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On the Differential Geometry of Inversion.

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On the Differential Geometry of Inversion,

by

KINNOSUKE OGURA in Sendai.

The main object of this note is to introduce two differential forms of a surface, which are appropriate to the investigation of invariantive properties of the surface for inversion.

Without any loss of generality we may suppose that the centre of inversion is the origin of coordinates x, y, z and its power is -1; that is, the inversion is defined by

$$\bar{x}=-\frac{x}{x^2+y^2+z^2}, \qquad \bar{y}=-\frac{y}{x^2+y^2+z^2}, \qquad \bar{z}=-\frac{z}{x^2+y^2+z^2}.$$

1. Let

$$\psi_1=E\,du^2+2\,F\,du\,dv+G\,dv^2,$$

$$\psi_2=L\,du^2+2\,M\,du\,dv+N\,dv^2$$

be the ordinary fundamental forms of a surface S. Then the six quantities

$$E_1=\frac{E}{\sqrt{EG-F^2}}, \qquad F_1=\frac{F}{\sqrt{EG-F^2}}, \qquad G_1=\frac{G}{\sqrt{EG-F^2}},$$

$$L_1=\frac{FL-EM}{\sqrt[4]{EG-F^2}^3}, \qquad 2M_1=\frac{GL-EN}{\sqrt[4]{EG-F^2}^3}, \qquad N_1=\frac{GM-FN}{\sqrt[4]{EG-F^2}^3}$$

are absolute invariants for the inversion([1]). Moreover, by a theorem due to Prof. R. Rothe, the above six quantities only are the essential surface-theoretic invariants for the inversion, which depend upon the fundamental quantities E, F, G, L, M, N alone; and all the invariants, which depend upon the fundamental quantities alone, can be expressed as functions of E_1, F_1, G_1, L_1, M_1, N_1.

Now if we put

$$(1) \qquad e=\mu E, \qquad f=\mu F, \qquad g=\mu G,$$

([1]) R. Rothe, "Über die Inversion einer Fläche, etc.", Math. Ann., 72 (1912), p. 57; A. R. Forsyth, Lectures on the differential geometry (1912), pp. 105–6.

$$l=\nu(FL-EM), \qquad 2m=\nu(GL-EN), \qquad n=\nu(GM-FN),$$

where

$$\mu=\frac{(GL-EN)^2-4(FL-EM)(GM-FN)}{(EG-F^2)^2},$$

$$\nu=\frac{\sqrt{(GL-EN)^2-4(FL-EM)(GM-FN)}}{\sqrt{EG-F^2}^{\,3}},$$

we have

$$e=4(M_1^2-L_1N_1)\,E_1, \qquad f=4(M_1^2-L_1N_1)\,F_1, \qquad g=4(M_1^2-L_1N_1)\,G_1,$$

$$l=2\sqrt{M_1^2-L_1N_1}.L_1, \quad m=2\sqrt{M_1^2-L_1N_1}.M_1, \quad n=2\sqrt{M_1^2-L_1N_1}.N_1\,;$$

or

$$(2)\qquad \begin{aligned} &E_1=\frac{e}{2\sqrt{m^2-ln}}, \qquad F_1=\frac{f}{2\sqrt{m^2-ln}}, \qquad G_1=\frac{g}{2\sqrt{m^2-ln}},\\ &L_1=\frac{l}{\sqrt{2\sqrt[4]{m^2-ln}}}, \quad M_1=\frac{m}{\sqrt{2\sqrt[4]{m^2-ln}}}, \quad N_1=\frac{n}{\sqrt{2\sqrt[4]{m^2-ln}}}. \end{aligned}$$

Now, for the sake of brevity, we will call

(I) $\qquad f_1=e\,du^2+2f\,du\,dv+g\,dv^2,$

(II) $\qquad f_2=l\,du^2+2\,m\,du\,dv+n\,dv^2$

the fundamental forms of the surface for inversion. The discriminants of these forms are

$$\Delta(f_1)=eg-f^2>0, \qquad \Delta(f_2)=ln-m^2=-\frac{1}{4}\Delta(f_1)<0$$

respectively, and the simultaneous invariant is

$$\theta(f_1, f_2)=en-2fm+gl=0.$$

The above definition leads us to the following theorem immediately :

The coefficients of the two fundamental forms for inversion only are the essential surface-theoretic invariants for inversion, which depend upon the fundamental quantities (in the ordinary sense) alone.

2. In order to consider the geometrical meaning of the fundamental forms for inversion f_1 and f_2, we will introduce the notion of semi-osculating sphere due to Prof. G. Demartres(¹).

