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On the Fourier Constants.

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On the Fourier Constants,

by

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I.

Prof. A. Hurwitz proved the following theorem⁽¹⁾: If in the interval ($0 \leq x \leq 2\pi$) the function $f(x)$ be finite and integrable and if all its Fourier constants be zero, then $f(x)$ is zero at every point of the interval at which it is continuous.

Now I will prove the following theorem:

Let $f(x)$ be any function which is absolutely integrable (or, more generally, integrable in the sense of Lebesgue) in the interval ($0 \leq x \leq 2\pi$) and is such that the corresponding Fourier series

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) dt + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos nx \int_0^{2\pi} f(t) \cos nt dt + \sin nx \int_0^{2\pi} f(t) \sin nt dt \right]$$

converges uniformly to the function $f(x)$ at every point of the interval ($a \leq x \leq b$, where $0 \leq a < b < 2\pi$ or $0 < a < b \leq 2\pi$) at which the function is continuous. Then there exists either (i) none or (ii) an infinite number of the inequalities

$$\int_a^b f(t) \cos mt dt \neq 0,$$

$$\int_a^b f(t) \sin nt dt \neq 0,$$

where m and n are integers.

And in the case (i) the function $f(x)$ is zero at every point of the interval ($a \leq x \leq b$) at which it is continuous⁽²⁾.

(1) A. Hurwitz, Über die Fourierschen Konstanten integrierbarer Funktionen, Math. Ann., 57 (1903), p. 425. For a generalization of this theorem, see Steklov, Sur la théorie de fermeture des systèmes de fonctions orthogonales, Mém. de l'Acad. St. Pétersbourg, (8) 30 (1911), p. 27.

(2) Prof. C. N. Moore proved that if in the interval ($a \leq x \leq b$) $f(x)$ is finite save for a finite number of points, and is integrable, and if

$$\int_a^b f(t) \cos nt dt = 0 \quad (n=0, 1, 2, \dots), \quad \int_a^b f(t) \sin nt dt = 0 \quad (n=1, 2, \dots),$$

Suppose that there is a finite number of positive integers $m_1, m_2, \dots, m_p; n_1, n_2, \dots, n_q$ such that

$$\int_a^b f(t) \cos m_\mu t dt \equiv A_{m_\mu} \neq 0 \quad \text{for } \mu=1, 2, \dots, p; \quad (1)$$

$$\int_a^b f(t) \cos mt dt = 0 \quad \text{for all non-negative integers } m \text{ except } m_1, m_2, \dots, m_p;$$

$$\int_a^b f(t) \sin n_\nu t dt \equiv B_{n_\nu} \neq 0 \quad \text{for } \nu=1, 2, \dots, q;$$

$$\int_a^b f(t) \sin nt dt = 0 \quad \text{for all positive integers } n \text{ except } n_1, n_2, \dots, n_q,$$

where $0 < a < b < 2\pi$.

If we consider the function $\psi(x)$ defined by

$$\begin{aligned} \psi(x) &= 0 & 0 \leq x < a, \\ &= f(x) & a \leq x \leq b, \\ &= 0 & b < x \leq 2\pi, \end{aligned}$$

then we see that the Fourier series corresponding to $\psi(x)$ converges uniformly to this function at every point of the interval $(0 \leq x \leq 2\pi)$ at which it is continuous⁽²⁾. But since

$$\int_0^{2\pi} \psi(t) \cos m_\mu t dt = A_{m_\mu} \quad \text{for } \mu=1, 2, \dots, p;$$

$$\int_0^{2\pi} \psi(t) \cos mt dt = 0 \quad \text{for all non-negative integers } m \text{ except } m_1, m_2, \dots, m_p;$$

$$\int_0^{2\pi} \psi(t) \sin n_\nu t dt = B_{n_\nu} \quad \text{for } \nu=1, 2, \dots, q;$$

