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Trajectories in the Irreversible Field of Force on the Surface.

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Trajectories in the Irreversible Field of Force on a Surface,

by

KINNOSUKE OGURA, Ôsaka.

Consider a dynamical system with two degrees of freedom which si reduced to the normal form (1):

$$(1) \qquad \ddot{p}+\lambda\dot{q}=\frac{\partial\gamma}{\partial p},$$

$$\ddot{q}-\lambda\dot{p}=\frac{\partial\gamma}{\partial q},$$

where p and q denote the two coordinates of the dynamical system, $\dot{p}$ and $\dot{q}$ their time derivatives, and λ, γ are functions of p and q. Then equations (1) admit the first integral

$$(2) \qquad \frac{1}{2}(\dot{p}^2+\dot{q}^2)=\gamma+h,$$

h being an arbitrary constant.

Throughout this paper I will consider the motion of a particle on a surface and confine myself to the case in which *h has a definite value.* A few properties of the trajectories have been obtained in my previous paper (2); but in the present I will deal with certain properties of a different nature, some of which may be considered as generalizations of those for the conservative field (that is, for the reversible field) (3).

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(1) Birkhoff, "Dynamical systems with two degrees of freedom," Trans Amer. Math. Soc., 18 (1917), p. 204.

(2) Ogura, "On a certain system of doubly infinite curves on a surface," Tôhoku Math. Journal, 8 (1915), p. 213; "A remark on the dynamical system with two degrees of freedom," Tôhoku Math. Journal, 15 (1919), p. 181. Hereafter these papers will be referred to as O_1 and O_2 respectively.

(3) Ogura, "Trajectories in the conservative field of force," Part I, Tôhoku Math. Journal, 7 (1915), p. 123; Part II, Tôhoku Math. Journal, 9 (1916), p. 134; "On the striped net of curves without ambages in dynamics," Proc. Tôkyô Math.-Phys. Soc., II 9 (1918), p. 284; "Note Supplementary to the paper 'on the striped net of curves without ambages in dynamics'," Proc. Tôkyô Math.-Phys. Soc., II 9 (1918), p. 409. Hereafter these papers will be referred to as O_3, O_4, O_5, O_6 respectively.

PART I. General Theorems.

The condition for ∞^1 given orbits.

1. Let the linear element of a surface S be

$$ds^2=\mu\,(dp^2+dq^2), \tag{3}$$

where isothermic parameters p and q are taken for parametrics. Then the differential equation of the orbits is [1]

$$q''=\left(\frac{\partial \log \sqrt{\gamma+h}}{\partial q}-\frac{\partial \log \sqrt{\gamma+h}}{\partial p}q'\right)(1+q'^2)+\frac{\lambda}{\sqrt{2\,(\gamma+h)}}(1+q'^2)^{\frac{3}{2}}, \tag{4}$$

where we have put

$$q'=\frac{dq}{dp}, \qquad q''=\frac{d^2q}{dp^2}.$$

Since $\gamma+h$ and μ are positive, we may put

$$\gamma+h=\frac{1}{2}\mu\,\varphi^2, \qquad \frac{\lambda}{\sqrt{2(\gamma+h)}}=\sqrt{\mu}\,\psi\,;$$

so that (4) becomes

$$q''=\left(\frac{\partial \log (\sqrt{\mu}\,\varphi)}{\partial q}-\frac{\partial \log (\sqrt{\mu}\,\varphi)}{\partial p}q'\right)(1+q'^2)+\sqrt{\mu}\,\psi.\ (1+q'^2)^{\frac{3}{2}}. \tag{5}$$

The integral curves of this differential equation [2] will be called *the orbits in the field* $(\varphi,\ \psi)$.

Now suppose that

$$f\,(p,q)=\text{const.}$$

form a family of ∞^1 orbits in the field $(\varphi,\ \psi)$. Since

$$q'=-\left(\frac{\partial f}{\partial p}\right)\cdot\left(\frac{\partial f}{\partial q}\right)^{-1},$$

$$q''=-\left[\left(\frac{\partial f}{\partial q}\right)^2\frac{\partial^2 f}{\partial p^2}-2\frac{\partial f}{\partial p}\,\frac{\partial f}{\partial q}\,\frac{\partial^2 f}{\partial p\,\partial q}+\left(\frac{\partial f}{\partial p}\right)^2\frac{\partial^2 f}{\partial q^2}\right]\cdot\left(\frac{\partial f}{\partial q}\right)^{-3},$$

---

(1) See O_2.

