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On the Sign and Magnitude of the Coefficients in the Fourier Series, the Sine Series and the Cosine Series.

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On the Sign and Magnitude of the Coefficients in the Fourier Series, the Sine Series and the Cosine Series,

by

KINNOSUKE OGURA, Osaka.

On the Fourier series.

1. Prof. Fejér[1] proved the following theorem:

Let $y=F(x)$ $(0<x<2\pi)$ be a single analytic arc joining the two points $[0,\ F(+0)]$ and $[2\pi,\ F(2\pi-0)]$, and let

$$\frac{b_0}{2}+\sum_{n=1}^{\infty}(a_n \sin nx+b_n \cos nx)$$

be the Fourier series corresponding to $F(x)$. If $F(+0)\neq F(2\pi-0)$, then for sufficiently large values of n all $a_n \neq 0$ and have the same sign and their absolute values decrease monotonously. If $F'(+0)\neq F'(2\pi-0)$, $F'(x)$ being the derivative of $F(x)$, a similar result holds for the coefficients b_n.

Here I will deal with such a problem more deeply.

Let $f(x)$ and all its derivatives be continuous and have limited total fluctuation in the interval $(0<x<2\pi)$ and have limits on the right and on the left at $x=0$ and at $x=2\pi$ respectively. Then the Fourier constants corresponding to the function $f(x)$ are

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt, \quad b_n=\frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt.$$

Hence

$$\frac{\pi}{2}\Delta_n=\int_0^{2\pi}\varphi(t)\cos\left(n+\frac{1}{2}\right)t\,dt,$$

where we have put

$$\Delta_n=a_{n+1}-a_n, \quad \varphi(t)=f(t)\sin\frac{t}{2}.$$

Integrating by parts we get

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(1) Fejér, Über die Fouriersche Reihe, Math. Ann., 64 (1907), p. 285.

$$\frac{\pi}{2}\Delta_n=\frac{1}{2}\cdot\frac{1}{\left(n+\frac{1}{2}\right)^2}\Big[f(2\pi-0)-f(+0)\Big]$$

$$-\frac{1}{\left(n+\frac{1}{2}\right)^2}\int_0^{2\pi}\varphi''(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

But by the fundamental lemma of Riemann-Lebesgue

$$\lim_{n\to\infty}\int_0^{2\pi}\varphi''(t)\cos\left(n+\frac{1}{2}\right)t\,dt=0\,;$$

so that for sufficiently large values of n

$$\Delta_n\gtreqless 0 \quad \text{according as} \quad f(2\pi-0)\gtreqless f(+0).$$

Since $\lim a_n=0$, it follows from the last inequalities that for sufficiently large values of n

$$a_n\gtreqless 0 \quad \text{according as} \quad f(2\pi-0)\gtreqless f(+0).$$

Also after a short calculation we can prove the inequalities ([1]):

$$\Delta_n\gtreqless\Delta_{n+1} \quad \text{according as} \quad f(2\pi-0)\gtreqless f(+0)\ ([2]).$$

If $f(2\pi-0)=f(+0)$, then integrating by parts we have

$$\frac{\pi}{2}\Delta_n=-\frac{3}{2}\,\frac{1}{\left(n+\frac{1}{2}\right)^4}\Big[f''(2\pi-0)-f''(+0)\Big]$$

$$+\frac{1}{\left(n+\frac{1}{2}\right)^4}\int_0^{2\pi}\varphi^{(4)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt\,;$$

so that

$$\Delta_n\gtreqless 0,\ a_n\gtreqless 0 \quad \text{according as} \quad f''(2\pi-0)\gtreqless f''(+0).$$

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(1) In what follows the phrase "*for sufficiently large values of n*" will be omitted.

(2) Our inequalities for the Fourier constants have certain similarities with those for the Lebesgue constants

$$\rho_n=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin(2n+1)t}{\sin t}\right|dt,$$

in which

$$\bar{\Delta}_n=\rho_{n+1}-\rho_n>0,\qquad \bar{\Delta}_n>\bar{\Delta}_{n+1}.$$

See Gronwall, Über die Lebesgueschen Konstanten bei den Fourierschen Reihen Math .Ann., 72 (1912), p. 244.

