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KINNOSUKE OGURA,

Determination of the Central Forces acting on a Particle whose Equations of Motion possess an Integral Quadratic in the Velocities.

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M. FUJIWARA, J. ISHIWARA, T. KUBOTA and S. KAKEYA.

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Determination of the Central Forces acting on a Particle whose Equations of Motion possess an Integral Quadratic in the Velocities,

by

KINNOSUKE OGURA, Ôsaka.

Darboux determined the *conservative* forces acting on a particle whose equations of motion possess an integral (other than the integral of energy) of the form

$$(1) \qquad P\dot{x}^2+Q\dot{x}\dot{y}+R\dot{y}^2+S\dot{x}+T\dot{y}+K=\text{const.},$$

where P, Q, R, S, T, K are functions of the position of the particle (x, y) (1). In this note I will treat the problem of similar nature for the *central* forces.

Let the equations of motion of a particle which is free to move under the central force F be

$$(2) \qquad \ddot{x}=-\frac{x}{\sqrt{x^2+y^2}}F(x,y), \quad \ddot{y}=-\frac{y}{\sqrt{x^2+y^2}}F(x,y).$$

It is required to find the function $F(x, y)$ in order that the differential equations (2) may possess an integral of the form (1), other than the integral of angular momentum.

Differentiating equation (1), and substituting for $\ddot{x}$ and $\ddot{y}$ from (2), we have

$$(3) \qquad \frac{\partial P}{\partial x}\dot{x}^3+\left(\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)\dot{x}^2\dot{y}+\left(\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial x}\right)\dot{x}\dot{y}^2+\frac{\partial R}{\partial y}\dot{y}^3$$
$$+\frac{\partial S}{\partial x}\dot{x}^2+\left(\frac{\partial S}{\partial y}+\frac{\partial T}{\partial x}\right)\dot{x}\dot{y}+\frac{\partial T}{\partial y}\dot{y}^2$$
$$+\frac{\partial K}{\partial x}\dot{x}+\frac{\partial K}{\partial y}\dot{y}$$

(1) Darboux, Archieves Néerlandaises, (2) 6 (1901), p. 371; Whittaker, Treatise on the analytical dynamics (2. ed., 1917), p. 332.

$$-\frac{F}{\sqrt{x^2+y^2}}\left[2Px\dot{x}+Q(x\dot{y}+y\dot{x})+2Ry\dot{y}+Sx+Ty\right]=0.$$

Equating to zero the term of the third degree in $\dot{x}$ and $\dot{y}$, we have

$$\frac{\partial P}{\partial x}=0,\quad \frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=0,\quad \frac{\partial Q}{\partial y}+\frac{\partial R}{\partial x}=0,\quad \frac{\partial R}{\partial y}=0,$$

from which it is easily seen that the terms of the second degree in the integral (1) must have the form

$$(ay^2+by+c)\,\dot{x}^2+(-2axy-bx-b'y+c_1)\,\dot{x}\dot{y}+(ax^2+b'x+c')\,\dot{y}^2,$$

where a, b, c, b', c', c_1 are constants.

Again equating to zero the terms of the second degree in $\dot{x}$ and $\dot{y}$ in (3),

$$\frac{\partial S}{\partial x}=0,\quad \frac{\partial S}{\partial y}+\frac{\partial T}{\partial x}=0,\quad \frac{\partial T}{\partial y}=0,$$

from which we have

$$S=my+p,\quad T=-mx+q,$$

where m, p, q are constants.

Further equating to zero the terms independent of $\dot{x}$ and $\dot{y}$ in (3), we have

$$Sx+Ty=0,$$

i.e.

$$px+qy=0.$$

This equation shows us that if p and q be different from zero, the force has the constant direction; for

$$\frac{\ddot{y}}{\ddot{x}}=\frac{y}{x}=-\frac{p}{q}.$$

Hence the constants p and q must each be zero.

Lastly equating to zero the terms of the first degree in $\dot{x}$ and $\dot{y}$ in (3),

$$\frac{\partial K}{\partial x}=\frac{F}{\sqrt{x^2+y^2}}(2Px+Qy),$$

$$\frac{\partial K}{\partial y}=\frac{F}{\sqrt{x^2+y^2}}(Qx+2Ry).$$

Differentiating the former with respect to y, and the latter with respect to x, and equating the two values of $\frac{\partial^2 K}{\partial x\,\partial y}$ thus obtained, we have

$$(2Px+Qy)\frac{\partial}{\partial y}\frac{F}{\sqrt{x^2+y^2}}+\left(2x\frac{\partial P}{\partial y}+y\frac{\partial Q}{\partial y}\right)\frac{F}{\sqrt{x^2+y^2}}$$

$$=(Qx+2Ry)\frac{\partial}{\partial x}\frac{F}{\sqrt{x^2+y^2}}+\left(x\frac{\partial Q}{\partial x}+2y\frac{\partial R}{\partial x}\right)\frac{F}{\sqrt{x^2+y^2}},$$

and replacing P, Q, R by their values as found above, we obtain

$$(4)\qquad (c_1x+2c'y-bx^2+b'xy)\frac{\partial}{\partial x}\log\frac{F}{\sqrt{x^2+y^2}}$$

$$+(-2cx-c_1y-bxy-b'y^2)\frac{\partial}{\partial y}\log\frac{F}{\sqrt{x^2+y^2}}$$