$$\int_0^{2\pi} \psi(t) \sin nt dt = 0 \quad \text{for all positive integers } n \text{ except } n_1, n_2, \dots, n_q,$$

in the interval $(0 \leq x \leq 2\pi)$ the Fourier series corresponding to $\psi(x)$ converges uniformly to the function $S(x)$, where

$$S(x) = \frac{1}{\pi} \sum_{\mu=1}^p A_{m_\mu} \cos m_\mu x + \frac{1}{\pi} \sum_{\nu=1}^q B_{n_\nu} \sin n_\nu x;$$

then $f(t)$ is zero at every point of the interval $(a \leq x \leq b)$ at which it is continuous. See C. N. Moore, On a certain constants analogous to Fourier's constants, Bull. Amer. Math. Soc., 14 (1908), p. 371.

(1) If $m_\mu=0$, then A_{m_μ} should be replaced by $\frac{A_0}{2}$.

(2) Vallée Poussin, Cours d'analyse infinitésimale, t. 2, (2. éd. 1912), p. 144.

so that

$$\begin{aligned} S(x) - \psi(x) &= \frac{1}{\pi} \sum_{\mu=1}^p A_{m_\mu} \cos m_\mu x + \frac{1}{\pi} \sum_{\nu=1}^q B_{n_\nu} \sin n_\nu x & 0 \leq x < a, \\ &= \frac{1}{\pi} \sum_{\mu=1}^p A_{m_\mu} \cos m_\mu x + \frac{1}{\pi} \sum_{\nu=1}^q B_{n_\nu} \sin n_\nu x - f(x) & a \leq x \leq b, \\ &= \frac{1}{\pi} \sum_{\mu=1}^p A_{m_\mu} \cos m_\mu x + \frac{1}{\pi} \sum_{\nu=1}^q B_{n_\nu} \sin n_\nu x & b < x \leq 2\pi. \end{aligned}$$

Since $\psi(x)$ is continuous in the intervals $(0 \leq x < a)$ and $(b < x \leq 2\pi)$, we must have

$$\sum_{\mu=1}^p A_{m_\mu} \cos m_\mu x + \sum_{\nu=1}^q B_{n_\nu} \sin n_\nu x = 0$$

at every point of the intervals $(0 \leq x < a)$ and $(b < x \leq 2\pi)$; consequently

$$A_{m_\mu} = 0 \quad (\mu=1, 2, \dots, p),$$

$$B_{n_\nu} = 0 \quad (\nu=1, 2, \dots, q).$$

And therefore

$$S(x) - \psi(x) = -f(x) \quad \text{for } a \leq x \leq b.$$

But since

$$S(x) = \psi(x)$$

at every point of the interval $(a \leq x \leq b)$ at which $\psi(x)$ is continuous.

Hence we must have

$$f(x) = 0$$

at every point of the interval $(a \leq x \leq b)$ at which $f(x)$ is continuous.

It is evident that these results hold good for the cases $0=a, b < 2\pi$; and $0 < a, b=2\pi$.

The method of proof may be applied to the case of developments in terms of any other normal functions, such as the Legendre polynomials⁽¹⁾, the Sturm-Liouville functions, etc.⁽²⁾, whenever we know that the series corresponding to any discontinuous function which satisfies a certain condition converges uniformly to that function at every point at which it is continuous.

(1) Hobson, On the representation of a function by a series of Legendre's functions, Proc. London Math. Soc. (2) 7 (1908), p. 24.

(2) For example, see Kneser, Die Theorie der Integralgleichungen und die Darstellung willkürlicher Funktionen in der mathematischen Physik, Math. Ann., 63 (1907), p. 477; Kneser, Die Integralgleichungen (1911); Juretzka, Die Entwicklung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen, Diss. Breslau (1909).

II.

Let $f(x)$ be the function defined by the series

$$(1) \quad a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

which converges uniformly at every point of the interval $(0 \leq x \leq 2\pi)$.

