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On the Theory of Representation of Surface.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 12, No. 4,
edited by TSURUICHI HAYASHI, College of Science,
Tôhoku Imperial University, Sendai, Japan,
with the collaboration of Messrs.
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December 1917

On the Theory of Representation of Surfaces,

by

KINNOSUKE OGURA, Ôsaka.

Introduction.

1. Let

$$f_1 \equiv Edu^2 + 2Fdudv + Gdv^2,$$
$$f_2 \equiv Ldu^2 + 2Mdudv + Ndv^2$$

be the first and second fundamental forms of a surface, and let $J(f_1, f_2)$ be the Jacobian of f_1, f_2. Then

$$f_1 = 0,\ f_2 = 0,\ J(f_1, f_2) = 0,\ J(f_1,\ J(f_1, f_2)) = 0,\ J(f_2,\ J(f_1, f_2)) = 0$$

are the equations to the minimal lines, the asymptotic lines, the lines of curvature, the lines of torsion and the characteristic lines respectively.

I have proved that the minimal lines, the lines of curvature and the lines of torsion form a cycle (the *first cycle*) in the sense that the directions of any one of these families are the double rays of the involution determined by the directions of the other two; similarly the asymptotic lines, the lines of curvature and the characteristic lines form the *second cycle*. And these five families of curves form the complete system(1). The characteristic properties of these five families and the five involutional systems and relations(2) among them are shown by the following table(3):

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(1) Ogura, Some theorems concerning binary, quadratic forms and their applications to the differential geometry, Science Reports of Tôhoku Imperial University, Series I, 5 (1916), p. 95.

(2) T. Hayashi, On the usual parametric curves on a surface, Science Reports of Tôhoku Imperial University, Ser. I, 5 (1916), p. 63. Compare with Ogura, On the T-system on a surface, Tôhoku Math. Jour., 9 (1916), p. 88, where the bibliography concerning the lines of torsion and the characteristic lines will be found.

(3) For example, when the parametric curves u=const., v=const. form an inverse-orthogonal system, we have

$$\frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = \frac{M}{F} \quad \text{or} \quad \frac{1}{T_u} - \frac{1}{T_v} = 0,$$

and conversely. (In this table we adopt the notation $\frac{1}{R}, \frac{1}{T}$ for the normal curvature and the geodesic torsion respectively). The lines of torsion are the double rays of this involutional system, and the minimal lines and the lines of curvature belong to this system.

	Inverse-orthogonal system([1]) $\frac{1}{2}\left(\frac{L}{E}+\frac{N}{G}\right)=\frac{M}{F}$	Orthogonal system $F=0$([2])	Isoclinal system $\frac{L}{E}=\frac{N}{G}$	Conjugate system $M=0$	Inverse-conjugate system $\frac{1}{2}\left(\frac{E}{L}+\frac{G}{N}\right)=\frac{F}{M}$
lines of torsion	**	*	*		
minimal lines	*	**	*		
lines of curvature	*	*	**	*	*
asymptotic lines			*	**	*
characteristic lines			*	*	**
	$\frac{1}{T_u}-\frac{1}{T_v}=0$	$\dfrac{\frac{1}{T_u}+\frac{1}{T_v}}{\frac{1}{R_u}-\frac{1}{R_v}}=0$	$\frac{1}{R_u}-\frac{1}{R_v}=0$	$\dfrac{\frac{R_u}{T_u}+\frac{R_v}{T_v}}{\frac{1}{R_u}-\frac{1}{R_v}}=0$	$\frac{R_u}{T_u}-\frac{R_v}{T_v}=$

2. In Part I we will treat the harmonic property concerning some surface-curves when the representation preserves any one of the five families above mentioned. Prof. P. Stäckel's theorems([3]) for the curves of equal normal curvature in the conformal and the conjunctive representations are generalized in several directions.

Considerable light has been thrown upon the theory of representation

([1]) Or the *complementary system.*

([2]) I take this opportunity to add a remark concerning *the converses of two theorems due to* Dupin. (i) Dupin proved that when the parametric curves form an orthogonal system,

$$\frac{1}{R_u}+\frac{1}{R_v}=\frac{1}{R_1}+\frac{1}{R_2},$$

R_1, R_2 being the radii of principal normal curvature. Conversely, if $2\infty^1$ curves satisfy this relation, they belong to an orthogonal system or an inverse-orthogonal system. (ii) Dupin also proved that when the parametric curves form a conjugate system,

$$R_u+R_v=R_1+R_2.$$

Conversely, if $2\infty^1$ curves satisfy this relation, they belong to a conjugate system or an inverse-conjugate system.

([3]) Stäckel, Beiträge zur Flächentheorie, Leipziger Berichte, 48 (1896), p. 489.

by the celebrated theorem due to Prof. Stäckel([1]): Any surface (excluding the so-called C-surface) admits no representation (excluding the similitude) which is conformal and conjunctive. We prove in Part II the fundamental theorem: The necessary and sufficient condition that any surface (excluding the C-surfaces) may be related to an infinite number of surfaces (which are not similar to one another) by the representations in which any two of the five families are preserved respectively or interchanged into each other is that the two families should belong to the same cycle. Thus the *conception of the cycle* is valuable in developing the theory of representation.

Part III deals with the necessary and sufficient condition that the relations between corresponding quantities such as normal curvatures or geodesic torsions, etc. in a representation should be independent of corresponding directions. As particular cases we give some formulas among the corresponding quantities for the representation in which two families of the same cycle are preserved respectively or interchanged into each other.

The secondary object of the present paper is to show how several important transformations in the literature (such as collineation, polar reciprocation, inversion, parallel transformation, Laguerre transformation, etc.) are included as special cases of our representation.

PART I.

3. It has been shown that the *harmonic property for certain surface-curves* is of fundamental importance in the theory of representation of surfaces([2]).

Now let us consider the representation of two surfaces S and S', in which the $2\infty^1$ curves

$$\phi_1\equiv A_1du^2+2A_2dudv+A_3dv^2=0,\ (\Delta(\phi_1)\equiv A_1A_3-A_2^2\neq 0),$$

on S correspond to the $2\infty^1$ curves

$$\phi_1'\equiv A_1'du^2+2A_2'dudv+A_3'dv^2=0,\ (\Delta(\phi_1')\equiv A_1'A_3'-A_2'^2\neq 0),$$

on S'. Then

$$A_1'=\lambda A_1,\ A_2'=\lambda A_2,\ A_3'=\lambda A_3.$$

([1]) Stäckel, loc. cit. See also Stäckel, Über Abbildungen, Math. Ann., 44 (1894), p. 553.

([2]) Lie, Untersuchungen über geodätische Curven, Math. Ann., 20 (1882), p. 553; Stäckel, Math. Ann., 44; Ogura, Notes on the representation of surfaces, Tôhoku Math. Journal, 10 (1916), p. 87.

The directions for which

$$\left(\frac{\phi_2'}{\phi_1'}\right)^2=\left(\frac{\phi_2}{\phi_1}\right)^2,$$

where

$$\phi_2\equiv B_1du^2+2B_2dudv+B_3dv^2,$$
$$\phi_2'\equiv B_1'du^2+2B_2'dudv+B_3'dv^2,$$

are given by

$$(B_1'\pm\lambda B_1)du^2+2(B_2'\pm\lambda B_2)dudv+(B_3'\pm\lambda B_3)dv^2=0.$$

In order that the two pairs of directions should be harmonic, it is necessary and sufficient that

$$(B_1'+\lambda B_1)(B_3'-\lambda B_3)+(B_3'+\lambda B_3)(B_1'-\lambda B_1)=2(B_2'+\lambda B_2)(B_2'-\lambda B_2);$$

and hence

$$B_1'B_3'-B_2'^2=\lambda^2(B_1B_3-B_2^2).$$

Consequently

$$\frac{B_1'B_3'-B_2'^2}{A_1'A_3'-A_2'^2}=\frac{B_1B_3-B_2^2}{A_1A_3-A_2^2},$$

i.e.

$$\frac{\Delta(\phi_2')}{\Delta(\phi_1')}=\frac{\Delta(\phi_2)}{\Delta(\phi_1)}.$$

The double rays of the involution determined by the two pairs of directions are

$$\begin{vmatrix} du^2 & -dudv & dv^2 \\ B_1 & B_2 & B_3 \\ B_1' & B_2' & B_3' \end{vmatrix}=0.$$

4. Now we proceed to show some applications of the above results. Throughout Part I we exclude the developable surface and the imaginary ruled surface having constant curvature([1]), besides the plane, the sphere and the minimal developable.

As the first case consider the *conformal representation* in which the minimal lines

$$f_1\equiv Edu^2+2Fdudv+Gdv^2=0$$

([1]) For this surface such quantities as R, T, etc. become fractional linear functions of $\frac{dv}{du}$.

correspond to the minimal lines

$$f_1'\equiv E'du^2+2F'dudv+G'dv^2=0.$$

(i) The normal curvature $\frac{1}{R}$ has the expression:

$$\frac{1}{R}=\frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}=\frac{f_2}{f_1}.$$

Hence in order that the two pairs of directions for which

$$\frac{1}{R'^2}=\frac{1}{R^2}$$

may be harmonic in the conformal representation, it is necessary and sufficient that

$$\frac{L'N'-M'^2}{E'G'-F'^2}=\frac{LN-M^2}{EG-F^2},$$

or

$$K'=K,$$

K, K' being the total curvatures at corresponding points.

The double rays of the involution determined by the above two pairs of directions form the *common conjugate system* of S and S'

$$\begin{vmatrix} du^2 & -dudv & dv^2 \\ L & M & N \\ L' & M' & N' \end{vmatrix}=0.$$

These results were already stated by Prof. Stäckel([2]).

