infty$ and is such that the integral

$$\int_{-\infty}^{+\infty} f(x_0+\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon\qquad (-\infty<x_0<+\infty)$$

exists. Then the integral gives *the probable value of* $f(x)$ at $x=x_0$.

Theorem XIX. *If $f(x)$ be continuous at $x=x_0$ $(-\infty<x_0<+\infty)$, the probable value of that function at that point converges uniformly to $f(x_0)$, when the measure of precision increases indefinitely.*

We have the identity

$$\int_{-\infty}^{+\infty} f(x_0+\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon=\int_{-\infty}^{+\infty} f(\varepsilon)\,\varphi_h(\varepsilon-x_0)\,d\varepsilon$$

and the condition (v) that

(1), (2) Helmert, loc. cit., p. 22.

$$\lim_{h=\infty}\int_0^c \varphi_h(\varepsilon)\,d\varepsilon=\frac{1}{2} \qquad (0<c<\infty)$$

uniformly. It follows therefore from Theorem III (§ 5) that

$$\lim_{h=\infty}\int_{-\infty}^{+\infty} f(x_0+\varepsilon)\,\varphi_h(\varepsilon)\,d\varepsilon=f(x_0)$$

uniformly.

For example, if we adopt Gauss' law, the probable value of $f(x)$ at $x=x_0$ is equal to

$$\frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(\varepsilon)\,e^{-h^2(\varepsilon-x_0)^2}\,d\varepsilon,$$

which is nothing but Weierstrass' integral (§ 7, II (iii)).

25. I. In n measurements of equal precision, if we adopt Gauss' law, the most probable values of observed quantities are given by the condition (the principle of least squares):

$$(28) \qquad \varepsilon_1^2+\varepsilon_2^2+\cdots\cdots\varepsilon_n^2=\text{min.},$$

$\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_n$ being the errors of the n measurements respectively.

In the same case, *if we assume the law*

$$(27) \qquad \varphi_h(\varepsilon)=\frac{(1-\varepsilon^2)^h}{2\int_0^1(1-\varepsilon^2)^h\,d\varepsilon} \qquad 0\leqq\varepsilon\leqq 1,$$
$$=0 \qquad 1<\varepsilon<\infty,$$

we have the condition

$$(1-\varepsilon_1^2)(1-\varepsilon_2^2)\cdots\cdots(1-\varepsilon_n^2)=\text{max.},$$

that is,

$$(29) \qquad \Sigma\,\varepsilon_i^2-\Sigma\,\varepsilon_i^2\varepsilon_j^2+\cdots\cdots\pm\varepsilon_1^2\varepsilon_2^2\cdots\cdots\varepsilon_n^2=\text{min.},$$

under the supposition

$$|\varepsilon_i|<1 \qquad (i=1, 2, \cdots\cdots, n).$$

Therefore in the case where the errors are extremely small, the condition (29) may be replaced by (28).

II. If we follow Gauss' law when n direct observations $l_1, l_2, \cdots\cdots, l_n$ on a single quantity are of equal precision, the arithmetic mean

$$\xi=\frac{l_1+l_2+\cdots\cdots+l_n}{n}$$

is the most probable value of the quantity.

On the other hand, *if we assume the law of errors* (27), the most probable value x is determined by

$$[1-(x-l_1)^2]\,[1-(x-l_2)^2]\cdots\cdots[1-(x-l_n)^2]=\text{max.},$$

where

$$(30) \qquad 1-(x-l_i)^2>0 \qquad (i=1, 2, \cdots\cdots, n).$$

Differentiating the left hand of the above equation and equating to zero

$$(31) \qquad F(x)\equiv\frac{x-l_1}{1-(x-l_1)^2}+\frac{x-l_2}{1-(x-l_2)^2}+\cdots\cdots$$
$$+\frac{x-l_n}{1-(x-l_n)^2}=0.$$

Let l and L be the minimum and the maximum among $l_1, l_2, \cdots\cdots, l_n$. Then remembering the inequalities (30) we obtain

$$F(l)<0, \qquad F(L)>0.$$

But since

$$\frac{d}{dx}F(x)=\frac{1+(x-l_1)^2}{[1-(x-l_1)^2]^2}+\cdots\cdots+\frac{1+(x-l_n)^2}{[1-[x-l_n)^2]^2}>0,$$

it follows that *there exists one and only one root of the equation* (31) *between l and L. This root gives the most probable value of the given quantity* (1).