(2) For some properties of the integral curves, see O_1.

we must have

$$\left[\left(\frac{\partial f}{\partial q}\right)^2\frac{\partial^2 f}{\partial p^2}-2\frac{\partial f}{\partial p}\,\frac{\partial f}{\partial q}\,\frac{\partial^2 f}{\partial p\,\partial q}+\left(\frac{\partial f}{\partial p}\right)^2\frac{\partial^2 f}{\partial q^2}\right]$$
$$+\frac{1}{2}\left(\frac{\partial \log\mu}{\partial p}\,\frac{\partial f}{\partial p}+\frac{\partial \log\mu}{\partial q}\,\frac{\partial f}{\partial q}\right)\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]$$
$$+\left(\frac{\partial \log\varphi}{\partial p}\,\frac{\partial f}{\partial p}+\frac{\partial \log\varphi}{\partial q}\,\frac{\partial f}{\partial q}\right)\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]$$
$$+\sqrt{\mu}\,\psi.\ \left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{\frac{3}{2}}=0.$$

But since the expression of the geodesic curvature $\frac{1}{\rho_f}$ of $f=$const.:

$$-\frac{1}{\rho_f}=\frac{1}{\mu}\left\{\frac{\partial}{\partial p}\left(\sqrt{\mu}\frac{\partial f}{\partial p}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}\right)\right.$$
$$\left.+\frac{\partial}{\partial q}\left(\sqrt{\mu}\frac{\partial f}{\partial q}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}\right)\right\}$$

gives

$$\left(\frac{\partial f}{\partial q}\right)^2\frac{\partial^2 f}{\partial p^2}-2\frac{\partial f}{\partial p}\,\frac{\partial f}{\partial q}\,\frac{\partial^2 f}{\partial p\partial q}+\left(\frac{\partial f}{\partial p}\right)^2\frac{\partial^2 f}{\partial q^2}$$
$$+\frac{1}{2}\left(\frac{\partial \log\mu}{\partial p}\,\frac{\partial f}{\partial p}+\frac{\partial \log\mu}{\partial q}\,\frac{\partial f}{\partial q}\right)$$
$$=-\frac{\sqrt{\mu}}{\rho_f}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{\frac{3}{2}};$$

whence we have

$$\frac{1}{\rho_f}-\frac{1}{\sqrt{\mu}}\left(\frac{\partial \log\varphi}{\partial p}\,\frac{\partial f}{\partial p}+\frac{\partial \log\varphi}{\partial q}\,\frac{\partial f}{\partial q}\right)\times\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}-\psi=0. \tag{6}$$

Now consider the differential parameters of the differential quadratic form

$$E\,du^2+2\,F\,du\,dv+G\,dv^2:$$

namely

$$\Delta_1\,U=\frac{E\left(\frac{\partial U}{\partial v}\right)^2-2\,F\,\frac{\partial U}{\partial u}\,\frac{\partial U}{\partial v}+G\left(\frac{\partial U}{\partial u}\right)^2}{E\,G-F^2},$$

$$\Delta_2 U = \frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\left(G\frac{\partial U}{\partial u} - F\frac{\partial U}{\partial v}\right)\Big/\sqrt{EG-F^2}\right]\right.$$

$$\left.+\frac{\partial}{\partial v}\left[\left(E\frac{\partial U}{\partial v} - F\frac{\partial U}{\partial u}\right)\Big/\sqrt{EG-F^2}\right]\right\},$$

$$\nabla(U, V) = \frac{E\dfrac{\partial U}{\partial v}\dfrac{\partial V}{\partial v} - F\left(\dfrac{\partial U}{\partial u}\dfrac{\partial V}{\partial v} + \dfrac{\partial U}{\partial v}\dfrac{\partial V}{\partial u}\right) + G\dfrac{\partial U}{\partial u}\dfrac{\partial V}{\partial u}}{EG-F^2}.$$

If we put

$$u=p, \qquad v=q;$$

$$E=\mu, \qquad F=0, \qquad G=\mu,$$

equation (6) may be written

$$(7) \qquad -\frac{1}{\rho_f} + \frac{1}{\sqrt{\Delta_1 f}}\nabla(f, \log\varphi) + \psi = 0,$$

or, by Beltrami's formula,

$$(8) \qquad \frac{\Delta_2 f}{\sqrt{\Delta_1 f}} + \nabla\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\sqrt{\Delta_1 f}}\nabla(f, \log\varphi) + \psi = 0.$$

This is the necessary and sufficient condition that $f(p, q)=\text{const.}$ should be orbits in the field (φ, ψ) on the surface having the linear element

$$ds^2 = \mu\,(dp^2 + dq^2).$$

Let us now apply the transformation

$$p = p\,(u, v), \qquad q = q\,(u, v)$$

and let

$$\mu\,(dp^2 + dq^2) = E\,du^2 + 2\,F\,du\,dv + G\,dv^2.$$

Putting

$$f\,(p, q) = \bar f\,(u, v), \qquad \varphi\,(p, q) = \bar\varphi\,(u, v), \qquad \psi\,(p, q) = \bar\psi\,(u, v)$$

and remembering the invariantive property of the differential parameters, (8) becomes

$$\frac{\Delta_2 \bar f}{\sqrt{\Delta_1 \bar f}} + \nabla\left(\bar f, \frac{1}{\sqrt{\Delta_1 \bar f}}\right) + \frac{1}{\sqrt{\Delta_1 \bar f}}\nabla\,(\bar f, \log\bar\varphi) + \bar\psi = 0.$$

Therefore we have the theorem:

In order that ∞^1 curves

$$f\,(u, v) = \text{const.}$$

may be orbits in the field $(\varphi\,(u, v), \psi\,(u, v))$ on the surface having the linear element

