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On Certain Mean Curves defined by the Series of Orthogonal Functions.

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On Certain Mean Curves defined by the Series of Orthogonal Functions,

by

KINNOSUKE OGURA, Ôsaka.

1. Let

$$\varphi_0(t),\ \varphi_1(t),\ \varphi_2(t), \cdots\cdots, \varphi_n(t), \cdots\cdots$$

be a system of orthogonal functions which are *limited* and *integrable* in the interval $(0 \leqq t \leqq 1)$ and are such that

$$\int_0^1 \varphi_m(t)\,\varphi_n(t)\,dt = 0 \quad (m \neq n),$$
$$= 1 \quad (m = n), \quad (m, n = 0, 1, 2, \cdots\cdots).$$

Let $f(\theta)$ be a function which is *limited* and *integrable* in the interval $(0 \leqq \theta \leqq 1)$, then the series of orthogonal functions $\varphi_n(\theta)$ corresponding to $f(\theta)$ is

$$(1) \qquad \sum_{n=0}^{\infty} \varphi_n(\theta) \int_0^1 f(t)\,\varphi_n(t)\,dt\,.$$

Now consider the plane curve C defined by

$$(2) \qquad x = x(\theta), \qquad y = y(\theta),$$

where the two functions $x(\theta)$ and $y(\theta)$ are *limited* and *integrable* in the interval $(0 \leqq \theta \leqq 1)$; and denote by D *the closed least convex domain which contains the curve C.*

If we put

$$(3) \qquad \begin{aligned} x_n^{(r)}(\theta) &= \int_0^1 x(t)\,K_n^{(r)}(\theta, t)\,dt, \\ y_n^{(r)}(\theta) &= \int_0^1 y(t)\,K_n^{(r)}(\theta, t)\,dt, \end{aligned}$$

where

(4) $K_n^{(r)}(\theta, t)$

$$=\varphi_0(\theta)\varphi_0(t)+\frac{n-1}{n+r-1}\varphi_1(\theta)\varphi_1(t)+\frac{(n-1)(n-2)}{(n+r-1)(n+r-2)}\varphi_2(\theta)\varphi_2(t)$$

$$+\cdots\cdots+\frac{(n-1)!}{(n+r-1)(n+r-2)\cdot(r+1)}\varphi_{n-1}(\theta)\varphi_{n-1}(t),$$

the curve $C_n^{(r)}$ defined by

(5) $$x=x_n^{(r)}(\theta), \quad y=y_n^{(r)}(\theta)$$

will be called *the n^{th} curve of the r^{th} mean (in Cesàro's sense) of C.*

Prof. Fejér proved that when

$$1, \quad \sqrt{2}\cos 2\pi t, \quad \sqrt{2}\sin 2\pi t, \cdots\cdots, \quad \sqrt{2}\cos 2n\pi t, \quad \sqrt{2}\sin 2n\pi t, \cdots\cdots$$

are taken as the system of orthogonal functions, then *all* the curves $C_n^{(1)} (n=1, 2, \cdots\cdots)$ of *any* curve C are contained in the domain D (¹).

In this note I propose to find *a necessary and sufficient condition that all the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ of any curve C should be contained in the domain D* (§ 4). Also I will give a theorem on the *summability* (the convergency of the sequence of $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$) in § 6, and lastly in § 7 remark that *all the sum curves $C_n^{(0)}$ $(n=1, 2, \cdots\cdots)$ of any curves C for the series of Haar's functions are contained in D.*

I.

2. I will begin with the *lemma*:

Let $K(t)$ be a given function which is limited and integrable in the interval $(0\leqq t\leqq 1)$. Then in order that the point

$$X=\int_0^1 x(t)\,K(t)\,dt, \qquad Y=\int_0^1 y(t)\,K(t)\,dt$$

may be contained in the domain D for *any* curve C

$$x=x(t), \qquad y=y(t),$$

it should be

$$K(t)\geqq 0$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous.

(¹) Fejér, "Über gewisse durch die Fouriersche und Laplacesche Reihe definierten Mittelkurven und Mittelflächen," Rend. Palermo, 38 (1914), p. 79.

Since $K(t)$ is limited and integrable in the interval $(0\leqq t\leqq 1)$, it must have points of continuity in any subinterval of $(0\leqq t\leqq 1)$. Let us denote by (A) all the subintervals in which $K(t)\geqq 0$ and by (B) those in which $K(t)<0$. If we take

$$x(t)=1 \quad \text{when } t \text{ belongs to } (B),$$
$$=0 \quad \text{when } t \text{ does not belong to } (B);$$
$$y(t)=\sin 2\pi t \quad \text{throughout } 0\leqq t\leqq 1,$$

then the domain D for the curve C is enclosed in the rectangle

$$0\leqq x\leqq 1, \qquad -1\leqq y\leqq 1.$$

But since

$$X=\int_{(B)} x(t)\,K(t)\,dt=\int_{(B)} K(t)\,dt<0,$$

the point (X, Y) lies outside the domain D, which proves the lemma.