(¹) Demartres, "Sur la torsion sphérique des courbes gauches et la torsion géodésique des lignes tracées sur une surface," Bulletin des Sciences math., II, 21 (1897), p. 182; Demartres, Cours de géométrie infinitésimale (1913), p. 263, p. 434.

Let C be the centre of normal curvature for the direction of the tangent to a surface-curve Γ at a point P on a surfaces S. Then the sphere σ, which passes through the point P and has the point C as its centre, is called *the semi-osculating sphere* at the point P for the curve Γ. For the inverse surface $\overline{S}$, if we denote by C' the centre of the normal curvature for the direction of the tangent to the inverse curve $\overline{\Gamma}$ at the inverse point $\overline{P}$, then the two centres C and C' lie on a right line with the centre of inversion(¹). Hence, for the inverse surfaces $\overline{S}$, the semi-osculating sphere $\overline{\sigma}$ at $\overline{P}$ for the curve $\overline{\Gamma}$ is the inverse of the sphere σ.

Suppose that $d\tau$ is the angle of geodesic torsion of the linear element $PP'=ds$ of Γ, and σ, σ' are the semi-osculating spheres at P and P' respectively. Also let R_1, R_2 be the principal radii of normal curvature at P ; and let σ_1, σ_2 be the semi-osculating spheres for the directions corresponding to R_1, R_2 at P(²), and σ_1', σ_2' be the corresponding spheres at the point P'. If φ be the angle between the tangent to Γ and the direction corresponding to R_1 at P ; and (σ, σ'), (σ_1, σ_1'), (σ_2, σ_2') be the angles between each pair of the spheres σ, σ' ; σ_1, σ_1'; σ_2, σ_2' respectively, then

$$(3)\qquad d\tau=PP'\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\sin\varphi\cos\varphi=(\sigma, \sigma'),$$

$$(4)\quad PP'\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\sin\varphi=(\sigma_1, \sigma_1'), \quad PP'\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\cos\varphi=(\sigma_2, \sigma_2');$$

so that

$$(5)\qquad PP'\left(\frac{1}{R_2}-\frac{1}{R_1}\right)=\frac{(\sigma_1, \sigma_1')\,(\sigma_2, \sigma_2')}{(\sigma, \sigma')},$$

$$(6)\qquad \frac{1}{(\sigma, \sigma')^2}=\frac{1}{(\sigma_1, \sigma_1')^2}+\frac{1}{(\sigma_2, \sigma_2')^2}.$$

Now since

$$\frac{1}{R_2}-\frac{1}{R_1}=\sqrt{\mu}, \qquad f_1=\mu\,ds^2,$$

we have from (5)

(¹) Salmon, A treatise on the analytic geometry of three dimensions, 5. ed., Vol. 2 (1915), p. 157.

(²) Since the lines of curvature on $\overline{S}$ correspond to the lines of curvature on S, two semi-osculating spheres $\overline{\sigma}_1, \overline{\sigma}_2$ for $\overline{S}$ are the inverses of the corresponding spheres σ_1, σ_2 for S.

$$(7)\qquad f_1=\frac{(\sigma_1,\sigma_1')^2\,(\sigma_2,\sigma_2')^2}{(\sigma,\sigma')^2}.$$

Comparing the following two expressions for the geodesic torsion $\frac{1}{T}$

$$\frac{1}{T}=\frac{d\tau}{ds}=\frac{1}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\sin 2\varphi,$$

$$\frac{1}{T}=\frac{\mu}{\nu\sqrt{EG-F^2}}\,\frac{f_2}{f_1}=\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\frac{f_2}{f_1},$$

we find

$$(8)\qquad \frac{f_2}{f_1}=\frac{1}{2}\sin 2\varphi \text{ (}^1\text{)}.$$

From (3), (5), (8) and (7) we have immediately

$$(\sigma,\sigma')=PP'\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\cdot\frac{\sin 2\varphi}{2}=\frac{(\sigma_1,\sigma_1')\,(\sigma_2,\sigma_2')}{(\sigma,\sigma')}\cdot\frac{f_2}{f_1}$$

$$=\frac{(\sigma_1,\sigma_1')\,(\sigma_2,\sigma_2')}{(\sigma,\sigma')}\cdot\frac{(\sigma,\sigma')^2}{(\sigma_1,\sigma_1')^2\,(\sigma_2,\sigma_2')^2}\cdot f_2;$$