Take any two fixed numbers a and b within the interval $(0, 2\pi)$, and form the $2n+1$ rowed determinant

$$\Delta_n = \begin{vmatrix} \int_a^b dx & \int_a^b \cos x dx & \int_a^b \sin x dx & \dots & \int_a^b \cos nx dx & \int_a^b \sin nx dx \\ \int_a^b \cos x dx & \int_a^b \cos^2 x dx & \int_a^b \cos x \sin x dx & \dots & \int_a^b \cos x \cos nx dx & \int_a^b \cos x \sin nx dx \\ \int_a^b \sin x dx & \int_a^b \sin x \cos x dx & \int_a^b \sin^2 x dx & \dots & \int_a^b \sin x \cos nx dx & \int_a^b \sin x \sin nx dx \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \int_a^b \cos nx dx & \int_a^b \cos nx \cos x dx & \int_a^b \cos nx \sin x dx & \dots & \int_a^b \cos nx \sin nx dx & \int_a^b \cos nx \sin^2 nx dx \\ \int_a^b \sin nx dx & \int_a^b \sin nx \cos x dx & \int_a^b \sin nx \sin x dx & \dots & \int_a^b \sin nx \cos nx dx & \int_a^b \sin nx \sin nx dx \end{vmatrix},$$

and let $\Delta_n^{(r)}$ be the determinant obtained from Δ_n by replacing the elements

of the $(r+1)$ -th column by the elements

$$\int_a^b f(x) dx, \quad \int_a^b f(x) \cos x dx, \quad \int_a^b f(x) \sin x dx, \dots$$

$$\dots, \quad \int_a^b f(x) \cos nx dx, \quad \int_a^b f(x) \sin nx dx.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\Delta_n^{(0)}}{\Delta_n} = a_0,$$

$$\lim_{n \rightarrow \infty} \frac{\Delta_n^{(r)}}{\Delta_n} = a_{\frac{r+1}{2}} \quad (r=1, 3, \dots, 2n-1),$$

$$= b_{\frac{r}{2}} \quad (r=2, 4, \dots, 2n),$$

the limits (the Fourier constants) being independent of a and b ⁽¹⁾.

Consider the identity due to Prof. I. Schur⁽²⁾:

$$(2) \quad \begin{vmatrix} \int_a^b \varphi_0(x) \psi_0(x) dx & \int_a^b \varphi_0(x) \psi_1(x) dx & \dots & \int_a^b \varphi_0(x) \psi_p(x) dx \\ \int_a^b \varphi_1(x) \psi_0(x) dx & \int_a^b \varphi_1(x) \psi_1(x) dx & \dots & \int_a^b \varphi_1(x) \psi_p(x) dx \\ \dots & \dots & \dots & \dots \\ \int_a^b \varphi_p(x) \psi_0(x) dx & \int_a^b \varphi_p(x) \psi_1(x) dx & \dots & \int_a^b \varphi_p(x) \psi_p(x) dx \end{vmatrix}$$

$$= \frac{1}{(p+1)!} \int_a^b \int_a^b \dots \int_a^b dx_0 dx_1 \dots dx_p$$

| | |
|------------------------------------------------------|---------------------------------------------|
| $\varphi_0(x_0) \varphi_0(x_1) \dots \varphi_0(x_p)$ | $\psi_0(x_0) \psi_0(x_1) \dots \psi_0(x_p)$ |
| $\varphi_1(x_0) \varphi_1(x_1) \dots \varphi_1(x_p)$ | $\psi_1(x_0) \psi_1(x_1) \dots \psi_1(x_p)$ |
| \dots | \dots |
| $\varphi_p(x_0) \varphi_p(x_1) \dots \varphi_p(x_p)$ | $\psi_p(x_0) \psi_p(x_1) \dots \psi_p(x_p)$ |

(1) This belongs to the case where the so-called "principe des réduites" is valid. See F. Riesz, Les systèmes d'équations linéaires à une infinité d'inconnues (1913), p. 7.