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$

it is necessary and sufficient that $f\,(u, v)$ should satisfy

$$(8) \qquad \frac{\Delta_2 f}{\sqrt{\Delta_1 f}} + \nabla\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\sqrt{\Delta_1 f}}\nabla\,(f, \log\varphi) + \psi = 0.$$

2. Let the two surfaces S, $\bar S$ having the linear elements

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$

$$ds^2 = \bar E\,du^2 + 2\,\bar F\,du\,dv + \bar G\,dv^2$$

be related by the conformal representation such that

$$\frac{\bar E}{E} = \frac{\bar F}{F} = \frac{\bar G}{G} = \varphi^2\,(u, v).$$

Then the differential parameters corresponding to the surface $\bar S$ have the expressions:

$$\bar\Delta_1 f = \frac{1}{\varphi^2}\Delta_1 f, \qquad \bar\Delta_2 f = \frac{1}{\varphi^2}\Delta_2 f,$$

$$\bar\nabla\left(f, \frac{1}{\sqrt{\bar\Delta_1 \bar f}}\right) = \bar\nabla\left(f, \frac{\varphi}{\sqrt{\Delta_1 f}}\right)$$

$$= \varphi.\,\bar\nabla\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\sqrt{\Delta_1 f}}.\,\bar\nabla\,(f, \varphi)$$

$$= \frac{1}{\varphi}.\,\nabla\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\varphi\sqrt{\Delta_1 f}}\nabla\,(f, \log\varphi);$$

so that (8) may be written

$$(9) \qquad \frac{\bar\Delta_2 f}{\sqrt{\Delta_1 f}}\bar\nabla + \left(f, \frac{1}{\sqrt{\bar\Delta_1 f}}\right) + \frac{\psi}{\varphi} = 0.$$

Consequently if $\dfrac{1}{\rho_f}$ be the geodesic curvature of $f=\text{const.}$ on the surface $\bar S$, we have

$$(10) \qquad \frac{1}{\bar{\rho}_f} = \frac{\psi}{\varphi}.$$

Thus we arrive at the theorem:

By the conformal representation

$$d\bar{s}^2 = \varphi^2 \, ds^2,$$

the orbits $f=$const. in the field (φ, ψ) on the surface S are transformed into the curves having the geodesic curvature

$$\frac{1}{\bar{\rho}_f} = \frac{\psi}{\varphi}$$

on the surface $\bar{S}$; and conversely.

In the particular case where $\varphi=$const., we have from (7)

$$\frac{1}{\rho_f} = \psi.$$

On the other hand, in the reversible field (that is, the case where $\psi=0$ identically), we have from (10)

$$\frac{1}{\bar{\rho}_f} = 0;$$

so that $f=$const. are geodesics on the surface $\bar{S}$, which is a well known result (1).

The condition for $2\infty^1$ given orbits.

3. Now we can infer from (8) the theorem immediately:

The necessary and sufficient condition that the $2\infty^1$ curves $u=$const., $v=$const. may be orbits in the field (φ, ψ) on the surface S is given by

$$(11) \qquad G\frac{\partial \log\varphi}{\partial u} - F\frac{\partial \log\varphi}{\partial v} = \frac{EG-F^2}{G}\begin{Bmatrix}22\\1\end{Bmatrix} - \sqrt{G}\sqrt{EG-F^2}\,\psi,$$

$$(12) \qquad -F\frac{\partial \log\varphi}{\partial u} + E\frac{\partial \log\varphi}{\partial v} = \frac{EG-F^2}{E}\begin{Bmatrix}11\\2\end{Bmatrix} - \sqrt{E}\sqrt{EG-F^2}\,\psi \;(2),$$

(1) Darboux, Théorie des surfaces, 2 (1 éd., 1889), p. 453; See also O_3, p. 178.

(2) In the reversible field ψ vanishes identically; so (11) and (12) become

$$G\frac{\partial \log\varphi}{\partial u} - F\frac{\partial \log\varphi}{\partial v} = \frac{EG-F^2}{G}\begin{Bmatrix}22\\1\end{Bmatrix},$$

$$-F\frac{\partial \log\varphi}{\partial u} - E\frac{\partial \log\varphi}{\partial v} = \frac{EG-F^2}{E}\begin{Bmatrix}11\\2\end{Bmatrix}$$

respectively. Compare with O_3 p. 179.

where $\begin{Bmatrix}11\\2\end{Bmatrix}$, $\begin{Bmatrix}22\\1\end{Bmatrix}$ denote Christoffel's symbols, that is,

$$\begin{Bmatrix}11\\2\end{Bmatrix} = \frac{-F\dfrac{\partial E}{\partial u} - E\dfrac{\partial E}{\partial v} + 2E\dfrac{\partial F}{\partial u}}{2(EG-F^2)},$$

$$\begin{Bmatrix}22\\1\end{Bmatrix} = \frac{-F\dfrac{\partial G}{\partial v} - G\dfrac{\partial G}{\partial u} + 2G\dfrac{\partial F}{\partial v}}{2(EG-F^2)}.$$

