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Geometry of the Field of Central Force.

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Geometry of the Field of Central Force,

by

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Part I. General Case.

1. Let the plane of the motion of a particle be the plane of reference and let the origin be at the centre of force. If F be the accelerating force at any point measured negatively towards the origin, then the equations of motion are

$$(1) \qquad \ddot{x}=\frac{x}{\sqrt{x^2+y^2}}\,F(x, y, \dot{x}, \dot{y}, t), \qquad \ddot{y}=\frac{y}{\sqrt{x^2+y^2}}\,F(x, y, \dot{x}, \dot{y}, t),$$

where dots denote derivatives with respect to the time t. Hence we have

$$(2) \qquad x\ddot{y}-y\ddot{x}=0.$$

This equation gives by integration

$$(3) \qquad x\dot{y}-y\dot{x}=h,$$

where h is an arbitrary constant whose value depends upon the initial conditions. Each motion corresponds to a definite value of the constant h; the motion may therefore be grouped according to the values of h. The totality of the orbits, corresponding to a given value of h, will be called the *central family*.

Space-time coordinates. Curves of a linear complex.

2. If we introduce the *space-time coordinates* (x, y, t), any orbit is the orthogonal projection of the corresponding space-time curves upon the plane of x, y.

Now the condition (3) for the central family may be written

$$(4) \qquad x\,dy-y\,dx=h\,dt;$$

so the corresponding space-time curves form the curves of the linear complex, the axis being the t-axis and the chief parameter being $-h$. Therefore we have the theorem:

The space-time curves corresponding to the central family are the curves of the linear complex, the axis being the time-axis and the chief parameter being the given value of the angular momentum with opposite sign. Conversely, *the parallel projection of the curves of a linear complex in the direction of its axis forms the central family, the centre of force being the projection of the axis and the angular momentum about the centre of force being equal to the chief parameter with opposite sign of the complex* (1).

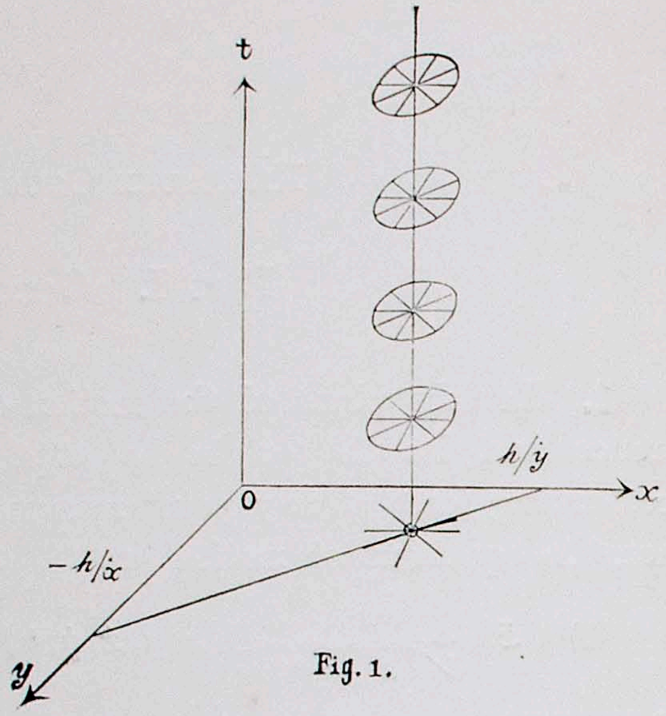

Fig. 1.

3. If the curve

$$x=p(\lambda), \quad y=q(\lambda)$$

belong to the central family (3),

$$h\,dt=(pq'-qp')\,d\lambda,$$

where accents denote differentiation with respect to the parameter λ; so that the orbit corresponds to a system of the space-time curves

$$x=p(\lambda), \quad y=q(\lambda), \quad t=\frac{1}{h}\int(pq'-qp')\,d\lambda+c,$$

where c is an arbitrary constant, and therefore the velocity-components and the force-components along the orbit are given by

$$\dot{x}=h\frac{p'}{pq'-qp'}, \qquad \dot{y}=h\frac{q'}{pq'-qp'};$$

$$\ddot{x}=-h^2\frac{p(p'q''-q'p'')}{(pq'-qp')^3}, \qquad \ddot{y}=-h^2\frac{q(p'q''-q'p'')}{(pq'-qp')^3}.$$

I. As a simple example, take the circle

$$x=a\cos\lambda, \quad y=a\sin\lambda, \quad (a \text{ being a constant})$$

(1) For the conservative field of force having the force-function $U(x, y)$, the equation of energy is