(2) I. Schur, Zur Theorie der linearen homogenen Integralgleichungen, Math. Ann. 67 (1909), p. 319; Richardson—W. A. Hurwitz, Note on determinants whose terms are certain integrals, Bull. Amer. Math. Soc., 16 (1909-10); Landsberg, Theorie der Elementarteiler linearer Integralgleichungen, Math. Ann. 69 (1910), p. 231. For an interesting application of this identity, see Fujiwara, Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der Schwarz'schen und der Tchebycheff'schen Ungleichungen für bestimmte Integrale, Tôhoku Math. Journal, 13 (1918), p. 231.

If we put

$$\begin{aligned}\varphi_0(x) &= \psi_0(x) = 1, \\ \varphi_1(x) &= \psi_1(x) = \cos x, \\ \varphi_2(x) &= \psi_2(x) = \sin x, \\ &\dots \dots \dots \\ \varphi_{2n-1}(x) &= \psi_{2n-1}(x) = \cos nx, \\ \varphi_{2n}(x) &= \psi_{2n}(x) = \sin nx;\end{aligned}$$

and

$$D_n(x_0, x_1, \dots, x_{2n}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \cos x_0 & \cos x_1 & \dots & \cos x_{2n} \\ \sin x_0 & \sin x_1 & \dots & \sin x_{2n} \\ \dots & \dots & \dots & \dots \\ \cos nx_0 & \cos nx_1 & \dots & \cos nx_{2n} \\ \sin nx_0 & \sin nx_1 & \dots & \sin nx_{2n} \end{vmatrix},$$

then (2) becomes

$$(3) \quad \Delta_n = \frac{1}{(2n+1)!} \int_a^b \int_a^b \dots \int_a^b [D_n(x_0, x_1, \dots, x_{2n})]^2 dx_0 dx_1 \dots dx_{2n}.$$

It is seen that $D_n(x_0, x_1, \dots, x_{2n})$ is not zero except

$$x_i = x_k, \quad (i, k = 0, 1, 2, \dots, 2n),$$

in virtue of the identity

$$D_n(x_0, x_1, \dots, x_{2n}) = 2^{2n^2} \prod \sin \frac{1}{2} (x_p - x_q)^{(1)},$$

where p, q are all duads from 0, 1, ..., 2n, ($p > q$).

Again if we put

$$\begin{aligned}\varphi_0(x) &= 1, \quad \psi_0(x) = f(x), \\ \varphi_1(x) &= \psi_1(x) = \cos x, \\ \varphi_2(x) &= \psi_2(x) = \sin x, \\ &\dots \dots \dots \\ \varphi_{2n-1}(x) &= \psi_{2n-1}(x) = \cos nx, \\ \varphi_{2n}(x) &= \psi_{2n}(x) = \sin nx,\end{aligned}$$

⁽¹⁾ Scott and Mathews, Theory of determinants (2, ed., 1904), p. 272.

(2) becomes

$$(4) \quad \Delta_n^{(0)} = \frac{1}{(2n+1)!} \int_a^b \int_a^b \dots \int_a^b dx_0 dx_1 \dots dx_{2n} D_n(x_0, x_1, \dots, x_{2n})$$

$$\begin{vmatrix} f(x_0) & f(x_1) & \dots & f(x_{2n}) \\ \cos x_0 & \cos x_1 & \dots & \cos x_{2n} \\ \dots & \dots & \dots & \dots \\ \sin nx_0 & \sin nx_1 & \dots & \sin nx_{2n} \end{vmatrix}$$

Since the series (1) converges uniformly to $f(x)$ at every point of the interval ($0 \leq x \leq 2\pi$), if we put