In order to interpret this condition geometrically, let us apply the conformal representation such that

$$d\bar{s}^2 = \varphi^2 \, ds^2.$$

Then (11) and (12) are equivalent to

$$\frac{1}{\bar{\rho}_u} = \frac{\psi}{\varphi}, \qquad \frac{1}{\bar{\rho}_v} = \frac{\psi}{\varphi}$$

respectively (§2); from which we find

$$(13) \qquad \frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v}.$$

Conversely, consider the parametric curves u, v on a surface $\bar{S}$, such that

$$\frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v} \; (=\Phi(u, v), \text{ say}).$$

Take any function $\varphi(u, v)$ and put $\psi = \varphi\,\Phi$. Then u, v will be orbits in the field (φ, ψ) on the surface S, which is obtained from $\bar{S}$ by the conformal transformation

$$\frac{1}{\varphi^2} d\bar{s}^2 = ds^2.$$

Thus we have the theorem:

In order that $2\infty^1$ curves u, v may be orbits on a surface S in an irreversible field of force, it is necessary and sufficient that the corresponding curves on the surface $\bar{S}$, obtained from S by a certain conformal transformation, should have the property

$$\frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v}.$$

4. Since (11), (12) may be written

$$(14)\qquad \frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\frac{\partial \log\varphi}{\partial u}-\frac{F}{\sqrt{G}}\frac{\partial \log\varphi}{\partial v}\right]=\frac{1}{\rho_u}-\psi,$$

$$(15)\qquad \frac{1}{\sqrt{EG-F^2}}\left[-\frac{F}{\sqrt{E}}\frac{\partial \log\varphi}{\partial u}+\sqrt{E}\frac{\partial \log\varphi}{\partial v}\right]=\frac{1}{\rho_v}-\psi$$

respectively, if we put

$$F_u=\frac{\varphi^2}{\sqrt{G}}\frac{\partial \log\varphi}{\partial v},\qquad F_v=\frac{\varphi^2}{\sqrt{E}}\frac{\partial \log\varphi}{\partial u},$$

$$F_{u'}=\frac{\varphi^2}{\rho_u}-\varphi^2\psi,\qquad F_{v'}=\frac{\varphi^2}{\rho_v}-\varphi^2\psi.$$

we can derive the two equations:

$$(16)\qquad F_u\cos\frac{\omega}{2}+F_{u'}\sin\frac{\omega}{2}=F_v\cos\frac{\omega}{2}+F_{v'}\sin\frac{\omega}{2},$$

$$(17)\qquad F_u\sin\frac{\omega}{2}-F_{u'}\cos\frac{\omega}{2}=-F_v\sin\frac{\omega}{2}+F_{v'}\cos\frac{\omega}{2},$$

ω being the angle between $u=$const. and $v=$const..

Now denote by $\mathfrak{F}_u$ the vector having F_u as its magnitude and the (positive) direction of the curve $u=$const. as its direction, and by $\mathfrak{F}_{u'}$ the vector having $F_{u'}$ as its magnitude and the direction to the centre of geodesic curvature of $u=$const. as its direction; also define $\mathfrak{F}_v$ and $\mathfrak{F}_{v'}$ in similar ways with respect to $v=$const.. Then it follows from (16) and (17) that the projections of the resultant of $\mathfrak{F}_u$ and $\mathfrak{F}_{u'}$ upon the two bisectors of the angles (internal and external) between $u=$const., $v=$const. are equal to those for $\mathfrak{F}_v$ and $\mathfrak{F}_{v'}$ upon the same bisectors respectively.

Consequently we arrive at the *dynamical interpretation* of the conditions (11) and (12):

A necessary and sufficient condition that $u=$const. and $v=$const. may be orbits in the field (φ, ψ) is that the resultant of the two vectors $\mathfrak{F}_u$ and $\mathfrak{F}_{u'}$ is equal to the resultant of $\mathfrak{F}_v$ and $\mathfrak{F}_{v'}$ (1).

In the reversible field $(\psi=0)$, if $\mathfrak{F}_{u0}$ $(\mathfrak{F}_{v0})$ and $\mathfrak{F}_{u'0}$ $(\mathfrak{F}_{v'0})$ be the components of force tangential and normal to the orbit $u=$const. ($v=$const.) respectively, we have the well known relations:

$$F_{u0}=\frac{\varphi^2}{\sqrt{G}}\frac{\partial \log\varphi}{\partial v},\qquad F_{v0}=\frac{\varphi^2}{\sqrt{E}}\frac{\partial \log\varphi}{\partial u},$$

(1) Compare with O_4, p. 140.

$$F_{u'0}=\frac{\varphi^2}{\rho_u},\qquad F_{v'0}=\frac{\varphi^2}{\rho_v}.$$

5. In this paragraph I will give three pairs of the formulae concerning the geodesic curvatures of $2\ \infty^1$ orbits $u=$const., $v=$const. and their related curves.