2. In general, when

$$f^{(2m)}(2\pi-0)=f^{(2m)}(+0),\quad (m=0,1,2,\cdots,p),$$

we have

$$\varphi^{(2p+3)}(2\pi-0)=-\frac{2p+3}{2}f^{(2p+2)}(2\pi-0),$$

$$\varphi^{(2p+3)}(+0)=\frac{2p+3}{2}f^{(2p+2)}(+0),$$

and

$$\frac{\pi}{2}\Delta_n=(-1)^{p+1}\,\frac{2p+3}{2}\,\frac{1}{\left(n+\frac{1}{2}\right)^{2p+4}}\Big[f^{(2p+2)}(2\pi-0)-f^{(2p+2)}(+0)\Big]$$

$$+(-1)^p\frac{1}{\left(n+\frac{1}{2}\right)^{2p+4}}\int_0^{2\pi}\varphi^{(2p+4)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

Therefore we arrive at the theorem:

Theorem I.

(i) *If* $f(2\pi-0)\gtreqless f(+0)$, *then* $a_n\gtreqless 0,\ \Delta_n\gtreqless 0,\ \Delta_n\gtreqless\Delta_{n+1}$.

(ii) *If* $f^{(2m)}(2\pi-0)=f^{(2m)}(+0)$, $(m=0,1,2,\cdots,p)$

and

$$\begin{cases} f^{(2p+2)}(2\pi-0)>f^{(2p+2)}(+0), \\ \qquad \textit{then}\ \left\{a_n\gtreqless 0,\ \Delta_n\lesseqgtr 0,\ \Delta_n\lesseqgtr\Delta_{n+1},\right. \quad \begin{matrix}\textit{for}\ \ p=0,2,\cdots,\\ \textit{for}\ \ p=1,3,\cdots,\end{matrix} \\ f^{(2p+2)}(2\pi-0)<f^{(2p+2)}(+0), \\ \qquad \textit{then}\ \left\{a_n\lesseqgtr 0,\ \Delta_n\gtreqless 0,\ \Delta_n\gtreqless\Delta_{n+1}.\right. \quad \begin{matrix}\textit{for}\ \ p=0,2,\cdots,\\ \textit{for}\ \ p=1,3,\cdots,\end{matrix} \end{cases}$$

From this theorem we can infer at once:

Theorem II. *For sufficiently large values of n, all the constants* a_n *have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p)}(2\pi-0)=f^{(2p)}(+0),\quad (p=0,1,2,\cdots,\infty).$$

In the particular case where $f(x)$ is analytic in the interval $(0\leqq x\leqq 2\pi)$, we can choose a fixed positive number ρ such that

$$f(x)=f(0)+f'(0)\frac{x}{1\,!}+f''(0)\frac{x^2}{2\,!}+f'''(0)\frac{x^3}{3\,!}+\cdots\cdots,\quad |x|\leqq\rho,$$

$$f(x+2\pi)=f(2\pi)+f'(2\pi)\frac{x}{1!}+f''(2\pi)\frac{x^2}{2!}+f'''(2\pi)\frac{x^3}{3!}+\cdots\cdots,$$
$$|x|\leqq\rho.$$

Whence

$$f(x+2\pi)-f(x)=\sum_{p=0}^{\infty}\left[f^{(p)}(2\pi)-f^{(p)}(0)\right]\frac{x^p}{p!},\quad |x|\leqq\rho.$$

Therefore, *in order that however great n may be, the constants a_n, corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq 2\pi)$, have no constant sign (or, become zero), $f(x+2\pi)-f(x)$ should be necessarily an odd function of x (or identically zero)* (1).

3. Now we come to consider the coefficients b_n. If we put

$$\delta_n=b_{n+1}-b_n,$$

we have

$$\frac{\pi}{2}\delta_n=-\int_0^{2\pi}\varphi(t)\sin\left(n+\frac{1}{2}\right)t\,dt.$$

Integrating by parts we get

$$\frac{\pi}{2}\delta_n=-\frac{1}{\left(n+\frac{1}{2}\right)^3}\left[f'(2\pi-0)-f'(+0)\right]$$
$$+\frac{1}{\left(n+\frac{1}{2}\right)^3}\int_0^{2\pi}\varphi^{(3)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt;$$

so that

$$\delta_n\lesseqgtr 0,\ b_n\gtreqless 0,\ \delta_n\lesseqgtr\delta_{n+1}\quad\text{according as}\quad f'(2\pi-0)\gtreqless f'(+0).$$

If $f'(2\pi-0)=f'(+0)$, then integrating by parts

$$\frac{\pi}{2}\delta_n=2.\frac{1}{\left(n+\frac{1}{2}\right)^5}\left[f'''(2\pi-0)-f'''(+0)\right]$$
$$-\frac{1}{\left(n+\frac{1}{2}\right)^5}\int_0^{2\pi}\varphi^{(5)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

(1) The converse of this theorem (or Theorem II) is not necessarily true. See Example IV in §4.