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x + \dots + a_n \cos nx + b_n \sin nx) + R_n(x),$$

then, corresponding to any positive number ε , it may be possible to find a positive integer N which is independent of x ($a \leq x \leq b$) and is such that

$$|R_n(x)| < \varepsilon, \quad n > N.$$

Also we have the identities

$$\begin{vmatrix} f(x_0) & f(x_1) & \dots & f(x_{2n}) \\ \cos x_0 & \cos x_1 & \dots & \cos x_{2n} \\ \dots & \dots & \dots & \dots \\ \sin nx_0 & \sin nx_1 & \dots & \sin nx_{2n} \end{vmatrix} = \begin{vmatrix} a_0 + R_n(x_0) & a_0 + R_n(x_1) & \dots & a_0 + R_n(x_{2n}) \\ \cos x_0 & \cos x_1 & \dots & \cos x_{2n} \\ \dots & \dots & \dots & \dots \\ \sin nx_0 & \sin nx_1 & \dots & \sin nx_{2n} \end{vmatrix}$$

$$= a_0 D_n(x_0, x_1, \dots, x_{2n}) + \begin{vmatrix} R_n(x_0) & R_n(x_1) & \dots & R_n(x_{2n}) \\ \cos x_0 & \cos x_1 & \dots & \cos x_{2n} \\ \dots & \dots & \dots & \dots \\ \sin nx_0 & \sin nx_1 & \dots & \sin nx_{2n} \end{vmatrix}.$$

Now let $D_n^{(r)}(x_0, x_1, \dots, x_{2n})$ be the determinant obtained from $D_n(x_0, x_1, \dots, x_{2n})$ by replacing the elements of the $(r+1)$ -th row by the elements

$$R_n(x_0), R_n(x_1), \dots, R_n(x_{2n}).$$

Then (4) becomes

$$\Delta_n^{(0)} = \frac{a_0}{(2n+1)!} \int_a^b \int_a^b \dots \int_a^b [D_n(x_0, x_1, \dots, x_{2n})]^2 dx_0 dx_1 \dots dx_{2n} {}^{(1)}$$

⁽¹⁾ The number of the signs of integration in these expressions is $2n+1$

$$+\frac{1}{(2n+1)!} \int_a^b \int_a^b \cdots \int_a^b D_n(x_0, x_1, \dots, x_{2n}) \cdot D_n^{(0)}(x_0, x_1, \dots, x_{2n}) dx_0 dx_1 \cdots dx_{2n};$$

so that we have from (2)

$$(5) \quad \Delta_n^{(0)}/\Delta_n$$

$$= a_0 + \frac{\int_a^b \int_a^b \cdots \int_a^b D_n(x_0, x_1, \dots, x_{2n}) \cdot D_n^{(0)}(x_0, x_1, \dots, x_{2n}) dx_0 dx_1 \cdots dx_{2n}}{\int_a^b \int_a^b \cdots \int_a^b [D_n(x_0, x_1, \dots, x_{2n})]^2 dx_0 dx_1 \cdots dx_{2n}}$$

Similarly we obtain

$$(5') \quad \Delta_n^{(r)}/\Delta_n$$

$$= a_{\frac{r+1}{2}} + \frac{\int_a^b \int_a^b \cdots \int_a^b D_n(x_0, x_1, \dots, x_{2n}) \cdot D_n^{(r)}(x_0, x_1, \dots, x_{2n}) dx_0 dx_1 \cdots dx_{2n}}{\int_a^b \int_a^b \cdots \int_a^b [D_n(x_0, x_1, \dots, x_{2n})]^2 dx_0 dx_1 \cdots dx_{2n}}$$

(r=1, 3, \dots, 2n-1),

$$= b_{\frac{r}{2}} + \frac{\int_a^b \int_a^b \cdots \int_a^b D_n(x_0, x_1, \dots, x_{2n}) \cdot D_n^{(r)}(x_0, x_1, \dots, x_{2n}) dx_0 dx_1 \cdots dx_{2n}}{\int_a^b \int_a^b \cdots \int_a^b [D_n(x_0, x_1, \dots, x_{2n})]^2 dx_0 dx_1 \cdots dx_{2n}}$$

(r=2, 4, \dots, 2n).

