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A Generalized Pascal Theorem on a Space Cubic.

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A Generalized Pascal Theorem on a Space Cubic,

by

KINNOSUKE OGURA, Ôsaka.

In this note I will give a *synthetic* proof of the following theorem on a space cubic, which may be considered as *a generalization*(1) *of the Pascal theorem on a conic:*

When 1, 2, 3,⋯⋯, 12 *are any twelve points on a space cubic, the four points*

$$\text{(I)}\qquad \left\{\begin{array}{l} [(1,2,3),\ (5,6,\ 7),\ (\ 9,10,11)],\\ [(2,3,4),\ (6,7,\ 8),\ (10,11,12)],\\ [(3,4,5),\ (7,8,\ 9),\ (11,12,\ 1)],\\ [(4,5,6),\ (8,9,10),\ (12,\ 1,\ 2)] \end{array}\right.$$

are in a plane(2).

(1) For other generalizations concerning a space cubic, see Encyclopédie des sciences mathématiques, (3) 4, fasc. 1 (1914), p. 128.

(2) Let θ_i $(i=1,2,\cdots,12)$ be any given quantities and $f_{i,j,k}$ denote the binary cubic forms $(x-\theta_i\, y)(x-\theta_j\, y)(x-\theta_k\, y)$. Then the theorem is equivalent to any one of the following two *algebraic* theorems:

I. *There exist the twelve constants* $\lambda, \mu, \nu, \cdots, \nu'''$, *for which we have the identities*

$$\lambda f_{1,2,3}+\mu f_{5,6,7}+\nu f_{9,10,11}\equiv\lambda' f_{2,3,4}+\mu' f_{6,7,8}+\nu' f_{10,11,12}$$
$$\equiv\lambda'' f_{3,4,5}+\mu'' f_{7,8,9}+\nu'' f_{11,12,1}\equiv\lambda''' f_{4,5,6}+\mu''' f_{8,9,10}+\nu''' f_{12,1,2}.$$

For the case of a conic, see Laguerre, Sur la représentation des formes binaires dans le plan et dans l'espace, Bull. de la Soc. Philomatique, (1) 40 (1872), p. 221 [=Oeuvres, II, p. 277].

II. *There exist the four constants* k_1, k_2, k_3, k_4, *for which we have the identity*

$$k_1\, K(f_{1,2,3};\, f_{5,6,7};\, f_{9,10,11})+k_2\, K(f_{2,3,4};\, f_{6,7,8};\, f_{10,11,12})$$
$$+k_3\, K(f_{3,4,5};\, f_{7,8,9};\, f_{11,12,1})+k_4\, K(f_{4,5,6};\, f_{8,9,10};\, f_{12,1,2})\equiv 0,$$

$K(f,\varphi,\psi)$ *standing for the determinant*

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 \varphi}{\partial x^2} & \dfrac{\partial^2 \psi}{\partial x^2} \\ \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 \varphi}{\partial x\,\partial y} & \dfrac{\partial^2 \psi}{\partial x\,\partial y} \\ \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 \varphi}{\partial y^2} & \dfrac{\partial^2 \psi}{\partial y^2} \end{vmatrix},$$

Let us suppose that $2, 3, 4, \cdots\cdots, 12$ are eleven fixed points on a given space cubic; and consider the correspondence between the points 1 and $\bar{1}$, for which the four points

$$\text{(II)} \qquad \left\{ \begin{aligned} A &\equiv [(1,2,3),\ (5,6,\ 7),\ (\ 9,10,11)], \\ B &\equiv [(2,3,4),\ (6,7,\ 8),\ (10,11,12)], \\ C &\equiv [(3,4,5),\ (7,8,\ 9),\ (11,12,\ \bar{1})], \\ D &\equiv [(4,5,6),\ (8,9,10),\ (12,\ \bar{1},\ 2)] \end{aligned} \right.$$

are in a plane.