(ii) The geodesic torsion $\frac{1}{T}$ has the expression:

$$\frac{1}{T}=-\frac{J(f_1,f_2)}{\sqrt{\Delta(f_1)}\cdot f_1}.$$

Hence in order that the two pairs of directions for which

$$\frac{1}{T'^2}=\frac{1}{T^2}$$

may be harmonic in the conformal representation, it is necessary and sufficient that

([1]) Stäckel, Leipziger Berichte, 48.

$$\frac{\Delta(J(f_1', f_2'))}{\Delta^2(f_1')} = \frac{\Delta(J(f_1, f_2))}{\Delta^2(f_1)},$$

i.e.

$$4K' - H'^2 = 4K - H^2,$$

H, H' being the mean curvature at corresponding points[1].

The double rays of the involution determined by the above two pairs of directions form the *common isoclinal system* of S and S'

$$\begin{vmatrix} du^2 & -2dudv & dv^2 \\ FL - EM & GL - EN & GM - FN \\ F'L' - E'M' & G'L' - E'N' & G'M' - F'N' \end{vmatrix} = 0.$$

(iii) Since

$$\frac{H}{2} - \frac{1}{R} = \frac{J(f_1, J(f_1, f_2))}{\Delta(f_1) \cdot f_1},$$

in order that the two pairs of directions for which

$$\left(\frac{H'}{2} - \frac{1}{R'}\right)^2 = \left(\frac{H}{2} - \frac{1}{R}\right)^2$$

may be harmonic in the conformal representation, it is necessary and sufficient that

$$\frac{\Delta(J(f_1', J(f_1', f_2')))}{\Delta^3(f_1')} = \frac{\Delta(J(f_1, J(f_1, f_2)))}{\Delta^3(f_1)},$$

i.e.

$$4K' - H'^2 = 4K - H^2.$$

Therefore if the two pairs of directions for which

$$\frac{1}{T'^2} = \frac{1}{T^2}$$

be harmonic in the conformal representation, then those for

$$\left(\frac{H'}{2} - \frac{1}{R'}\right)^2 = \left(\frac{H}{2} - \frac{1}{R}\right)^2$$

are also harmonic; and conversely.

The double rays of the involution determined by the two pairs of directions form the *common inverse-orthogonal system* of S and S'

(1) Ogura, Tôhoku Math Journal, 10, p. 87.

$$\begin{vmatrix} du^2 & -2dudv & dv^2 \\ \begin{vmatrix} E & EM - FL \\ F & \frac{1}{2}(EN - GL) \end{vmatrix} & \begin{vmatrix} E & EM - FL \\ G & FN - GM \end{vmatrix} & \begin{vmatrix} F & \frac{1}{2}(EN - GL) \\ G & FN - GM \end{vmatrix} \\ \begin{vmatrix} E' & E'M' - F'L' \\ F' & \frac{1}{2}(E'N' - G'L') \end{vmatrix} & \begin{vmatrix} E' & E'M' - F'L' \\ G' & F'N' - G'M' \end{vmatrix} & \begin{vmatrix} F' & \frac{1}{2}(E'N' - G'L') \\ G' & F'N' - G'M' \end{vmatrix} \end{vmatrix} = 0.$$

(iv) Since

$$\frac{H}{2R} - K = -\frac{J(f_2, J(f_1, f_2))}{\Delta(f_1) \cdot f_1},$$

in order that the two pairs of directions for which

$$\left(\frac{H'}{2R'} - K'\right)^2 = \left(\frac{H}{2R} - K\right)^2$$

may be harmonic in the conformal representation, it is necessary and sufficient that

$$\frac{\Delta(J(f_2', J(f_1', f_2')))}{\Delta^3(f_1')} = \frac{\Delta(J(f_2, J(f_1, f_2)))}{\Delta^3(f_1)},$$

i.e.

$$K'(4K' - H'^2) = K(4K - H^2).$$

The double rays of the involution determined by the two pairs of directions form the *common inverse-conjugate system* of S and S'

$$\begin{vmatrix} du^2 & -2dudv & dv^2 \\ \begin{vmatrix} L & EM - FL \\ M & \frac{1}{2}(EN - GL) \end{vmatrix} & \begin{vmatrix} L & EM - FL \\ N & FN - GM \end{vmatrix} & \begin{vmatrix} M & \frac{1}{2}(EN - GL) \\ N & FN - GM \end{vmatrix} \\ \begin{vmatrix} L' & E'M' - F'L' \\ M' & \frac{1}{2}(E'N' - G'L') \end{vmatrix} & \begin{vmatrix} L' & E'M' - F'L' \\ N' & F'N' - G'M' \end{vmatrix} & \begin{vmatrix} M' & \frac{1}{2}(E'N' - G'L') \\ N' & F'N' - G'M' \end{vmatrix} \end{vmatrix} = 0.$$

5. Next, we take the *conjunctive representation* in which the asymptotic lines $f_2 = 0$ correspond to the asymptotic lines $f_2' = 0$.

(i) In order that the two pairs of directions for which

$$\frac{1}{R'^2} = \frac{1}{R^2}$$

may be harmonic in the conjunctive representation, it is necessary and sufficient that

$$K'=K.$$

The double rays of the involution determined by these directions form the *common orthogonal system* (the *principal curves* of Tissot) of S and S'

$$\begin{vmatrix} du^2 & -dudv & dv^2 \\ E & F & G \\ E' & F' & G' \end{vmatrix}=0\ (^1).$$

(ii) Since

$$\frac{R}{T}=-\frac{J(f_1,f_2)}{\sqrt{\Delta(f_1)\cdot f_2}},$$

in order that the two pairs of directions for which

$$\left(\frac{R'}{T'}\right)^2=\left(\frac{R}{T}\right)^2$$

may be harmonic in the conjunctive representation, it is necessary and sufficient that

$$\frac{1}{K'}(4K'-H'^2)=\frac{1}{K}(4K-H^2).$$

The double rays of the involution determined by these directions form the common isoclinal system of S and S'.

(iii) Since

$$\left(\frac{H}{2}-\frac{1}{R}\right)R=-\frac{J(f_1,J(f_1,f_2))}{\Delta(f_1)\cdot f_2},$$

in order that the two pairs of directions for which

$$\left(\frac{H'}{2}-\frac{1}{R'}\right)^2 R'^2=\left(\frac{H}{2}-\frac{1}{R}\right)R^2$$

may be harmonic in the conjunctive representation, it is necessary and sufficient that

$$\frac{1}{K'}(4K'-H'^2)=\frac{1}{K}(4K-H^2).$$

The double rays of the involution determined by these directions form the common inverse-orthogonal system of S and S'.

(iv) Since

(1) Stäckel, loc. cit.

$$\left(\frac{H}{2R}-K\right)R=-\frac{J(f_2,J(f_1,f_2))}{\Delta(f_1)\cdot f_2},$$

in order that the two pairs of directions for which

$$\left(\frac{H'}{2R'}-K'\right)^2 R'^2=\left(\frac{H}{2R}-K\right)^2 R^2$$

may be harmonic in the conjunctive representation, it is necessary and sufficient that

$$4K'-H'^2=4K-H^2.$$

The double rays of the involution determined by these directions form the common inverse-conjugate system of S and S'.

6. Again, we consider *the representation in which the lines of curvature*

$$J(f_1,f_2)=0$$

correspond to the lines of curvature

$$J(f_1',f_2')=0.$$

(i) In order that the two pairs of directions for which

$$\frac{1}{T'^2}=\frac{1}{T^2}$$

may be harmonic in this representation, it is necessary and sufficient that

$$4K'-H'^2=4K-H^2.$$

The double rays of the involution determined by these directions form the common orthogonal system.

(ii) In order that the two pairs of directions for which

$$\left(\frac{R'}{T'}\right)^2=\left(\frac{R}{T}\right)^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$\frac{1}{K'}(4K'-H'^2)=\frac{1}{K}(4K-H^2).$$

The double rays of the involution determined by these directions form the common conjugate system.

(iii) *The two pairs of directions for which*

$$\left(\frac{H'}{2}-\frac{1}{R'}\right)^2 T'^2=\left(\frac{H}{2}-\frac{1}{R}\right)^2 T^2$$

are harmonic in this representation. For, since

$$\left(\frac{H}{2}-\frac{1}{R}\right)T=-\frac{J(f_1,\,J(f_1,f_2))}{\sqrt{\Delta(f_1)}\cdot J(f_1,f_2)},$$

the condition for the harmonic pencils becomes

$$\frac{\Delta^3(f_1')\cdot(4K'-H'^2)}{\Delta^3(f_1')\cdot(4K'-H'^2)}=\frac{\Delta^3(f_1)\cdot(4K-H^2)}{\Delta^3(f_1)\cdot(4K-H^2)},$$

which is an identity.

The double rays of the involution determined by these directions form the common inverse-orthogonal system.

(iv) Since

$$\left(\frac{H}{2R}-K\right)T=\frac{J(f_2,\,J(f_1,f_2))}{\sqrt{\Delta(f_1)}\cdot J(f_1,f_2)},$$

in order that the two pairs of directions for which

$$\left(\frac{H'}{2R'}-K'\right)^2T'^2=\left(\frac{H}{2R}-K\right)^2T^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$K'=K.$$

The double rays of the involution determined by these directions form the common inverse-conjugate system.