On the conduction of heat.

26. Firstly I will consider *the linear flow in a doubly-infinite solid.* Let x be the distance, t the time, $u(x, t)$ the temperature and a a positive constant (a^2 being the diffusivity of the substance); and let

(1) The following simplest cases will serve to interpret the relation between x and the arithmetic mean ξ.

(i) When $n=2$, $x=\xi$.

(ii) Let $n=3$ and let, for example, $\xi-l_1=\varepsilon'_1>0$, $\xi-l_2=\varepsilon'_2>0$. Then $\xi-l_3=\varepsilon'_3=-(\varepsilon'_1+\varepsilon'_2)<0$; and consequently

$$F(\xi)=\left(\frac{1}{1-\varepsilon_1'^2}-\frac{1}{1-\varepsilon_3'^2}\right)\varepsilon'_1+\left(\frac{1}{1-\varepsilon_2'^2}-\frac{1}{1-\varepsilon_3'^2}\right)\varepsilon'_2<0.$$

Therefore x lies between the arithmetic mean ξ and l_3 (that is, the observation is in the greatest difference from the arithmetic mean).

(32) $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2},$ for $0<t<+\infty,$ $-\infty<x<+\infty;$

(33) $u(x,0)=f(x)$ for $-\infty\leqq x\leqq+\infty,$

(34) $\lim\limits_{x=\pm\infty} u(x,t)=\lim\limits_{x=\pm\infty} f(x)$ for $t>0;$

where $f(x)$ is limited in the interval $-\infty\leqq x\leqq+\infty$ and continuous for $-\infty<x<+\infty$. Then *we have*

$$\lim_{t=+0} u(x,t)=f(x) \quad \textit{for} \quad -\infty<x<+\infty$$

uniformly (1).

It is well known that the function $u(x,t)$, which satisfies all the above conditions, is uniquely determined by

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(x+2a\xi\sqrt{t})\,e^{-\xi^2}d\xi, \qquad 0\leqq t<+\infty.$$

In the case where $t>0$, if we put $\eta=x+2a\xi\sqrt{t}$ the above expression becomes

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(\eta)\,e^{-\frac{(\eta-x)^2}{4a^2t}}d\eta;$$

so that we have

$$\lim_{t=+0} u(x,t)=f(x), \qquad -\infty<x<+\infty$$

uniformly (§ 7, II (iii)).

It will be noticed that *if the boundary condition* (34) *be dropped the above result does not hold good in general* (2). In order to show this, take the function:

$$u(x,t)=x\lim_{n=+\infty}\frac{nt}{1+nt}.$$

Then

(1) Compare with Weierstrass, loc. cit.; Sommerfeld, "Randwertaufgaben. u. s. w.," Encyk. d. math. Wiss., II 1, p. 535.

(2) H. Weber stated *incorrectly* that $u(x,t)$ is uniquely determined by (32) and (33) only, in his Part. Dif. Gleichungen d. Math. Physik, II (1901), p. 92.

$$u(x,t)=x \quad (t>0); \qquad \frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \quad (t>0);$$

$$\lim_{x=\pm\infty} u(x,t)=\pm\infty \quad (t>0);$$

$$u(x,0)=0 \quad (-\infty<x<+\infty);$$

and

$$\lim_{t=+0} u(x,t)=x.$$

27. Next we consider *the linear flow in a semi-infinite solid.* Let

(35) $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}$ for $0<t<+\infty,$ $0<x<+\infty;$

