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On the Automecoic Curves of Two Surfaces.

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On the Automecoic Curves of Two Surfaces,

by

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Recently we have had an occasion to consider the automecoic curves of two surfaces ([1]). The present note aims mainly to deal with some properties of these curves.

I.

1. Let the curve-lengths of the automecoic curves of two surfaces S_1 and S_2 be taken for the parameters u and v. Then the linear elements of these surfaces are of the form

$$(1)\qquad \begin{aligned} ds_1^2 &= du^2 + 2\,F_1\,du\,dv + dv^2, \\ ds_2^2 &= du^2 + 2\,F_2\,du\,dv + dv^2, \end{aligned}$$

respectively. Hence the automecoic curves $u = \text{const.}$, $v = \text{const.}$ form an equidistantial system on each surface. Conversely, the corresponding curves belonging to any equidistantial systems on two surfaces are automecoic.

Therefore *the problem of finding the automecoic curves of two surfaces and their curve-lengths is equivalent to that of finding the equidistantial systems on these surfaces.*

2. Now consider the case where the automecoic curves of two surfaces S_1 and S_2 are the asymptotic lines of these surfaces respectively. If the curve-lengths of the asymptotic lines be taken for parameters u, v, then the linear elements of these surfaces are given by (1). But according to a theorem of Prof. A. Voss ([2]), when the asymptotic lines on a surface form an equidistantial system, the surface is necessarily pseudospherical. Hence in our case both S_1 and S_2 must be pseudospherical.

Conversely, it is well known that the linear elements of S_1 and S_2

([1]) K. Ogura, "Notes on the representation of surfaces," Tôhoku Math. Journal, 10 (1916), p. 87.

([2]) A. Voss, "Über ein neues Princip der Abbildung krümmer Oberflächen," Math. Annalen, 19 (1882), p. 1.

having constant negative curvatures $-1/\rho_1^2$ and $-1/\rho_2^2$ respectively take the forms

$$ds_1^2=\rho_1^2\,(U_1\,du_1^2+2\,F_1\,du_1\,dv_1+V_1\,dv_1^2),$$

$$ds_2^2=\rho_2^2\,(U_2\,du_2^2+2\,F_2\,du_2\,dv_2+V_2\,dv_2^2),$$

where these surfaces are referred to their asymptotic lines u_1, v_1; u_2, v_2 respectively, and U_1, V_1, U_2, V_2 are functions of u_1, v_1, u_2, v_2 alone respectively. Hence if we put

$$u=\pm\rho_1\int\sqrt{U_1}\,du_1, \qquad v=\pm\rho_1\int\sqrt{V_1}\,dv_1 \qquad \text{for } S_1$$

and

$$u=\pm\rho_2\int\sqrt{U_2}\,du_2, \qquad v=\pm\rho_2\int\sqrt{V_2}\,dv_2 \qquad \text{for } S_2,$$

the asymptotic lines u, v are automecoic.

Therefore *in order that the asymptotic lines of two surfaces may be automecoic, these two surfaces should be pseudospherical; and we can find the correspondence between any two pseudospherical surfaces such that the asymptotic lines are automecoic.*

For an example, let us consider a surface S whose centro-surfaces S_1, S_2 have the corresponding and automecoic asymptotic lines. Since the asymptotic lines of S_1, S_2 correspond, S must be a Weingarten surface, and therefore we have

$$K_1=-\frac{1}{(R_1-R_2)^2}\frac{d\,R_2}{d\,R_1}, \qquad K_2=-\frac{1}{(R_1-R_2)^2}\frac{d\,R_1}{d\,R_2},$$

$$K_1\,K_2=\frac{1}{(R_1-R_2)^4},$$

where R_1, R_2 denote the principal radii of normal curvature of S and K_1, K_2 the Gauss curvatures of S_1, S_2 respectively. Also since the asymptotic lines of S_1, S_2 are automecoic, K_1 and K_2 must be negative constants; so that we find

$$R_1-R_2=\text{constant}.$$

Conversely, it was already shown by Lie that the asymptotic lines of the centro-surfaces of the surface for which $R_1-R_2=$const. are automecoic [1].