Now let $(\xi_0, \xi_1, \dots, \xi_{2n})$ be the rectangular point coordinates in space of $2n+1$ dimensions, and consider the $2n+1$ planes

$$\xi_0 + \xi_1 \cos x_0 + \xi_2 \sin x_0 + \dots + \xi_{2n} \sin nx_0 = R_n(x_0),$$

$$\xi_0 + \xi_1 \cos x_1 + \xi_2 \sin x_1 + \dots + \xi_{2n} \sin nx_1 = R_n(x_1),$$

.....

$$\xi_0 + \xi_1 \cos x_{2n} + \xi_2 \sin x_{2n} + \dots + \xi_{2n} \sin nx_{2n} = R_n(x_{2n}).$$

The angle $\theta_n(x_i, x_k)$ between the i -th and k -th planes is given by

$$\begin{aligned} \cos \theta_n(x_i, x_k) &= \frac{1 + \sum_{p=1}^n (\cos px_i \cos px_k + \sin px_i \sin px_k)}{\sqrt{1 + \sum_{p=1}^n (\cos^2 px_i + \sin^2 px_i)} \sqrt{1 + \sum_{p=1}^n (\cos^2 px_k + \sin^2 px_k)}} \\ &= \left\{ 1 + \sum_{p=1}^n \cos p(x_i - x_k) \right\} \div (1+n) \end{aligned}$$

$$= \frac{\cos \frac{n}{2}(x_i - x_k) \cdot \sin \frac{n+1}{2}(x_i - x_k)}{(1+n) \sin \frac{1}{2}(x_i - x_k)}$$

Hence $\cos \theta_n(x_i, x_k)$ converges uniformly to zero, that is, the angle $\theta_n(x_i, x_k)$ converges uniformly to $\frac{\pi}{2}$, as n tends to infinity, when x_i, x_k lie in the interval (a, b) except $|x_i - x_k| < \delta$. Next the distance of i -th plane from the origin of coordinates is

$$\frac{R_n(x_i)}{\sqrt{1+n}}$$

which converges uniformly to zero as n tends to infinity. Consequently the common point of the $2n+1$ planes tends uniformly to the origin as n tends to infinity, except

$$|x_i - x_k| < \delta \quad (i, k=0, 1, 2, \dots)$$

But the coordinates of the common point of these planes are

$$\xi_r(x_0, x_1, \dots, x_{2n}) = \frac{D_n^{(r)}(x_0, x_1, \dots, x_{2n})}{D_n(x_0, x_1, \dots, x_{2n})}, \quad (r=0, 1, 2, \dots, 2n).$$

Hence if x_0, x_1, \dots, x_{2n} lie in the interval (a, b) , excluding the domain defined by

$$|x_i - x_k| < \delta \quad (i, k=0, 1, 2, \dots, 2n),$$

then, corresponding to any positive numbers ε_r ($r=0, 1, 2, \dots, 2n$), it may be possible to find a positive integer N , which is independent of x_0, x_1, \dots, x_{2n} and is such that

$$|\xi_r(x_0, x_1, \dots, x_{2n})| < \varepsilon_r, \quad n > N.$$

But

$$\begin{aligned} &\frac{\int_a^b \cdots \int_a^b D_n(x_0, \dots, x_{2n}) D_n^{(r)}(x_0, \dots, x_{2n}) dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \\ &= \frac{\int_a^{*b} \cdots \int_a^{*b} \xi_r(x_0, \dots, x_{2n}) \cdot [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \end{aligned}$$

$$+\frac{\int_a^b \cdots \int_a^b D_n(x_0, \dots, x_{2n}) D_n^{(r)}(x_0, \dots, x_{2n}) dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}},$$

where \int^* denotes the integration excluding the domains $|x_i - x_k| < \delta$, $(i, k = 0, 1, \dots, 2n)$, and \int_* that over these domains only.