$$=3(bx-b'y).$$

This partial differential equation can be integrated in the following way. The characteristics are determined by

$$\frac{dx}{c_1x+2c'y-bx^2+b'xy}=\frac{dy}{-2cx-c_1y-bxy+b'y^2}=\frac{d\log F/\sqrt{x^2+y^2}}{3(bx-b'y)}$$

so that putting

$$\frac{F}{\sqrt{x^2+y^2}}=\frac{1}{z^3}$$

and

$$L=c_1x+2c'y,\quad M=-2cx-c_1y,\quad N=bx-b'y,$$

we get

$$(5)\qquad \frac{dx}{Nx-L}=\frac{dy}{Ny-M}=\frac{dz}{Nz},$$

which belongs to the type treated by Fouret[1]. Consequently it follows that if we regard x, y, z as the rectangular point coordinates in space, *this system represents (special) W-curves in space.*

Now since

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(1) Fouret, Comptes Rendus, Paris (1876); Wilczynski, Projective differential geometry of curves and ruled surfaces (1906), p. 282.

$$\frac{dx}{Nx-L}=\frac{dy}{Ny-M}$$

is a Jacobi equation, it can be integrated by the ordinary method, and the integral is

$$u^{2\lambda}\, v^{-\lambda}\, w^{-\lambda}=\text{const.},$$

i. e.

$$\frac{u^2}{v\,w}=\text{const.},$$

where

$$u\equiv \alpha\, x+\beta\, y-1,\ \ v\equiv 2\,c\,x+(c_1-\lambda)\,y,\ \ w\equiv 2\,c\,x+(c_1+\lambda)\,y\,;$$
$$\alpha\equiv(b\,c_1+2\,b'\,c)\,\lambda^{-2},\ \ \beta\equiv(b'\,c_1+2\,b\,c')\,\lambda^{-2},\ \ \lambda\equiv\sqrt{c_1^{\,2}-4\,c\,c'}\,.$$

Next in virtue of the identity

$$\alpha\, L+\beta\, M=N,$$

we obtain from (5)

$$\frac{\alpha\, dx+\beta\, dy}{\alpha\, x+\beta\, y-1}=\frac{dz}{z}\,,$$

whose integral is

$$\frac{z}{u}=\text{const.}\,.$$

Therefore the general integral of (4) is

$$F=\frac{\sqrt{x^2+y^2}}{u^3}\,\Psi\left(\frac{u^2}{v\,w}\right), \tag{6}$$

where Ψ is an arbitrary function. Thus we have arrived at the theorem:

The only cases of the motion of a particle, under the action of the central forces, which possess an integral quadratic in the velocities other than the integral of angular momentum, are those for which the force has the form

$$F=\frac{\sqrt{x^2+y^2}}{u^3}\,\Psi\left(\frac{u^2}{v\,w}\right),$$

where Ψ is an arbitrary function, and then the integral has the form

$$(a\,y^2+b\,y+c)\,\dot{x}^2+(-2\,a\,x\,y-b\,x-b'\,y+c_1)\,\dot{x}\,\dot{y}+(a\,x^2+b'\,x+c')\,\dot{y}^2$$
$$+m\,(y\,\dot{x}-x\,\dot{y})+K\,(x,\,y)=\text{const.}\ (^1),$$

$K(x,y)$ standing for the integral

$$\int\frac{\Psi}{u^3}\left[(N\,y-M)\,dx-(N\,x-L)\,dy\right].$$

Here we add some particular cases:

I. If we put

$$b=b'=0,\quad c=c'=\frac{1}{2},\quad c_1=0,$$

then

$$F=-r\,\Psi\left(\frac{1}{r^2}\right),\quad (r=\sqrt{x^2+y^2})\,;$$

and the first integral becomes

$$\frac{1}{2}\,(\dot{x}^2+\dot{y}^2)-\int\Psi\left(\frac{1}{r^2}\right)r\,dr=\text{const.},$$

which is nothing but the equation of energy.

II. If we put

$$\Psi\,(\xi)=\text{const.},$$

then

$$F=\frac{r}{(A\,x+B\,y+C)^3}\,, \tag{7}$$

A, B, C being arbitrary constants.

Again, if we put

$$\Psi\,(\xi)=\xi^{\frac{3}{2}}\,,$$

we have

$$F=\frac{r}{(A_1x^2+B_1xy+C_1y^2)^{\frac{3}{2}}} \tag{8}$$

A_1, B_1, C_1 being arbitrary constants.

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(1) If we use the integral of angular momentum

$$y\,\dot{x}-x\,\dot{y}=k,\quad (k,\ \text{any constant}),$$

the above equation may be written

$$c\,\dot{x}^2+c_1\,\dot{x}\,\dot{y}+c'\,\dot{y}^2+k\,(b\,\dot{x}-b'\,\dot{y})+K\,(x,\,y)=\text{const.}\,.$$

Now remembering that (7) and (8) give the laws of force discovered by Darboux and Halphen in the Bertrand problem (1), we infer the theorem:

When a particle describes a conic for any initial condition under a central force, the equations of motion have an integral quadratic in the velocities.

Ikeda near Osaka, May 1918.

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(1) Darboux, Comptes Rendus, Paris, 84 (1877), p. 936; Halphen, ibid., p. 939; Appell, Traité de mécanique rationnelle, t. 1 (3. éd., 1909), p. 400.

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