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[1] Bianchi, Vorlesungen über Differentialgeometrie, 1. Aufl. (1899), p. 244.

II.

3. Suppose that any corresponding points P_1 and P_2 of automecoic curves of S_1 and S_2 respectively are at a constant distance $2\,t$. If (x_1, y_1, z_1) be the coordinates of P_1 on S_1, the coordinates of P_2 on S_2 are given by

(2) $$x_2=x_1+2\,t\,\mathfrak{X}, \qquad y_2=y_1+2\,t\,\mathfrak{Y}, \qquad z_2=z_1+2\,t\,\mathfrak{Z},$$

where $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ denote the direction-cosines of the straight line P_1P_2. The totality of P_1P_2 forms a line-congruence which will be called the *congruence Γ*, and the ruled surfaces of Γ which intersect with S_1 and S_2 along the automecoic curves will be called the *ruled surfaces Σ*.

If E_1, F_1, G_1 and E_2, F_2, G_2 be the fundamental coefficients of the first order of S_1 and S_2 respectively, we have

$$E_2=E_1+4\,t\sum\frac{\partial x_1}{\partial u}\frac{\partial\mathfrak{X}}{\partial u}+4\,t^2\sum\left(\frac{\partial\mathfrak{X}}{\partial u}\right)^2,$$

$$G_2=G_1+4\,t\sum\frac{\partial x_1}{\partial v}\frac{\partial\mathfrak{X}}{\partial v}+4\,t^2\sum\left(\frac{\partial\mathfrak{X}}{\partial v}\right)^2;$$

so that it is necessary and sufficient that

$$\sum\frac{\partial x_1}{\partial u}\frac{\partial\mathfrak{X}}{\partial u}+t\sum\left(\frac{\partial\mathfrak{X}}{\partial u}\right)^2=0, \qquad \sum\frac{\partial x_1}{\partial v}\frac{\partial\mathfrak{X}}{\partial v}+t\sum\left(\frac{\partial\mathfrak{X}}{\partial v}\right)^2=0,$$

to secure that the surfaces S_1, S_2 should be referred to the automecoic curves.

Now introducing Kummer's fundamental quantities

$$\mathfrak{E}=\sum\left(\frac{\partial\mathfrak{X}}{\partial u}\right)^2, \qquad \mathfrak{F}=\sum\frac{\partial\mathfrak{X}}{\partial u}\frac{\partial\mathfrak{X}}{\partial v}, \qquad \mathfrak{G}=\sum\left(\frac{\partial\mathfrak{X}}{\partial v}\right)^2,$$

$$\mathfrak{L}=\sum\frac{\partial x_1}{\partial u}\frac{\partial\mathfrak{X}}{\partial u}, \qquad \mathfrak{M}=\sum\frac{\partial x_1}{\partial v}\frac{\partial\mathfrak{X}}{\partial u},$$

$$\mathfrak{M}'=\sum\frac{\partial x_1}{\partial u}\frac{\partial\mathfrak{X}}{\partial v}, \qquad \mathfrak{N}=\sum\frac{\partial x_1}{\partial v}\frac{\partial\mathfrak{X}}{\partial v},$$

the above condition becomes

(3) $$\frac{\mathfrak{L}}{\mathfrak{E}}=\frac{\mathfrak{N}}{\mathfrak{G}}=-t\ (\text{a constant}).$$

This gives *a necessary and sufficient condition that Σ should be the parametric ruled surfaces of the congruence Γ.*

4. Let Σ be taken for the parametric ruled surfaces $u=$const.,

$v=$const.. If we denote by P the point where the shortest distance of the two consecutive straight lines $P_1P_2\,(u, v)$ and $P_1'P_2'\,(u+du,\ v+dv)$ meets the line $P_1P_2\,(u, v)$, the distance r of P from the surface S_1, measured along P_1P_2, is given by

$$r=-\frac{\mathfrak{L}\,du^2+(\mathfrak{M}+\mathfrak{M}')\,du\,dv+\mathfrak{N}\,dv^2}{\mathfrak{E}\,du^2+2\,\mathfrak{F}\,du\,dv+\mathfrak{G}\,dv^2};$$

and therefore (3) becomes

$$(4)\qquad r_{(u=\text{const.})}=r_{(v=\text{const.})}=t\ (^1).$$

Conversely, in the congruence for which

$$r_{(u=\text{const.})}=r_{(v=\text{const.})}=\text{const.},$$

the parametric ruled surfaces are Σ.