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)=U(x, y)+k, \quad (k, \text{ the energy constant});$$

so that the space-time curves form the integral curves of the Monge equation

$$dx^2+dy^2-2(U+k)\,dt^2=0.$$

as a central orbit. Then the space-time curves form a system of the helices

$$x=a\cos\lambda, \quad y=a\sin\lambda, \quad t=\frac{a^2}{h}\lambda+c;$$

and therefore the force-components along the given circle are given by

$$\ddot{x}=-\frac{h^2}{a^3}\cos\lambda=-\frac{h^2}{a^4}x=\frac{h}{a^2}\dot{y}=-\frac{h^2}{a^3}\cos\frac{h}{a^2}(t-c)=\cdots\cdots,$$

$$\ddot{y}=-\frac{h^2}{a^3}\sin\lambda=-\frac{h^2}{a^4}y=\frac{h}{a^2}\dot{x}=-\frac{h^2}{a^3}\sin\frac{h}{a^2}(t-c)=\cdots\cdots.$$

II. As the second example, take the parabola

$$x=\lambda, \quad y=\frac{\lambda^2}{2}$$

as a central orbit. Then the space-time curves form a system of the space cubics

$$x=\lambda, \quad y=\frac{\lambda^2}{2}, \quad t=\frac{\lambda^3}{6h}+c;$$

and therefore the force-components along the given parabola are

$$\ddot{x}=-\frac{4h}{\lambda^3}=-\frac{2h}{xy}=-\dot{x}\dot{y}=\cdots\cdots,$$

$$\ddot{y}=-\frac{2h}{\lambda^2}=-\frac{2h}{x^2}=-\dot{x}=\cdots\cdots.$$

III. Prof. E. Picard (1) proved that a necessary and sufficient condition that a plane rational curve of the m^{th} degree (in general)(2) should be considered as the projection of a space rational curve of the m^{th} degree (in general) belonging to the linear complex whose axis is perpendicular to the plane of the plane curve is that the plane curve has m points of inflexion at infinity. Consequently we have the theorem: *A necessary and sufficient condition that a rational curve of the m^{th} degree (in general) belong to the central family and one of the*

(1) Picard, Application de la théorie des complexes linéaires à l'étude des surfaces et des courbes gauches, Annales de l'École Normale Supérieure, II, 2 (1877), p. 341.

(2) The parametric equations of this curve are of the form

$$x=\frac{P(\lambda)}{R(\lambda)}, \qquad y=\frac{Q(\lambda)}{R(\lambda)},$$

where $P(\lambda)$, $Q(\lambda)$ and $R(\lambda)$ denote polynomials of the m th degree with respect to λ.

corresponding space-time curve be a rational curve of the m^{th} degree (in general) is that the plane curve has m points of inflexion at infinity.

4. I. By means of the theorem in § 2 we can interpretate some geometrical theorems for the curves of a linear complex from the standpoint of dynamics, and conversely. For example, from (2)

$$\frac{y}{\dddot{y}}=\frac{x}{\dddot{x}}=\frac{\dot{x}y-\dot{y}x}{\dot{x}\ddot{y}-\dot{y}\ddot{x}};$$

so that we have from (3)

$$\frac{-y}{-\ddot{y}}=\frac{x}{\ddot{x}}=\frac{-h}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}; \tag{5}$$

but since $-\ddot{y}$, $\ddot{x}$, $\dot{x}\ddot{y}-\dot{y}\ddot{x}$ are proportional to the direction-cosines of the binormal at (x, y, t), it is seen that all the complex curves through any point (x, y, t) have the same osculating plane at that point, which is nothing but the theorem due to Lie and Prof. Appell[1]. Thus the well known fact that for the central orbit the angular momentum about the centre of force is constant corresponds to Lie-Appell's theorem for the curves of a linear complex.

II. We see from (5) that when the position (x, y) of a particle is given, the binormals to the space-time curves at the corresponding points are parallel to one another. And when the position (x, y) and the velocity-components $(\dot{x}, \dot{y})$ are given, the principal normals to the space-time curves at the corresponding points and directions are parallel.

Next the torsion of any complex curve (4) is

$$\frac{1}{T}=\frac{h}{x^2+y^2+h^2}\,(^2); \tag{6}$$

consequently when the position of a particle is given, the torsions of the space-time curves at the corresponding points are the same.

