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Binary Forms and Duality.

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Binary Forms and Duality,

by

Kinnosuke Ogura, Ôsaka.

Clebsch([1]) established the relation among 3 binary n-ic forms and the Jacobians of the Jacobians formed from the given forms; this result has been extended by Rosanes([2]) to the case of 4 binary n-ic forms and also by Lindemann([3]) to the case of 4 binary cubic forms. In this short note I will generalize these results to the case of r $(3 \leqq r \leqq n+1)$ binary n-ic forms, *the method of proof being based entirely upon the principle of duality*, the starting point of Clebsch.

1. Take n binary n-ic forms:

$$f_1(\xi_1, \xi_2), \quad f_2(\xi_1, \xi_2), \quad \cdots\cdots, \; f_n(\xi_1, \xi_2)$$

and form from them the following determinants:

$$K(f_{i_1}, f_{i_2}, \cdots, f_{i_m}) \equiv K_{i_1, i_2, \cdots, i_m} \quad (i_1, i_2, \cdots, i_n = 1, 2, \cdots, n;\; 2 \leqq m \leqq n)$$

$$\equiv \begin{vmatrix} \dfrac{\partial^{m-1} f_{i_1}}{\partial \xi_1^{m-1}} & \dfrac{\partial^{m-1} f_{i_1}}{\partial \xi_1^{m-2}\,\partial \xi_2} & \cdots\cdots & \dfrac{\partial^{m-1} f_{i_1}}{\partial \xi_2^{m-1}} \\ \dfrac{\partial^{m-1} f_{i_2}}{\partial \xi_1^{m-1}} & \dfrac{\partial^{m-1} f_{i_2}}{\partial \xi_1^{m-2}\,\partial \xi_2} & \cdots\cdots & \dfrac{\partial^{m-1} f_{i_2}}{\partial \xi_2^{m-1}} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ \dfrac{\partial^{m-1} f_{i_m}}{\partial \xi_1^{m-1}} & \dfrac{\partial^{m-1} f_{i_m}}{\partial \xi_1^{m-2}\,\partial \xi_2} & \cdots\cdots & \dfrac{\partial^{m-1} f_{i_m}}{\partial \xi_2^{m-1}} \end{vmatrix}.$$

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(1) Clebsch, Über eine Eigenschaft von Funktionaldeterminanten, Journ. f. Math., 69 (1868), p. 355. See also Pascal, Die Determinanten (1909), p. 236.

(2) Rosanes, Über Funktionen, welche ein den Funktionaldeterminanten analoges Verhalten zeigen, Journ. f. Math., 75 (1873), p. 166.

(3) Lindemann, Über die Darstellung binärer Formen und ihrer Covarianten durch geometrische Gebilde im Raume, Math. Ann., 23 (1884), p. 111. He stated "....die in....aufgestellten entsprechenden Relationen für ein System von....cubischen Formen....auf Systeme binärer Formen $n$ter Ordnung erweitern lassen. Es würde mehr umständlich als schwierig sein,....und es mag dies deshalb unterbleiben."

K_{i_1, i_2} is the Jacobian $J(f_{i_1}, f_{i_2})$ of f_{i_1} and f_{i_2}; and $K_{i_1, i_2, \cdots, i_n}$ is the covariant which was called the Rosanesian by Prof. T. Hayashi(¹). For the sake of brevity, when

$$f_{i_1}=K(f_{i_{11}}, f_{i_{12}}, \cdots, f_{i_{1r}}), \quad \cdots \; \cdots, \quad f_{i_m}=K(f_{i_{m1}}, f_{i_{m2}}, \cdots, f_{i_{mr}}),$$

we will denote $K(f_{i_1}, f_{i_2}, \cdots, f_{i_m})$ by

$$K_{i_{11}\, i_{12} \cdots\, i_{1r}, \; \cdots \; \cdots, \; i_{m1}\, i_{m2} \cdots\, i_{mr}}.$$