When P_1, P_2 are corresponding points of the automecoic curves of S_1, S_2, the locus S of the middle point of P_1P_2 will be called the *automecoic mean surface* of S_1 and S_2.

As a consequence of this definition we can infer from (4) the theorem: *The intersection of the automecoic mean surface S and a ruled surface Σ is the line of striction of the surface Σ* (²).

(¹) The ruled surfaces Σ are isoclinal with respect to the principal surfaces of the congruence Γ.

(²) I take this opportunity to give a *characteristic property of the line of striction* of a ruled surface.

If the coordinates of a point A_0 of the directrix of a ruled surface be x_0, y_0, z_0, expressed in terms of the arc v measured from a point of it, and p, q, r be the direction-cosines of the generator through A_0, then the equations of the ruled surface are

$$x=x_0+pu,\qquad y=y_0+qu,\qquad z=z_0+ru,$$

where u is the distance from A_0 to a point A on the generator through A_0. Hence the linear element of the surface takes the form

$$ds^2=du^2+2\cos\theta\,du\,dv+(P^2u^2+2\,Q\,u+1)\,dv^2,$$

where θ, P, Q are functions of v only.

Since the elements of arc of the curves

$$u=\text{const.}=t\quad\text{and}\quad u=-t$$

are

$$(P^2t^2+2\,Q\,t+1)\,dv^2\quad\text{and}\quad(P^2t^2-2\,Q\,t+1)\,dv^2$$

respectively, these two curves have equal lengths when and only when $Q=0$.

Therefore *a necessary and sufficient condition that a curve on a ruled surface should be the line of striction is that the two curves described by end points of segments of a constant length, measured from the given curve, taken upon generators on both sides of the given curve have equal lengths.*

For example, on the focal surfaces of a pseudospherical congruence, the asymptotic lines correspond and they are automecoic, and also the corresponding points are at a constant distance. But since in this case the automecoic mean surface of the focal surfaces is nothing but the middle surface of the congruence, we have the theorem: *The ruled surfaces Σ of a pseudospherical congruence Γ are generated by the straight lines joining corresponding points of the asymptotic lines on the focal surfaces; and the line of striction of a ruled surface Σ is the intersection of Σ and the middle surface of the congruence Γ.*

III.

5. Consider the two derived surfaces S_1 and S_2 obtained by measuring along the normal two variable distances t and $-t$ from the surface S.

If X, Y, Z be the direction-cosines of the normal, we have

$$\Sigma\,X\,dx=0,\qquad \Sigma\,X^2=1,\qquad \Sigma\,X\,dX=0;$$

so that the linear elements of S_1, S_2 are given by

$$(5)\qquad \begin{aligned} ds_1^2&=ds^2+2\,t\,\Sigma\,dx\,dX+t^2\,\Sigma\,dX^2+dt^2,\\ ds_2^2&=ds^2-2\,t\,\Sigma\,dx\,dX+t^2\,\Sigma\,dX^2+dt^2,\end{aligned}$$

ds standing for the linear element of S.

When the corresponding curves on S_1 and S_2 are automecoic, it follows that

$$\Sigma\,dx\,dX=0,$$

that is, the corresponding curves on S are necessarily the asymptotic ines; and conversely.

Therefore *a necessary and sufficient condition that the corresponding curves on the two derived surfaces S_1 and S_2 obtained by measuring along the positive and negative directions of the normal equal distances from the surfaces S should be automecoic is that the corresponding curves on S be the asymptotic lines. When this condition is satisfied, the lines of striction of the ruled surfaces generated by the straight lines joining corresponding points of the automecoic curves of S_1, S_2 are the asymptotic lines of S.*

As a particular case we have the theorem: *The necessary and sufficient condition that the corresponding curves on the centro-surfaces of a minimal surface should be automecoic is that the corresponding curves on the minimal surface be the asymptotic lines.*

6. *In order that the corresponding curves of S, S_1 and S_2 are* *automecoic, all the surfaces must be parallel planes and then the automecoic curves of these surfaces are straight lines* [1].