In general, when

$$f^{(2m-1)}(2\pi-0)=f^{(2m-1)}(+0),\quad (m=1,2,\cdots,p),$$

we have

$$\varphi^{(2p+2)}(2\pi-0)=-(p+1)f^{(2p+1)}(2\pi-0),$$
$$\varphi^{(2p+2)}(+0)=(p+1)f^{(2p+1)}(+0),$$

and then

$$\frac{\pi}{2}\delta_n=(-1)^{p+1}(p+1).\frac{1}{\left(n+\frac{1}{2}\right)^{2p+3}}\left[f^{(2p+1)}(2\pi-0)-f^{(2p+1)}(+0)\right]$$
$$+(-1)^p\frac{1}{\left(n+\frac{1}{2}\right)^{2p+3}}\int_0^{2\pi}\varphi^{(2p+3)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

Therefore we arrive at the theorems:

Theorem I'.

(i) *If* $f'(2\pi-0)\gtreqless f'(+0)$, *then* $b_n\gtreqless 0,\ \delta_n\lesseqgtr 0,\ \delta_n\lesseqgtr\delta_{n+1}$.

(ii) *If* $f^{(2m+1)}(2\pi-0)=f^{(2m+1)}(+0),\quad (m=0,1,2,\cdots,p)$

and

$$\begin{cases} f^{(2p+3)}(2\pi-0)>f^{(2p+3)}(+0), \\ \qquad\text{then}\ \{b_n\lesseqgtr 0,\ \delta_n\gtreqless 0,\ \delta_n\gtreqless\delta_{n+1}\ \ \text{for } p=0,2,\cdots;\ \text{for } p=1,3,\cdots \\ f^{(2p+3)}(2\pi-0)<f^{(2p+3)}(+0), \\ \qquad\text{then}\ \{b_n\gtreqless 0,\ \delta_n\lesseqgtr 0,\ \delta_n\lesseqgtr\delta_{n+1}\ \ \text{for } p=0,2,\cdots;\ \text{for } p=1,3,\cdots \end{cases}$$

Theorem II'. *For sufficiently large values of n, all the constants b_n have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p+1)}(2\pi+0)=f^{(2p+1)}(+0),\quad (p=0,1,2,\cdots,\infty).$$

Particularly, *in order that however great n may be, the constants b_n, corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq 2\pi)$, have no constant sign (or, become zero), $f(x+2\pi)-f(x)$ should be necessarily an even function (or a constant)* (1).

Consequently we can infer the theorem immediately: *In order that however great n may be, both a_n and b_n, corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq 2\pi)$, have no constant signs (or, become zero), the function $f(x)$ should have the period 2π* (2).

(1), (2) The converse of these (and Theorem II') is not necessarily true. See Example IV in §4.

4. Lastly we add some simple examples:

I. $$f(x)=e^x.$$

In this case

$$f(2\pi)>f(0),\quad f'(2\pi)>f'(0);$$

so that

$$a_n<0,\quad b_n>0.$$

In fact

$$e^x\sim\frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2}-\frac{1}{1+1^2}\sin x+\frac{1}{1+1^2}\cos x-\frac{2}{1+2^2}\sin 2x\right.$$
$$\left.+\frac{1}{1+2^2}\cos 2x-\cdots\cdots-\frac{n}{1+n^2}\sin nx+\frac{1}{1+n^2}\cos nx-\cdots\cdots\right].$$

II. $$f(x)=e^x\sin x.$$

In this case

$$f(2\pi)=f(0),\quad f'(2\pi)>f'(0),\quad f''(2\pi)>f''(0);$$

so that

$$a_n>0,\quad b_n>0.$$

In fact

$$a_n=\frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{1+(n-1)^2}-\frac{1}{1+(n+1)^2}\right]>0,\quad (n\geqq 1),$$

$$b_n=\frac{e^{2\pi}-1}{2\pi}\left[\frac{n+1}{1+(n-1)^2}-\frac{n+1}{1+(n+1)^2}\right]>0,\quad (n>1).$$