I. Let $ds(u_1)$ and $ds(v_1)$ be the arc elements of the curves $u_1=$const. and $v_1=$const. which are the bisectors of the angles, external, between $u=$const. and $v=$const. respectively. Then

$$(18)\begin{cases}2\sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)=\dfrac{1}{\sqrt{E}}\dfrac{\partial}{\partial u}(\log\varphi)-\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial v}(\log\varphi),\\[2ex] 2\cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)=\dfrac{1}{\sqrt{E}}\dfrac{\partial}{\partial u}(\log\varphi)+\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial v}(\log\varphi).\end{cases}$$

Therefore by means of (18) we find from (14) and (15) the following *formulae* of importance:

$$(19)\begin{cases}\cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)=\dfrac{1}{2}\left(\dfrac{1}{\rho_u}-\dfrac{1}{\rho_v}\right),\\[2ex] \sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)=\dfrac{1}{2}\left(\dfrac{1}{\rho_u}+\dfrac{1}{\rho_v}\right)-\psi.\end{cases}$$

II. Next suppose that $u'=$const., $v'=$const. are the orthogonal trajectories of $u=$const., $v=$const. respectively. Then we have

$$(20)\begin{cases}\dfrac{G}{E}\begin{Bmatrix}11\\2\end{Bmatrix}+\dfrac{F}{G}\begin{Bmatrix}22\\1\end{Bmatrix}=\dfrac{\sqrt{G}}{\rho_{u'}}-\dfrac{G}{\sqrt{EG-F^2}}\dfrac{\partial\omega}{\partial u},\\[2ex] \dfrac{F}{E}\begin{Bmatrix}11\\2\end{Bmatrix}+\dfrac{E}{G}\begin{Bmatrix}22\\1\end{Bmatrix}=\dfrac{\sqrt{E}}{\rho_{v'}}-\dfrac{E}{\sqrt{EG-F^2}}\dfrac{\partial\omega}{\partial v}\ (^1).\end{cases}$$

Now from (11), (12) and (20) we get

$$\frac{1}{\sqrt{G}}\frac{\partial \log\varphi}{\partial v}=\frac{1}{\rho_{u'}}-\frac{\sqrt{G}}{\sqrt{EG-F^2}}\frac{\partial\omega}{\partial u}-\frac{\sqrt{EG}+F}{\sqrt{EG-F^2}}\psi,$$

$$\frac{1}{\sqrt{E}}\frac{\partial \log\varphi}{\partial u}=\frac{1}{\rho_{v'}}-\frac{\sqrt{E}}{\sqrt{EG-F^2}}\frac{\partial\omega}{\partial v}-\frac{\sqrt{EG}+F}{\sqrt{EG-F^2}}\psi.$$

Applying (18) to these two equations we find the *formulae*:

(1) For an application of these formulae, see Ogura, "On a generalization of the Bonnet-Darboux theorem concerning the line of striction," Proc. Tôkyô Math.-Phys., Soc., II, 9 (1918), p. 304; where these formulae have been printed incorrectly.

$$
(21)\begin{cases}
\sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)-\dfrac{1}{2}\sec\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(u_1)}=-\dfrac{1}{2}\left(\dfrac{1}{\rho_{u'}}-\dfrac{1}{\rho_{v'}}\right),\\
\cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)+\dfrac{1}{2}\operatorname{cosec}\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(v_1)}=\dfrac{1}{2}\left(\dfrac{1}{\rho_{u'}}+\dfrac{1}{\rho_{v'}}\right)\\
\qquad\qquad -\cot\dfrac{\omega}{2}\cdot\psi.
\end{cases}
$$

III. Lastly applying the following formulae of Prof. Lilienthal (1)

$$
\frac{2}{\rho(u_1)}=\sin\frac{\omega}{2}\cdot\left(\frac{1}{\rho_u}+\frac{1}{\rho_v}\right)+\cos\frac{\omega}{2}\cdot\left(\frac{1}{\rho_{u'}}+\frac{1}{\rho_{v'}}\right),
$$

$$
\frac{2}{\rho(v_1)}=-\cos\frac{\omega}{2}\cdot\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right)+\sin\frac{\omega}{2}\cdot\left(\frac{1}{\rho_{u'}}-\frac{1}{\rho_{v'}}\right)
$$

to (19) and (21), we obtain the *formulae :*

$$
(22)\begin{cases}
\dfrac{\partial}{\partial s(u_1)}(\log\varphi)-\dfrac{1}{2}\cot\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(u_1)}=-\dfrac{1}{\rho(v_1)},\\
\dfrac{\partial}{\partial s(v_1)}(\log\varphi)+\dfrac{1}{2}\cot\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(v_1)}=\dfrac{1}{\rho(u_1)}-\operatorname{cosec}\dfrac{\omega}{2}\cdot\psi.
\end{cases}
$$

PART II. Some Particular Cases.

2 ∞^1 orbits comprising a constant angle.

6. Hereafter we will consider some system of 2 ∞^1 orbits in the field (φ, ψ), which are of geometrical interest.

We begin with the case where *the angle ω between u=const., v=const. is constant.*

In this case (19), (21) and (22) give

$$
\cos\frac{\omega}{2}\cdot\frac{\partial}{\partial s(u_1)}(\log\varphi)=\frac{1}{2}\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right),
$$

$$
\sin\frac{\omega}{2}\cdot\frac{\partial}{\partial s(u_1)}(\log\varphi)=-\frac{1}{2}\left(\frac{1}{\rho_{u'}}-\frac{1}{\rho_{v'}}\right),
$$

$$
\frac{\partial}{\partial s(u_1)}(\log\varphi)=-\frac{1}{\rho(v_1)}\,;
$$

so that we find

(1) Lilienthal, Vorlesungen über Differentialgeometrie, II $_1$ (1913), pp. 240–241.