(36) $u(x,0)=F(x)$ for $0<x\leqq+\infty;$

(37) $\begin{cases} u(0,t)=F(t) & \text{for} \quad 0<t\leqq+\infty, \\ \lim\limits_{x=+\infty} u(x,t)=\lim\limits_{x=+\infty} F(x) & \text{for} \quad 0<t<+\infty, \end{cases}$

where $u(\xi,0)$ and $u(0,\xi)$ are limited in the interval $0\leqq\xi\leqq+\infty$ and continuous for $0<\xi<+\infty$. Then *we have*

$$\lim_{t=+0} u(x,t)=F(x) \quad \textit{for} \quad 0<x<+\infty$$

uniformly, and

$$\lim_{x=+0} u(x,t)=f(t) \quad \textit{for} \quad 0<t<+\infty$$

uniformly (1).

(i) Let $\Psi(x)$ be a function such that

$$\Psi(x)=F(x),$$
$$\Psi(-x)=-F(x), \quad \text{for} \quad x>0,$$
$$\Psi(0)=0;$$

and let us put

$$u_1(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\Psi(x+2a\xi\sqrt{t})\,e^{-\xi^2}d\xi \quad \text{for} \quad t\geqq 0.$$

This function $u_1(x,t)$ satisfies the differential equation (35) for $x>0$, $t>0$; and

(1) If the latter of the boundary conditions (37) be dropped this result does not hold in general, which is seen from the function defined by

$$u(x,t)=x\lim_{n=+\infty}\frac{nt}{1+nt}.$$

$$u_1(x,0)=\Psi(x)=F(x) \qquad \text{for} \qquad x>0,$$

$$\lim_{x=+\infty}(x,t)=\lim_{x=+\infty}F(x) \qquad \text{for} \qquad t>0.$$

Also in the case where $t>0$, $u_1(x,t)$ may be written

$$u_1(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\Psi(\eta)\,e^{-\frac{(\eta-x)^2}{4a^2t}}d\eta\,;$$

so that

$$u_1(0,t)=\frac{1}{2a\sqrt{\pi t}}\left[\int_0^{+\infty}\Psi(-\eta)\,e^{-\frac{\eta^2}{4a^2t}}d\eta+\int_0^{+\infty}\Psi(\eta)\,e^{-\frac{\eta^2}{4a^2t}}d\eta\right]$$

$$=0, \qquad t>0.$$

Now we have from § 7, II (iii) that

$$\lim_{t=+0}u_1(x,t)=\Psi(x)$$

$$=F(x), \qquad 0<x<+\infty$$

uniformly. Also it is easily seen that

$$\lim_{x=+0}u_1(x,t)=\lim_{x=+0}\frac{1}{2a\sqrt{\pi t}}\int_0^{+\infty}\Psi(\eta)\left[e^{-\frac{(\eta-x)^2}{4a^2t}}-e^{-\frac{(\eta+x)^2}{4a^2t}}\right]d\eta$$

$$=0, \qquad 0<t<+\infty$$

uniformly.

(ii) Next let us put

$$\Phi(t)=f(t), \qquad t>0,$$

$$\Phi(0)=0\,;$$

and take

$$u_2(x,t)=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty}\Phi\left(t-\frac{x^2}{4a^2\xi^2}\right)e^{-\xi^2}d\xi \quad \text{for} \quad t>0,\ x\geqq 0.$$

Then the function $u_2(x,t)$ satisfies the differential equation (35) for $x>0$, $t>0$; and

$$u_2(0,t)=\frac{2}{\sqrt{\pi}}\int_0^{+\infty}\Phi(t)\,e^{-\xi^2}d\xi$$

$$=f(t) \qquad \text{for} \qquad t>0.$$

Also in the case where $x>0$, $u_2(x,t)$ may be written (1)

$$u_2(x,t)=\int_0^t\Phi(\eta)\,\varphi\left(\frac{1}{x},\eta-t\right)d\eta,$$

where

$$\varphi\left(\frac{1}{x},\xi\right)=\frac{x}{2a\sqrt{\pi}}(-\xi)^{-\frac{3}{2}}e^{\frac{x^2}{4a^2\xi}} \qquad \text{for} \quad \xi<0,$$