Also the curvature of any complex curve (4) is

$$\frac{1}{R}=\frac{\sqrt{\ddot{x}^2+\ddot{y}^2+(\dot{x}\ddot{y}-\dot{x}\ddot{y})^2}}{\sqrt{1+\dot{x}^2+\dot{y}^2}^{\,3}};$$

But we have from (1) and (5)

$$\frac{-y}{-\ddot{y}}=\frac{x}{\ddot{x}}=\frac{-h}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}=\frac{\sqrt{x^2+y^2}}{F(x, y, \dot{x}, \dot{y}, t)};$$

(1) Lie-Scheffers, Geometrie der Berührungstransformationen, I (1896), p. 230 Picard, Traité d'analyse, I (2. éd., 1901), p. 380.

(2) Lie-Scheffers, ibid, p. 231.

whence

$$\frac{1}{R}=\frac{\sqrt{x^2+y^2+h^2}}{\sqrt{x^2+y^2}\,\sqrt{1+\dot{x}^2+\dot{y}^2}^{\,3}}F(x, y, \dot{x}, \dot{y}, t). \tag{7}$$

If the force do not contain the time t explicitly, R is independent of t. Therefore for the central family in which the force does not contain the time explicitly, the centres of curvature of the space-time curves, corresponding to the given position and velocity-components of a particle, lie on a straight line parallel to the time-axis.

5. The ∞^3 straight lines

$$x=at+b, \quad y=ct+d,$$

where a, b, c, d are arbitrary constants such that $bc-ad=h$, are tangents to the curves of linear complex (4). Hence on the plane of x, y, the ∞^2 straight lines

$$cx-ay=h$$

are tangents to the central family. But this equation is nothing but (3), which may be written

$$\frac{x}{\frac{h}{\dot{y}}}+\frac{y}{-\frac{h}{\dot{x}}}=1;$$

so that the tangent to the central family at the point (x, y) in the direction $\frac{dy}{dx}$ cuts off the intercepts

$$\frac{h}{\dot{y}}, \quad -\frac{h}{\dot{x}}$$

on the x-and y-axes respectively. From this consideration we can find geometrically the velocity-components of a particle at the given position in the given direction[1]. (See Fig. 1.)

This may be proved directly as follows: From (3) and

$$\frac{dy}{dx}\equiv y'=\frac{\dot{y}}{\dot{x}} \tag{8}$$

we have

$$\dot{x}=\frac{h}{xy'-y}, \quad \dot{y}=\frac{hy'}{xy'-y}; \tag{9}$$

so that the tangent to the central family

(1) For another method, see § 6.

$$Y-y=y'(X-x),$$

X, Y being current coordinates, cuts off the intercepts

$$\frac{xy'-y}{y'}=\frac{h}{\dot{y}}, \quad y-xy'=-\frac{h}{\dot{x}}$$

on the x-and y-axes respectively.

Hodograph.

6. Let x_1, y_1 be the point-coordinates in the plane of the hodograph. Then the definition of the hodograph gives

$$x_1=\dot{x}, \quad y_1=\dot{y}.$$

For the central family we have from (9)

$$(10) \qquad x_1=\frac{h}{xy'-y}, \quad y_1=\frac{hy'}{xy'-y};$$

whence

$$y'_1\equiv\frac{dy_1}{dx_1}=\frac{\ddot{y}}{\ddot{x}}=\frac{y}{x}.$$

But since

$$dx_1=-h\frac{y'\,dx+x\,dy'-dy}{(xy'-y)^2},$$

$$dy_1=-h\frac{y\,dy'+y'^2\,dx-y'\,dy}{(xy'-y)^2},$$

it follows that

$$dy_1-y'_1\,dx_1=\frac{h^2}{x(xy'-y)}(dy-y'\,dx).$$

Therefore *each of the central family and its hodograph is derived from the other by the contact transformation*

$$(11) \qquad x_1=\frac{h}{xy'-y}, \quad y_1=\frac{hy'}{xy'-y}, \quad y'_1=\frac{y}{x},$$

or

$$x=-\frac{h}{x_1y'_1-y_1}, \quad y=-\frac{hy_1'}{x_1y'_1-y_1}, \quad y'=\frac{y_1}{x_1}.$$

Hence we have, as in Fig. 2, the parallelogram $OPQP_1$ whose area is equal to h. For, equation (3) may be written

$$\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x_1 & y_1 & 1 \end{vmatrix}=h.$$

Therefore we obtain the *dual relation* between the central family and its hodograph that the point P in the plane of the central family corresponds to the line P_1Q in the plane of the hodograph, and the point P_1 in the latter plane corresponds to the line PQ in the former plane.