III. $$f(x)=-\frac{x}{2}.$$

In this case

$$f(2\pi)<f(0),\quad f^{(2p+1)}(2\pi)=f^{(2p+1)}(0),\quad (p=0,1,2,\cdots,\infty);$$

so that $a_n>0$ and b_n have no constant sign (or, become zero). In fact

$$-\frac{x}{2}\sim\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\cdots\cdots.$$

IV. Each of the following four functions has the period 2π.

$$e^{\cos x}\cos(\sin x)\sim 1+\frac{1}{1!}\cos x+\frac{1}{2!}\cos 2x+\frac{1}{3!}\cos 3x+\cdots\cdots,$$

$$e^{-\cos x}\cos(\sin x)\sim 1-\frac{1}{1!}\cos x+\frac{1}{2!}\cos 2x-\frac{1}{3!}\cos 3x+\cdots\cdots,$$

$$e^{\cos x}\sin(\sin x)\sim \frac{1}{1!}\sin x+\frac{1}{2!}\sin 2x+\frac{1}{3!}\sin 3x+\cdots\cdots,$$

$$e^{-\cos x}\sin(\sin x)\sim \frac{1}{1!}\sin x+\frac{1}{2!}\sin 2x+\frac{1}{3!}\sin 3x-\cdots\cdots.$$

Also we have

$$\cosh(\cos x)\cos(\sin x)\sim 1+\frac{1}{2!}\cos 2x+\frac{1}{4!}\cos 4x+\cdots\cdots,$$

$$e^{\cos x}\cos(\sin x)+e^{-\cos x}\sin(\sin x)\sim 1+\frac{1}{1!}\sin x-\frac{1}{2!}\sin 2x$$
$$+\frac{1}{3!}\sin 3x-\cdots\cdots$$
$$+\frac{1}{1!}\cos x+\frac{1}{2!}\sin 2x$$
$$+\frac{1}{3!}\cos 3x+\cdots\cdots,$$

etc.

On the sine series.

5. Let $f(x)$ and all its derivatives be continuous and have limited total fluctuation in the interval $(0<x<\pi)$ and have limits on the right and on the left at $x=0$ and at $x=\pi$ respectively. If

$$\sum_{n=1}^{\infty} a_n\sin nx$$

be the sine series corresponding to the function $f(x)$, then

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt;$$

so that

$$\frac{\pi}{4}\Delta_n=\int_0^{\pi}\varphi(t)\cos\left(n+\frac{1}{2}\right)t\,dt,$$

where

$$\Delta_n = a_{n+1} - a_n, \quad \varphi(t) = f(t) \sin \frac{t}{2}.$$

Integrating by parts we get

$$\frac{\pi}{4} \Delta_n = (-1)^n \frac{1}{n+\frac{1}{2}} f(\pi-0) - \frac{1}{n+\frac{1}{2}} \int_0^\pi \varphi'(t) \sin\left(n+\frac{1}{2}\right) t\, dt;$$

whence

$$\Delta_{2m} \gtrless 0, \ \Delta_{2m+1} \lessgtr 0 \quad \text{according as} \quad f(\pi-0) \gtrless 0.$$

If $f(\pi-0)=0$, then

$$\frac{\pi}{4} \Delta_n = -\frac{1}{2} \frac{1}{\left(n+\frac{1}{2}\right)^2} f(+0) - \frac{1}{\left(n+\frac{1}{2}\right)^2} \int_0^\pi \varphi''(t) \cos\left(n+\frac{1}{2}\right) t\, dt;$$

so that

$$\Delta_n \lessgtr 0 \ (\text{and consequently } a_n \gtrless 0) \ \text{according as} \ f(+0) \gtrless 0.$$

Again if $f(\pi-0)=0$, $f(+0)=0$, then

$$\frac{\pi}{4} \Delta_n = (-1)^{n+1} \frac{1}{\left(n+\frac{1}{2}\right)^3} f''(\pi-0)$$

$$+ \frac{1}{\left(n+\frac{1}{2}\right)^3} \int_0^\pi \varphi^{(3)}(t) \sin\left(n+\frac{1}{2}\right) t\, dt;$$

whence

$$\Delta_{2m} \lessgtr 0, \ \Delta_{2m+1} \gtrless 0 \quad \text{according as} \quad f''(\pi-0) \gtrless 0.$$