It will be seen, by successive applications of Euler's formulae for homogeneous functions, that the following identities hold good:

$$\begin{vmatrix} f_{i_1} & df_{i_1} \\ f_{i_2} & df_{i_2} \end{vmatrix} = \begin{vmatrix} f_{i_1} & \dfrac{\partial f_{i_1}}{\partial \xi_1} d\xi_1 + \dfrac{\partial f_{i_1}}{\partial \xi_2} d\xi_2 \\ f_{i_2} & \dfrac{\partial f_{i_2}}{\partial \xi_1} d\xi_1 + \dfrac{\partial f_{i_2}}{\partial \xi_2} d\xi_2 \end{vmatrix} = \frac{1}{n} K_{i_1, i_2} \begin{vmatrix} \xi_1 & d\xi_1 \\ \xi_2 & d\xi_2 \end{vmatrix}$$

$$= \frac{1}{n} (\xi_1\, d\xi_2 - \xi_2\, d\xi_1)\, K_{i_1, i_2},$$

$$\begin{vmatrix} f_{i_1} & df_{i_1} & d^2 f_{i_1} \\ f_{i_2} & df_{i_2} & d^2 f_{i_2} \\ f_{i_3} & df_{i_3} & d^2 f_{i_3} \end{vmatrix} = \frac{1}{n(n-1)^2} K_{i_1, i_2, i_3} \begin{vmatrix} \xi_1^2 & 2\xi_1\, d\xi_1 & d\xi_1^2 \\ \xi_1\, \xi_2 & \xi_2\, d\xi_1 + \xi_1\, d\xi_2 & d\xi_1\, d\xi_2 \\ \xi_2^2 & 2\xi_2\, d\xi_2 & d\xi_2^2 \end{vmatrix},$$

$$= \frac{1}{n(n-1)} (\xi_1\, d\xi_2 - \xi_2\, d\xi_1)^3 .\, K_{i_1, i_2, i_3},$$

....

$$(1) \quad \begin{vmatrix} f_{i_1} & df_{i_1} & \cdots\cdots & d^{m-1} f_{i_1} \\ f_{i_2} & df_{i_2} & \cdots\cdots & d^{m-1} f_{i_2} \\ \cdots & \cdots & \cdots\cdots & \cdots\cdots \\ f_{i_m} & df_{i_m} & \cdots\cdots & d^{m-1} f_{i_m} \end{vmatrix} = \frac{\binom{m-1}{1}\binom{m-1}{2}\cdots\binom{m-1}{m-2} K_{i_1, i_2, \cdots, i_m}}{(m-1)^{m-2} . n(n-1)^2 \cdots (n-m+2)^{m-1}}$$

$$\times \begin{vmatrix} \xi_1^{m-1} & \binom{m-1}{1} \xi_1^{m-2}\, d\xi_1 & \cdots\cdots\cdots & d\xi_1^{m-1} \\ \xi_1^{m-2}\, \xi_2 & (m-2)\, \xi_1^{m-3}\, \xi_2\, d\xi_1 + \xi_1^{m-2}\, d\xi_2 & \cdots\cdots\cdots & d\xi_1^{m-2}\, d\xi_2 \\ \xi_1^{m-3}\, \xi_2^3 & (m-3)\, \xi_1^{m-4}\, \xi_2^2\, d\xi_1 + 2\xi_1^{m-3}\, \xi_2\, d\xi_2 & \cdots\cdots\cdots & d\xi_1^{m-3}\, d\xi_2^2 \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \xi_2^{m-1} & \binom{m-1}{1} \xi_2^{m-2}\, d\xi_2 & \cdots\cdots\cdots & d\xi_2^{m-1} \end{vmatrix}$$

$$= \frac{\binom{m-1}{1}\binom{m-1}{2}\cdots\cdots\binom{m-1}{m-2}}{(m-1)^{m-2} . n(n-1)^2 \cdots (n-m+2)^{m-1}} \cdot (\xi_1\, d\xi_2 - \xi_2\, d\xi_1)^{\frac{1}{2}m(m-1)}\, K_{i_1, i_2, \cdots, i_m}.$$

(¹) Hayashi, Some theorems on binary forms, Science Reports of Tôhoku Imperial University, 6 (1917), p. 123.

2. Now since the case of 3 *forms* has been treated by Clebsch(¹), we begin with the case of the 4 *forms*:

$$(2) \qquad x_1 \equiv f_1(\xi_1, \xi_2), \quad x_2 \equiv f_2(\xi_1, \xi_2), \quad x_3 \equiv f_3(\xi_1, \xi_2), \quad x_4 \equiv f_4(\xi_1, \xi_2).$$

If we regard x_1, x_2, x_3, x_4 as the homogeneous point coordinates in space, (2) represents a rational curve of degree n, ξ_1, ξ_2 being homogeneous parameters. Then it follows from (1) that the plane coordinates of the osculating plane at the point (ξ_1, ξ_2) are given by

$$(3) \quad \begin{cases} \rho u_1 = K_{2, 3, 4} = K(x_2, x_3, x_4), \\ \rho u_2 = -K_{3, 4, 1} = -K(x_3, x_4, x_1), \\ \rho u_3 = K_{4, 1, 2} = K(x_4, x_1, x_2), \\ \rho u_4 = -K_{1, 2, 3} = -K(x_1, x_2, x_3). \end{cases}$$