Now

$$\begin{aligned} & \left| \frac{\int_a^{*b} \cdots \int_a^{*b} \xi_r(x_0, \dots, x_{2n}) \cdot [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \right| \\ & < \frac{\int_a^{*b} \cdots \int_a^{*b} |\xi_r(x_0, \dots, x_{2n})| \cdot [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}}{\int_a^{*b} \cdots \int_a^{*b} [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \\ & < \varepsilon_r \cdot \frac{\int_a^{*b} \cdots \int_a^{*b} [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}}{\int_a^{*b} \cdots \int_a^{*b} [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \\ & = \varepsilon_r, \quad n > N, \quad (r = 0, 1, 2, \dots, 2n). \end{aligned}$$

On the other hand we may put

$$D_n^{(0)}(x_0, \dots, x_{2n}) = A_0 \cdot R_n(x_0) + A_1 \cdot R_n(x_1) + \cdots + A_{2n} \cdot R_n(x_{2n}),$$

where A_p is the algebraic complement of $R_n(x_p)$ in the determinant $D_n^{(0)}(x_0, \dots, x_{2n})$, i.e.,

$$A_0 = \begin{vmatrix} \cos x_1 & \cos x_2 & \cdots & \cos x_{2n} \\ \sin x_1 & \sin x_2 & \cdots & \sin x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cos nx_1 & \cos nx_2 & \cdots & \cos nx_{2n} \\ \sin nx_1 & \sin nx_2 & \cdots & \sin nx_{2n} \end{vmatrix},$$

But we have

$$|A_0| = 2^{2n^2-2n+1} \cdot \prod \left| \sin \frac{1}{2}(x_i - x_k) \right| \cdot |S|^{\frac{1}{2}},$$

(1) Scott and Mathews, loc. cit., p. 272.

where

$$S = \sum \cos \frac{1}{2}(x_1 + x_2 + \cdots + x_n - x_{n+1} - \cdots - x_{2n})$$

is formed by dividing the $2n$ angles x_1, x_2, \dots, x_{2n} into two sets of n angles in all possible ways and taking the cosine of half the difference of the sums of these sets; so that there exists the inequality

$$|S| < (2n)!/(n!)^2,$$

which becomes, by virtue of Stirling's formula,

$$|S| < \frac{1}{\sqrt{n\pi}} 2^{2n} + \lambda_n, \quad (\lim_{n \rightarrow \infty} \lambda_n = 0).$$

If η be any fixed positive number smaller than 1, we may take δ such that

$$\left| \sin \frac{1}{2}(x_i - x_k) \right| < \eta \quad \text{for } |x_i - x_k| < \delta, \quad (i, k = 0, 1, 2, \dots, 2n)$$

Hence

$$|A_0| < 2^{2n^2+1} \left(\frac{1}{\sqrt{n\pi}} + \frac{\lambda_n}{2^{2n}} \right) \eta^n = 2^{2n^2} \eta^n \lambda'_n \quad \text{for } |x_i - x_k| < \delta,$$

where

$$\lim_{n \rightarrow \infty} \lambda'_n = 0.$$

By similar ways we have

$$|A_p| < 2^{2n^2} \eta^n \lambda'_n \quad (p = 1, 2, \dots, 2n) \quad \text{for } |x_i - x_k| < \delta.$$

Consequently

$$|D_n^{(0)}(x_0, \dots, x_{2n})| < \sum_{p=0}^{2n} |A_p| \cdot |R_n(x_p)| < (2n+1) 2^{2n^2} \eta^n \lambda'_n \varepsilon, \quad \text{for } |x_i - x_k| < \delta, \quad n > N.$$