whence it follows that

$$(\mathrm{II}')\qquad f_2=(\sigma_1,\sigma_1')\,(\sigma_2,\sigma_2').$$

Also we have from (7) and (II′)

$$\frac{f_2^2}{f_1}=(\sigma,\sigma')^2;$$

but since

$$\frac{1}{(\sigma,\sigma')^2}=\frac{(\sigma_1,\sigma_1')^2+(\sigma_2',\sigma_2')^2}{(\sigma_1,\sigma_1')^2\,(\sigma_2,\sigma_2')^2},\qquad f_2^2=(\sigma_1,\sigma_1')^2\,(\sigma_2,\sigma_2')^2,$$

we obtain

$$(\mathrm{I}')\qquad f_1=(\sigma_1,\sigma_1')^2+(\sigma_2,\sigma_2')^2.$$

Therefore we may take

(¹) Compare with

$$\frac{\psi_2}{\psi_1}=\frac{1}{R},$$

where R denotes the radius of normal curvature.

$$(9)\qquad \begin{aligned}(\sigma_1,\sigma_1')^2&=\frac{1}{2}\left(f_1+\varepsilon\sqrt{f_1^2-4f_2^2}\right),\\ (\sigma_2,\sigma_2')^2&=\frac{1}{2}\left(f_1-\varepsilon\sqrt{f_1^2-4f_2^2}\right),\end{aligned}$$

where $\varepsilon=+1$ or -1 according as $(\sigma_1,\sigma_1')^2>$ or $<(\sigma_2,\sigma_2')^2$.

Thus *by* (I′) *and* (II′) *we have found the geometrical meaning of the fundamental forms for inversion.*

3. The Jacobian of f_1 and f_2, viz.,

$$J(f_1,f_2)=\begin{vmatrix} e\,du+f\,dv & f\,du+g\,dv\\ l\,du+m\,dv & m\,du+n\,dv\end{vmatrix}$$

is connected with f_1 and f_2 by the well known identity

$$J^2(f_1,f_2)=-f_1^2\,\Delta(f_2)+f_1f_2\,\Theta(f_1,f_2)-f_2^2\,\Delta(f_1)$$

$$=\frac{eg-f^2}{4}\,(f_1^2-4f_2^2).$$

Whence

$$(10)\qquad \frac{J(f_1,f_2)}{\sqrt{eg-f^2}}=\pm\frac{(\sigma_1,\sigma_1')^2-(\sigma_2,\sigma_2')^2}{2}.$$

But since

$$(\sigma_2,\sigma_2')^2-(\sigma_1,\sigma_1')^2=\overline{PP'}^2\left(\frac{1}{R_2}-\frac{1}{R_1}\right)^2(\cos^2\varphi-\sin^2\varphi)=f_1\cos 2\varphi,$$

we have

$$(11)\qquad \frac{J(f_1,f_2)}{\sqrt{eg-f^2}.f_1}=\mp\frac{1}{2}\cos 2\varphi.$$

Consequently, if θ be the angle between the tangent to Γ and the direction to the first principal radius of geodesic curvature T_1 (¹), then

$$\theta=\varphi-\frac{\pi}{4};$$

and hence

$$(12)\qquad \frac{J(f_1,f_2)}{\sqrt{eg-f^2}.f_1}=\mp\frac{1}{2}\sin 2\theta \text{ (}^2\text{)}.$$

(¹) T. Hayashi, "On the usual parametric curves on a surface," Science reports of the Tôhoku Imperial University, Series I, 5 (1916), pp. 65-66.

(²) Compare with

$$\frac{J(\psi_1,\psi_2)}{\sqrt{EG-F^2}.\psi_1}=-\frac{1}{T}.$$

The ambiguity of signs in equations (10), (11) and (12) depends upon the selection of parametric curves. For instance, suppose that the directions corresponding to R_1, R are taken for $v=$const., $u=$const. respectively. In this case when φ increases from 0 to $\frac{\pi}{4}$, the values of $\sin 2\varphi$ and $\frac{dv}{du}$ [1] increase monotonously; also since

$$(8) \qquad \frac{f_2}{f_1}=\frac{1}{2}\sin 2\varphi,$$

$\frac{f_2}{f_1}$ must increase monotonously. Differentiating this equation (8) with respect to φ, we see that $J(f_1, f_2)$ must be positive within the interval $0<\varphi<\frac{\pi}{4}$. On the other hand, in this interval $\cos 2\varphi$ is positive; hence by (11) in this interval the lower sign must be taken. In exactly the same way, we can show that when the parametric curves are chosen as above, we must take the lower sign in the whole interval $0\leqq\varphi<2\pi$.