Applying this lemma to (3) we have *the first necessary condition*: In order that *all* the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ of *any* curve C should be contained in D, we must have

(6) $$K_n^{(r)}(\theta, t)\geqq 0, \quad (n=1, 2, \cdots\cdots)$$

at every point of the domain $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which they are continuous.

3. Next I will prove *the second lemma*:

In order that the curve $C_1^{(r)}$ of *any* curve C may be contained in D, it is necessary that

$$\varphi_0(t)=1 \quad \text{or} \quad -1$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous.

(i) Firstly we have from (6)

$$\varphi_0(\theta)\,\varphi_0(t)\geqq 0$$

at every point of the domain $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which $\varphi_0(\theta)\,\varphi_0(t)$ is continuous; so that $\varphi_0(t)$ must have a definite sign at all points of the interval $(0\leqq t\leqq 1)$ at which it is continuous. (For the sake of brevity, this set of points will be donoted by E).

Next since

$$\int_0^1 \varphi_0^2(t)\,dt=1,$$

$\varphi_0(t)$ can not vanish identically throughout E. So we may suppose that this function is non-negative throughout E.

Also from the last identity we can infer that $\varphi_0(t)$ is neither always greater nor always smaller than 1 throughout E.

(ii) Now let us denote by M the absolute maximum of $\varphi_0(t)$ in E, and suppose

$$\varphi_0(\theta_1)=M>1,$$

where θ_1 belongs to E. Since $\varphi_0(t)$ can not be equal to M identically throughout E, we have

$$\int_0^1 M\,\varphi_0(t)\,dt>\int_0^1 \varphi_0^2(t)\,dt=1.$$

If we take the curve

$$\begin{aligned} x&=x(t)=t \qquad (0\leqq t\leqq 1);\\ y&=y(t)=0 \qquad (t=0),\\ &=M \qquad (0<t\leqq 1), \end{aligned}$$

the domain D becomes the triangle having the vertices $(0,0)$, $(0,M)$, $(1,M)$. On the other hand, since

$$y_1^{(r)}(\theta)=\varphi_0(\theta)\int_0^1 M\,\varphi_0(t)\,dt>\varphi_0(\theta),$$

we have

$$y_1^{(r)}(\theta_1)>\varphi_0(\theta_1)=M;$$

so that the domain D can not contain the whole part of the curve $C_1^{(r)}$.

Consequently we must have $M=1$, from which it follows that $\varphi_0(t)=1$ throughout E.

This lemma leads us to *the second necessary condition*: In order that *all* the curves $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ of *any* curve C should be contained in D, it must be

$$(7) \qquad \varphi_0(t)=1 \quad \text{or} \quad -1$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous[1].

---

[1] By kind information of Prof. Fujiwara we add here the following remark: It may be directly proved in order that the point (X, Y) for any curve C should be contained in D it is necessary and sufficient that

4. Thus we have had two necessary conditions. Here I will prove that *these two conditions are sufficient*.

By the second condition we have

$$\varphi_0(t)=1 \quad \text{or} \quad -1$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous, so that

$$\int_0^1 \varphi_n(t)\,dt=\pm\int_0^1 \varphi_0(t)\,\varphi_n(t)\,dt$$
$$=0, \qquad (n=1,2,\cdots\cdots).$$

But since (4) gives

$$K_n^{(r)}(\theta,t)=1+\frac{n-1}{n+r-1}\varphi_1(\theta)\varphi_1(t)+\cdots+\frac{(n-1)!}{(n+r-1)\cdots(r+1)}\varphi_{n-1}(\theta)\,\varphi_{n-1}(t),$$

we obtain *the identities*:

$$(8) \qquad \int_0^1 K_n^{(r)}(\theta,t)\,dt=1, \qquad (n=1,2,\cdots\cdots).$$

Therefore equations (3) may be written

$$x_n^{(r)}(\theta)=\frac{\int_0^1 x(t)\,K_n^{(r)}(\theta,t)\,dt}{\int_0^1 K_n^{(r)}(\theta,t)\,dt},$$
$$(n=1,2,\cdots\cdots),$$
$$y_n^{(r)}(\theta)=\frac{\int_0^1 y(t)\,K_n^{(r)}(\theta,t)\,dt}{\int_0^1 K_n^{(r)}(\theta,t)\,dt}.$$

But by the first condition we have

$$(6) \qquad K_n^{(r)}(\theta,t)\geqq 0 \qquad (n=1,2,\cdots\cdots)$$

at every point of the domain $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which they are

---

$$1^\circ \quad K(t)\geqq 0; \quad 2^\circ \quad \int_0^1 K(t)\,dt=1.$$

For the present case the condition 2° is equivalent to $\varphi_0^2=1$; it is, however, proved in the text that 2° is a consequence of 1°.

continuous; consequently, by a theorem due to Weierstrass, all the curves $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ lie in the domain D.