Since

$$\lim_{n \rightarrow \infty} (2n+1) \eta^n = 0,$$

it may be possible to find the positive integer N' such that

$$\varepsilon \lambda'_n \cdot (2n+1) \eta^n < \varepsilon', \quad n > N' > N$$

corresponding to any positive number ε' ; whence we obtain

$$|D_n^{(0)}(x_0, \dots, x_{2n})| < 2^{2n^2} \varepsilon', \quad n > N', \quad |x_i - x_k| < \delta.$$

Similarly

$$|D_n^{(r)}(x_0, \dots, x_{2n})| < 2^{2n} \epsilon', \quad n > N', \quad |x_i - x_k| < \delta, \\ (r=1, 2, \dots, 2n).$$

Therefore

$$\left| \frac{\int_a^b \cdots \int_a^b D_n(x_0, \dots, x_{2n}) D_n^{(r)}(x_0, \dots, x_{2n}) dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} \right| \\ = \left| \frac{\int_a^b \cdots \int_a^b \Pi \sin \frac{1}{2}(x_i - x_k) \cdot D_n^{(r)}(x_0, \dots, x_{2n}) dx_0 \cdots dx_{2n}}{2^{2n} \int_a^b \cdots \int_a^b \left[\Pi \sin \frac{1}{2}(x_i - x_k) \right]^2 dx_0 \cdots dx_{2n}} \right| \\ < \epsilon' \div \int_a^b \cdots \int_a^b \left[\Pi \sin \frac{1}{2}(x_i - x_k) \right]^2 dx_0 \cdots dx_{2n}, \quad n > N', \quad (r=0, 1, 2, \dots, 2n).$$

Thus we arrive at the identities

$$\lim_{n \rightarrow \infty} \frac{\int_a^b \cdots \int_a^b D_n(x_0, \dots, x_{2n}) D_n^{(r)}(x_0, \dots, x_{2n}) dx_0 \cdots dx_{2n}}{\int_a^b \cdots \int_a^b [D_n(x_0, \dots, x_{2n})]^2 dx_0 \cdots dx_{2n}} = 0, \\ (r=0, 1, 2, \dots, 2n).$$

Consequently it follows from (5) and (5') that

$$\lim_{n \rightarrow \infty} \Delta_n^{(0)} / \Delta_n = a_0, \\ \lim_{n \rightarrow \infty} \Delta_n^{(r)} / \Delta_n = a_{\frac{r+1}{2}} \quad (r=1, 3, \dots, 2n-1), \\ = b_{\frac{r}{2}} \quad (r=2, 4, \dots, 2n).$$

Lastly we remark that the method of proof may be applied to any function defined by the series, which is uniformly convergent in the interval ($0 \leq x \leq 2\pi$), of orthogonal functions $\varphi_n(x)$, such that

$$\int_0^1 \varphi_m(x) \varphi_n(x) dx = 0 \quad m \neq n$$

$$= 1 \quad m = n,$$

and have one of the forms

$$\varphi_n(x) = k \cos(n\pi x + k') + \frac{\omega(n, x)}{n},$$

$$\varphi_n(x) = k \cos(2n\pi x + k') + \frac{\omega(n, x)}{n},$$

$$\varphi_n(x) = k \cos\left(\frac{2n+1}{n}\pi x + k'\right) + \frac{\omega(n, x)}{n} \quad (1),$$

.....

where k, k' are constants and $|\omega(n, x)|$ is smaller than a finite number independent of n and x .

In such a case, if we take any two numbers a and b within the interval (0, 1), the theorem similar to the above holds good also. The series of the Sturm-Liouville functions, that occurring in the theory of cooling of a sphere, that in the theory of lateral vibration of a bar, etc. belong to this case⁽²⁾.

Ikeda near Ōsaka, April 1918.

(1) In these three forms we may replace the cosines by the sines respectively.

(2) Kneser, loc. cit.; Juretzka, loc. cit.; Ogura, Note on the representation of an arbitrary function in mathematical physics, Tôhoku Math. Journal, 1 (1911-12), p. 120

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