Further if $f(\pi-0)=f(+0)=0$, $f''(\pi-0)=0$, then

$$\frac{\pi}{4} \Delta_n = \frac{3}{2} \frac{1}{\left(n+\frac{1}{2}\right)^4} f''(+0) + \frac{1}{\left(n+\frac{1}{2}\right)^4} \int_0^\pi \varphi^{(4)}(t) \cos\left(n+\frac{1}{2}\right) t\, dt;$$

so that

$$\Delta_n \gtrless 0, \ a_n \lessgtr 0 \quad \text{according as} \quad f''(+0) \gtrless 0,$$

and so on. Thus we arrive at the theorem:

Theorem III.

(i) *If* $f(\pi-0) \gtrless 0$, *then* $\Delta_{2m} \gtrless 0$, $\Delta_{2m+1} \lessgtr 0$.

If $f(\pi-0)=0$ *and* $f(+0) \gtrless 0$, *then* $\Delta_n \lessgtr 0$, $a_n \gtrless 0$.

(ii) *If* $f^{(2\nu)}(\pi-0)=f^{(2\nu)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

and $\begin{cases} f^{(2p+2)}(\pi-0)>0, & \textit{then} \ \left\{ \Delta_{2m} \lessgtr 0, \ \Delta_{2m+1} \gtrless 0, \ \begin{matrix} \textit{for} \ p=0, 2, \cdots, \\ \textit{for} \ p=1, 3, \cdots, \end{matrix} \right. \\ f^{(2p+2)}(\pi-0)<0, & \textit{then} \ \left\{ \Delta_{2m} \gtrless 0, \ \Delta_{2m+1} \lessgtr 0, \ \begin{matrix} \textit{for} \ p=0, 2, \cdots, \\ \textit{for} \ p=1, 3, \cdots. \end{matrix} \right. \end{cases}$

If $f^{(2\nu)}(\pi-0)=f^{(2\nu)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

$f^{(2p+2)}(\pi-0)=0$,

and $\begin{cases} f^{(2p+2)}(+0)>0, & \textit{then} \ \left\{ \Delta_n \gtrless 0, \ a_n \lessgtr 0, \ \begin{matrix} \textit{for} \ p=0, 2, \cdots, \\ \textit{for} \ p=1, 3, \cdots, \end{matrix} \right. \\ f^{(2p+2)}(+0)<0, & \textit{then} \ \left\{ \Delta_n \lessgtr 0, \ a_n \gtrless 0, \ \begin{matrix} \textit{for} \ p=0, 2, \cdots, \\ \textit{for} \ p=1, 3, \cdots. \end{matrix} \right. \end{cases}$

6. Next we proceed to discuss the expression $\Delta_n + \Delta_{n+1}$. Since

$$\Delta_n + \Delta_{n+1} = a_{n+2} - a_n$$

$$= \frac{4}{\pi} \int_0^\pi \psi(t) \cos(n+1) t\, dt,$$

where

$$\psi(t) = f(t) \sin t,$$

integrating by parts we have

$$\frac{\pi}{4}(a_{n+2}-a_n) = \frac{2}{(n+1)^2} \Big[(-1)^n f(\pi-0) - f(+0) \Big]$$

$$- \frac{1}{(n+1)^2} \int_0^\pi \psi''(t) \cos(n+1) t\, dt.$$

Hence

$a_{2m} \lessgtr a_{2m+2}$, (and consequently $a_{2m} \lessgtr 0$), according as

$$f(\pi-0)-f(+0) \gtrless 0;$$

$a_{2m+1} \gtrless a_{2m+3}$, (and consequently $a_{2m+1} \gtrless 0$), according as

$$f(\pi+0)+f(+0) \gtrless 0$$

I. If

$$f(\pi-0)-f(+0)=0,$$

then

$$\frac{\pi}{4}(a_{2m+2}-a_{2m})=-\frac{3}{(2m+1)^4}\Big[f''(\pi-0)-f''(+0)\Big] +\frac{1}{(2m+1)^4}\int_0^\pi \psi^{(4)}(t)\cos(2m+1)t\,dt\,;$$

so that

$$a_{2m}\gtreqless a_{2m+2},\quad a_{2m}\gtreqless 0,\quad \text{according as } f''(\pi-0)-f''(+0)\gtreqless 0.$$