$$=0 \qquad \text{for} \quad \xi\geqq 0.$$

Thus we have

$$u_2(x,t)=\int_0^{+\infty}\Phi(\eta)\,\varphi\left(\frac{1}{x},\eta-t\right)d\eta,$$

and consequently

$$u_2(x,0)=0 \qquad \text{for} \qquad x>0\,;$$

and

$$\lim_{x=+\infty}u_2(x,t)=0 \qquad \text{for} \qquad t>0.$$

Now it is easily seen that

$$\lim_{t=+0}u_2(x,t)=0 \qquad \text{for} \qquad 0<x<+\infty$$

uniformly.

Again

$$\varphi\left(\frac{1}{x},\xi\right)\geqq 0, \qquad x>0.$$

Since

$$\varphi\left(\frac{1}{x},\xi\right)=\frac{d}{d\xi}\left[\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2a\sqrt{-\xi}}}e^{-\zeta^2}d\zeta\right] \quad \text{for} \quad \xi<0,\ x>0,$$

we have

$$\int_{-\infty}^{\xi}\varphi\left(\frac{1}{x},\xi\right)d\xi=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2a\sqrt{-\xi}}}e^{-\zeta^2}d\zeta-\frac{2}{\sqrt{\pi}}\int_{-\infty}^{0}e^{-\zeta^2}d\zeta$$

$$=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2a\sqrt{-\xi}}}e^{-\zeta^2}d\zeta-1 \quad \text{for} \quad -\infty<\xi<0,$$

(1) Compare, with H. Weber, loc. cit., p. 105.

and

$$\int_{-\infty}^{\xi} \varphi\left(\frac{1}{x}, \xi\right) d\xi = \int_{-\infty}^{0} \varphi\left(\frac{1}{x}, \xi\right) d\xi + \int_{0}^{\xi} \varphi\left(\frac{1}{x}, \xi\right) d\xi$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\zeta^2} d\zeta - 1$$

$$= 1 \qquad \text{for} \qquad 0 \leqq \xi < +\infty.$$

Therefore for any constants c, d, such that

$$-\infty < -c < 0 < d < +\infty,$$

we have

$$\int_{-c}^{d} \varphi\left(\frac{1}{x}, \xi\right) d\xi = \int_{-\infty}^{d} \varphi\left(\frac{1}{x}, \xi\right) d\xi - \int_{-\infty}^{-c} \varphi\left(\frac{1}{x}, \xi\right) d\xi$$

$$= 1 - \left[\frac{2}{\pi} \int_{-\infty}^{\frac{x}{2a\sqrt{-\xi}}} e^{-\zeta^2} d\zeta - 1\right],$$

so that

$$\lim_{\frac{1}{x}=+\infty} \int_{-c}^{d} \varphi\left(\frac{1}{x}, \xi\right) d\xi = 2 - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-\zeta^2} d\zeta$$

$$= 1$$

uniformly.

Consequently we have from Theorem III (§ 5) that

$$\lim_{\frac{1}{x}=+\infty} \int_{0}^{+\infty} \Phi(\eta)\, \varphi\left(\frac{1}{x}, \eta - t\right) d\eta = \Phi(t)$$

uniformly for $0 < t < +\infty$; that is,

$$\lim_{x=+0} u_2(x, t) = f(t) \qquad \text{for} \quad 0 < t < +\infty$$

uniformly.

(iii) Lastly if we put

$$u(x, t) = u_1(x, t) + u_2(x, t),$$

then $u(x, t)$ satisfies all the conditions (35), (36), (37), and moreover

$$\lim_{t=+0} u(x, t) = F(x) \qquad 0 < x < +\infty,$$

$$\lim_{x=+0} u(x, t) = f(t) \qquad 0 < t < +\infty$$

uniformly. But since $u(x, t)$ is uniquely determined by those three conditions, the theorem has been proved.

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