Since (3) may be considered as the equations of the space curve (2) in the plane coordinates, the principle of duality shows us that

$$K(u_2, u_3, u_4), \quad -K(u_3, u_4, u_1), \quad K(u_4, u_1, u_2), \quad -K(u_1, u_2, u_3)$$

must be proportional to

$$x_1, \quad x_2, \quad x_3, \quad x_4$$

respectively. Hence

$$(4) \quad \frac{K_{341,\,412,\,123}}{f_1} = \frac{K_{412,\,123,\,234}}{f_2} = \frac{K_{123,\,234,\,341}}{f_3} = \frac{K_{234,\,341,\,412}}{f_4} = k_I^{(4)}.$$

Next we have from (1) the radial coordinates of the tangent to the space curve (2) at the point (ξ_1, ξ_2):

$$\sigma\, p_{ik} = K_{i,k} = K(x_i, x_k), \qquad (i, k = 1, 2, 3, 4);$$

and the axial coordinates of the tangent are

$$\sigma' q_{12} = p_{43}, \quad \sigma' q_{23} = p_{41}, \quad \sigma' q_{31} = p_{42},$$
$$\sigma' q_{41} = p_{23}, \quad \sigma' q_{42} = p_{31}, \quad \sigma' q_{43} = p_{12}.$$

(¹) Clebsch's result is

$$\frac{K_{31 \cdot 12}}{f_1} = \frac{K_{12 \cdot 23}}{f_2} = \frac{K_{23 \cdot 31}}{f_3} = k^{(3)}.$$

When f_1, f_2, f_3 are quadratic forms, the proportional factor $k^{(3)}$ becomes constant. If binary quadratic forms be referred to the *normal curve in a plane*, the above equations are equivalent to the following theorem, which is *self-evident*: In a plane, the polar triangle of the polar triangle of a given triangle, with respect to a fixed conic (the normal curve), coincides with the given triangle.

But by the principle of duality, q_{ik} must be proportional to $K(u_i, u_k)$ so that

$$(5) \qquad \frac{K_{234,\,341}}{K_{4,\,3}} = \frac{K_{341,\,412}}{K_{4,\,1}} = \frac{K_{412,\,234}}{-K_{4,2}}$$

$$= \frac{K_{123,\,234}}{K_{2,\,3}} = \frac{K_{123,\,341}}{-K_{3,\,1}} = \frac{K_{123,\,412}}{K_{1,\,2}} = k_{II}^{(4)}.$$

When f_i $(i=1, 2, 3, 4)$ are cubic forms, $k_I^{(4)}$ and $k_{II}^{(4)}$ become constant[1]. If 4 binary cubic forms be referred to the *normal curve in space*[2], (4) is equivalent to the following theorem, which is *self-evident*: The polar tetrahedron of the polar tetrahedron of a given tetrahedron, with respect to a fixed space cubic (the normal curve), coincides with the given tetrahedron. The given tetrahedron and its polar tetrahedron are mutually inscribed.

3. We pass now to consider the 5 *forms*:

$$(6) \qquad x_1 \equiv f_1(\xi_1, \xi_2), \quad x_2 \equiv f_2(\xi_1, \xi_2), \quad x_3 \equiv f_3(\xi_1, \xi_2),$$
$$x_4 \equiv f_4(\xi_1, \xi_2), \quad x_5 \equiv f_5(\xi_1, \xi_2).$$

If we regard x_1, x_2, x_3, x_4, x_5 as the homogeneous point coordinates in space of 4 dimensions, (6) represents a rational curve of degree n, ξ_1, ξ_2 being homogeneous parameters. Then it follows from (1) that the hyperplane coordinates of the osculating hyperplane at the point (ξ_1, ξ_2) are given by

$$(7) \qquad \begin{cases} \rho u_1 = K_{2,\,3,\,4,\,5} = K(x_2, x_3, x_4, x_5), \\ \rho u_2 = K_{3,\,4,\,5,\,1} = K(x_3, x_4, x_5, x_1), \\ \rho u_3 = K_{4,\,5,\,1,\,2} = K(x_4, x_5, x_1, x_2), \\ \rho u_4 = K_{5,\,1,\,2,\,3} = K(x_5, x_1, x_2, x_3), \\ \rho u_5 = K_{1,\,2,\,3,\,4} = K(x_1, x_2, x_3, x_4). \end{cases}$$

But by the principle of duality,

$$K(u_2, u_3, u_4, u_5), \quad K(u_3, u_4, u_5, u_1), \quad K(u_4, u_5, u_1, u_2),$$
$$K(u_5, u_1, u_2, u_3), \quad K(u_1, u_2, u_3, u_4)$$

(1) For this case Lindemann gave

$$\frac{K_{123,\,234,\,341}}{f_3} = k_I^{(4)}, \qquad \frac{K_{123,\,234}}{K_{2,\,3}} = k_{II}^{(4)};$$

and determined these two constants.