7. Further, we consider *the representation in which the lines of torsion*

$$J(f_1,\,J(f_1,f_2))=0$$

correspond to the lines of torsion

$$J(f_1',\,J(f_1',f_2'))=0\,.$$

(i) *The two pairs of directions for which*

$$\left(\frac{H'}{2}-\frac{1}{R'}\right)^2T'^2=\left(\frac{H}{2}-\frac{1}{R}\right)^2T^2$$

are harmonic in this representation. And the double rays of the involution determined by these pairs form the common isoclinal system.

(ii) In order that the two pairs of directions for which

$$\left(\frac{H'}{2}-\frac{1}{R'}\right)^2=\left(\frac{H}{2}-\frac{1}{R}\right)^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$4K'-H'^2=4K-H^2.$$

The double rays of the involution determined by these directions form the common orthogonal system.

(iii) In order that the two pairs of directions for which

$$\left(\frac{H'}{2}-\frac{1}{R'}\right)^2R'^2=\left(\frac{H}{2}-\frac{1}{R}\right)^2R^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$\frac{1}{K'}(4K'-H'^2)=\frac{1}{K}(4K-H^2)\,.$$

The double rays of the involution determined by these directions form the common conjugate system.

(iv) Since

$$\frac{2KR-H}{2-HR}=\frac{J(f_2,\,J(f_1,f_2))}{J(f_1,\,J(f_1,f_2))},$$

in order that the two pairs of directions for which

$$\left(\frac{2K'R'-H'}{2-H'R'}\right)^2=\left(\frac{2KR-H}{2-HR}\right)^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$K'=K\,.$$

The double rays of the involution determined by these directions form the common inverse-conjugate system.

8. Lastly we consider *the representation in which the characteristic lines*

$$J(f\,,\,J(f_1,f_2))=0$$

correspond to the characteristic lines

$$J(f_2',\,J(f_1',f_2'))=0\,.$$

(i) In order that the two pairs of directions for which

$$\left(\frac{H'}{2R'}-K'\right)^2T'^2=\left(\frac{H}{2R}-K\right)^2T^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$K'=K\,.$$

The double rays of the involution determined by these directions form the common isoclinal system.

(ii) In order that the two pairs of directions for which

$$\left(\frac{H'}{2R'}-K'\right)^2=\left(\frac{H}{2R}-K\right)^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$K'(4K'-H'^2)=K(4K-H^2)\,.$$

The double rays of the involution determined by these directions form the common orthogonal system.

(iii) In order that the two pairs of directions for which

$$\left(\frac{H'}{2R'}-K'\right)^2 R'^2=\left(\frac{H}{2R}-K\right)^2 R^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$4K'-H'^2=4K-H^2.$$

The double rays of the involution determined by these directions form the common conjugate system.

(iv) In order that the two pairs of directions for which

$$\left(\frac{2K'R'-H'}{2-H'R'}\right)^2=\left(\frac{2KR-H}{2-HR}\right)^2$$

may be harmonic in this representation, it is necessary and sufficient that

$$K'=K\,.$$

The double rays of the involution determined by these directions form the common inverse-orthogonal system.

It is thus apparent that *the four Weingarten surfaces in which*

(α) $K=$const. (i.e. $R_1R_2=$const.)

(β) $4K-H^2=$const. (i.e. $\dfrac{1}{R_1}-\dfrac{1}{R_2}=$const.)(1),

(γ) $\dfrac{1}{K}(4K-H^2)=$const. (i.e. $\dfrac{R_1}{R_2}=$const.)(2),

(δ) $K(4K-H^2)=$const. (i.e. $\dfrac{1}{\sqrt{R_1R_2}}\left(\dfrac{1}{R_1}-\dfrac{1}{R_2}\right)=$const.(3)).

respectively are worthy of peculiar notice in the theory of representation.

(1) This surface is characterised by the property: *The lines of torsion on this surface correspond to the asymptotic lines on both sheets of the evolute; and conversely.*

For another property, see Mannheim, Principes et développements de géométrie cinématique (1894), p. 236–7.

(2) This surface, belonging to the C-surface, was studied by Prof. Stäckel in Leipziger Berichte, 48 (1896), p. 491; 50 (1898), p. 10. See also Ribaucour, Comptes rendus, Paris, 74 (1872), p. 1399; Mannheim, loc. cit.

(3) This surface seems as yet to have received no attention.

PART II.

A. Representations of the same cycle.

9. In this Part II we consider the surfaces only, excluding the plane, the sphere and the minimal developable.

(i) When two surfaces S, S' are represented in such a way that the lines of torsion on S correspond to the lines of torsion on S' and the minimal lines on S correspond to the minimal lines on S', the isoclinal system (the involution determined by the lines of torsion and the minimal lines) on S corresponds to the isoclinal system on S'; and therefore the lines of curvature (the double lines of the isoclinal system) on S correspond to the lines of curvature on S'.

If the lines of curvature be taken as the parametric curves on both surfaces, then

$$F=0\,,\quad M=0\,,\quad F'=0\,,\quad M'=0\,.$$

In order that the minimal lines

$$Edu^2+Gdv^2=0\,,\quad E'du^2+G'dv^2=0$$

on these two surfaces may correspond, we must have

$$\frac{E'}{E}=\frac{G'}{G}\,;$$

and also in order that the lines of torsion

$$Edu^2-Gdv^2=0,\quad E'du^2-G'dv^2=0$$

may correspond, we must have

$$\frac{E'}{E}=\frac{G'}{G}\,,$$

which is the same relation as before.

As the analysis is manifestly reversible we get the result: The representation, with preservation of the lines of torsion and the minimal lines respectively, preserves the lines of curvature also. The necessary and sufficient condition that two surfaces may be represented in such a way is

$$(1)\quad \begin{cases} F=0\,,\quad M=0\,,\quad F'=0\,,\quad M'=0\,, \\ \dfrac{G}{E}=\dfrac{G'}{E'}\,, \end{cases}$$

the lines of curvature being taken for the parametric curves.

(ii) Next, when two surfaces are represented in such a way that the minimal lines and the lines of curvature on a surface correspond to the minimal lines and the lines of curvature respectively on the other, the inverse-orthogonal system and the lines of torsion on a surface correspond to the inverse-orthogonal system and the lines of torsion on the other respectively. If the lines of torsion be taken as the parametric curves on both surfaces, the condition for such a representation is given by

$$(2)\qquad F=0\,,\ F'=0\,,\ \frac{G}{E}=\frac{G'}{E'}=\frac{N}{L}=\frac{N'}{L'}.$$

(iii) Lastly, when two surfaces are represented in such a way that the lines of curvature and the lines of torsion on a surface correspond to the lines of curvature and the lines of torsion on the other respectively, the orthogonal system and the minimal lines on a surface correspond to the orthogonal system and the minimal lines on the other respectively. If the minimal lines be taken as the parametric curves on both surfaces, the condition for such a representation is given by

$$(3)\qquad E=0\,,\quad G=0\,,\quad E'=0\,,\quad G'=0\,,\qquad \frac{N}{L}=\frac{N'}{L'}.$$

Recalling the definition of the cycle in § 1, we arrive at the theorem:

The representation with preservation of any two families of the first cycle preserves the third family of this cycle also; this representation preserves all involutional systems belonging to this cycle.

Such a representation will be called the *representation of the first cycle.* The *inversion*(1) is the most well known example of this representation.

10. Now we proceed to prove the theorem:

Any surface admits of a continuous representation of the first cycle.

Let the fundamental quantities of a given surface S be

$$E=0,\ F=F,\ G=0,\ L=L,\ M=M,\ N=N.$$

Then these quantities satisfy the equations of Gauss and Codazzi

(1) Compare with Ogura, On the differential geometry of inversion, Tôhoku Math. Journal, 9 (1916), p. 216.

$$(4)\qquad \begin{cases} LN-M^2=\dfrac{\partial^2 F}{\partial u\partial v}-\dfrac{1}{F}\dfrac{\partial F}{\partial u}\dfrac{\partial F}{\partial v}, \\ \dfrac{\partial L}{\partial v}-\dfrac{\partial M}{\partial u}=-\dfrac{M}{F}\dfrac{\partial F}{\partial u}, \\ \dfrac{\partial N}{\partial u}-\dfrac{\partial M}{\partial v}=-\dfrac{M}{F}\dfrac{\partial F}{\partial v}. \end{cases}$$

If S and S' be related by our representation, it follows from (3) that the fundamental quantities of S' must take the form

$$(5)\qquad E'=0,\ F'=F',\ G'=0,\ L'=\lambda L,\ M'=M',\ N'=\lambda N.$$

Hence the three functions λ, F', M' must satisfy the equations

$$\lambda^2 LN-M'^2=\frac{\partial^2 F'}{\partial u\partial v}-\frac{1}{F'}\frac{\partial F'}{\partial u}\frac{\partial F'}{\partial v},$$

$$\lambda\frac{\partial L}{\partial v}+L\frac{\partial \lambda}{\partial v}-\frac{\partial M'}{\partial u}=-\frac{M'}{F'}\frac{\partial F'}{\partial u},$$

$$\lambda\frac{\partial N}{\partial u}+N\frac{\partial \lambda}{\partial u}-\frac{\partial M'}{\partial v}=-\frac{M'}{F'}\frac{\partial F'}{\partial v};$$

or by (4)

$$(6)\qquad \begin{cases} \dfrac{\partial^2 F'}{\partial u\partial v}-\dfrac{1}{F'}\dfrac{\partial F'}{\partial u}\dfrac{\partial F'}{\partial v}+M'^2=\lambda^2\Big(\dfrac{\partial^2 F}{\partial u\partial v}-\dfrac{1}{F}\dfrac{\partial F}{\partial u}\dfrac{\partial F}{\partial v}+M^2\Big), \\ \dfrac{\partial M'}{\partial u}-\dfrac{M'}{F'}\dfrac{\partial F'}{\partial u}=\lambda\Big(\dfrac{\partial M}{\partial u}-\dfrac{M}{F}\dfrac{\partial F}{\partial u}\Big)+L\dfrac{\partial \lambda}{\partial v}, \\ \dfrac{\partial M'}{\partial v}-\dfrac{M'}{F'}\dfrac{\partial F'}{\partial v}=\lambda\Big(\dfrac{\partial M}{\partial v}-\dfrac{M}{F}\dfrac{\partial F}{\partial v}\Big)+N\dfrac{\partial \lambda}{\partial u}. \end{cases}$$

Since this system of three partial differential equations in three unknown functions is satisfied by

$$\lambda=1\,,\quad F'=F,\quad M'=M,$$

and moreover has the complete solution (λ, F', M') which contains arbitrary functions, the theorem has been proved.