Again if

$$f(\pi-0)=f(+0),\quad f''(\pi-0)=f''(+0),$$

then

$$\frac{\pi}{4}(a_{2m+2}-a_{2m})=\frac{5}{(2m+1)^6}\Big[f^{(4)}(\pi-0)-f^{(4)}(+0)\Big] -\frac{1}{(2m+1)^6}\int_0^\pi \psi^{(6)}(t)\cos(2m+1)t\,dt\,;$$

so that

$$a_{2m}\lesseqgtr a_{2m+2},\quad a_{2m}\lesseqgtr 0\quad \text{according as } f^{(4)}(\pi-0)-f^{(4)}(+0)\gtreqless 0\,;$$

and so on.

II. Similarly if

$$f(\pi-0)+f(+0)=0,$$

then

$$\frac{\pi}{4}(a_{2m+3}-a_{2m+1})=\frac{3}{(2m+2)^4}\Big[f''(\pi-0)+f''(+0)\Big] +\frac{1}{(2m+2)^4}\int_0^\pi \psi^{(4)}(t)\cos(2m+2)t\,dt\,;$$

so that

$$a_{2m+1}\lesseqgtr a_{2m+3},\quad a_{2m+1}\lesseqgtr 0\quad \text{according as } f''(\pi-0)+f''(+0)\gtreqless 0\,;$$

and so on.

Therefore we can state the theorem:

Theorem IV.

I.

(i) *If* $f(\pi-0)-f(+0)\gtreqless 0$, *then* $a_{2m}\lesseqgtr 0$, $a_{2m}\lesseqgtr a_{2m+2}$.

(ii) *If* $f^{(2\nu)}(\pi-0)-f^{(2\nu)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;

and
$$\begin{cases} f^{(2p+2)}(\pi-0)-f^{(2p+2)}(+0)>0, \\ \qquad \textit{then}\ \{a_{2m}\gtreqless 0,\quad a_{2m}\gtreqless a_{2m+2}, \quad \begin{matrix}\textit{for}\ p=0,2,\cdots, \\ \textit{for}\ p=1,3,\cdots;\end{matrix} \\ f^{(2p+2)}(\pi-0)-f^{(2p+2)}(+0)<0, \\ \qquad \textit{then}\ \{a_{2m}\lesseqgtr 0,\quad a_{2m}\lesseqgtr a_{2m+2}, \quad \begin{matrix}\textit{for}\ p=0,2,\cdots, \\ \textit{for}\ p=1,3,\cdots.\end{matrix} \end{cases}$$

II.

(i) *If* $f(\pi-0)+f(+0)\gtreqless 0$, *then* $a_{2m+1}\gtreqless 0$, $a_{2m+1}\gtreqless a_{2m+3}$.

(ii) *If* $f^{(2\nu)}(\pi-0)+f^{(2\nu)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;

and
$$\begin{cases} f^{(2p+2)}(\pi-0)+f^{(2p+2)}(+0)>0, \\ \qquad \textit{then}\ \{a_{2m+1}\lesseqgtr 0,\quad a_{2m+1}\lesseqgtr a_{2m+3}, \quad \begin{matrix}\textit{for}\ p=0,2,\cdots, \\ \textit{for}\ p=1,3,\cdots;\end{matrix} \\ f^{(2p+2)}(\pi-0)+f^{(2p+2)}(+0)<0, \\ \qquad \textit{then}\ \{a_{2m+1}\gtreqless 0,\quad a_{2m+1}\gtreqless a_{2m+3}, \quad \begin{matrix}\textit{for}\ p=0,2,\cdots, \\ \textit{for}\ p=1,3,\cdots\end{matrix} \end{cases}$$

It is apparent that this theorem contains some parts of Theorem III. Also from Theorem IV we can infer the following theorem immediately:

Theorem V. *For sufficiently large values of m, all the coefficients a_{2m} of even terms (the coefficients a_{2m+1} of odd terms) have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p)}(\pi-0)-f^{(2p)}(+0)=0,\quad \{f^{(2p)}(\pi-0)+f^{(2p)}(+0)=0\},\ (p=0,1,2,\cdots,\infty).$$

Therefore, *in order that however great m may be, the coefficients a_{2m} (or a_{2m+1}), corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq\pi)$, have no constant sign or become zero, $f(x+\pi)-f(x)$ {or $f(x+\pi)+f(x)$} should be necessarily an odd function of x or identically zero* (1).