(2) Fr. Meyer, Apolarität und rationale Curven (1883), p. 46. *If f_1, f_2, f_3 be given points in space, $K_{1,\,2,\,3}$ represents the pole of the plane passing through the given 3 points.* For another interpretation of $K_{1,\,2,\,3}$, see Hayashi, loc. cit.

must be proportional to

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5$$

respectively; so that

$$(8) \qquad \frac{K_{3451,\,4512,\,5123,\,1234}}{f_1} = \frac{K_{4512,\,5123,\,1234,\,2345}}{f_2} = \frac{K_{5123,\,1234,\,2345,\,3451}}{f_3}$$

$$= \frac{K_{1234,\,2345,\,3451,\,4512}}{f_4} = \frac{K_{2345,\,3451,\,4512,\,5123}}{f_5} = k_I^{(5)}.$$

Next we have from (1) the radial coordinates of the tangent to the curve at the point (ξ_1, ξ_2):

$$\sigma p_{ik} = K_{i,k} = K(x_i, x_k), \qquad (i, k = 1, 2, 3, 4, 5).$$

Further let the plane

$$\begin{cases} v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4 + v_5 x_5 = 0, \\ w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + w_5 x_5 = 0 \end{cases}$$

have the contact of the second order to the curve (6) at the point (ξ_1, ξ_2) and let us put

$$\pi_{ik} = \begin{vmatrix} v_i & v_k \\ w_i & w_k \end{vmatrix}.$$

Then it follows from (1), by aid of Grassmann's theorem[1], that

$$\lambda\pi_{12} = K_{3,\,4,\,5}, \quad \lambda\pi_{23} = K_{1,\,4,\,5}, \quad \lambda\pi_{34} = K_{1,\,3,\,5}, \quad \lambda\pi_{45} = K_{1,\,2,\,3}.$$
$$\lambda\pi_{13} = -K_{2,\,4,\,5}, \quad \lambda\pi_{24} = -K_{1,\,3,\,5}, \quad \lambda\pi_{35} = -K_{1,\,2,\,4};$$
$$\lambda\pi_{14} = K_{2,\,3,\,5}, \quad \lambda\pi_{25} = K_{1,\,3,\,4};$$
$$\lambda\pi_{15} = -K_{2,\,3,\,4}.$$

But by the principle of duality, $\pi_{ik}(x)$ must be proportional to $p_{ik}(u)$ and $p_{ik}(u)$ to $\pi_{ik}(x)$; consequently

$$(9) \qquad \frac{K_{4512,\,5123,\,1234}}{K_{1,\,2}} = \frac{K_{3451,\,5123,\,1234}}{-K_{1,\,3}} = \frac{K_{3451,\,4512,\,1234}}{K_{1,\,4}}$$

$$= \frac{K_{3451,\,4512,\,5123}}{-K_{1,\,5}} = \frac{K_{2345,\,5123,\,1234}}{K_{2,\,3}} = \cdots\cdots = k_{II}^{(5)},$$

and

(1) See Fr. Meyer, loc. cit., p. 1.

$$(10) \qquad \frac{K_{2345,\,3451}}{K_{3,\,4,\,5}} = \frac{K_{2345,\,4512}}{-K_{2,\,4,\,5}} = \frac{K_{2345,\,5123}}{K_{2,\,3,\,5}}$$

$$= \frac{K_{2345,\,1234}}{-K_{2,\,3,\,4}} = \frac{K_{3451,\,4512}}{K_{1,\,4,\,5}} = \cdots\cdots = k_{III}^{(5)}.$$

When f_i $(i=1, 2, 3, 4, 5)$ are quartic forms, $k_I^{(5)}$, $k_{II}^{(5)}$ and $k_{III}^{(5)}$ become constant. If 5 binary quartic forms be referred to the *normal curve in space of* 4 *dimensions*, (8) is equivalent to the following theorem, which is *self-evident:* The polar pentahedron of the polar pentahedron of a given pentahedron, with respect to a fixed space quartic (the normal curve), coincides with the given pentahedron.

These results can be easily extended to the systems of 6, 7, ……, or $n+1$ *forms* respectively.

Takedao, January 8, 1918.

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