11. Here we add some examples:

(i) It is easily seen from equations (1) that *the surface which is related to an isothermic surface*(1) *by the representation of the first cycle should be necessarily isothermic.*

Moreover, *any two given isothermic surfaces can be related by the re-*

(1) For example, quadrics, surfaces of revolution and surfaces of constant mean curvature are isothermic surfaces.

presentation of the first cycle. For, by the suitable choice of parameters (u, v) on S we have

$$F=0, \quad M=0, \quad E=G;$$

and similarly for (u', v') on S',

$$F'=0, \quad M'=0, \quad E'=G'.$$

Hence if we put

$$u'=u, \qquad v'=v,$$

a representation of the first cycle is obtained.

An excellent example of such a representation is the *problem of Christoffel*(1).

(ii) In general, the necessary and sufficient condition that any two given surfaces may be related by a representation of the first cycle is already obtained by Prof. R. Rothe(2) in an analytical manner.

(iii) A well known particular problem concerning the representation of the first cycle is to determine the surfaces which can be deformed with preservation of their lines of curvature(3).

12. Now we pass to consider the representation which interchanges two families belonging to the first cycle.

Any surface admits of an infinite number of representations which interchange any two families belonging to the first cycle.

(i) Firstly we will begin with the representation which interchanges the lines of curvature and the lines of torsion. If the lines of curvature be taken as the parametric curves on S and the lines of torsion be taken as those on S', then

$$F=0, \qquad M=0, \qquad F'=0, \qquad \frac{L'}{E'}=\frac{N'}{G'}.$$

Since the equations to the lines of torsion on S is

$$Edu^2-Gdv^2=0$$

and that to the lines of curvature on S' is

$$E'du^2-G'dv^2=0,$$

it must be

(1) Christoffel, Über einige allgemeine Eigenschaften der Minimumsflächen, Journal für Math., 67 (1867), p. 218; Darboux, Théorie des surfaces, 2 (1889), p. 239.

(2) R. Rothe, Über die Inversion einer Flächen, u.s.w., Math. Ann., 72 (1912), p. 57.

(3) Bonnet-Hazzidakis' problem.

$$\frac{E'}{E}=\frac{G'}{G}.$$

But since the equations of the minimal lines on S and S' are

$$Edu^2+Gdv^2=0, \qquad E'du^2+G'dv^2=0$$

respectively, the minimal lines are preserved.

When the minimal lines are taken as the parametric curves on both surfaces,

$$(7) \qquad E=0, \qquad G=0, \qquad E'=0, \qquad G'=0;$$

and the equations to the lines of curvature and the lines of torsion on S are

$$Ldu^2-Ndv^2=0, \qquad Ldu^2+Ndv^2=0$$

respectively and those on S' are

$$L'du^2-N'dv^2=0, \qquad L'du^2+N'dv^2=0$$

respectively. Hence we must have

$$(7)' \qquad \frac{L'}{L}=-\frac{N'}{N}.$$

Now let the fundamental quantities of the given surface S be

$$E=0, \quad F=F, \quad G=0, \quad L=L, \quad M=M, \quad N=N;$$

then those for S' must take the form

$$E'=0, \quad F'=F', \quad G'=0, \quad L'=\lambda L, \quad M'=M', \quad N'=-\lambda N.$$

In a similar way as in §10 we see that the three functions λ, F', M' are determined as the solution of the three partial differential equations:

$$(8) \quad \begin{cases} \dfrac{\partial^2 F'}{\partial u \partial v}-\dfrac{1}{F'}\dfrac{\partial F'}{\partial u}\dfrac{\partial F'}{\partial v}+M'^2=-\lambda^2\left(\dfrac{\partial^2 F}{\partial u\partial v}-\dfrac{1}{F}\dfrac{\partial F}{\partial u}\dfrac{\partial F}{\partial v}+M^2\right), \\ \dfrac{\partial M'}{\partial u}-\dfrac{M'}{F'}\dfrac{\partial F'}{\partial u}=\lambda\left(\dfrac{\partial M}{\partial u}-\dfrac{M}{F}\dfrac{\partial F}{\partial u}\right)+L\dfrac{\partial \lambda}{\partial v}, \\ \dfrac{\partial M'}{\partial v}-\dfrac{M'}{F'}\dfrac{\partial F'}{\partial v}=-\lambda\left(\dfrac{\partial M}{\partial v}-\dfrac{M}{F}\dfrac{\partial F}{\partial v}\right)-N\dfrac{\partial \lambda}{\partial u}. \end{cases}$$

Since the complete solution contains arbitrary functions, there exists an infinite number of S'.

In similar ways we can treat

(ii) the representation which interchanges the lines of curvature and the minimal lines, that is,

$$(9) \qquad F=0, \quad F'=0, \quad \frac{G}{E}=-\frac{G'}{E'}=\frac{N}{L}=-\frac{N'}{L'};$$

or

(iii) the representation which interchanges the minimal lines and the lines of torsion, that is,

$$(10) \qquad F=0, \quad M=0, \quad F'=0, \quad M'=0, \quad \frac{E'}{E}=-\frac{G'}{G}.$$

For an example let us consider two isothermic surfaces S and S'. When the lines of curvature are taken as the parameters (u, v) and (u', v'), we may put

$$F(u, v)=0, \qquad M(u, v)=0, \qquad ds^2=\phi(u, v)\,(du^2+dv^2),$$
$$F'(u', v')=0, \qquad M'(u', v')=0, \qquad ds'^2=\phi'(u', v')\,(du'^2+dv'^2),$$

ds, ds' standing for the linear elements. If we apply the correspondence

$$u'=u, \qquad v'=\sqrt{-1}\,v,$$

then

$$F'(u, v)=0, \quad M'(u, v)=0, \quad ds'^2=\phi'(u, \sqrt{-1}\,v)(du^2-dv^2).$$

Consequently it follows from (10) that *any two isothermic surfaces may be related by the representation which interchanges the minimal lines and the lines of torsion.*

13. When two surfaces S and S' are represented in such a way that the asymptotic lines and the characteristic lines on both surfaces correspond respectively, the lines of curvature on both surfaces correspond also. If the lines of curvature be taken as the parametric curves on both surfaces, we have the conditions for the representation

$$(11) \qquad F=0, \quad M=0, \quad F'=0, \quad M'=0, \quad \frac{L'}{L}=\frac{N'}{N}.$$

Now we can prove the theorem: *Any surface admits of a continuous representation in which the asymptotic lines and the characteristic lines are preserved respectively.*

Let the fundamental quantities of the given surface S be

$$E=E, \quad F=0, \quad G=G, \quad L=L, \quad M=0, \quad N=N.$$

If S and S' be related by such a representation, the fundamental quantities of S' must take the form

$$E'=E', \quad F'=0, \quad G'=G', \quad L'=\lambda L, \quad M'=0, \quad N'=\lambda N.$$

But the three functions λ, E', G' must satisfy the equations of Gauss and Codazzi which may be written

$$(12)\left\{\begin{aligned}
&2\left(\frac{\partial^2 E'}{\partial v^2}+\frac{\partial^2 G'}{\partial u^2}\right)-\frac{1}{E'}\left\{\left(\frac{\partial E'}{\partial v}\right)^2+\frac{\partial E'}{\partial u}\frac{\partial G'}{\partial u}\right\}\\
&\qquad -\frac{1}{G'}\left\{\left(\frac{\partial G'}{\partial u}\right)^2+\frac{\partial E'}{\partial v}\frac{\partial G'}{\partial v}\right\}=\lambda^2\left[2\left(\frac{\partial^2 E}{\partial v^2}+\frac{\partial^2 G}{\partial u^2}\right)\right.\\
&\qquad \left.-\frac{1}{E}\left\{\left(\frac{\partial E}{\partial v}\right)^2+\frac{\partial E}{\partial u}\frac{\partial G}{\partial u}\right\}-\frac{1}{G}\left\{\left(\frac{\partial G}{\partial u}\right)^2+\frac{\partial E}{\partial v}\frac{\partial G}{\partial v}\right\}\right],\\
&\frac{\partial E'}{\partial v}\left(\frac{L}{E'}+\frac{N}{G'}\right)-2L\frac{1}{\lambda}\frac{\partial \lambda}{\partial v}=\frac{\partial E}{\partial v}\left(\frac{L}{E}+\frac{N}{G}\right),\\
&\frac{\partial G'}{\partial u}\left(\frac{L}{E'}+\frac{N}{G'}\right)-2N\frac{1}{\lambda}\frac{\partial \lambda}{\partial u}=\frac{\partial G}{\partial u}\left(\frac{L}{E}+\frac{N}{G}\right).
\end{aligned}\right.$$

Since this system of three partial differential equations in three unknowns is satisfied by

$$\lambda=1, \qquad E'=E, \qquad G'=G$$

and has the complete solution (λ, E', G') which contains arbitrary functions, the theorem has been proved.