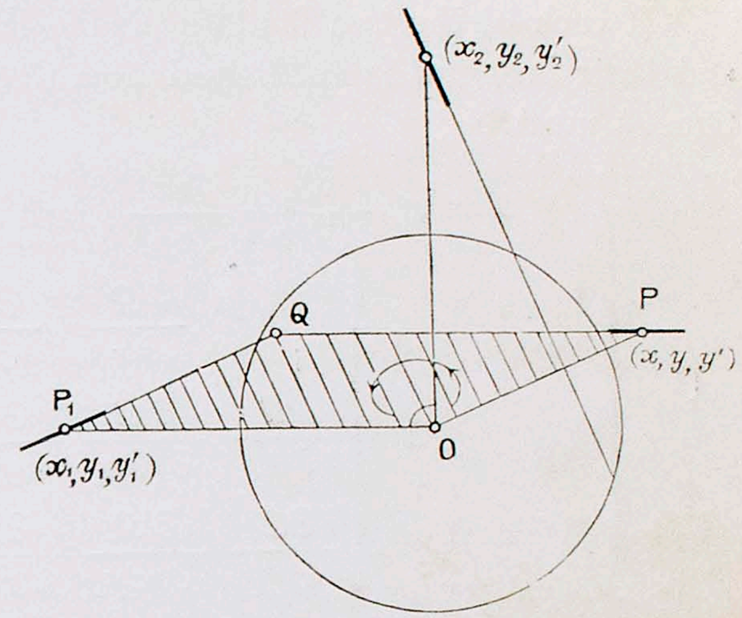

Fig. 2.

Also we can easily show that the contact transformation (11) is nothing but the combination of the polar reciprocation

$$x_2=\frac{hy'}{xy'-y}, \quad y_2=-\frac{h}{xy'-y}, \quad y'_2=-\frac{x}{y}$$

with respect to the circle

$$x^2+y^2=h$$

and the rotation through the angle 90° about the origin

$$x_1=-y_2, \quad y_1=x_2, \quad y'_1=-\frac{1}{y'_2},$$

which was already stated by Routh(1).

7. Now consider the first integral of the equations of motion in the general field of force

$$\ddot{x}=\phi(x, y, \dot{x}, \dot{y}, t), \quad \ddot{y}=\psi(x, y, \dot{x}, \dot{y}, t).$$

If the first integral contain the time explicitly, the hodograph loses its proper meaning. For, since the proper hodograph has the form

(1) Routh, Treatise on dynamics of a particle (1898), p. 253. He did not introduce the notion of contact transformation only, but did not give the analytical expressions of the hodograph.

$$x_1 = X(x, y, y'), \quad y_1 = Y(x, y, y'),$$

by elimination of y' we must have a relation among x, y, x_1, y_1. Hence we assume that the first integral has the form

$$(12) \qquad \Omega(x, y, x_1, y_1) = c,$$

where c is an arbitrary constant.

If each of the two families of the orbits for the given value of c and their hodographs be derived from the other by a contact transformation, it must be

$$(13) \begin{cases} \Omega = 0, \quad \dfrac{\partial \Omega}{\partial x} + \dfrac{\partial \Omega}{\partial y} y' = 0, \quad \dfrac{\partial \Omega}{\partial x_1} + \dfrac{\partial \Omega}{\partial y_1} y'_1 = 0, \\ \Delta \equiv \begin{vmatrix} 0 & \dfrac{\partial \Omega}{\partial x} & \dfrac{\partial \Omega}{\partial y} \\ \dfrac{\partial \Omega}{\partial x_1} & \dfrac{\partial^2 \Omega}{\partial x \partial x_1} & \dfrac{\partial^2 \Omega}{\partial y \partial x_1} \\ \dfrac{\partial \Omega}{\partial y_1} & \dfrac{\partial^2 \Omega}{\partial x \partial y_1} & \dfrac{\partial^2 \Omega}{\partial y \partial y_1} \end{vmatrix} \neq 0 \text{ for } \Omega = c \, (^1), \end{cases}$$

under the conditions

$$(14) \qquad y' = \frac{y_1}{x_1}, \quad y'_1 = \frac{\psi(x, y, x_1, y_1, t)}{\phi(x, y, x_1, y_1, t)}.$$

Therefore the generating equation (the first integral)

$$\Omega = c$$

must satisfy

$$(15) \qquad x_1 \frac{\partial \Omega}{\partial x} + y_1 \frac{\partial \Omega}{\partial y} = 0, \quad \phi \frac{\partial \Omega}{\partial x_1} + \psi \frac{\partial \Omega}{\partial y_1} = 0.$$

