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Note on the Representation of Surfaces.

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Note on the Representation of Surfaces,

by

Kinnosuke Ogura in Sendai.

Conformal representation and the principal radii of normal curvature.

1. Bonnet solved the problem of finding applicable surfaces with equal radii of principal curvature at corresponding points. Here we proceed to consider the necessary and sufficient condition that the two surfaces S and S', which are conformally related to each other, should have either one of the relations

$$(1)\qquad \left.\begin{matrix} R'_1=R_1 \\ R'_2=R_2 \end{matrix}\right\},\quad \left.\begin{matrix} R'_1=-R_1 \\ R'_2=-R_2 \end{matrix}\right\},\quad \left.\begin{matrix} R'_1=R_2 \\ R'_2=R_1 \end{matrix}\right\},\quad \left.\begin{matrix} R'_1=-R_2 \\ R'_2=-R_1 \end{matrix}\right\},$$

where R_1, R_2; R'_1, R'_2 respectively denote the principal radii of normal curvature of S and S' at corresponding points.

In order that there may exist either one of the relations (1), it must be

$$(2)\qquad \frac{1}{R'_1\,R'_2}=\frac{1}{R_1\,R_2}$$

and

$$(3)\qquad \left(\frac{1}{R'_1}-\frac{1}{R'_2}\right)^2=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2.$$

But we have the following theorem, due to Prof. Stäckel(1): If two surfaces S, S' be conformally related to each other and have the equal Gauss measures of curvature K, K' at corresponding points, then the (four) directions, for which the squares of corresponding normal curvatures $\frac{1}{R}$, $\frac{1}{R'}$ are equal, form a harmonic pencil at any point on any one of the given surfaces; and conversely.

Hence it is sufficient for our purpose to find the condition that

(1) P. Stäckel, "Beiträge zur Flächentheorie," Leipziger Berichte, 48 (1896), pp. 498–9.

these surfaces S, S' should have the relation (3) at corresponding points.

2. Let us suppose that corresponding points of S and S' have the same parameters u and v. Since the correspondence is conformal, we have

$$(4)\qquad E'=\kappa E,\quad F'=\kappa F,\quad G'=\kappa G,$$

where E, F, G; $E', F'\ G'$ are the fundamental quantities of the first order of S, S' respectively, and κ is a function of u, v.

Now the equation to the directions, for which the squares of corresponding geodesic torsions $\frac{1}{T}$, $\frac{1}{T'}$ are equal, takes the form

$$(5)\qquad \left\{\frac{(F'L'-E'M')\,dv^2+(G'L'-E'N')\,du\,dv+(G'M'-F'N')\,dv^2}{(E'\,du^2+2\,F\,du\,dv+G'\,dv^2)\sqrt{E'G'-F'^2}}\right\}^2$$
$$=\left\{\frac{(FL-EM)\,du^2+(GL-EN)\,du\,dv+(GM-FN)\,dv^2}{(E\,du^2+2\,F\,du\,dv+G\,dv^2)\sqrt{EG-F^2}}\right\}^2,$$

L, M, N; L', M', N' being the fundamental quantities of the second order of S, S' respectively. In consequence of (4), equation (5) reduces to the two quadratic equations in $dv:du$

$$(6)\qquad [(FL'-EM')\pm\kappa\,(FL-EM)]\,du^2+[(GL'-EN'\pm\kappa\,(GL-EN)]\,du\,dv$$
$$+[(GM'-FN')\pm\kappa\,(GM-FN)]\,dv^2=0\ (^1).$$

In order that the four directions given by (6) may be harmonic, it is necessary and sufficient that

$$[(FL'-EM')+\kappa\,(FL-EM)]\,[(GM'-FN')-\kappa\,(GM-FN)]$$
$$+[(FL'-EM')-\kappa\,(FL-EM)]\,[(GM'-FN')+\kappa\,(GM-FN)]$$
$$-2[(GL'-EN')+\kappa\,(GL-EN)]\,[(GL'-EN')-\kappa\,(GL-EN)]=0,$$

that is,

$$(FL'-EM')\,(GM'-FN')-(GL'-EN')^2$$
$$=\kappa^2\,[(FL-EM)\,(GM-FN)-(GL-EN)^2];$$

and hence

(1) The tangents to the curves (6) belong to the involution whose double rays are

$$\begin{vmatrix} dv^2 & -2\,du\,dv & du^2 \\ FL-EM & GL-EN & GM-FN \\ F'L'-E'M' & G'L'-E'N' & G'M'-F'N' \end{vmatrix}=0\,.$$

The tangents to the lines of curvature also belong to this involution. See Stäckel, "Über Abbildungen," Math. Ann., 44 (1894), p. 553.