Also the reasoning we have used in § 9 leads to the theorem: *The representation which preserves any two families belonging to the second cycle preserves the third family of this cycle also; this representation preserves any involutional system belonging to the second cycle.*

Such a representation will be called the *representation of the second cycle.*

14. Recently Prof. A. E. Young[1] has dealt with the two kinds of surfaces whose lines of curvature are isothermic conjugate or associate isothermic conjugate. For the sake of brevity we will call the former the *Young surface of the first type*[2] and the latter the *Young surface of the second type*[3].

It is easily seen from equations (11) that *the surface which is related*

(1) A. E. Young, On the determination of a certain class of surface, American Journal of Mathamatics, 39 (1917), p. 75.

(2) For example, the quadric having positive curvature, the surface of revolution having positive curvature and the surface of constant positive curvature belong to this type.

(3) For example, the quadric having negative curvature, the surface of revolution having negative curvature and the surface of constant negative curvature belong to this type.

to a Young surface by the representation of the second cycle should be necessarily the Young surface of the same type.

Moreover *any two Young surfaces S, S' of the same type may be related by the representation of the second cycle.* For, by the suitable choice of parameters (u, v) on S it may be written

$$F=0, \quad M=0, \quad L=N \quad (\text{or } L=-N);$$

and similarly for (u', v') on S'

$$F'=0, \quad M'=0, \quad L'=N' \quad (\text{or } L'=-N').$$

Hence if we put

$$u'=u, \quad v'=v,$$

a representation of the second cycle is obtained.

An excellent example of such a representation is *the Bäcklund transformation for pseudospherical surfaces*(1).

15. In a similar way as in § 12 we can prove the theorem:

Any surface admits of an infinite number of representations which interchange any two families belonging to the second cycle.

For example, *the surface which is related to a Young surface by the representation which interchanges the asymptotic lines and the characteristic lines should be necessarily the Young surface of the different type.* Moreover *any two Young surfaces of different types may be related by the representation which interchanges the asymytotic lines and the characteristic lines.*

Prof. A. E. Young has stated the particular case of this theorem(2): All isothermic surfaces S and S' connected by the Bour-Darboux theorem(3) are associates of one another, the asymptotic lines of one corresponding to the characteristic lines on the other, and vice versa(4).

B. Representations of different cycles.

16. I. Now we proceed to prove the theorem:

(1) See Bianchi, Vorlesungen über Differentialgeometrie, 1. Aufl. (1899), pp. 451–455.

(2) A. E. Young, loc. cit.

(3) Darboux, loc. cit., p. 243.

(4) By the line-sphere transformation of Lie, the asymptotic lines on a surface S correspond to the lines of curvature on the transformed surface S'; but the lines of curvature on S do not correspond to the asymptotic lines on S'.

There exists, in general, no representation, (excluding the similitude) which preserves any two families(1) *belonging to different cycles.*

If there exist the representation which preserves the lines of torsion and the asymptotic lines, it preserves the isoclinal system (the involutional system determined by the lines of torsion and the asymptotic lines) and consequently the lines of curvature. When the lines of curvature are taken as the parametric curves on both surfaces S, S', the condition for the representation becomes

$$(13) \quad \begin{cases} F=0, \quad M=0, \quad F'=0, \quad M'=0, \\ \dfrac{E'}{E}=\dfrac{G'}{G}, \quad \dfrac{L'}{L}=\dfrac{N'}{N}. \end{cases}$$

In quite similar ways we have the same condition as above for the representation which preserves any other pair of two families (except the lines of curvature) belonging to different cycles. But Prof. Stäckel(2) showed that any surface (excluding the so-called C-surface)(3) admits no representation (excluding the similitude) which preserves the minimal lines and the asymptotic lines. Hence equations (13) are inconsistent in general. Thus the theorem has been proved.

Moreover, *let two involutional systems I_1 and I_2 belong to different cycles. Then there exists, in general, no representation (excluding the similitude) in which $2\infty^1$ curves Γ_1 of I_1 and $2\infty^1$ curves Γ_2 of I_2 correspond.*

If such a representation may exist, it preserves the common curves I_{12} of the two involutional systems I_1, I_2. But it follows from § 1 that the common curves I_{12} are the lines of curvature. Now the double rays of the involution determined by Γ_1 and I_{12} are preserved, and hence the double rays D_1 of I_1 are preserved; similarly the double rays D_2 of I_2 are preserved. This is impossible, for D_1 and D_2 are two families of curves belonging to different cycles.

II. On the contrary we can state the theorem concerning the C-surface:

Any C-surface and only this admits of the representation (excluding the similitude) in which all the five families of curves and all the five involutional systems are preserved.

(1) Of course we can not take the lines of curvature.

(2) Stäckel, Leipziger Berichte, 48.

(3) For example, the surface whose asymptotic lines form infinitesimal rhombi having constant angles (that is, the surface (γ) in § 8) and especially the minimal surface belong to the C-surface. Compare with Young, loc. cit., especially pp. 80–81.

This theorem is readily proved from the consideration of the representation in which the minimal lines and the asymptotic lines are preserved.

A well known example of such a representation is the *Goursat transformation for minimal surfaces*(1).

17. Now we prove the theorem:

There exists, in general, no representation which interchanges the minimal lines and the asymptotic lines.

To prove this it is sufficient to consider the developable surface S, where

$$LN-M^2=0.$$

If S' be related to S by a representation which interchanges the minimal lines and the asymptotic lines, then

$$\frac{E'}{L}=\frac{F'}{M}=\frac{G'}{N};$$

and hence

$$E'G'-F'^2=0.$$

Therefore the surface S' must be the minimal developable which is excluded.

Moreover, *although we exclude the developable surface*, the following theorem may be stated:

Any surface S (excluding the C-surface) can not be related to an infinite number of surfaces S' (unless all these are similar) by the representation which interchanges the minimal lines and the asymptotic lines.

If S' be one of the surfaces related to a given surface S by such a representation, we should have the relation

$$\frac{E'}{L}=\frac{F'}{M}=\frac{G'}{N}=\lambda, \qquad \frac{L'}{E}=\frac{M'}{F}=\frac{N'}{G}=\mu.$$

Now two cases arise:

I. λ and μ may be written

$$\lambda=l.f(u,v), \qquad \mu=m.\varphi(u,v),$$

where l, m are *arbitrary* constants, and f, φ are *definite* functions which contain neither arbitrary constants nor arbitrary functions;

(1) Goursat, Acta Mathematica, 11 (1888), p. 135.

II. One of f and φ, at least, may contain an *arbitrary* constant (which is not the multiplier) or an *arbitrary* function at least.

We will consider *Case II* first. Let S_1' and S_2' be the two surfaces which correspond to any two values of the arbitrary constant (or any two forms of the arbitrary function). Then S_1' and S_2' are related to each other by a representation which preserves the minimal lines and the asymptotic lines. But since the representation is not a similitude, S_1' and S_2' must be C-surfaces. Consequently all the surfaces S' belong to the C-surface. Now suppose that we apply the representation of this type II to the surface S_1'. Then, by the similar reasoning, all the surfaces related to S_1' by this representation should belong to the C-surface.

Hence *S is necessarily a C-surface.*

In order to prove the *existence* of such a representation, let us take the general minimal surface. Then the fundamental quantities may be written

$$E=G, \qquad F=0, \qquad L=0, \qquad N=0.$$

Hence the fundamental quantities of S' must take the form

$$E'=0, \qquad F'=F', \qquad G'=0,$$
$$L'=\lambda L, \qquad M'=0, \qquad N'=\lambda N.$$

Since the Codazzi equations are reducible to

$$\frac{\partial(\lambda E)}{\partial u}=0, \qquad \frac{\partial(\lambda E)}{\partial v}=0,$$

it follows that

$$\lambda E=\text{arbitr. const.}=k, \text{ say.}$$

And therefore the function F' is determined by the Gauss equation

$$\frac{\partial^2 \log F'}{\partial u \partial v}=\frac{k^2}{F'},$$

which may be written

$$\frac{\partial^2 \log\left(\frac{1}{F'}\right)}{\partial u \partial v}=-k^2\left(\frac{1}{F'}\right).$$

This equation is of the Liouville type, and its complete solution is of the form

$$F'=-\frac{k^2}{2}\cdot\frac{\{\varphi(u)+\psi(v)\}^2}{\varphi'(u)\,\psi'(v)},$$

where $\varphi(u)$ is an arbitrary function of u alone and $\psi(v)$ of v alone; and $\varphi'(v)$, $\psi'(v)$ denote the derivatives with respect to the corresponding arguments.

It is easily seen that S' is the general minimal surface. Hence *a minimal surface admits of an infinite number of representations which interchange the minimal lines and the asymptotic lines; and all the corresponding surfaces are of the same kind.*

This result can be generalized for the surface whose asymptotic lines form infinitesimal rhombi having constant angles.

18. Let us now consider *Case I.* This case may occur, even when S does not belong to the C-surface. But in this case all the surfaces S' are similar.

