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KINNOSUKE OGURA,

On the Integral Inequalities between Two Systems of Orthogonal Functions.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 14, Nos. 1, 2.
edited by TSURUICHI HAYASHI, College of Science,
Tôhoku Imperial University, Sendai, Japan,
with the collaboration of Messrs.
M. FUJIWARA, J. ISHIWARA, T. KUBOTA and S. KAKEYA.

August 1918

On Integral Inequalities between Two Systems of Orthogonal Functions,

by

KINNOSUKE OGURA, Ôsaka.

Let

$$f_1(x),\quad f_2(x),\quad \cdots\cdots,\quad f_n(x),\quad \cdots\cdots;$$
$$\varphi_1(x),\quad \varphi_2(x),\quad \cdots\cdots,\quad \varphi_n(x),\quad \cdots\cdots$$

be two different systems of orthogonal functions in the same interval $(a \leqq x \leqq b)$, such that

$$\int_a^b f_n^2(x)\,dx = \int_a^b \varphi_n^2(x)\,dx = 1, \quad (n=1,2,\cdots\cdots);$$

$$\int_a^b f_m(x) f_n(x)\,dx = \int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0, \quad (m \neq n;\ m, n = 1, 2, \cdots\cdots).$$

If we put

$$f_n(x) = \varphi_n(x) + \psi_n(x),$$

then

$$\int_a^b f_n^2(x)\,dx = \int_a^b \varphi_n^2(x)\,dx + 2\int_a^b \varphi_n(x)\,\psi_n(x)\,dx + \int_a^b \psi_n^2(x)\,dx,$$

so that

$$\int_a^b \psi_n^2(x)\,dx = -2\int_a^b \varphi_n(x)\,\psi_n(x)\,dx.$$

Similarly

$$\int_a^b \psi_m^2(x)\,dx = -2\int_a^b \varphi_m(x)\,\psi_m(x)\,dx.$$

Next, since

$$\int_a^b f_m(x) f_n(x)\,dx = \int_a^b \varphi_m(x)\,\varphi_n(x)\,dx + \int_a^b \varphi_m(x)\,\psi_n(x)\,dx$$
$$+\int_a^b \varphi_n(x)\,\psi_m(x)\,dx + \int_a^b \psi_m(x)\,\psi_n(x)\,dx,$$

we have

$$\int_a^b \psi_m(x)\,\psi_n(x)\,dx = -\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx + \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right].$$

Hence the Schwarz inequality

$$\left[\int_a^b \psi_m(x)\,\psi_n(x)\,dx\right]^2 \leqq \int_a^b \psi_m^2(x)\,dx \cdot \int_a^b \psi_n^2(x)\,dx$$

becomes

$$\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx + \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right]^2$$
$$\leqq 4\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx,$$

which may be written

$$\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx - \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right]^2$$
$$\leqq 4\left[\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_n(x)\,dx\right.$$
$$\left.-\int_a^b \varphi_m(x)\,\psi_n(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right].$$

Therefore we have the identities:

$$\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx \geqq 0,$$
$$\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx$$
$$-\int_a^b \varphi_m(x)\,\psi_n(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_m(x)\,dx \geqq 0,$$

the latter belonging to the type treated by Prof. M. Fujiwara (1).

It follows from these inequalities, by a short calculation, that *there exist the following integral inequalities among* $f_m(x), f_n(x), \varphi_m(x), \varphi_n(x)$:

$$1-\int_a^b f_m(x)\,\varphi_m(x)\,dx - \int_a^b f_n(x)\,\varphi_n(x)\,dx$$

(1) Fujiwara, Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen für bestimmte Integrale, Tôhoku Math. Journal, 13 (1918), p. 228.

$$+\int_a^b f_m(x)\,\varphi_m(x)\,dx \cdot \int_a^b f_n(x)\,\varphi_n(x)\,dx \geqq 0,$$
$$1-\int_a^b f_m(x)\,\varphi_m(x)\,dx - \int_a^b f_n(x)\,\varphi_n(x)\,dx$$
$$+\int_a^b f_m(x)\,\varphi_m(x)\,dx \cdot \int_a^b f_n(x)\,\varphi_n(x)\,dx$$
$$-\int_a^b f_m(x)\,\varphi_n(x)\,dx \cdot \int_a^b f_n(x)\,\varphi_m(x)\,dx \geqq 0;$$
$$(m \neq n;\ m, n = 1, 2, \ldots.)$$

which may be written respectively

$$\left[1-\int_a^b f_m(x)\,\varphi_m(x)\,dx\right]\left[1-\int_a^b f_n(x)\,\varphi_n(x)\,dx\right] \geqq 0\ (^1);$$
$$1-\int_a^b [f_m(x)\,\varphi_m(x) + f_n(x)\,\varphi_n(x)]\,dx$$
$$+\frac{1}{2}\int_a^b\int_a^b \begin{vmatrix} f_m(x), & f_m(y) \\ f_n(x), & f_n(y) \end{vmatrix} \cdot \begin{vmatrix} \varphi_m(x), & \varphi_m(y) \\ \varphi_n(x), & \varphi_n(y) \end{vmatrix}\,dx\,dy \geqq 0,$$
$$(m \neq n;\ m, n = 1, 2, \ldots\ldots).$$

Ikeda near Ôsaka, May 1918.

(1) This is an immediate consequence of the Schwarz inequality: for from

$$\left[\int_a^b f_n(x)\,\varphi_n(x)\,dx\right]^2 \leqq \int_a^b f_n^2(x)\,dx \cdot \int_a^b \varphi_n^2(x)\,dx = 1$$

we have

$$-1 \leqq \int_a^b f_n(x)\,\varphi_n(x)\,dx \leqq 1.$$

THE TÔHOKU MATHEMATICAL JOURNAL.

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Price per volume (consisting of four numbers) payable in advance: 3 yen=6 shillings=6 Mark=7.50 francs=1.50 dollars. Postage inclusive.