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On Special Systems of Linear Equations Having Infinite Unknowns.

Extracted from

THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 16, No. 1, 2.

edited by TSURUICHI HAYASHI, College of Science,

Tôhoku Imperial University, Sendai, Japan,

with the collaboration of Messrs.

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June 1919

On special Systems of linear Equations having infinite Unknowns,

by

KINNOSUKE OGURA, Ôsaka.

Poincaré treated a system of linear equations having infinite unknowns x_n $(n=1, 2, \cdots\cdots)$ of the form [1]

$$\sum_{n=1}^{\infty} a_n^i x_n = 0 \qquad (i=0, 1, 2, \cdots\cdots),$$

where

$$|a_n| < |a_{n+1}| \text{ and } \lim_{n=\infty} |a_n| = \infty.$$

But his method can not be employed in the case of the following system [2]:

$$\text{(A)} \qquad \sum_{n=1}^{\infty} n^{2i} x_n = 0 \qquad (i=0, 1, 2, \cdots\cdots).$$

In this small contribution I propose to find the *general* solution of the system (A) and moreover that of

$$\text{(B)} \qquad \sum_{n=1}^{\infty} n^{2i+1} y_n = 0 \qquad (i=0, 1, 2, \cdots\cdots).$$

(i) Let the two systems (A) and (B) be given. In order that all the series $\Sigma n^{2i} x_n$ and $\Sigma n^{2i+1} y_n$ $(i=0, 1, 2, \cdots\cdots)$ may converge, it should be

$$\lim_{n=\infty} n^p x_n = 0, \qquad \lim_{n=\infty} n^p y_n = 0,$$

p being any positive integer. Hence if we put

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(1) Appell, "Sur une méthode élémentaire pour obtenir les développements en séries trigonométriques des fonctions elliptiques," Bull. Soc. math. de France, 13 (1885), pp. 13–18; Poincaré, "Remarques sur l'emploi de la méthode précédente," Ibid, pp. 19–27. See also F. Riesz, Les systèmes d'équations linéaires à une infinité d'inconnues (1913), pp. 15–16.

(2) Applying Poincaré's method to this system formally, we obtain

$$x_n = (-1)^n n^2, \qquad (n=1, 2, \cdots\cdots)$$

up to any constant multiple; so that all the series $\Sigma n^{2i} x_n$ $(i=0, 1, 2, \cdots\cdots)$ do not converge.

(1) $$f(\theta) \equiv \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta) \qquad -\pi \leqq \theta \leqq +\pi,$$

the function $f(\theta)$ is differentiable indefinitely and its successive derivatives become

$$f^{(2i)}(\theta) = (-1)^i \sum_{n=1}^{\infty} n^{2i} (x_n \cos n\theta + y_n \sin n\theta),$$

$$f^{(2i+1)}(\theta) = (-1)^i \sum_{n=1}^{\infty} n^{2i+1} (x_n \sin n\theta - y_n \cos n\theta).$$

Therefore we must have

(2) $$\int_{-\pi}^{+\pi} f(\theta)\, d\theta = 0\,;$$

(3) $$f^{(i)}(-\pi) = f^{(i)}(+\pi), \qquad (i = 0, 1, 2, \cdots\cdots);$$

and from (A) and (B)

(4) $$f^{(i)}(0) = 0, \qquad (i = 0, 1, 2, \cdots\cdots).$$

Lastly we have from (1) that

(5) $$x_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \cos n\theta\, d\theta,$$

(6) $$y_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \sin n\theta\, d\theta, \qquad (n = 1, 2, \cdots\cdots).$$

(ii) Conversely, let $f(\theta)$ be *any* function which is differentiable indefinitely in the interval $-\pi \leqq \theta \leqq +\pi$ and satisfies the conditions (2), (3) and (4). Then the quantities x_n, y_n defined by (5) satisfy

(7) $$\lim_{n=\infty} n^p x_n = 0, \qquad \lim_{n=\infty} n^p y_n = 0,$$

where p denotes any positive integer [1].

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(1) For, we have

$$\int_{-\pi}^{+\pi} f'(\theta) \sin n\theta\, d\theta + n \int_{-\pi}^{+\pi} f(\theta) \cos n\theta\, d\theta = \Big[f(\theta) \sin n\theta \Big]_{-\pi}^{+\pi} = 0,$$

$$\int_{-\pi}^{+\pi} f''(\theta) \cos n\theta\, d\theta - n \int_{-\pi}^{+\pi} f'(\theta) \sin n\theta\, d\theta = \Big[f'(\theta) \cos n\theta \Big]_{-\pi}^{+\pi} = (-1)^n \Big[f'(+\pi) - f'(-\pi) \Big] = 0,$$

and so on; whence

$$\int_{-\pi}^{+\pi} f(\theta) \cos n\theta\, d\theta = -\frac{1}{n} \int_{-\pi}^{+\pi} f'(\theta) \sin n\theta\, d\theta = -\frac{1}{n^2} \int_{-\pi}^{+\pi} f''(\theta) \cos n\theta\, d\theta$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots = \pm \frac{1}{n^p} \int_{-\pi}^{+\pi} f^{(p)}(\theta) \begin{matrix}\cos \\ \sin\end{matrix}\, n\theta\, d\theta.$$