In order to prove the *existence of the case,* consider the *imaginary ruled surface S of constant curvature*[1], which does not belong to the C-surface. When the asymptotic lines are taken as the parametric curves, the fundamental quantities of S are of the form:

$$E=0, \qquad F=-\frac{2}{(u+v)^2}, \qquad G=g(v),$$

$$L=0, \qquad M=-\frac{2}{(u+v)^2}, \qquad N=0,$$

where $g(v)$ is any function of v alone. If S' be related to the surface S by the representation which interchanges the minimal lines and the asymptotic lines, the fundamental quantities of S' must take the form:

$$E'=0, \qquad F'=F', \qquad G'=0,$$

$$L'=0, \qquad M'=-\frac{2}{(u+v)^2}e^{w}, \qquad N'=g(v)\,e^{w},$$

where $e^{w}\neq 0$, since we exclude the case in which S' is a plane. Then the equations of Gauss and Codazzi for S' are

$$(14) \qquad \frac{1}{F'}\frac{\partial F'}{\partial u}\frac{\partial F'}{\partial v}-\frac{\partial^2 F'}{\partial u\partial v}=\frac{4}{(u+v)^4}e^{2w},$$

$$(15) \qquad \frac{1}{F'}\frac{\partial F'}{\partial u}=\frac{\partial w}{\partial u}-\frac{2}{u+v},$$

$$(16) \qquad \frac{1}{F'}\frac{\partial F'}{\partial v}=\frac{\partial w}{\partial v}-\frac{2}{u+v}+\frac{1}{2}g(v).(u+v)^2\frac{\partial w}{\partial u}.$$

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(1) Stäckel, Leipziger Berichte. 48 (1896); Scheffers, Einführung in die Theorie der Flächen, 1. Aufl. (1902), p. 113, p. 227.

Integrating (15) we have

$$(17) \qquad F'=\frac{e^{w}}{(u+v)^2}\phi(v),$$

$\phi(v)$ being an arbitrary function of v alone. From (16) and (17) we get

$$(18) \qquad \frac{1}{2}g(v)\cdot(u+v)^2\frac{\partial w}{\partial u}=\frac{1}{\phi(v)}\frac{d\phi(v)}{dv}.$$

I. Now we assume that $\phi(v)$ is not a constant.

Integrating (18)

$$(19) \qquad w=-\frac{2}{g(v)\phi(v)}\frac{d\phi(v)}{dv}\cdot\frac{1}{u+v}+\varphi(v),$$

$\varphi(v)$ being an arbitrary function of v alone. Hence w, $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ are rational functions of $u+v$.

From (17)

$$\frac{\partial F'}{\partial u}=e^{w}\phi(v)\left[\frac{\partial w}{\partial u}(u+v)^{-2}-2(u+v)^{-3}\right]$$

$$\equiv e^{w}\cdot \Psi(u+v, v),$$

$$\frac{\partial F'}{\partial v}=e^{w}\cdot \Psi_1(u+v, v),$$

$$\frac{\partial^2 F'}{\partial u\partial v}=e^{w}\cdot \Psi_2(u+v, v),$$

where Ψ, Ψ_1, Ψ_2 are rational functions of $u+v$. Hence (14) takes the form

$$(20) \qquad e^{\Psi_3(u+v,\,v)}=\Psi_4(u+v, v),$$

where Ψ_3, Ψ_4 are rational functions of $u+v$, and

$$\Psi_3\equiv-\frac{2}{g(v)\phi(v)}\frac{d\phi(v)}{dv}\frac{1}{u+v}+\varphi(v), \qquad \left(\frac{d\phi(v)}{dv}\neq 0\right).$$

Therefore whatever forms $g(v)$, $\phi(v)$, and $\varphi(v)$ may take, (20) can not be an identity.

II. Hence we must consider the case where $\phi(v)$ is constant.

In this case

$$w=\varphi(v),$$

$\varphi(v)$ being any function of v alone. Hence by (17)

(21) $$F'=\frac{1}{(u+v)^2}\xi(v),$$

$\xi(v)$ being a function of v alone. From (16) and (21) we obtain

$$\frac{1}{\xi(v)}\frac{d\xi(v)}{dv}=\frac{dw}{dv},$$

so that

$$e^w=-\frac{1}{2k}\xi(v),$$

k being an arbitrary constant. Hence (14) becomes

$$\frac{1}{2k^2}\xi^2(v)+\xi(v)=0.$$

Since $\xi(v)\neq 0$,

$$\xi(v)=-2k^2, \quad e^w=k.$$

Therefore the fundamental quantities of S' are

$$E'=0, \quad F'=\frac{-2k^2}{(u+v)^2}, \quad G'=0,$$

$$L'=0, \quad M'=\frac{-2k}{(u+v)^2} \quad N'=-k\,g(v)\ (^1);$$

so that a system of the asymptotic lines $v=$const. coincide with a system of the minimal lines, and the total curvature is

$$K'=\frac{1}{k^2}.$$

Hence S' is an imaginary ruled surface having constant curvature also.

19. *Quite the same results hold good for the representation which interchanges the lines of torsion and the characteristic lines.*

If such a representation may exist, let us take the lines of torsion as the parametric curves on S and the characteristic lines as those on S'. Then

$$F=0, \ \frac{L}{E}=\frac{N}{G}, \ M'=0, \ \frac{L'}{E'}=\frac{N'}{G'}.$$

---

(1) Therefore

$$\frac{E'}{L}=\frac{F'}{M}=\frac{G'}{N}=k^2, \quad \frac{L'}{E}=\frac{M'}{F}=\frac{N'}{G}=k.$$

Since the characteristic lines

$$EMdu^2+(GL+EN)dudv+GMdv^2=0$$

on S correspond to the lines of torsion

$$F'L'du^2+(G'L'+E'N')dudv+F'N'dv^2=0$$

on S', we must have

$$\frac{L'}{E}=\frac{N'}{G},$$

unless both S and S' are spheres, in which case $M=0$, $F'=0$. Therefore the lines of curvature

$$Edu^2-Gdv^2=0$$

on S correspond to the lines of curvature

$$L'du^2-N'dv^2=0$$

on S'.

If the lines of curvature be taken as the parametric curves on both surfaces, the equations to the lines of torsion and the characteristic lines are

$$Edu^2-Gdv^2=0, \quad Ldu^2-Ndv^2=0$$

on S respectively and

$$E'du^2-G'dv^2=0, \quad L'du^2-N'dv^2=0$$

on S' respectively. Hence it must be

$$\frac{L'}{E}=\frac{N'}{G}, \quad \frac{L}{E'}=\frac{N}{G'}.$$

But since the equations to the minimal lines and the asymptotic lines are

$$Edu^2+Gdv^2=0, \quad Ldu^2+Ndv^2=0$$

on S respectively and

$$E'du^2+G'dv^2=0, \quad L'du^2+N'dv^2=0$$

on S' respectively, the two families must be interchanged, which is the case treated in § 17.

20. *There exists, in general, no representation which interchanges the lines of torsion and the asymptotic lines.*

If such a representation may exist, take the asymptotic lines as the parametric curves on S and the lines of torsion as those on S'. Then

$$L=0, \quad N=0,$$

$$(22) \qquad F'=0, \quad \frac{L'}{E'}=\frac{N'}{G'}.$$

Since the lines of torsion

$$EFdu^2+2EGdudv+FGdv^2=0$$

on S correspond to the asymptotic lines

$$L'du^2+2M'dudv+N'dv^2=0$$

on S', we must have

$$(23) \qquad \frac{EF}{L'}=\frac{EG}{M'}=\frac{FG}{N'}.$$

Now let us take for S the imaginary ruled surface of constant curvature whose fundamental quantities are

$$E=0, \quad F=-\frac{2}{(u+v)^2}, \quad G=g(v),$$

$$L=0, \quad M=-\frac{2}{(u+v)^2}, \quad N=0.$$

Then we have from (23)

$$(24) \qquad L'=0, \quad M'=0, \quad N'\neq 0.$$

Hence it follows from (22) and (24) that

$$E'=0,$$

so that

$$E'G'-F'^2=0,$$

and consequently S' is the minimal developable which must be excluded.

Moreover, *although we exclude such an imaginary surface*, the following theorem can be established in a similar manner as in §§ 17–18:

Any surface (excluding the C-surface) can not be related to an infinite number of surfaces (unless all these are similar) by the representation which interchanges the lines of torsion and the asymptotic lines.

21. Lastly we have *quite the same results for the representation which interchanges the minimal lines and the characteristic lines.*

If such a representation may exist, let us take the minimal lines as the parametric curves on S and the characteristic lines as those on S'.

Then

$$E=0, \quad G=0, \quad M'=0, \quad \frac{L'}{E'}=\frac{N'}{G'}.$$

Since the characteristic lines

$$LMdu^2+2LNdudv+MNdv^2=0$$

on S correspond to the minimal lines

$$E'du^2+2F'dudv+G'dv^2=0$$

on S', we must have

$$\frac{L}{E'}=\frac{N}{G'},$$

unless S is a minimal surface where $M=0$. Therefore the lines of curvature

$$Ldu^2-Ndv^2=0$$

on S correspond to the lines of curvature

$$E'du^2-G'dv^2=0$$

on S'.

If the lines of curvature be taken as the parametric curves on both surfaces, in order that the minimal lines and the characteristic lines should be interchanged, it should be

$$\frac{L}{E'}=-\frac{N}{G'}, \quad \frac{E}{L'}=-\frac{G}{N'}.$$

But since the equations to the lines of torsion and the asymptotic lines are

$$Edu^2-Gdv^2=0, \quad Ldu^2+Ndv^2=0$$

on S and

$$E'du^2-G'dv^2=0, \quad L'du^2+N'dv^2=0$$

on S' respectively, these two families must be interchanged which is the case treated in § 20.

22. It may be convenient to make a summary statement of the main results obtained in §§ 10, 12, 13, 15, 16, 17, 19, 20 and 21.