8. I. Particularly, for the central family and its hodograph, we have had the generating equation

$$\Omega \equiv x y_1 - y x_1 = c.$$

In this case

$$x_1 \frac{\partial \Omega}{\partial x} + y_1 \frac{\partial \Omega}{\partial y} \equiv x_1 y_1 - y_1 x_1 = 0,$$

$$\phi \frac{\partial \Omega}{\partial x_1} + \psi \frac{\partial \Omega}{\partial y_1} \equiv \frac{F(x, y, x_1, y_1, t)}{\sqrt{x^2 + y^2}} (-yx + xy) = 0,$$

$$\Delta \equiv y x_1 - y_1 x = -c \neq 0.$$

(1) Lie-Scheffers, ibid, p. 54.

We have already shown that the generating equation determines a particular case of general duality of Möbius (§ 6). But we can prove the theorem:

The only orbit, which is derived from its hodograph by the general duality, belongs to a central family.

For, by the assumption the function

$$\Omega \equiv (a_1 x + b_1 y + c_1) x_1 + (a_2 x + b_2 y + c_2) y_1 + (a_3 x + b_3 y + c_3)$$

must satisfy the identities

$$(a_1 x_1 + a_2 y_1 + a_3) x_1 + (b_1 x_1 + b_2 y_1 + b_3) y_1 = 0,$$

$$(a_1 x + b_1 y + c_1) \phi + (a_2 x + b_2 y + c_2) \psi = 0.$$

The former gives

$$a_1 = 0, \; a_3 = 0, \; b_2 = 0, \; b_3 = 0, \; a_2 = -b_1;$$

and hence

$$\Delta \equiv -b_1^2 [b_1 (x_1 y - x y_1) + (c_1 x_1 + c_2 y_1)] = -b_1^2 c \neq 0;$$

consequently the latter becomes

$$\frac{\psi}{\phi} = \frac{b_1 y + c_1}{b_1 x - c_2},$$

which shows us that the force passes through the fixed point

$$\left(\frac{c_2}{b_1}, \; -\frac{c_1}{b_1} \right).$$

II. For the conservative field of force having the force-function $U(x, y)$,

$$\Omega \equiv \frac{1}{2}(x_1^2 + y_1^2) - U(x, y) = c.$$

In this case, since

$$\Delta = 0,$$

we have no contact transformation.

If the central force be conservative, that is, the force depends upon the distance from the centre only, we have

$$x y_1 - y x_1 = h, \quad \frac{1}{2}(x_1^2 + y_1^2) - U(x^2 + y^2) = k.$$

Hence the hodograph is derived from the orbit by the *point transformation*

$$x_1 = \frac{-hy \pm x\sqrt{2(x^2+y^2)(U+k)-h^2}}{x^2+y^2},$$

$$y_1 = \frac{hx \pm y\sqrt{2(x^2+y^2)(U+k)-h^2}}{x^2+y^2}.$$

Part II. Positional Force.

Geometrical characterization of the central family.

9. In what follows we will confine ourselves to consider the most important case where the central force is *positional*. Then the equations of motion are

$$(16) \qquad \ddot{x} = \frac{x}{\sqrt{x^2+y^2}} F(x, y), \quad \ddot{y} = \frac{y}{\sqrt{x^2+y^2}} F(x, y).$$

Now since

$$\frac{d^2y}{dx^2} \equiv y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3},$$

we have from (2) and (1)

$$\ddot{x} = -\frac{h^2\, xy''}{(xy'-y)^3}, \quad \ddot{y} = -\frac{h^2\, yy''}{(xy'-y)^3}.$$

Hence it follows that

$$(17) \qquad h^2 y'' + \frac{F(x, y)}{\sqrt{x^2+y^2}} (x\, y' - y)^3 = 0,$$

which is the differential equation of the central family for the positional force.