In fact, equations (5) may be replaced by

$$ds^2=E\,du^2+2\,F\,du\,dv+G\,dv^2,$$

$$ds_1^{\,2}=\left[E-2\,Lt+e\,t^2+\left(\frac{\partial t}{\partial u}\right)^2\right]du^2$$

$$+2\left[F-2\,Mt+f\,t^2+\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}\right]du\,dv+\left[G-2\,Nt+gt^2+\left(\frac{\partial t}{\partial v}\right)^2\right]dv^2,$$

$$ds_2^{\,2}=\left[E+2\,Lt+e\,t^2+\left(\frac{\partial t}{\partial u}\right)^2\right]du^2$$

$$+2\left[F+2\,Mt+f\,t^2+\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}\right]du\,dv+\left[G+2\,Nt+gt^2+\left(\frac{\partial t}{\partial v}\right)^2\right]dv^2,$$

where $L\,du^2+2\,M\,du\,dv+N\,dv^2$ is the fundamental form of the second order of S, and $e\,du^2+2f\,du\,dv+g\,dv^2$ the square of the linear element of the spherical representation of S.

If the automecoic curves be taken for the parameters, we must have

$$E=E-2\,Lt+e\,t^2+\left(\frac{\partial t}{\partial u}\right)^2=E+2\,Lt+e\,t^2+\left(\frac{\partial t}{\partial u}\right)^2,$$

$$G=G-2\,Nt+g\,t^2+\left(\frac{\partial t}{\partial v}\right)^2=G+2\,Nt+g\,t^2+\left(\frac{\partial t}{\partial v}\right)^2;$$

from which we get

$$(6)\qquad L=0,\qquad N=0,$$

and

$$(7)\qquad et^2+\left(\frac{\partial t}{\partial u}\right)^2=0,\qquad gt^2+\left(\frac{\partial t}{\partial v}\right)^2=0.$$

From (6) it follows that $u=$const., $v=$const. are the asymptotic lines on S. And then since

$$e=\frac{EM^2}{EG-F^2}\geqq 0,\qquad g=\frac{GM^2}{EG-F^2}\geqq 0,$$

we have from (7)

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[1] All curves and surfaces are assumed to be real.

$$(8)\qquad M=0$$

and

$$(9)\qquad t=\text{constant}.$$

It is seen from (6) and (8) that S must be a plane, and from (9) that S_1 and S_2 must be parallel to S. Lastly the fundamental quantities of S_1 and S_2 are

$$E_1=E_2=E,\qquad F_1=F_2=F,\qquad G_1=G_2=G,$$

$$L_1=L_2=0,\qquad M_1=M_2=0,\qquad N_1=N_2=0.$$

Thus our theorem has been proved.

IV.

7. The Weingarten surfaces for which

(i) $R_1-R_2=$const., (ii) $R_1+R_2=0$ (i.e. the minimal surface),

are remarkable in virtue of the property that the centro-surfaces of each of them have equal curvatures at corresponding points respectively. The centro-surfaces of the former have constant negative curvatures and have the automecoic curves which are the asymptotic lines (Art. 2). The centro-surfaces of the latter have positive curvatures and have the automecoic curves which correspond to the asymptotic lines of the original surface (Art. 5).

Here we proceed to prove the theorem: *The automecoic curves of the centro-surfaces of a minimal surface form isothermal conjugate systems.*

If we take the lines of curvature of a minimal surfaces S for the parametric curves $u=$const., $v=$const., we may put

$$E=G,\qquad F=0.$$

Since $R_1+R_2=0$, the fundamental quantities of the centro-surfaces S_1, S_2 of S are

$$(10)\quad\begin{cases} E_1=\left(\dfrac{\partial R_1}{\partial u}\right)^2, & F_1=\dfrac{\partial R_1}{\partial u}\dfrac{\partial R_1}{\partial v}, & G_1=\left(\dfrac{\partial R_1}{\partial v}\right)^2+4\,E, \\ E_2=\left(\dfrac{\partial R_1}{\partial u}\right)^2+4\,E, & F_2=\dfrac{\partial R_1}{\partial u}\dfrac{\partial R_1}{\partial v}, & G_2=\left(\dfrac{\partial R_1}{\partial v}\right)^2, \\ L_1=\dfrac{\sqrt{E}}{R_1}\dfrac{\partial R_1}{\partial u}, & M_1=0, & N_1=\dfrac{\sqrt{E}}{R_1}\dfrac{\partial R_1}{\partial u}, \end{cases}$$