$$
(23)\qquad \frac{2\cos\frac{\omega}{2}}{\rho(v_1)}=-\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right),
$$

$$
(24)\qquad \frac{2\sin\frac{\omega}{2}}{\rho(v_1)}=\frac{1}{\rho_{u'}}-\frac{1}{\rho_{v'}}.
$$

Similarly

$$
(25)\qquad \frac{2\sin\frac{\omega}{2}}{\rho(u_1)}=\frac{1}{\rho_u}+\frac{1}{\rho_v},
$$

$$
(26)\qquad \frac{2\cos\frac{\omega}{2}}{\rho(u_1)}=\frac{1}{\rho_{u'}}+\frac{1}{\rho_{v'}}\ (1).
$$

Since the product of the right hand sides of (23) and (25) is equal to that for (24) and (26), we have

$$
\frac{1}{\rho_u^2}+\frac{1}{\rho_{u'}^2}=\frac{1}{\rho_v^2}+\frac{1}{\rho_{v'}^2}.
$$

Adding the squares of (23), (24), (25) and (26),

$$
2\left(\frac{1}{\rho^2(u_1)}+\frac{1}{\rho^2(v_1)}\right)=\frac{1}{\rho_u^2}+\frac{1}{\rho_v^2}+\frac{1}{\rho_{u'}^2}+\frac{1}{\rho_{v'}^2}.
$$

From the last two equations we obtain

$$
\frac{1}{\rho^2(u_1)}+\frac{1}{\rho^2(v_1)}=\frac{1}{\rho_u^2}+\frac{1}{\rho_{u'}^2}=\frac{1}{\rho_v^2}+\frac{1}{\rho_{v'}^2}.
$$

7. As a simple application, let us consider the 2 ∞^1 curves u, v which make a constant angle and may be orbits in the irreversible field determined by

(1) These four formulae have certain similarities with the following four concerning *a net of curves without ambages* $\bar{u}$=const., $\bar{v}$=const. :

$$
\frac{4\cos\frac{\varpi}{2}}{\rho(\bar{v}_1)}=-\left(\frac{1}{\rho\bar{u}}-\frac{1}{\rho\bar{v}}\right),\qquad \frac{2\cos\varpi}{\rho(\bar{v}_1)}=-\sin\frac{\varpi}{2}\cdot\left(\frac{1}{\rho\bar{u}'}-\frac{1}{\rho\bar{v}'}\right),
$$

$$
\frac{4\sin\frac{\varpi}{2}}{\rho(\bar{u}_1)}=\frac{1}{\rho\bar{u}}+\frac{1}{\rho\bar{v}},\qquad \frac{2\cos\varpi}{\rho(\bar{u}_1)}=\cos\frac{\varpi}{2}\cdot\left(\frac{1}{\rho\bar{u}'}+\frac{1}{\rho\bar{v}'}\right),
$$

ϖ being the angle between $\bar{u}$ and $\bar{v}$. See Lilienthal, loc. cit., pp 244-245.

$\varphi=$const., $\varphi=$const..

From (19) we have

$$\frac{1}{\rho_u}=\frac{1}{\rho_v}=\text{const. } (=k, \text{ say});$$

so that from (23) and (25)

$$\frac{1}{\rho(v_1)}=0, \qquad \frac{1}{\rho(u_1)}=\frac{k}{\sin\frac{\omega}{2}}=\text{const.}.$$

Consequently by well known theorems the surface must be a pseudospherical surface of revolution of the parabolic type or a surface applicable to it, and $u=$const., $v=$const. make constant angles ($\frac{\omega}{2}$ and $-\frac{\omega}{2}$ respectively) with the geodesic parallels $u_1=$const.. Conversely if a surface have the linear element

$$ds^2=du_1^2+c^2 e^{\frac{2u_1}{a}} dv_1^2,$$

a, c being constants, we have

$$\frac{1}{\rho(v_1)}=0, \qquad \frac{1}{\rho(u_1)}=\frac{1}{a}.$$

Hence equations (22) are satisfied by

$$\omega=\text{const.}, \ \varphi=\text{const.}, \ \psi=\text{const.}, \text{ where } \operatorname{cosec}\frac{\omega}{2}\cdot\psi=\frac{1}{a}.$$

Therefore *in order that the* $2\,\infty^1$ *curves* $u=$const., $v=$const. *which make a constant angle and may be orbits in an irreversible field of the form* ($\varphi=$const., $\psi=$const.), *it is necessary and sufficient that the surface should be a pseudospherical surface of revolution of the parabolic type or a surface applicable to it, and moreover the bisectors of the external angles of u, v should be geodesic parallels.*

The isothermal system as orbits.