4. Let ω be *the angle between the two linear elements* PP' and PP'' whose directions are determined by $dv:du$ and $\delta v:\delta u$ respectively at P. If these two directions make the angles φ' and φ'' respectively with the direction corresponding to the first radius of normal curvature R_1 at P, then

$$\cos\omega=\cos(\varphi'-\varphi'')=\cos\varphi'\cos\varphi''+\sin\varphi'\sin\varphi''.$$

Hence we have from (4) and (I') the expression

$$(13) \qquad \cos\omega=\frac{(\sigma_1, \sigma_1')(\sigma_1, \sigma_1'')+(\sigma_2, \sigma_2')(\sigma_2, \sigma_2'')}{\sqrt{(\sigma_1, \sigma_1')^2+(\sigma_2, \sigma_2')^2}\sqrt{(\sigma_1, \sigma_2'')^2+(\sigma_2, \sigma_2'')^2}}.$$

Also, since

$$\cos\omega=\frac{e\,du\,\delta u+f(du\,\delta v+dv\,\delta u)+g\,dv\,\delta v}{\sqrt{e\,du^2+f\,du\,dv+g\,dv^2}\sqrt{e\,\delta u^2+2f\,\delta u\,\delta v+g\,\delta v^2}},$$

it follows that

$$(14) \qquad e\,du\,\delta u+f(du\,\delta v+dv\,\delta u)+g\,dv\,\delta v$$
$$=(\sigma_1, \sigma_1')(\sigma_1, \sigma_1'')+(\sigma_2, \sigma_2')(\sigma_2, \sigma_2'').$$

(1) $\mathrm{tg}\varphi=\sqrt{\frac{g}{e}}\frac{dv}{du}$.

5. Lastly we will mention some invariantive systems of surface-curves for inversion.

I. If the two directions $dv:du$ and $\delta v:\delta u$ be *orthogonal*, then

$$(15) \qquad 2\,du\,\delta u+f(du\,\delta v+dv\,\delta u)+g\,dv\,\delta v=0.$$

The self-orthogonal system consists of *the minimal lines*

$$(16) \qquad f_1=0,$$

which is equivalent to

$$(16)' \qquad (\sigma_1, \sigma_1')^2+(\sigma_2, \sigma_2')^2=0.$$

II. If the two directions $dv:du$ and $\delta v:\delta u$ be *isoclinal* (with respect to the lines of curvature), then

$$(17) \qquad l\,du\,\delta u+m(du\,\delta v+dv\,\delta u)+n\,dv\,\delta v=0 \text{ (1)}.$$

The self-isoclinal system consists of *the lines of curvature*

$$(18) \qquad f_2=0,$$

which is equivalent to

$$(18)' \qquad (\sigma_1, \sigma_1')(\sigma_2, \sigma_2')=0 \quad \text{or} \quad (\sigma, \sigma')=0 \text{ (2)}.$$

III. If the two directions $dv:du$ and $\delta v:\delta u$ be *inverse-orthogonal* (3) (or *complementary*), then

$$(19) \qquad (fn-mg)\,du\,\delta u+\frac{1}{2}(en-gl)(du\,\delta v+dv\,\delta u)$$
$$+(em-fl)\,dv\,\delta v=0.$$

The self-complementary system consists of *the lines of torsion*

$$(20) \qquad J(f_1, f_2)=0,$$

which is equivalent to

$$(21)' \qquad (\sigma_1, \sigma_1')^2-(\sigma_2, \sigma_2')^2=0.$$

IV. Since

$$\theta(f_1, f_2)=0,$$

we have

(1) K. Ogura, "On the T-system on a surface," Tôhoku Math. Journal, 9 (1916), p. 88; T. Hayashi, loc. cit..

(2) Demartres, loc. cit..

(3) T. Hayashi, loc. cit..

$$J(f_1, J(f_1, f_2)) = -\Delta(f_1) \cdot f_2, \qquad J(f_2, J(f_1, f_2)) = \Delta(f_2) \cdot f_1 \text{ (}^1\text{)}.$$

Therefore the lines of torsion, the minimal lines and the lines of curvature form the only orthogonal-isoclinal system, the only isoclinal-complementary system and the only orthogonal-complementary system respectively.

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(1) K. Ogura, "Some theorems concerning binary quadratic forms and their applications to the differential geometry," Science reports of the Tôhoku Imperial University, Series I, 5, (1916), p. 95.

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