Let us exclude the C-surface and the similitude. Then the fundamental theorem can be expressed as follows:

The necessary and sufficient condition that any surface may be related to an infinite number of surfaces by the representations in which any two of the five families——the lines of torsion, the minimal lines, the lines of curvature, the asymptotic lines and the characteristic lines——are preserved respectively or interchanged into each other is that the two families should belong to the same cycle.

We hope that attention will thus be directed to the importance of the conception of cycles in the theory of representation.

PART III.

23. Let us define the quantity P for S by the expression

$$(25)\qquad P=\frac{A_1du^2+2A_2dudv+A_3dv^2}{B_1du^2+2B_2dudv+B_3dv^2},$$

and P' for S' by

$$(26)\qquad P'=\frac{A_1'du^2+2A_2'dudv+A_3'dv^2}{B_1'du^2+2B_2'dudv+B_3'dv^2}.$$

Eliminating du, dv among (25), (26) and

$$(27)\qquad C_1du^2+2C_2dudv+C_3dv^2=0,$$

we have

$$\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1-PB_1 & A_2-PB_2 & A_3-PB_3 \\ A_1'-P'B_1' & A_2'-P'B_2' & A_3'-P'B_3' \end{vmatrix}=0,$$

i.e.

$$(28)\quad \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ A_1' & A_2' & A_3' \end{vmatrix} - P\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1' & A_2' & A_3' \end{vmatrix} - P'\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1' & B_2' & B_3' \end{vmatrix} + PP'\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ B_1' & B_2' & B_3' \end{vmatrix}=0.$$

Hence the relation between P and P' depends, in general, upon the corresponding curves (27). We require under what conditions this relation becomes independent of corresponding curves.

In this case the four determinants

$$\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ A_1' & A_2' & A_3' \end{vmatrix},\quad \begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1' & A_2' & A_3' \end{vmatrix},\quad \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1' & B_2' & B_3' \end{vmatrix},\quad \begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ B_1' & B_2' & B_3' \end{vmatrix}$$

should have a common factor of the form $C_1X+C_2Y+C_3Z$; for all these determinants are linear with respect to C_1, C_2, C_3. Hence the four systems of linear equations

$$\begin{cases} A_1X+A_2Y+A_3Z=0, \\ A_1'X+A_2'Y+A_3'Z=0, \end{cases}\quad \begin{cases} B_1X+B_2Y+B_3Z=0, \\ A_1'X+A_2'Y+A_3'Z=0, \end{cases}$$

$$\begin{cases} A_1X+A_2Y+A_3'Z=0, \\ B_1'X+B_2'Y+B_3'Z=0, \end{cases}\quad \begin{cases} B_1X+B_2Y+B_3Z=0, \\ B_1'X+B_2'Y+B_3Z=0 \end{cases}$$

must have the common solution (X, Y, Z) other than $X=Y=Z=0$. Consequently the rank of the matrix

$$\begin{Vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ A_1' & A_2' & A_3' \\ B_1' & B_2' & B_3' \end{Vmatrix}$$

is smaller than 3. Therefore we can choose the eight constants λ, μ, ν, ρ; $\lambda', \mu' u', \nu', \rho'$ such that

$$\frac{\lambda' A_1'+\mu' B_1'}{\lambda A_1+\mu B_1}=\frac{\lambda' A_2'+\mu' B_2'}{\lambda A_2+\mu B_2}=\frac{\lambda' A_3'+\mu' B_3'}{\lambda A_3+\mu B_3},$$

$$\frac{\nu' A_1'+\rho' B_1'}{\nu A_1+\rho B_1}=\frac{\nu' A_2'+\rho' B_2'}{\nu A_2+\rho B_2}=\frac{\nu' A_3'+\rho' B_3'}{\nu A_3+\rho B_3}.$$

Hence the two families of $2\infty^1$ curves

$$(\lambda A_1+\mu B_1)du^2+2(\lambda A_2+\mu B_2)dudv+(\lambda A_3+\mu B_3)dv^2=0,$$

$$(\nu A_1+\rho B_1)du^2+2(\nu A_2+\rho B_2)dudv+(\nu A_3+\rho B_3)dv^2=0,$$

belonging to the involution determined by

$$(29)\qquad \begin{cases} A_1du^2+2A_2dudv+A_3dv^2=0, \\ B_1du^2+2B_2dudv+B_3dv^2=0 \end{cases}$$

on S must correspond to the two families

$$(\lambda' A_1'+\mu' B_1')du^2+2(\lambda' A_2'+\mu' B_2')dudv+(\lambda' A_3'+\mu' B_3')dv^2=0,$$

$$(\nu' A_1' + \rho' B_1')du^2 + 2(\nu' A_2' + \rho' B_2')dudv + (\nu' A_3' + \rho' B_3')dv^2 = 0$$

belonging to the involution determined by

$$\text{(30)} \qquad \begin{cases} A_1'du^2 + 2A_2'dudv + A_3'dv^2 = 0, \\ B_1'du^2 + 2A_2'dudv + B_3'dv^2 = 0 \end{cases}$$

on S' respectively.

Conversely, if two families of $2\infty^1$ curves belonging to the involution determined by (29) correspond to two families of $2\infty^1$ curves belonging to the involution determined by (30), the rank of the above matrix is smaller than 3; so the four determinants have the common factor of the form $C_1X + C_2Y + C_3Z$. Consequently equation (28) becomes

$$\text{(31)} \qquad a + bP + cP' + dPP' = 0,$$

a, b, c, d being functions of A's and B's alone.

Hence we have the theorem: *The necessary and sufficient condition that P and P' should have the relation independent of corresponding directions is that two families of $2\infty^1$ curves belonging to the involution determined by (29) correspond to two families of $2\infty^1$ curves belonging to the involution determined by (30). When the condition is fulfilled, P and P' have the bilinear relation (31).*

Particularly, *if the double rays of the two involutions correspond*, we may put

$$A_2 = A_2' = B_2 = B_2' = 0;$$

and therefore *(31) becomes*

$$\text{(32)} \quad \begin{vmatrix} A_1 & A_3 \\ A_1' & A_3' \end{vmatrix} - P \begin{vmatrix} B_1 & B_3 \\ A_1' & A_3' \end{vmatrix} - P' \begin{vmatrix} A_1 & A_3 \\ B_1' & B_3' \end{vmatrix} + PP' \begin{vmatrix} B_1 & B_3 \\ B_1' & B_3' \end{vmatrix} = 0.$$

24. Now we will show some applications of the above theorem. Throughout this Part III we exclude the developable surface, the imaginary ruled surface of constant curvature, besides the plane, the sphere and the minimal developable.

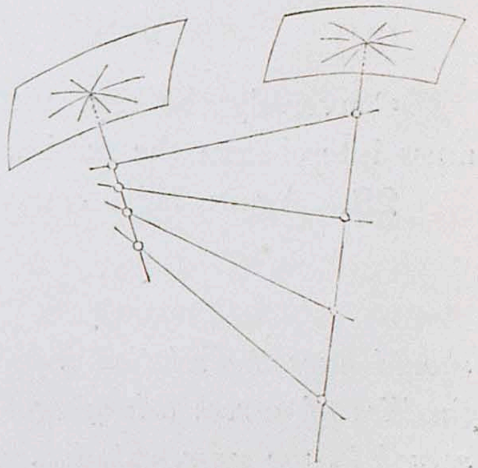
Consider the relation between corresponding normal curvatures $\dfrac{1}{R}$ and $\dfrac{1}{R'}$. Since

$$\frac{1}{R} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2},$$

$$\frac{1}{R'} = \frac{L'du^2 + 2M'dudv + N'dv^2}{E'du^2 + 2F'dudv + G'dv^2},$$

the above theorem becomes:

The necessary and sufficient condition that the normal curvatures should have the definite relation at corresponding points, independent of corresponding directions, is that two families of curves belonging to the isoclinal system should be preserved in the representation. When the condition is fulfilled, the centres of normal curvature at corresponding normal sections form projective ranges. Hence the straight lines which join the centres of normal curvature of corresponding normal sections at corresponding points form a system of generators of a quadric.



Particularly for the representation in which the lines of curvature are preserved, the above quadric may exist. *The Laguerre transformation*(1) belongs to this case.

For *the parallel transformation* the corresponding projective ranges have the same base, and the relation between normal curvatures becomes

$$aKRR' + (1 - aH)R - (1 - a^2K)R' + a(2 - aH) = 0,$$

a being the distance between the parallel surfaces S and S'.

For *the inversion* the corresponding projective ranges become perspective, the centre of perspective being the centre of inversion(1).

25. Next, *the necessary and sufficient condition that the geodesic torsions $\dfrac{1}{T}$ and $\dfrac{1}{T'}$ should have the definite relation (the bilinear relation independent of corresponding directions) is that two families of curves belonging to the inverse-orthogonal system should be preserved in the representation.*

(i) For the representation of the first cycle (for example, the inversion), we may put

$$F = 0, \qquad F' = 0, \qquad \frac{G}{E} = \frac{G'}{E'} = \frac{N}{L} = \frac{N'}{L'}.$$

Hence the bilinear relation becomes

(1) Darboux, Théorie des surfaces, 1 (1887), p. 253.

$$\sqrt{H^2-4K}\cdot T=\sqrt{H'^2-4K'}\cdot T'. \tag{33}$$

Conversely the representation having the *absolute invariant*

$$\sqrt{H^2-4K}\cdot T$$

must be of the first cycle.