8. For the reversible field of force (i. e. $\psi=0$ identically), I have proved the theorem ([1]): A necessary and sufficient condition that $2\,\infty^1$ curves belonging to an orthogonal system may be orbits in a reversible

([1]) O_1, p. 168, p. 173, p. 180.

field of force is that these curves form an isothermal system. Now I proceed to establish the following theorem concerning the irreversible field of force (φ, ψ):

Consider an orthogonal system $u=$const., $v=$const.. *If any three of the four propositions be true:*

(i) *that u, v form an isothermal system;*

(ii) *that u, v form a net of curves without ambages;*

(iii) *that* ψ *is constant along each bisector* $u_1=$const. *of the external angles of u, v;*

(iv) *that u, v may be orbits in the field of the form* (φ, ψ)*; then the fourth is true also.*

In virtue of $F=0$, the necessary and sufficient conditions for (i), (ii), (iii), (iv) are given by

$$\text{(i)} \qquad \frac{\partial^2 \log E}{\partial u\,\partial v}=\frac{\partial^2 \log G}{\partial u\,\partial v};$$

$$\text{(ii)} \qquad \frac{\partial\sqrt{E}}{\partial v}=\frac{\partial\sqrt{G}}{\partial u}\ (^1);$$

$$\text{(iii)} \qquad \sqrt{G}\frac{\partial\psi}{\partial u}=\sqrt{E}\frac{\partial\psi}{\partial v};$$

$$\text{(iv)} \qquad \begin{cases}\dfrac{\partial\log\varphi}{\partial u}=-\dfrac{1}{2}\dfrac{\partial\log G}{\partial u}-\sqrt{E}\cdot\psi,\\[2mm] \dfrac{\partial\log\varphi}{\partial v}=-\dfrac{1}{2}\dfrac{\partial\log E}{\partial v}-\sqrt{G}\cdot\psi\end{cases}$$

respectively.

Firstly suppose that (i), (ii), (iii) are true. Then we have

$$\frac{1}{2}\frac{\partial^2\log G}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{E}}{\partial v}+\sqrt{E}\frac{\partial\psi}{\partial v}$$
$$=\frac{1}{2}\frac{\partial^2\log E}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{G}}{\partial u}+\sqrt{G}\frac{\partial\psi}{\partial u},$$

which shows us that

$$\left(\frac{1}{2}\frac{\partial\log G}{\partial u}+\sqrt{E}\cdot\psi\right)du+\left(\frac{1}{2}\frac{\partial\log E}{\partial v}+\sqrt{G}\cdot\psi\right)dv$$

([1]) See, for example, R. Rothe, "Bemerkungen über die Gewebe (Kurvennetze ohne Umwege) auf einer Fläche," Jahresb. Deuts. Math. Ver., 17 (1903), p. 325.

is an exact differential. Hence if we define the function φ by

$$\log\varphi=-\int\left(\frac{1}{2}\frac{\partial\log G}{\partial u}+\sqrt{E}\,.\,\psi\right)du+\left(\frac{1}{2}\frac{\partial\log E}{\partial v}+\sqrt{G}\,.\,\psi\right)dv,$$

equations (iv) are satisfied; that is, u, v may be orbits in the field (φ,ψ).

Next suppose that (i), (iii), (iv) are true. Then (iv) gives the condition of integrability

$$\frac{1}{2}\frac{\partial^2\log G}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{E}}{\partial v}+\sqrt{E}\frac{\partial\psi}{\partial v}$$
$$=\frac{1}{2}\frac{\partial^2\log E}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{G}}{\partial u}+\sqrt{G}\frac{\partial\psi}{\partial u},$$

from which we find, by means of (i) and (iii)

$$\frac{\partial\sqrt{E}}{\partial v}=\frac{\partial\sqrt{G}}{\partial u}.$$

In similar ways we can prove other two cases.

The striped net of curves without ambages as orbits.

9. It is well known that $2\infty^1$ geodesics u, v may be orbits in a reversible field when and only when φ is constant (1). A generalization of this can be obtained from (19) immediately:

If $2\infty^1$ curves u=const., v=const. such that

$$\frac{1}{\rho_u}=\frac{1}{\rho_v}$$

may be orbits in an irreversible field, the function φ must be constant along u_1=const.. When the condition is fulfilled, ψ is determined by

$$\psi=\frac{1}{\rho}-\sin\frac{\omega}{2}\cdot\frac{\partial}{\partial s(v_1)}(\log\varphi),$$

where we have put

$$\frac{1}{\rho}=\frac{1}{\rho_u}=\frac{1}{\rho_v}.$$

Already I have dealt with the condition that a striped net of curves

(1) O_3, p. 133, p. 181.

without ambages should be orbits in a reversible field (1). Now we can prove the theorem:

Let u=const., v=const. be a striped net of curves without ambages. If any two of the three propositions be true:

(i) *that each of u_1=const. has a constant geodesic curvature;*

(ii) *that ψ is constant along u_1=const.;*

(iii) *that u, v may be orbits in the field of the form (φ,ψ);*

then the third is true also; and φ is constant along u_1=const.

I. Firstly we will deduce (iii) from (i) and (ii). Since u, v form a striped net of curves without ambages, we may take

$$(27)\qquad E=\left(\frac{\partial u_1}{\partial u}\right)^2,\qquad F=\frac{\partial u_1}{\partial u}\frac{\partial u_1}{\partial v}\cos\omega,\qquad G=\left(\frac{\partial u_1}{\partial v}\right),$$

where ω is a function of u_1 alone and u_1=const. are the bisectors of the external angles between u and v (2). Moreover we have

$$(28)\qquad \rho_u=\rho_v\,(=\rho,\ \text{say})\ (3).$$

But I have proved the following *lemma* (4): When u, v form a striped net of curves without ambages, each of u_1=const. has a constant geodesic curvature $\frac{1}{\rho(u_1)}$ when and only when ρ is constant along u_1=const.