If (X, Y) be the centre of curvature at (x, y),

$$y' = -\frac{X-x}{Y-y}, \qquad y'' = \frac{(X-x)^2+(Y-y)^2}{(Y-y)^3};$$

hence the locus of the centres of curvature of all curves belonging to the central family at the point (x, y) is the cubic curve

$$(18) \quad [x(X-x)+y(Y-y)]^3 = \frac{h^2\sqrt{x^2+y^2}}{F(x, y)} [(X-x)^2+(Y-y)^2].$$

If we put

$$(19) \quad \xi = \frac{x(X-x)+y(Y-y)}{\sqrt{x^2+y^2}}, \quad \eta = \frac{-y(X-x)+x(Y-y)}{\sqrt{x^2+y^2}},$$

(18) becomes

$$(20) \qquad \xi^3 = \omega\,(\xi^2+\eta^2),$$

where

$$(21) \qquad \omega = \frac{h^2}{(x^2+y^2)\, F}.$$

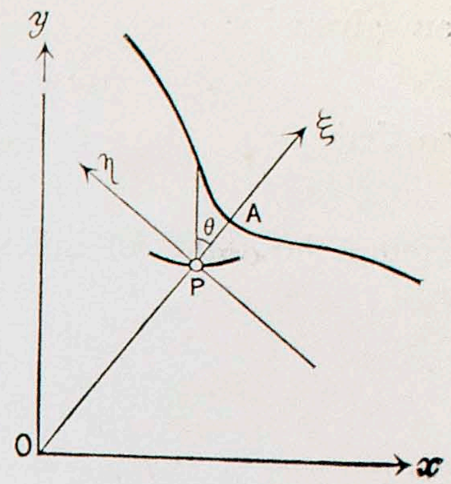

Fig. 3.

This cubic is nothing but the *cubical duplicatrix* of Longchamps [1]. The point $P\,(\xi=0, \eta=0)$ will be called the *fundamental point* [2] and the point $A(\xi=\omega, \eta=0)$ the *vertex* of the curve [3]. These definitions and equations (19) lead us to the result: The fundamental point and the vertex are collinear with the centre of force.

Therefore we have arrived at the theorem:

The centres of curvature of all curves of the central family passing through any point P lie on the cubical duplicatrix, the fundamental point being at P and the vertex A being collinear with P and the centre of force O.

Conversely it is easily seen that *if the centres of curvature of the* ∞^2 *curves passing through any point P lie on the cubical duplicatrix, the fundamental point being at P and the vertex A being collinear with P and a fixed point O (independent of the position of P), then these* ∞^2 *curves form the central family, the centre of force being the fixed point O.* Thus the property given by the above theorem is completely characteristic to the central family for positional force.

10. If we put

$$\eta = \xi \operatorname{tg}\theta = \lambda\,\xi,$$

the cubical duplicatrix (20) may be represented parametrically

$$\xi = \omega(1+\lambda^2), \qquad \eta = \omega(1+\lambda^2)\,\lambda.$$

The intersections $\lambda_1, \lambda_2, \lambda_3$ of this curve and any straight line

$$a\,\xi + b\,\eta + c = 0$$

(1) Longchamps, Essai sur la géométrie de la règle et de l'équerre (1890), p. 92–94. See also Loria, Spezielle ebene Kurven, 2. Aufl., Bd. 1 (1910), p. 93; Wieleitner, Spezielle ebene Kurven (1908), p. 371.

(2) This point is the isolate point of the curve.

(3) $\overrightarrow{PA} = \omega$.

are given by

$$b\,\omega\,\lambda^3+a\,\omega\,\lambda^2+b\,\omega\,\lambda+(a\,\omega+c)=0,$$

so that

$$\lambda_1\,\lambda_2+\lambda_2\,\lambda_3+\lambda_3\,\lambda_1=1.$$

Hence the points of inflexion (¹) are determined by

$$\lambda^2=\frac{1}{3}, \qquad \text{i.e.} \qquad \theta=\pm 30^\circ.$$

It is remarkable that *the magnitude of this angle is independent of the magnitude of force.*

11. Here we add some properties of the curve (20).

I. If

$$F=F_1+F_2,$$

then

$$\frac{1}{\omega}=\frac{1}{\omega_1}+\frac{1}{\omega_2}.$$

II. In order that the vertex A coincide always with the centre of force O, it is necessary and sufficient that

$$\frac{h^2}{(x^2+y^2)\,F}=-\sqrt{x^2+y^2},$$

i.e.

$$F=-\frac{h^2}{r^3}, \qquad (r^2=x^2+y^2).$$

In general the necessary and sufficient condition that the vertex A divide the segment OP in a constant ratio is that the central force be proportional to the inverse cube of the distance from its centre.

III. The necessary and sufficient condition that $\omega=$const. $(=c$, say) is

$$F=\frac{h^2}{c\,r^2}.$$

In this case the curves (20) for all positions of P are congruent, and conversely.

But the central family for the force which is proportional to the inverse square of the distance is consisted of the homofocal conics whose common focus is the centre of force, and conversely(²). Therefore we obtain the geometrical theorem :

(¹) The third point of inflexion is at infinity.

(²) Routh, ibid, p. 216.