$$\frac{(F'L'-E'M')\,(G'M'-F'N')-(G'L'-E'N')^2}{(E'G'-F'^2)^2}$$
$$=\frac{(FL-EM)\,(GM-FN)-(GL-EN)^2}{(EG-F'^2)^2},$$

i.e.

$$(3)\qquad \left(\frac{1}{R'_1}-\frac{1}{R'_2}\right)^2=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2.$$

As the analysis is manifestly reversible, we get the theorem: *If two surfaces S and S', which are conformally related to each other, have the relation*

$$\left(\frac{1}{R'_1}-\frac{1}{R'_2}\right)^2=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2$$

at corresponding points, then the directions, for which the squares of corresponding geodesic torsions are equal, form a harmonic pencil; and conversely.

Combining this with Stäckel's theorem, we arrive at the following result:

When two surfaces S and S' are conformally related to each other, it is necessary and sufficient that the directions for which

$$R^2=R'^2$$

should form a harmonic pencil and moreover the directions for which

$$T^2=T'^2$$

should form another harmonic pencil, in order to secure that these two surfaces should have either one of the relations

$$\left.\begin{matrix}R'_1=R_1\\ R'_2=R_2\end{matrix}\right\},\quad \left.\begin{matrix}R'_1=-R_1\\ R'_2=-R_2\end{matrix}\right\},\quad \left.\begin{matrix}R'_1=R_2\\ R'_2=R_1\end{matrix}\right\},\quad \left.\begin{matrix}R'_1=-R_2\\ R'_2=-R_1\end{matrix}\right\}$$

at corresponding points.

Equivalent representation and the Gauss measure of curvature.

3. Consider any two surfaces S and S' which are not conformally related, and let ds and ds' be corresponding linear elements of these surfaces respectively.

The $2\infty^1$ curves defined by

$$(7) \qquad ds'^2 = ds^2,$$

i.e.

$$(8) \qquad (E'-E)\,du^2 + 2\,(F'-F)\,du\,dv + (G'-G)\,dv^2 = 0$$

will be called the *automecoic curves*, and these defined by

$$(9) \qquad ds'^2 = -ds^2,$$

i.e.

$$(10) \qquad (E'+E)\,du^2 + 2\,(F'+F)\,du\,dv + (G'+G)\,dv^2 = 0$$

the *anti-automecoic curves*.

Now in order that the automecoic curves passing through any point (u, v) may be harmonic conjugate with the anti-automecoic curves passing through that point, it is necessary and sufficient that

$$(E'-E)\,(G'+G) + (E'+E)\,(G'-G) - 2\,(F'-F)\,(F'+F) = 0,$$

that is,

$$E'G' - F'^2 = EG - F^2.$$

Consequently we have the theorem:

The necessary and sufficient condition that any non-conformal representation should be equivalent is that the automecoic curves and the anti-automecoic curves passing through any point form a harmonic pencil.

It will be noticed that the directions of the automecoic curves and the anti-automecoic curves belong to the involution whose double rays are given by

$$(11) \qquad \begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ E' & F' & G' \end{vmatrix} = 0,$$

that is, the *principal curves* of Tissot. The directions of the minimal lines of the two surfaces S, S' respectively

$$E\,du^2 + 2\,F\,du\,dv + G\,dv^2 = 0 \quad \text{and} \quad E'\,du^2 + 2\,F'\,du\,dv + G'\,dv^2 = 0$$

belong also to this involution.

4. Next we consider the spherical representations of S and S', and let their linear elements be

$$d\sigma^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2, \quad d\sigma'^2 = e'\,du^2 + 2f'\,du\,dv + g'\,dv^2$$

respectively.

I. When

$$e' = \pm e, \qquad f' = \pm f, \qquad g' = \pm g,$$

we have

$$e'g' - f'^2 = eg - f^2.$$

II. In the other case, we can prove that when and only when

$$e'g' - f'^2 = eg - f^2,$$

the curves

$$d\sigma'^2 = d\sigma^2 \quad \text{and} \quad d\sigma'^2 = -d\sigma^2$$

form a harmonic pencil at any point (u, v).