Also, *in order that however great m may be, both a_{2m} and a_{2m+1}, corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq\pi)$, have no constant signs or become zero, both $f(x)$ and $f(x+\pi)$ should be necessarily odd functions of x.*

Lastly suppose that

$$f(\pi-0)>0,\quad f(\pi-0)-f(+0)>0.$$

Then

(1) The converse of this theorem (or Theorem V) is not necessarily true. See Example VII in §7.

$$|\Delta_{2m+1}|-|\Delta_{2m}|=-\Delta_{2m+1}-\Delta_{2m}$$
$$=a_{2m}-a_{2m+2}$$
$$<0.$$

In similar ways we can prove the theorem:

Theorem VI. *For sufficiently large values of n, there exist the following inequalities between the absolute values of* Δ_n *and* Δ_{n+1}:

	$f(\pi-0)>0,$	$f(\pi-0)<0,$
$f(\pi-0)-f(+0)\gtreqless 0,$	$\vert\Delta_{2m}\vert\gtreqless\vert\Delta_{2m+1}\vert,$	$\vert\Delta_{2m}\vert\lesseqgtr\vert\Delta_{2m+1}\vert,$
$f(\pi-0)+f(+0)\gtreqless 0,$	$\vert\Delta_{2m+1}\vert\gtreqless\vert\Delta_{2m+2}\vert,$	$\vert\Delta_{2m+1}\vert\gtreqless\vert\Delta_{2m+2}\vert;$

and so on.

7. Here we add some examples:

I. $$f(x)=\frac{\pi-x}{2}.$$

In this case
$$f(\pi)=0,\quad f(0)>0;$$
so that
$$a_n>0.$$
In fact
$$\frac{\pi-x}{2}\sim\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\cdots\cdots.$$

II. $$f(x)=\frac{x}{2}.$$

In this case
$$f(\pi)=\frac{\pi}{2},\quad f(0)=0,\quad f(\pi)-f(0)>0,\quad f(\pi)+f(0)>0;$$
so that
$$a_{2m}<0,\quad a_{2m+1}>0,\quad |\Delta_{2m}|>|\Delta_{2m+1}|>|\Delta_{2m+2}|.$$
In fact

$$\frac{x}{2}\sim\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\cdots\cdots.$$

III. The following example belongs to the same case as II.
$$\frac{\pi}{4}-\frac{\pi}{2}\cos x\sim\sin x-\frac{4}{3}\sin 2x+\frac{1}{3}\sin 3x-\frac{8}{15}\sin 4x$$
$$+\cdots+\frac{1}{2m-1}\sin(2m-1)x-\frac{4m}{4m^2-1}\sin 2mx+\cdots\cdots,$$
in which we see that
$$|a_{2m-1}|<|a_{2m}|,\quad |\Delta_{2m}|>|\Delta_{2m+1}|>|\Delta_{2m+2}|.$$

IV. $$f(x)=\frac{\pi}{4}+\frac{\pi}{2}\cos x.$$

In this case
$$f(\pi)=-\frac{\pi}{4},\quad f(0)=\frac{3}{4}\pi,\quad f(\pi)-f(0)<0,\quad f(\pi)+f(0)>0;$$
so that
$$a_{2m}>0,\quad a_{2m+1}>0,\quad |\Delta_{2m}|>|\Delta_{2m+1}|,\quad |\Delta_{2m+1}|<|\Delta_{2m+2}|.$$
In fact
$$\frac{\pi}{4}+\frac{\pi}{2}\cos x\sim\sin x+\frac{4}{3}\sin 2x+\frac{1}{3}\sin 3x+\cdots\cdots$$
$$+\frac{1}{2m-1}\sin(2m-1)x+\frac{4m}{4m^2-1}\sin 2mx+\cdots,$$
where
$$|a_{2m-1}|<|a_{2m}|.$$