The centres of curvature of all homofocal conics passing through any point P lie on the cubical duplicatrix, the fundamental point being at P and the vertex A being collinear with P and the common focus O; and these cubical duplicatrix for all positions of P are congruent with one another. Conversely, *if the centres of curvature of all the ∞^2 curves passing through any point P lie on the cubical duplicatrix, the fundamental point being at P and the vertex A being collinear with P and a fixed point O (independent of the position of P), and these cubical duplicatrix for all positions of P be congruent, then the ∞^2 curves must be the homofocal conics whose common focus is at the fixed point O.*

12. We will give the characteristic property of the space-time curves corresponding to the central family for positional force.

If α, β, γ be the direction-cosines of the tangent to the space-time curves,

$$\gamma=\frac{1}{\sqrt{1+\dot{x}^2+\dot{y}^2}};$$

whence the radius of curvature

$$(7) \qquad \frac{1}{R}=\frac{\sqrt{x^2+y^2+h^2}}{\sqrt{x^2+y^2}\,\sqrt{1+\dot{x}^2+\dot{y}^2}^{\,3}}\,F(x,y,\dot{x},\dot{y},t)$$

gives

$$(22) \qquad R\gamma^3=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+h^2}}\cdot\frac{1}{F(x,y,\dot{x},\dot{y},t)}.$$

Therefore *a necessary and sufficient condition that the central force be positional is that the quantity $R\gamma^3$ of the space-time curves corresponding to the central family depend upon the position of the particle only.*

Also we have from (6) and (22) the remarkable formula

$$F=\frac{r}{R\gamma^3\sqrt{hT}}, \qquad (r=\sqrt{x^2+y^2}).$$

13. Lastly we add the case where *the force is parallel*, that is, the centre of force is at infinity.

The equations of motion are then of the form

$$\ddot{x}=\phi\,(x,y), \qquad \ddot{y}=0;$$

so that the first integral becomes

$$\dot{y}=\text{const.}\ (=a,\ \text{say}).$$

Whence the differential equation of the orbits having the given value a is

$$y'' + \frac{\phi(x, y)}{a^2} y'^3 = 0.$$

Let (X, Y) be the centre of curvature at (x, y), and let

$$\xi = X - x, \qquad \eta = Y - y.$$

Then we have also the cubical duplicatrix

$$\xi^3 = \omega(\xi^2 + \eta^2),$$

where

$$\omega = \frac{a^2}{\phi(x, y)}.$$

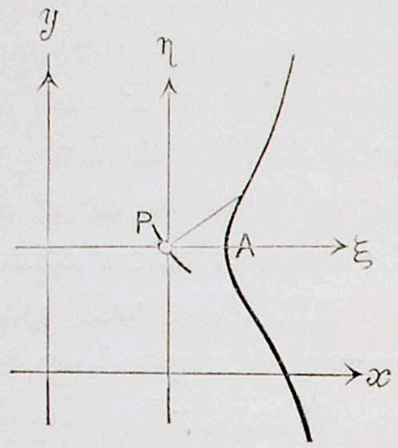

Fig. 4.

Collineation.

14. Let the equations of motion be

$$(23) \quad \ddot{x} = \phi(x, y) = F(x, y) \frac{x}{\sqrt{x^2+y^2}}, \quad \ddot{y} = \psi(x, y) = F(x, y) \frac{y}{\sqrt{x^2+y^2}}.$$

Then the differential equation of the central family is

$$(17) \qquad h^2 y'' + \frac{F(x, y)}{\sqrt{x^2+y^2}} (x y' - y)^3 = 0.$$

Now we proceed to prove the theorem:

By a collineation the central family is transformed into a central family.

By a collineation the origin is transformed into either a point at finite distance or a point at infinity.

In the former case we may suppose that the origin is invariant, without any loss of generality. Then the collineation takes the form

$$(24) \qquad x_1 = \frac{a x + b y}{a'' x + b'' y + c''}, \quad y_1 = \frac{a' x + b' y}{a'' x + b'' y + c''},$$

or

$$(24)' \qquad x = \frac{A x_1 + A' y_1}{C x_1 + C' y_1 + C''}, \quad y = \frac{B x_1 + B' y_1}{C x_1 + C' y_1 + C''},$$

where $a, b, \cdots\cdots$ are arbitrary constants and $A, B, \cdots\cdots$ the algebraic complements of $a, b, \cdots\cdots$ in the determinant