Hence recalling the relation

$$eg - f^2 = K^2\,(EG - F^2),$$

we have the theorem:

The necessary and sufficient condition that two surfaces S and S', which are connected by an equivalent representation, should have the relation

$$K^2 = K'^2$$

between the Gauss curvatures at corresponding points, is that either

I. *all the corresponding curves of the spherical representations of S, S' are automecoic or anti-automecoic*; *or*

II. *the automecoic curves and the anti-automecoic curves of the spherical representations of S, S' form a harmonic pencil at corresponding points.*

Also, *the squares of Gauss curvatures of any two surfaces are equal at corresponding points, when the automecoic curves and the anti-automecoic curves form a harmonic pencil and moreover the automecoic curves and the anti-automecoic curves of the spherical representations form another harmonic pencil.*

5. Here we will give some examples.

I. Let us take the two surfaces of translation (1)

(1) S. Nakagawa, "Zur Theorie des Gauss'schen Krümmungsmasses," Proceedings of the Tôkyô Mathematico-Physical Society, II, 4 (1907-8), p. 183. In this example, corresponding normals to the two surfaces are parallel. For such an equivalent representation, see C. Guichard, "Sur les surfaces qui se correspondent avec parallélisme des plans tangents et conservation des aires," Comptes Rendus, (1903), p. 151.

$$(EG-F^2)\lambda^4-(EG'+GE'-2FF')\lambda^2+(E'G'-F'^2)=0.$$

Hence

$$\lambda_1^2\lambda_2^2=\frac{E'G'-F'^2}{FG-F^2}, \tag{15}$$

$$\frac{1}{2}(\lambda_1^2+\lambda_2^2)=\frac{EG'+GE'-2FF'}{2(EG-F^2)}. \tag{16}$$

We will call these quantities the *total quadratic magnification* and the *mean quadratic magnification* respectively.

Thus *the necessary and sufficient condition that any non-conformal representation should be equivalent is that the total quadratic magnification is equal to unity.* (For the analogous case with respect to the mean quadratic magnification, see § 8).

7. Particularly, let us suppose that the spherical representation of S has been taken for the surface S'. Then the linear element of S' is given by

$$ds'^2=d\sigma^2=e\,du^2+2f\,du\,dv+g\,dv^2$$

$$=(HL-KE)\,du^2+2(HM-KF)\,du\,dv+(HN-KG)\,dv^2,$$

where K and H denote the total and mean curvatures of S respectively. Hence the differential equation of the principal curves becomes

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix}=0\,(^1);$$

and we have

$$\lambda_1^2\lambda_2^2=\frac{eg-f^2}{EG-F^2}=K=\frac{1}{R_1R_2}, \tag{17}$$

$$\frac{1}{2}(\lambda_1^2+\lambda_2^2)=\frac{Eg+Ge-2Ff}{2(EG-F^2)}=\frac{1}{2}H^2-K=\frac{1}{2}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right). \tag{18}$$

Therefore we arrive at the theorem:

The directions corresponding to the extremes of the quadratic magnification of the spherical representation of a surface (excluding the minimal surface) with respect to the given surface are the tangents to the lines of curvature of the given surface.

(1) When S is a minimal surface, the principal curves become indeterminate.

The total quadratic magnification and the mean quadratic magnification of the spherical representation with respect to a surface are the Gauss curvature and the Casorati curvature (1) *of the given surface respectively.*

Infinitesimal equivalent representation.

8. Let (x, y, z) and (x', y', z') respectively be the co-ordinates of two surfaces S and S', the latter being obtained from the former by a very small deformation. If we put

$$x'=x+\varepsilon x_1,\quad y'=y+\varepsilon y_1,\quad z'=z+\varepsilon z_1, \tag{19}$$

where ε denotes a small constant and x_1, y_1, z_1 are functions of u and v, then

$$E'=E+2\varepsilon\sum\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial u}+\varepsilon^2E_1,\quad G'=G+2\varepsilon\sum\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial v}+\varepsilon^2G_1,$$

$$F'=F+\varepsilon\sum\left(\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}\right)+\varepsilon^2F_1;$$

so that

$$E'G'-F'^2=EG-F^2+2\varepsilon\Big[G\sum\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial u}+E\sum\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial v}$$
$$-F\sum\left(\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}\right)\Big]+\text{terms of higher orders in }\varepsilon.$$

If

$$G\sum\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial u}+E\sum\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial v}=F\sum\left(\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}\right), \tag{20}$$

corresponding small areas of S and S' are equal to within terms of the second order in ε.

When ε is taken so small that ε^2 may be neglected, the surface S' defined by (19) and (20) will be said to have been derived from S by an *infinitesimal equivalent representation.*

In consequence of

(1) Casorati, "Mesure de la courbure des surfaces suivant l'idée commune," Acta Mathematica, 14 (1890-91), p. 95.

$$2\varepsilon\sum\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial u}=E'-E,\quad 2\varepsilon\sum\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial v}=G'-G,$$

$$\sum\left(\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}\right)=F'-F,$$

equation (20) becomes

$$(21)\qquad GE'+EG'-2\,FF'=2\,(EG-F^2),$$

which is nothing but the necessary and sufficient condition that the representation (19) should be infinitesimally equivalent. Thus we have from § 6 the theorem :

When S' is derived from S by a very small deformation, a necessary and sufficient condition that S and S' should be infinitesimally equivalent is that the mean quadratic magnification of S' with respect to S is equal to unity.