V. $$f(x)=\frac{\pi}{4}.$$

In this case
$$f(\pi)+f(0)>0,\quad f^{(2p)}(\pi)-f^{(2p)}(0)=0,\quad (p=0,1,2,\cdots,\infty);$$
so that $a_{2m+1}>0$ and a_{2m} have no constant sign. In fact
$$\frac{\pi}{4}\sim\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\cdots.$$

VI. $$f(x)=\cos(\cos x)\ \sinh(\sin x).$$

In this case $f(x)$ and $f(x+\pi)$ are odd functions of x. And

$$\cos(\cos x)\sinh(\sin x)\sim\frac{1}{1\,!}\sin x-\frac{1}{3\,!}\sin 3x+\frac{1}{5\,!}\sin 5x-\cdots\cdots.$$

VII. $$f(x)=e^{\cos x}\sin(\sin x).$$

In this case $f(x+\pi)+f(x)$ is an odd function, but

$$e^{\cos x}\sin(\sin x)\sim\frac{1}{1\,!}\sin x+\frac{1}{2\,!}\sin 2x+\frac{1}{3\,!}\sin 3x+\cdots\cdots.$$

On the cosine series.

8. Let $f(x)$ be any function which satisfies the conditions in §5, and let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

be the cosine series corresponding to that function. Then we can treat this series in similar ways as in the sine series; and the following are some typical results:

I.

(i) *If* $f'(\pi-0)\gtrless 0$, *then* $\Delta_{2m}\lessgtr 0$, $\Delta_{2m+1}\gtrless 0$.

If $f'(\pi-0)=0$, $f'(+0)\gtrless 0$, *then* $\Delta_n\gtrless 0$, $a_n\lessgtr 0$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)=f^{(2\nu+1)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)>0, & \text{then } \{\Delta_{2m}\gtrless 0,\quad \Delta_{2m+1}\lessgtr 0,\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots;\end{matrix}\\ f^{(2p+3)}(\pi-0)<0, & \text{then } \{\Delta_{2m}\lessgtr 0,\quad \Delta_{2m+1}\gtrless 0,\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots.\end{matrix}\end{cases}$$

If $f^{(2\nu+1)}(\pi-0)=f^{(2\nu+1)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;
$f^{(2p+3)}(\pi-0)=0$,

$$\text{and}\begin{cases} f^{(2p+3)}(+0)>0, & \text{then } \{\Delta_n\lessgtr 0,\quad a_n\gtrless 0,\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots;\end{matrix}\\ f^{(2p+3)}(+0)<0, & \text{then } \{\Delta_n\gtrless 0,\quad a_n\lessgtr 0,\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots.\end{matrix}\end{cases}$$

II.

(i) *If* $f'(\pi-0)-f'(+0)\gtrless 0$, *then* $a_{2m}\gtrless 0$, $a_{2m}\gtrless a_{2m+2}$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)-f^{(2\nu+1)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)-f^{(2p+3)}(+0)>0,\\ \qquad\text{then } \{a_{2m}\lessgtr 0,\quad a_{2m}\lessgtr a_{2m+2},\quad \begin{matrix}\text{for } p=0,2,\cdots;\\ \text{for } p=1,3,\cdots\end{matrix}\\ f^{(2p+2)}(\pi-0)-f^{(2p+3)}(+0)<0,\\ \qquad\text{then } \{a_{2m}\gtrless 0,\quad a_{2m}\gtrless a_{2m+2},\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots.\end{matrix}\end{cases}$$

III.

(i) *If* $f'(\pi-0)+f'(+0)\gtrless 0$, *then* $a_{2m+1}\lessgtr 0$, $a_{2m+1}\lessgtr a_{2m+3}$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)+f^{(2\nu+1)}(+0)=0$, $(\nu=0,1,2,\cdots,p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)+f^{(2p+3)}(+0)>0,\\ \qquad\text{then } \{a_{2m+1}\gtrless 0,\quad a_{2m+1}\gtrless a_{2m+3},\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots;\end{matrix}\\ f^{(2p+3)}(\pi-0)+f^{(2p+3)}(+0)<0,\\ \qquad\text{then } \{a_{2m+1}\lessgtr 0,\quad a_{2m+1}\lessgtr a_{2m+3},\quad \begin{matrix}\text{for } p=0,2,\cdots,\\ \text{for } p=1,3,\cdots.\end{matrix}\end{cases}$$

Ikeda near Ôsaka, June 1918.

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