(ii) For the representation which interchanges the minimal lines and the lines of curvature we may put

$$F=0,\quad F'=0,\quad \frac{G}{E}=\frac{N}{L}=-\frac{G'}{E'}=-\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$\sqrt{H^2-4K}\ T\cdot\sqrt{H'^2-4K'}\ T'=1. \tag{34}$$

Conversely the representation in which the relation (34) holds good must interchange the minimal lines and the lines of curvature.

26. Again, *the necessary and sufficient condition that*

$$\frac{T}{R}\ \textit{and}\ \frac{T'}{R'}$$

should have the relation independent of corresponding directions is that two families of curves belonging to the inverse-conjugate system should be preserved in the representation.

(i) For the representation of the second cycle, we may put

$$M=0,\quad M'=0,\quad \frac{G}{E}=\frac{N}{L}=\frac{G'}{E'}=\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$\frac{1}{2}\sqrt{\frac{H^2-4K}{K}}\ \frac{T}{R}=\frac{1}{2}\sqrt{\frac{H'^2-4K'}{K'}}\ \frac{T'}{R'}.$$

Conversely the representation in which the quantity

$$\frac{1}{2}\sqrt{\frac{H^2-4K}{K}}\ \frac{T}{R}$$

is an absolute invariant must be of the second cycle.

(ii) For the representation which interchanges the lines of curvature and the asymptotic lines, we may put

$$M=0,\quad M'=0,\quad \frac{G}{E}=\frac{N}{L}=-\frac{G'}{E'}=-\frac{N'}{L'}.$$

(1) Salmon, Treatise on the analytic geometry of three dimensions, 5. ed., 2 (1915) p. 157.

Hence the bilinear relation becomes

$$\frac{1}{2}\sqrt{\frac{H^2-4K}{K}}\ \frac{T}{R}\cdot\frac{1}{2}\sqrt{\frac{H'^2-4K'}{K'}}\ \frac{T'}{R'}=1. \tag{35}$$

Conversely the representation in which the relation (35) holds good must interchange the lines of curvature and the asymptotic lines.

27. Further, *the necessary and sufficient condition that*

$$\left(\frac{H}{2}-\frac{1}{R}\right)T\ \textit{and}\ \left(\frac{H'}{2}-\frac{1}{R'}\right)T'$$

should have the relation independent of corresponding directions is that two families of curves belonging to the orthogonal system should be preserved in the representation.

(i) For the representation of the first cycle (for example, the inversion), we may put

$$E=0,\quad G=0,\quad E'=0,\quad G'=0,\quad \frac{N}{L}=\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$\left(\frac{H}{2}-\frac{1}{R}\right)T=\left(\frac{H'}{2}-\frac{1}{R'}\right)T'.$$

Conversely the representation in which the quantity

$$\left(\frac{H}{2}-\frac{1}{R}\right)T$$

is an absolute invariant must be of the first cycle.

(ii) For the representation which interchanges the lines of curvature and the lines of torsion, we may put

$$E=0,\quad G=0,\quad E'=0,\quad G'=0,\quad \frac{N}{L}=-\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$\left(\frac{H}{2}-\frac{1}{R}\right)T\cdot\left(\frac{H'}{2}-\frac{1}{R'}\right)T'=1. \tag{36}$$

Conversely the representation in which the relation (36) holds good must interchange the lines of torsion and the lines of curvature.

28. Further, *the necessary and sufficient condition that*

$$\left(\frac{H}{2R}-K\right)T\ \textit{and}\ \left(\frac{H'}{2R'}-K'\right)T'$$

should have the relation independent of corresponding directions is that two *families of curves belonging to the conjugate system should be preserved in the representation.*

It is noteworthy that *collineation* and *polar reciprocation* belong to this case.

(i) For the representation of the second cycle, we may put

$$L=0,\quad N=0,\quad L'=0,\quad N'=0,\quad \frac{G}{E}=\frac{G'}{E'}.$$

Hence the bilinear relation becomes

$$\left(\frac{H}{2R}-K\right)\frac{T}{\sqrt{-K}}=\left(\frac{H'}{2R'}-K'\right)\frac{T'}{\sqrt{-K'}}.$$

Conversely the representation in which the quantity

$$\left(\frac{H}{2R}-K\right)\frac{T}{\sqrt{-K}}$$

is an absolute invariant must be of the second cycle.

(ii) For the representation which interchanges the lines of curvature and the characteristic lines we may put

$$L=0,\quad N=0,\quad L'=0,\quad N'=0,\quad \frac{G}{E}=-\frac{G'}{E'}.$$

Hence the bilinear relation becomes

$$(37)\qquad \left(\frac{H}{2R}-K\right)\frac{T}{\sqrt{-K}}\cdot\left(\frac{H'}{2R'}-K'\right)\frac{T}{\sqrt{-K'}}=1.$$

Conversely the representation in which the relation (37) holds good must interchange the lines of curvature and the characteristic lines.

29. Further, *the necessary and sufficient condition that*

$$\frac{H}{2}-\frac{1}{R'}\quad \textit{and}\quad \frac{H'}{2}-\frac{1}{R'}$$

should have the relation independent of corresponding directions is that two families of curves belonging to the isoclinal system should be preserved in the representation.

For example, the inversion and the parallel transformation belong to this case.

(i) For the representation of the first cycle, we may put

$$F=0,\quad M=0,\quad F'=0,\quad M'=0,\quad \frac{G}{E}=\frac{G'}{E'}.$$

Hence the bilinear relation becomes

$$\frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)=\frac{1}{\sqrt{H'^2-4K'}}\left(\frac{H'}{2}-\frac{1}{R'}\right).$$

Conversely the representation in which the quantity

$$\frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)$$

is an absolute invariant must be of the first cycle.

(ii) For the representation which interchanges the minimal lines and the lines of torsion we may put

$$F=0,\quad M=0,\quad F'=0,\quad M'=0,\quad \frac{G}{E}=-\frac{G'}{E'}.$$

Hence the bilinear relation becomes

$$(38)\qquad \frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)\cdot\frac{1}{\sqrt{H'^2-4K'}}\left(\frac{H'}{2}-\frac{1}{R'}\right)=1.$$

Conversely the representation in which the relation (38) holds good must interchange the minimal lines and the lines of torsion.

30. Lastly, *the necessary and sufficient condition that*

$$\left(\frac{H}{2R}-K\right)R\quad \textit{and}\quad \left(\frac{H'}{2R'}-K'\right)R'$$

should have the relation independent of corresponding directions is that two families of curves belonging to the isoclinal system should be preserved in the representation.

(i) For the representation of the second cycle we may put

$$F=0,\quad M=0,\quad F'=0,\quad M'=0,\quad \frac{N}{L}=\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$\frac{1}{2\sqrt{4K-H^2}}\left(\frac{H}{2R}-K\right)R=\frac{1}{\sqrt{4K'-H'^2}}\left(\frac{H'}{2R'}-K'\right)R'.$$

Conversely the representation in which the quantity

$$\frac{1}{2\sqrt{4K-H^2}}\left(\frac{H}{2R}-K\right)R$$

is an absolute invariant must be of the second cycle.

(ii) For the representation which interchanges the asymptotic lines and the characteristic lines we may put

$$F=0,\quad M=0,\quad F'=0,\quad M'=0,\quad \frac{N}{L}=-\frac{N'}{L'}.$$

Hence the bilinear relation becomes

$$(39)\quad \frac{1}{2\sqrt{4K-H^2}}\left(\frac{H}{2R}-K\right)R\cdot\frac{1}{2\sqrt{4K'-H'^2}}\left(\frac{H'}{2R'}-K'\right)R'=1.$$

Conversely the representation in which the relation (39) holds good must interchange the asymptotic lines and the characteristic lines.

31. All absolute invariants obtained in §§ 25–30 may be arranged as follows:

	minimal lines	lines of curvature	asymptotic lines	characteristic lines
lines of torsion	$\frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)$	$\left(\frac{H}{2}-\frac{1}{R}\right)T$		
minimal lines		$\sqrt{H^2-4K}\,T$		
lines of curvature			$\frac{1}{2}\sqrt{\frac{H^2-4K}{K}}\,\frac{T}{R}$	$\frac{1}{\sqrt{-K}}\left(\frac{H}{2R}-K\right)T$
asymptotic lines				$\frac{1}{2\sqrt{4K-H^2}}\left(\frac{H}{2R}-K\right)R$

It will be noticed that we have the relation

$$\frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)\cdot\sqrt{H^2-4K}\;T=\left(\frac{H}{2}-\frac{1}{R}\right)T\ (^1)$$

among the absolute invariants for the first cycle and

$$\frac{1}{2\sqrt{4K-H^2}}\left(\frac{H}{2R}-K\right)\cdot\frac{1}{2}\sqrt{\frac{H^2-4K}{K}}\;\frac{T}{R}=\frac{1}{\sqrt{-K}}\left(\frac{H}{2R}-K\right)T$$

for the second cycle.

Ikeda near Ôsaka, July 1917.

(1) The geometrical meaning of the absolute invariants of the first cycle is easily seen from the identities:

$$\frac{1}{\sqrt{H^2-4K}}\left(\frac{H}{2}-\frac{1}{R}\right)=-\frac{1}{2}\cos 2\theta,\quad \left(\frac{H}{2}-\frac{1}{R}\right)T=-\cot 2\theta,\quad \sqrt{H^2-4K}\,T=2\,\mathrm{cosec}\,2\theta,$$

where θ denotes the angle between the directions whose radii of normal curvature are R and R_1.

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Price per volume (consisting of four numbers) payable in advance: 3 yen = 6 shillings = 6 Mark = 7.50 francs = 1.50 dollars. Postage inclusive.