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

If we put, for convenience,

$$\xi_1 = A x_1 + A' y_1, \quad \eta_1 = B x_1 + B' y_1, \quad \zeta_1 = C x_1 + C' y_1 + C'',$$
$$\xi'_1 = A + A' y'_1, \quad \eta'_1 = B + B' y'_1, \quad \zeta'_1 = C + C' y'_1,$$

where

$$y'_1 = \frac{dy_1}{dx_1}, \qquad y''_1 = \frac{d^2 y_1}{dx_1^2},$$

we have

$$y' = \frac{\zeta_1 \eta'_1 - \eta_1 \zeta'_1}{\zeta_1 \xi'_1 - \xi_1 \zeta'_1}, \qquad y'' = \frac{D_1 \zeta_1^3 y''_1}{(\zeta_1 \xi'_1 - \xi_1 \zeta'_1)^3},$$

D_1 standing for the determinant

$$\begin{vmatrix} A & A' & 0 \\ B & B' & 0 \\ C & C' & C'' \end{vmatrix}.$$

Whence the differential equation (17) becomes

$$h^2 D_1 \zeta_1^2 y''_1 + \frac{F(x, y)}{\sqrt{\xi_1^2+\eta_1^2}} \Big[(B \xi_1 - A \eta_1) + (B' \xi_1 - A' \eta_1) y'_1 \Big]^3 = 0,$$

or

$$h^2 y''_1 + \frac{c''^4 (a b' - a' b)}{\zeta_1^2 \sqrt{\xi_1^2+\eta_1^2}} F(x, y) \cdot (x_1 y'_1 - y_1)^3 = 0.$$

Consequently, if we put

$$(25) \quad \begin{aligned} \phi_1(x_1, y_1) &= \frac{c''(a'' x + b'' y + c'')^2}{(a b' - a' b)^2} \Big[a \phi(x, y) + b \psi(x, y) \Big], \\ \psi_1(x_1, y_1) &= \frac{c''(a'' x + b'' y + c'')^2}{(a b' - a' b)^2} \Big[a' \phi(x, y) + b' \psi(x, y) \Big] \end{aligned}$$

and

$$(26) \quad \phi_1(x_1, y_1) = F_1(x_1, y_1) \frac{x_1}{\sqrt{x_1^2+y_1^2}}, \quad \psi_1(x_1, y_1) = F_1(x_1, y_1) \frac{y_1}{\sqrt{x_1^2+y_1^2}},$$

the above equation becomes

$$h^2 y''_1 + \frac{F_1(x_1, y_1)}{\sqrt{x_1^2+y_1^2}} (x_1 y'_1 - y_1)^3 = 0,$$

which has the same form as (17). Thus it is seen that *the central family* (17) *is transformed into the other central family having the force-components determined by* (25).

Since

$$\frac{\psi_1(x_1, y_1)}{\phi_1(x_1, y_1)} = \frac{a' + b' \dfrac{\psi(x, y)}{\phi(x, y)}}{a + b \dfrac{\psi(x, y)}{\phi(x, y)}},$$

the direction of the force is transformed projectively, which is an obvious result.

In a similar way we can treat the case where the origin is transformed into a point at infinity.

15. Now we consider the accompanied transformation of the time t. Since from (24)

$$\frac{dx_1}{dt_1} = \frac{dx_1}{dt}\frac{dt}{dt_1} = \left[C'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + B'\frac{dx}{dt} - A'\frac{dy}{dt}\right]\frac{1}{(a''x + b''y + c'')^2}\frac{dt}{dt_1}$$

$$= \left(C'h + B'\frac{dx}{dt} - A'\frac{dy}{dt}\right)\frac{1}{(a''x + b''y + c'')^2}\frac{dt}{dt_1},$$

$$\frac{dy_1}{dt_1} = -\left(Ch + B\frac{dx}{dt} - A\frac{dy}{dt}\right)\frac{1}{(a''x + b''y + c'')^2}\frac{dt}{dt_1},$$

we obtain

$$x_1\frac{dy_1}{dt_1} - y_1\frac{dx_1}{dt_1} = \frac{C''}{(a''x + b''y + c'')^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{dt}{dt_1}$$

$$= \frac{hC''}{(a''x + b''y + c'')^2}\frac{dt}{dt_1}.$$

Consequently, in order that the transformed orbits may be the central family having the constant h, it is necessary and sufficient that

$$dt_1 = \frac{ab' - a'b}{(a''x + b''y + c'')^2}\,dt.$$

Thus we infer the theorem: *By* Appell's *transformation* ([1])

([1]) Appell, De l'homographie en mécanique, American Journal of Mathematics, 12 (1890), p. 103.

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