Hence it follows from (i) that ρ is a function of u_1 only. Moreover we see from (ii) that ψ is also a function of u_1 only. If we define the function φ by

$$\log\varphi=\int\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)du_1,$$

then

$$\frac{\partial\log\varphi}{\partial u}=\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)\frac{\partial u_1}{\partial u},$$

$$\frac{\partial\log\varphi}{\partial v}=\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)\frac{\partial u_1}{\partial v}.$$

Consequently we have from (27), (28) and the last two equations

$$(14)\qquad \frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\frac{\partial\log\varphi}{\partial u}-\frac{F}{\sqrt{G}}\frac{\partial\log\varphi}{\partial v}\right]=\frac{1}{\rho_u}-\psi,$$

(1) O_5, pp. 284–286; consult with O_6 also.
(2), (3) Rothe, loc. cit.
(4) See O_6.

$$(15) \qquad \frac{1}{\sqrt{EG-F^2}}\left[-\frac{F}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}+\sqrt{E}\frac{\partial \log \varphi}{\partial v}\right]=\frac{1}{\rho_v}-\psi.$$

Thus the proposition (iii) has been obtained.

II. Next we will deduce (ii) from (i) and (iii). Since u, v form a striped net of curves without ambages, (28) must hold. Hence in order that (iii) is true, by the first theorem in this paragraph, φ must be a function of u_1 only. But from (27) and (iii) we have

$$\frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\frac{\partial \log \varphi}{\partial u}-\frac{F}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v}\right]=\mathrm{tg}\frac{\omega(u_1)}{2}\cdot\frac{d}{du_1}\log \varphi(u_1)$$

$$=\frac{1}{\rho}-\varphi.$$

Moreover we see from (i) and the lemma above mentioned that ρ is a function of u_1 only; so that ψ must be a function of u_1 only.

III. Lastly we will deduce (i) from (ii) and (iii). From (iii) we get

$$\mathrm{tg}\frac{\omega(u_1)}{2}\cdot\frac{1}{du_1}\log \varphi(u_1)=\frac{1}{\rho}-\psi.$$

But by (ii) ψ is a function of u_1 only; so that ρ must be a function of u_1 alone. Therefore by the above lemma, each of $u_1=\text{const.}$ has a constant geodesic curvature.

10. Now we see, from the last theorem, that if a striped net of curves without ambages $u=\text{const.}$, $v=\text{const.}$ may be orbits in the field $(\varphi, \psi(u_1))$, then each of $u_1=\text{const.}$ should have a constant geodesic curvature. On the other hand, since u, v form a striped net of curves without ambages, $u_1=\text{const.}$ are geodesic parallels (¹). Hence it follows that the linear element must take the form

$$ds^2=du_1^2+U_1(u_1)\,dv_1^2,$$

$U_1(u_1)$ being a function of u_1 alone; and the surface should be applicable to a surface of revolution.

Conversely, if a surface be applicable to a surface of revolution and u, v form a striped net of curves without ambages, then we must have

$$ds^2=\sec^2\frac{\omega}{2}du_1^2+\mu^2\sin^2\frac{\omega}{2}dv_1^2$$

$$=\sec^2\frac{\omega}{2}du_1^2+f_1(u_1)f_2(v_1)\,dv_1^2,$$

(¹) Rothe, lac. cit.

where ω and $f_1(u_1)$ are functions of u_1 alone and $f_2(v_1)$ of v_1 alone; so that μ takes the form $F_1(u_1)F_2(v_1)$. But in virtue of the formula

$$-\frac{2}{\rho(u_1)}=\sin\omega\cdot\frac{\partial}{\partial u_1}\log\left(\mu\sin^2\frac{\omega}{2}\right),\ (^1)$$

$\rho(u_1)$ should be a function of u_1 alone. Therefore, by the last theorem, u, v may be orbits in the field of the form $(\varphi, \psi(u_1))$. Thus we have theorem:

In order that a surface may have a striped net of curves without ambages u, v which can be orbits in the field $(\varphi, \psi(u_1))$, it is necessary and sufficient that the surface should be applicable to a surface of revolution.

Moreover we can state the following theorem (²):

In order that u, v constitute a striped net of curves without ambages which can be orbits in the field of the form $(\varphi, \psi(u_1))$, it is necessary and sufficient that the linear element should take the form

$$ds^2=\xi(u+v)\left[du^2+2\eta(u+v)\,du\,dv+dv^2\right],$$

where ξ and η are functions of $u+v$ only.

Lastly we add the theorem:

If a striped net of curves without ambages u, v be orbits in the reversible field $(\varphi, 0)$, then the net may be orbits in the irreversible field of the form $(\varphi, \psi(u_1))$ also.

Ikeda near Ôsaka, June 1918.

(¹) Rothe, loc. cit.
(²) For the method of proof, see O_5, pp. 287–288.

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