$$L_2=\frac{\sqrt{E}}{R_1}\,\frac{\partial R_1}{\partial v},\qquad M_2=0,\qquad N_2=\frac{\sqrt{E}}{R_1}\,\frac{\partial R_1}{\partial v}.$$

Hence if the asymptotic lines $\bar{u}$, $\bar{v}$ on S be taken as the new parametric curves, we may put

$$\bar{u}=u+v,\qquad \bar{v}=u-v;$$

and then

$$\bar{E}_1=\bar{E}_2=\left(\frac{\partial R_1}{\partial u}+\frac{\partial R_1}{\partial v}\right)^2+4\,E,\qquad \bar{G}_1=\bar{G}_2=\left(\frac{\partial R_1}{\partial u}-\frac{\partial R_1}{\partial v}\right)^2+4E,$$

$$\bar{F}_1=\left(\frac{\partial R_1}{\partial u}\right)^2-\left(\frac{\partial R_1}{\partial v}\right)^2-4\,E,\qquad \bar{F}_2=\left(\frac{\partial R_1}{\partial u}\right)^2-\left(\frac{\partial R_1}{\partial v}\right)^2+4\,E,$$

$$\bar{L}_1=\bar{N}_1=-2\frac{\sqrt{E}}{R_1}\,\frac{\partial R_1}{\partial u},\qquad \bar{L}_2=\bar{N}_2=-2\frac{\sqrt{E}}{R_1}\,\frac{\partial R_1}{\partial v},$$

$$\bar{M}_1=0,\qquad \bar{M}_2=0.$$

Therefore $\bar{u}=$const., $\bar{v}=$const. are the automecoic curves of S_1, S_2 (which coincide with the result obtained in Art. 5); and they form isotherma conjugate systems on S_1 and S_2.

8. Lastly we add a theorem on Weingarten surfaces.

Since equations (10) give

$$L_1=N_1,\quad M_1=0;\qquad L_2=N_2,\quad M_2=0,$$

the lines of curvature on a minimal surface correspond to isotherma conjugate systems of its centro-surfaces. Here we will prove the theorem:

A necessary condition that the curves on the centro-surfaces S_1, S_2 of a surface S, corresponding to the lines of curvature on S, form isothermal conjugate systems is that S be a Weingarten surface.

For, if the lines of curvature on S be taken for the parametric curves, the fundamental quantities of the second order of S_1 and S_2 have the expressions:

$$L_1=\frac{\sqrt{E}}{R_1}\,\frac{\partial R_1}{\partial u},\qquad M_1=0,\qquad N_1=-\frac{G}{\sqrt{E}}\,\frac{R_1}{R_2^2}\,\frac{\partial R_2}{\partial u},$$

$$L_2=-\frac{E}{\sqrt{G}}\,\frac{R_2}{R_1^2}\,\frac{\partial R_1}{\partial v},\qquad M_2=0,\qquad N_2=\frac{\sqrt{G}}{R_2}\,\frac{\partial R_2}{\partial v}.$$

From the assumption we have

$$L_1=N_1,\qquad L_2=N_2;$$

consequently

$$\frac{E}{R_1^2}\,\frac{\partial R_1}{\partial u}+\frac{G}{R_2^2}\,\frac{\partial R_2}{\partial u}=0,\qquad \frac{E}{R_1^2}\,\frac{\partial R_1}{\partial v}+\frac{G}{R_2^2}\,\frac{\partial R_2}{\partial v}=0.$$

From these equations it follows that

$$\begin{vmatrix} \dfrac{\partial R_1}{\partial u} & \dfrac{\partial R_2}{\partial u} \\ \dfrac{\partial R_1}{\partial v} & \dfrac{\partial R_2}{\partial v} \end{vmatrix}=0,$$

which proves the theorem.

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