9. If the automecoic curves of S and S'

$$(E'-E)\,du^2+2\,(F'-F)\,du\,dv+(G'-G)\,dv^2=0$$

and the minimal lines of S

$$E\,du^2+2\,F\,du\,dv+G\,dv^2=0$$

be harmonic at any point $(u,\ v)$, it must be

$$(E'-E)\,G+(G'-G)\,E-2\,(F'-F)\,F=0,$$

i.e.

$$GE'+EG'-2\,FF'=2\,(EG-F^2)\,;$$

and conversely.

Hence, *when S' is obtained from S by a very small deformation, a necessary and sufficient condition that S and S' should be infinitesimally equivalent is that the automecoic curves of S, S' and the minimal lines on S form a harmonic pencil at any point $(u,\ v)$.*

Here we add an example of a family of surfaces of translation, which are infinitesimally equivalent to each other :

$$x=a\,U_1+b\,V_1,\qquad y=b\,U_1-a\,V_1,\qquad z=U_2\,;$$

$$x'=x+\varepsilon\,(b\,U_3+a\,V_2),\quad y'=y+\varepsilon\,(-a\,U_3+b\,V_2),\quad z'=z+\varepsilon\,V_3,$$

where a and b are constants, and the U's and V's are functions of u and v alone respectively.

10. I. It is well known that for the infinitesimal isometric representation the Gauss curvature is invariantive to within terms of the

second order in ε. But the converse is not necessarily true, even when the representation is infinitesimally equivalent.

For example, from the general surface of revolution

$$x=u\cos v,\quad y=u\sin v,\quad z=\phi(u)$$

we can derive the helicoid

$$x'=u\cos v,\quad y'=u\sin v,\quad z'=\phi(u)+\varepsilon\,v$$

by the infinitesimal equivalent representation. Although these two surfaces have the equal Gauss curvatures up to terms of the second order in ε, they are not infinitesimally isometric ; for

$$E=1+\phi_u^2(u),\quad F=0,\qquad G=u^2\,;$$

$$E'=1+\phi_u^2(u),\quad F'=\varepsilon\,\phi_u(u),\quad G'=u^2+\varepsilon^2\,;$$

$$EG'+GE'-2\,FF'=2\,(EG-F^2),$$

$$K=\frac{\phi_u(u)\,\phi_{uu}(u)}{u\,[1+\phi_u^2(u)]^2},\qquad K'=\frac{u^3\,\phi_u(u)\,\phi_{uu}(u)-\varepsilon^2}{[u^2+u^2\,\phi_u^2(u)+\varepsilon^2]^2}.$$

As the second example, we can take

$$x=au,\qquad y=bv,\qquad z=a\,\phi(u)+b\,\psi(v),$$

$$x'=a\,(1+\varepsilon)\,u,\quad y'=b\,(1-\varepsilon)\,v,\quad z'=a\,(1+\varepsilon)\,\phi(u)+b\,(1-\varepsilon)\,\psi(v),$$

a, b being constants.

II. Now we consider the parallel surface S' obtained by measuring along the normal an infinitesimally small distance ε from the surface S. If X, Y, Z be the direction-cosines of the normal to S, then

$$x'=x+\varepsilon\,X,\quad y'=y+\varepsilon\,Y,\quad z'=z+\varepsilon\,Z\,;$$

and therefore

$$E'G'-F'^2=(1-\varepsilon\,H+\varepsilon^2\,K)^2\,(EG-F^2),$$

$$K'=\frac{K}{1-\varepsilon\,H+\varepsilon^2\,K}.$$

Hence *a necessary and sufficient condition that the parallel surface S' at an infinitesimally small distance ε from the surface S should have the same Gauss curvature as S is that S' may be obtained from S by an infinitesimal equivalent representation ; and such a case occurs when and only when S is a minimal surface.*

11. Lastly, let us consider a surface S which can be obtained from its spherical representation S' by a very small deformation. Then

$$x'=X=x+\varepsilon x_1, \quad y'=Y=y+\varepsilon y_1, \quad z'=Z=z+\varepsilon z_1.$$

If S' may be obtained from S by an infinitesimal equivalent representation, we must have by (21)

$$Eg+Ge-2Ff=2(EG-F^2),$$

that is,

$$\frac{1}{2}\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)=1;$$

and conversely.

Therefore *it is necessary and sufficient that the Casorati curvature of a surface should be equal to unity, in order to secure that the surface should be infinitesimally equivalent to its spherical representation.*

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