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A Remark on the Dynamical System with Two Degrees of Freedom.

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A Remark on the Dynamical System with Two Degrees of Freedom,

by

KINNOSUKE OGURA, Ôsaka.

1. In his great memoir "Dynamical systems with two degrees of freedom"(1), Prof. G. D. Birkhoff proved that the equations of motion of any dynamical system with two degrees of freedom can be reduced to *the normal form :*

$$\ddot{x}+\lambda\,\dot{y}=\frac{\partial\gamma}{\partial x},\qquad \ddot{y}-\lambda\,\dot{x}=\frac{\partial\gamma}{\partial y}, \tag{1}$$

where x, y denote the two coordinates of the dynamical system, $\dot{x}$, $\dot{y}$ their time derivatives, and λ, γ are real analytic functions of x and y.

The dynamical system is *reversible* or *irreversible* according as λ vanishes identically or not.

The equations (1) admit the first integral

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)=\gamma+h, \tag{2}$$

h being an arbitrary constant. In this short note I will confine myself to the case in which h has a definite value.

Now if we interpret (x, y) as the rectangular point coordinates of a plane, and eliminate the four quantities $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$ among (1), (2) and the identities

$$y'\equiv\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}},\qquad y''\equiv\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3},$$

we have *the differential equation of the orbits*

$$y''=(Ay'-B)(1+y'^2)+C(1+y'^2)^{\frac{3}{2}}, \tag{3}$$

where we have put

$$A=-\frac{1}{2}\,\frac{\partial\log(\gamma+h)}{\partial x},\quad B=-\frac{1}{2}\,\frac{\partial\log(\gamma+h)}{\partial y},\quad C=\frac{\lambda}{\sqrt{2(\gamma+h)}}.$$

(1) Trans. Amer. Math. Soc., 18 (1917), p. 204.

This is nothing but the differential equation already treated by me, and the totality of its integral curves has been called *the N-system*([1]). Consequently we have the theorem :

The totality of the orbits of any dynamical (normal) system with two degrees of freedom consists of an N-system ; and conversely any N-system is composed of all the orbits of a certain normal system with two degrees of freedom.

2. Applying the results previously obtained in this case, we see that the locus of the centres of curvature of all the orbits (3) at any point $P(x, y)$ is the conic having P as a focus, and $\frac{2}{\lambda}\sqrt{2(\gamma+h)}$ as the parameter, and

$$\frac{\partial \log \sqrt{\gamma+h}}{\partial x}(\xi-x)+\frac{\partial \log \sqrt{\gamma+h}}{\partial y}(\eta-y)=1$$

as the directrix corresponding to P, where (ξ, η) are the current coordinates ; and the converse is also true.

But it is known that the directrix is the locus of the centres of curvature of all the orbits, at the point P, in the reversible system

$$y''=(Ay'-B)(1-y'^2);$$

and conversely. On the other hand, Prof. Kasner([2]) proved that a system of ∞^2 curves in a plane, one for each direction at each point of the plane, will constitute the totality of all the orbits for a reversible system (so called the natural family), when and only when, it possesses the two properties that (i) the osculating circles at any given point must form a pencil, and (ii) the two hyperosculating circles contained in such a pencil must be orthogonal.

Therefore we can state the following theorem which gives *the geometrical characterization* for the normal system with two degrees of freedom :

A system of ∞^2 curves in a plane, one for each direction at each point of the plane, will constitute the totality for any normal system with two degrees of freedom, when and only when, it possesses the two properties such that

(i) *the centre of curvature at any given point must lie on a conic having that point as a focus; and*

(ii) *the directrix of the conic corresponding to the focus must coincide with the straight line which is the locus of the centres of curvature of all the orbits at the given point for any reversible system (characterized geometrically by* Kasner's *theorem).*

3. Moreover we have *the analytical characterization :*

A necessary and sufficient condition that

$$y''=\phi(x, y, y')$$

may be the differential equation to the orbits of the normal system with two degrees of freedom is that the expression

$$\frac{\partial \phi}{\partial y'}-\frac{3\phi y'}{1+y'^2}$$

should be the exact differential quotient of a function of x and y with respect to x. When the condition is satisfied, we have

$$\gamma+h=k \exp.\left\{-2\int\left(\frac{\partial \phi}{\partial y'}-\frac{3\phi y'}{1+y'^2}\right)dx\right\},$$

k being an arbitrary constant.

Lastly we remark that Prof. Ph. Frank([1]) and Prof. Whittaker([2]) mentioned *erroneously* the above as the necessary and *sufficient* condition for *the reversible system.*

Ikeda near Ôsaka, May 1918.

([1]) Ogura, "On the integral curves of ordinary differential equations of the second order of a certain type," Tôhoku Math. Journal, 8 (1915), p. 93. See also Mr. K. Kurosu: paper of the same title, Tôhoku Math. Journal, 12 (1917), p. 197; and Ogura, "On a certain system of doubly infinite curves on a surface," Tôhoku Math. Journal, 8 (1915) p. 213.

([2]) Kasner, "Natural families of trajectories." Trans. Amer. Math. Soc., 10 (1909) p. 201, Consult with Ogura, "Trajectories in the conservative field of force," Part. I, Tôhoku Math. Journal, 7 (1915), p. 123.

([1]) Frank, "Über die Bahnkurven der Mechanik," Crelle's Journal, 134 (1908), p, 156.

([2]) Whittaker, Treatise on the analytical dynamics (2. ed., 1917), p. 407.

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