Therefore we arrive at the theorem:

Theorem I. *A necessary and sufficient condition that all the curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *for any curve* C *may lie in the domain* D *is that the orthogonal functions* $\varphi_n(t)$ *should satisfy*

$$\varphi_0(t)=\pm 1,$$

$$K_n^{(r)}(\theta,t)\geqq 0, \qquad (n=2,3,\cdots\cdots)(^1)$$

at every point of the domain $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which $\varphi_0(t)$ and $K_n^{(r)}(\theta,t)$ *are continuous.*

II.

5. As the first application of the above theorem we consider the relation between the curves $C_n^{(r)}$ and $C_n^{(r+1)}$.

By means of the formula, which is easily seen,

$$K_n^{(r+1)}(\theta,t)=\frac{A_1^{(r)}K_1^{(r)}(\theta,t)+A_2^{(r)}K_2^{(r)}(\theta,t)+\cdots+A_n^{(r)}K_n^{(r)}(\theta,t)}{A_1^{(r)}+A_2^{(r)}+\cdots+A_n^{(r)}},$$

where

$$A_n^{(r)}=\frac{(r+1)(r+2)\cdots(r+n-1)}{(n-1)!},$$

it follows that if

$$K_n^{(r)}(\theta,t)\geqq 0 \qquad (n=1,2,\cdots\cdots),$$

then

$$K_n^{(r+1)}(\theta,t)\geqq 0 \qquad (n=1,2,\cdots\cdots).$$

Consequently we have the theorem:

Theorem II. *When all the curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *for any curve* C *lie in the domain* D, *all the curves* $C_n^{(r+1)}$ $(n=1,2,\cdots\cdots)$ *have the same nature.*

6. As the second application we come to the problem of summability.

When the function $f(t)$ is limited and integrable in the interval

(1) From $\varphi_0(t)=\pm 1$, it follows that

$$K_1^{(r)}(\theta,t)=1>0.$$

$(0\leqq t\leqq 1)$ and moreover there exist the n constants $c_0, c_1,\cdots\cdots c_{n-1}$, such that

$$|f(t)-c_0\varphi_0(t)-c_1\varphi_1(t)-\cdots\cdots-c_{n-1}\varphi_{n-1}(t)|<\delta, \quad (0\leqq t\leqq 1),$$

corresponding to any given positive number δ, $f(t)$ is said to lie within *the domain of the orthogonal functions* $\varphi_n(t)$.

Now applying the method of Prof. Haar(1) to the series (1), we can show that when there exists a fixed positive number M, independent of n and θ $(0\leqq\theta\leqq 1)$, such that

$$\int_0^1 |K_n^{(r)}(\theta,t)|\,dt<M,$$

the r^{th} mean of the series (1) corresponding to any function $f(t)$, lying within the domain of the orthogonal functions $\varphi_n(t)$, converges to $f(\theta)$ at every point of the interval $(0\leqq\theta\leqq 1)$ at which it is continuous.

But if all the curves $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ for any curve C lie in the domain D, then

$$(6) \qquad K_n^{(r)}(\theta,t)\geqq 0, \qquad (n=1,2,\cdots\cdots)$$

at every point of the domain $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which they are continuous; and

$$(8) \qquad \int_0^1 K_n^{(r)}(\theta,t)\,dt=1, \qquad (n=1,2,\cdots\cdots);$$

so that

$$\int_0^1 |K_n^{(r)}(\theta,t)|\,dt=1.$$

Hence we can infer the theorem:

Theorem III. *If all the curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *for any curve* C:

$$x=x(\theta), \qquad y=y(\theta), \qquad (0\leqq\theta\leqq 1),$$

where $x(\theta)$ *and* $y(\theta)$ *are functions lying within the domain of the orthogonal functions* $\varphi_n(\theta)$, *lie in the domain* D, *then the sequence of curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *converges to that curve* C *at every point of the curve at which it is continuous.*

7. Lastly we will give some simple examples.

I. If we take the system of orthogonal functions such as

(1) Haar, "Zur Theorie der orthogonalen Funktionensysteme," 1. Mitteilung, Math. Ann., 69 (1910), p. 331.

(i) $$\varphi_n(t)=\sqrt{2}\sin n\pi t, \quad (n=0, 1, 2, \cdots\cdots);$$

or

(ii) $$\varphi_n(t)=J(\sqrt{\lambda_n}\, t)\left[\int_0^1 J^2(\sqrt{\lambda_n}\, t)\,dt\right]^{-\frac{1}{2}}, \quad (n=0, 1, 2, \cdots\cdots),$$

J being the Bessel function of the 0^{th} order and λ_n the n^{th} roots of the equation

$$J(\sqrt{\lambda})=0,$$

then the identity

$$\varphi_0(t)=1 \quad \text{or} \quad -1$$

does not hold.