In virtue of (2), (3) and (7), $f(\theta)$ and all its successive derivatives are expansible in the Fourier series such that

$$f(\theta) = \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta),$$

$$f^{(2i)}(\theta) = (-1)^i \sum_{n=1}^{\infty} n^{2i} (x_n \cos n\theta + y_n \sin n\theta),$$

$$f^{(2i+1)}(\theta) = (-1)^i \sum_{n=1}^{\infty} n^{2i+1} (x_n \sin n\theta - y_n \cos n\theta),$$

$$(-\pi \leqq \theta \leqq +\pi).$$

Since all these series converge uniformly in the given interval, we obtain from (4) that

(A) $$\sum_{n=1}^{\infty} n^{2i} x_n = 0,$$

(B) $$\sum_{n=1}^{\infty} n^{2i+1} y_n = 0, \qquad (i = 0, 1, 2, \cdots\cdots).$$

Thus we arrive at the theorem:

The general solution of

(A) $$\sum_{n=1}^{\infty} n^{2i} x_n = 0 \qquad (i = 0, 1, 2, \cdots\cdots)$$

and

(B) $$\sum_{n=1}^{\infty} n^{2i+1} y_n = 0 \qquad (i = 0, 1, 2, \cdots\cdots)$$

are the Fourier constants of $f(\theta)$:

(5) $$x_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \cos n\theta\, d\theta,$$

(6) $$y_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \sin n\theta\, d\theta,$$

respectively, where $f(\theta)$ *is any function which is differentiable indefinitely in the interval* $-\pi \leqq \theta \leqq +\pi$ *and is such that*

Consequently by Riemann-Lebesgue's theorem

$$\lim_{n=\infty} n^p \int_{-\pi}^{+\pi} f(\theta) \cos n\theta\, d\theta = \pm \lim_{n=\infty} \int_{-\pi}^{+\pi} f^{(p)}(\theta) \begin{matrix}\cos \\ \sin\end{matrix}\, n\theta\, d\theta = 0.$$

Similarly

$$\lim_{n=\infty} n^p \int_{-\pi}^{+\pi} f(\theta) \sin n\theta\, d\theta = 0.$$

(2) $$\int_{-\pi}^{+\pi} f(\theta)\, d\theta = 0,$$

(4) $$f^{(i)}(0) = 0,$$

(3) $$f^{(i)}(-\pi) = f^{(i)}(+\pi), \qquad (i = 0, 1, 2, \cdots\cdots).$$

Thus *the solution of* (A) [or (B)] *is not unique;* and *any linear combination of particular solutions of* (A) [or (B)] *is also a solution of that system.*

It will be noticed that *the function $f(\theta)$ used in the above theorem is not analytic at $\theta = 0$,* that is, it can not be developed in the Maclaurin series; for we have

$$f^{(i)}(0) = 0, \qquad (i = 0, 1, 2, \cdots\cdots).$$

Lastly, as an example, I will find a particular solution of (A) and that of (B). We can easily see that both

$$\begin{aligned} \varphi_1(\theta) &= e^{-\frac{1}{\sin^2\theta}} && \text{for } \theta \neq -\pi,\ 0,\ +\pi, \\ &= 0 && \text{for } \theta = -\pi,\ 0,\ +\pi, \end{aligned}$$

and

$$\begin{aligned} \varphi_2(\theta) &= e^{-\frac{1}{\sin^2\frac{\theta}{2}}} && \text{for } \theta \neq 0, \\ &= 0 && \text{for } \theta = 0 \end{aligned}$$

are differentiable indefinitely in the interval $-\pi \leqq \theta \leqq +\pi$ and moreover satisfy the conditions (3) and (4). Consequently if we put

$$f(\theta) \equiv \varphi_1(\theta) \cdot \int_{-\pi}^{+\pi} \varphi_2(\theta)\, d\theta - \varphi_2(\theta) \cdot \int_{-\pi}^{+\pi} \varphi_1(\theta)\, d\theta,$$

then $f(\theta)$ satisfies all the conditions imposed in the theorem; so that (5) gives a solution of (A) and (6) that of (B).

December 1918, Ikeda near Ôsaka.

THE TÔHOKU MATHEMATICAL JOURNAL.

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