When 1 is given, the two points A and B are fixed and lie on the fixed line $AB(\equiv l)$. Also C is on the fixed line

$$m \equiv \{(3,4,5),\ (7,8,9)\},$$

and D on the fixed line

$$n \equiv \{(4,5,6),\ (8,9,10)\};$$

and the line CD intersects the line l. Let us take any point $1'$ on the cubic, and let C' be the point of intersection of m and the plane $(11, 12, 1')$, and D' be that of n and the line cutting l and n and passing through C'. If $1''$ be the third point of intersection of the cubic and the plane $(12, D', 2)$, then there exists a one-one correspondence between $1'$ and $1''$; so that there are two self-corresponding points (that is, the points $\bar{1}$, for which the four points (II) are in a plane). Denote these points by $\bar{1}_1$ and $\bar{1}_2$.

Conversely, when $\bar{1}_1$ is given the three points B, C, D are determined uniquely. Hence the plane, passing through 2, 3 and the point of intersection of the plane (B, C, D) and the line $\{(5,6,7),\ (9,10,11)\}$, cuts the cubic at the third point 1. When $\bar{1}_2$ is given, a similar result will be obtained.

It follows that we have a one-two correspondence between 1 and $\bar{1}$; so that there are three or ∞^1 self-corresponding points (that is, the points 1, for which the four points (I) are in a plane).

But we can prove that *the locus of the points 1 in space, for which*

which was treated by Profs. Rosanes, Lindemann and Hayashi. (See Ogura, Binary forms and duality, Tôhoku Math. Journ., (1918), p. 290.)

For the case of a conic, see Hesse, Zur Involution, Crelle's Journal, 63 (1864) [=Werke, p. 515]; and Fr. Meyer, Allgemeine Formen- und Invariantentheorie, 1 (1909), p. 361, where Prof. Meyer proposed to solve the question "....Wie lautet die entsprechende Übertragung auf kubische Raumkurven?"

the four points (I) are in a plane (P), is a cubic surface. Consider the tetrahedron having the faces

$$(P);\quad (1,2,3);\quad (11,12,1);\quad (12,1,2),$$

which contain

	the fixed point	$[(2,3,4),\ (6,7,8),\ (10,11,12)]$;
	the fixed line	$\{\ 2,\ 3\}$;
	the fixed line	$\{11, 12\}$;
and	the fixed line	$\{12,\ 2\}$

respectively. Since the three edges

$$\{(P), (1,2,3)\};\quad \{(P), (11,12,1)\};\quad \{(P), (12,1,2)\}$$

cut the three fixed lines

$$\{(5,6,7),\ (9,10,11)\};\ \{(3,4,5),\ 7,8,9)\};\ \{(4,5,6),\ (8,9,10)\}$$

respectively, we obtain three trilinear point ranges. Hence the three faces

$$(1,2,3);\qquad (11,12,1);\qquad (12,1,2)$$

form three trilinear axial pencils; whence the locus of the vertex 1 is a cubic surface(1).

Therefore we have, at least, nine points 1 on the space cubic, for which the four points (I) are in a plane; and consequently any point on the cubic can be taken as the point 1. Thus the theorem has been established.

Lastly we remark that *a similar theorem holds good for the rational curve in space of n dimensions*:

$$\rho x_1 = \theta^n,\quad \rho x_2 = \theta^{n-1}, \cdots\cdots\ \cdots\cdots,\ \rho x_n = \theta,\quad \rho x_{n+1} = 1,$$

θ being the parameter.

Ikeda near Ôsaka, March 1918.

(1) F. August, De superficiebus tertii ordinis, Diss. Berlin 1862: F. London, Zur Theorie der trilinearen Verwandtschaft dreier einstufiger Grundgebilde, Math. Ann., 44 (1894), p. 405; R. Sturm, Die Lehre von den geometrischen Verwandtschaften, 1 (1908), p. 324.

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