In such cases we can choose the curve C for which the domain D can not contain all the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ *however great* r *is taken* (See § 3).

II. As the second example, we consider the system of Haar's functions (1):

$$\begin{aligned}
\varphi_0(t)&=1 && 0\leqq t\leqq 1;\\
\varphi_n^{(k)}(t)&=0 && 0\leqq t<(2k-2)\,2^{-n},\\
&=\frac{1}{2}\sqrt{2^{n-1}}, && t=(2k-2)\,2^{-n},\\
&=\sqrt{2^{n-1}}, && (2k-2)\,2^{-n}<t<(2k-1)\,2^{-n},\\
&=0, && t=(2k-1)\,2^{-n},\\
&=-\sqrt{2^{n-1}}, && (2k-1)\,2^{-n}<t<2k\cdot 2^{-n},\\
&=-\frac{1}{2}\sqrt{2^{n-1}}, && t=2k\cdot 2^{-n},\\
&=0, && 2k\cdot 2^{-n}<t\leqq 1,\\
& && (k=1, 2, \cdots\cdots, 2^{n-1}).
\end{aligned}$$

Then

$$\varphi_0(t)=1,\ \varphi_1(t),\ \varphi_2^{(1)}(t),\ \varphi_2^{(2)}(t),\ \varphi_3^{(1)}(t),\ \varphi_3^{(2)}(t), \cdots\cdots$$

form a system of orthogonal functions, and it is seen, as Prof. Haar proved, that

$$K_n^{(0)}(\theta, t)\geqq 0, \quad (n=2, 3, \cdots\cdots),$$
$$0\leqq\theta\leqq 1, \quad 0\leqq t\leqq 1.$$

(1) Haar, loc. cit.

Hence we have the theorem:

Theorem IV. *All the sum curves* $C_n^{(0)}$ $(n=1, 2, \cdots\cdots)$ *for any curve* C, *corresponding to the series of Haar's function, lie in the domain* D.

III. As the third example, we take the system:

$$\varphi_0(t)=1, \quad \varphi_{2n-1}(t)=\sqrt{2}\cos 2n\pi t, \quad \varphi_{2n}(t)=\sqrt{2}\sin 2n\pi t.$$

In this case from the well known inequalities

$$K_n^{(1)}(\theta, t)\geqq 0, \quad (n=1, 2, \cdots\cdots),$$

we arrive at the theorem due to Prof. Fejér (§ 1).

Lastly we add a remark: For the series of orthogonal functions $\varphi_n(t)$ in general, *the inequalities*

$$K_n^{(1)}(\theta, t)\geqq 0, \quad (n=2, 3, \cdots\cdots),$$

where

$$\varphi_0(t)=+1 \quad \text{or} \quad -1$$

hold good if there exist the inequalities of the doubly symmetric determinants (1):

$$\begin{vmatrix}
2 & \varphi_1(\theta)\varphi_1(t) & \varphi_2(\theta)\varphi_2(t) & \cdots & \varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-1}(\theta)\varphi_{n-1}(t)\\
\varphi_1(\theta)\varphi_1(t) & 2 & \varphi_1(\theta)\varphi_1(t) & \cdots & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \varphi_{n-2}(\theta)\varphi_{n-2}(t)\\
\varphi_2(\theta)\varphi_2(t) & \varphi_1(\theta)\varphi_1(t) & 2 & \cdots & \varphi_{n-4}(\theta)\varphi_{n-4}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t)\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots\\
\varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \varphi_{n-4}(\theta)\varphi_{n-4}(t) & \cdots & 2 & \varphi_1(\theta)\varphi_1(t)\\
\varphi_{n-1}(\theta)\varphi_{n-1}(t) & \varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \cdots & \varphi_1(\theta)\varphi_1(t) & 2
\end{vmatrix}\geqq 0,$$

$$(n=2, 3, \cdots\cdots).$$

This is an immediate consequence of the two theorems of Prof. Fejér (2) and Prof. Toeplitz (3).

Ikeda near Ôsaka, May 1918.

(1) That is, symmetric with respect to the principal and second diagonals. This determinant is centrosymmetric, and also orthosymmetric with respect to the second diagonal.

(2) Fejér, loc. cit., p. 89 (Theorem VI).

(3) Toeplitz, "Über die *Fourier*sche Entwickelung positiver Funktionen," Rend. Palermo, 32 (1911), p. 191; Carathéodory, "Über die Variabilitätsbereich der *Fourier*'schen Konstanten von positiven harmonischen Funktionen," Rend. Palermo, 32 (1911), p. 193.

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