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Trajectries in the Conservative Field of Force, Part II.

(Continued from Vol. 8, p. 203.)

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by

KINNOSUKE OGURA in Sendai.

CHAPTER VI.

Further Investigations for the Two Families of Orbits on a Surface.

Actions corresponding to an orthogonal system of orbits.

43. P. G. Tait (1) stated that "A very interesting plane example, which has elegant applications in fluid motion, and in the conduction of electric currents in plates of uniform thickness, is furnished by assuming

$$A = \log r, \quad \text{or} \quad A' = \theta \ (^2),$$

where r and θ are the polar co-ordinates of the moving particle.

"In the former, where the curves of equal action are circles with the origin as centre, ... the paths are radii vectors described with velocity $\frac{1}{r}$. Also .... *the force is central, and its value is* .... $-\frac{1}{r^3}$. In the second case, where the curves of equal action are radii drawn from the pole ...... The kinetic energy is still $\frac{1}{2r^2}$, and *the central force* $-\frac{1}{r^3}$, but the paths are circles with the origin as centre (3).

"Thus *the lines of equal action and the paths of individual particles are convertible* ...... A and A' not only satisfy the same actional equation, but are elementary solutions of the partial differential equation

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0 \qquad \text{(I)}$$

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(1) Tait, Dynamics (1895), pp. 220–224.

(2) A and A' denote the actions.

(3) Compare with Art. 28 in this Journal, Vol. 7 (1915), p. 176.

and *they are conjugate*, in the sense that

$$\frac{\partial A}{\partial x}=\frac{\partial A'}{\partial y}, \quad \frac{\partial A}{\partial y}=-\frac{\partial A'}{\partial x}.$$

From this reason the paths belonging to the two systems are everywhere *orthogonal* to one another.

"Also, as the differential equation (I) for A is linear, any linear function of particular integrals is an integral. Thus, for instance, we may take (p being any constant)

$$A=\log r-p\,\theta, \quad \text{with} \quad A'=p\log r+\theta.$$

These, representing orthogonal sets of logarithmic spirals, possess the same properties with regard to actions as did the concentric circles and their radii.... We may pursue the subject much further, by combining particular solutions like those given but taken from different origins ......"

And then he showed some particular examples, but did not consider these matters in a general way. Here we propose to establish the following theorem:

The following three conditions are equivalent to one another:

I. *The condition that the* $2\infty^1$ *curves belonging to an orthogonal system on a surface can be orbits in one and the same conservative field of force;*

II. *The condition that the two actions, corresponding to two families of* ∞^1 *orbits in a conservative field of force respectively, are conjugate functions of a point of the surface;*

III. *The condition that one family of* ∞^1 *orbits and the corresponding* ∞^1 *curves of equal action are convertible.*

44. (i) When there exists a conservative force under which two orthogonal families of curves can be orbits, these families must form an isothermal system (Arts. 23–27 or Art. 31). Now let us suppose that the two orthogonal families u=const., v=const. on a surface can be orbits, and that the linear element of the surface is

$$ds^2=E\,(du^2+dv^2),$$

where u and v are taken for the isothermal parameters. Then the force-function is given by

$$U+h=\frac{k^2}{2E},$$

where k denotes an arbitrary constant, so that the linear element takes the form

$$(114) \qquad ds^2=\frac{k^2}{2(U+h)}\,(du^2+dv^2).$$

Now if we denote by A_u the action corresponding to the family of orbits v=const., then

$$A_u=\int\sqrt{2\,(U+h)}\,ds \qquad \text{along } v=\text{const.},$$

$$=\int k\,du \qquad \text{by (114)},$$

$$=k\,u+\alpha, \quad (\alpha \text{ being an arbitrary constant}).$$

Similarly the action A_v corresponding to the orbits u=const. is given by

$$A_v=k\,v+\beta, \quad (\beta \text{ being an arbitrary constant}).$$

Hence we obtain

$$\frac{\partial A_u}{\partial u}=k, \quad \frac{\partial A_u}{\partial v}=0;$$

$$\frac{\partial A_v}{\partial u}=0, \quad \frac{\partial A_v}{\partial v}=k.$$

Whence

$$\frac{\partial A_u}{\partial u}=\frac{\partial A_v}{\partial v}, \quad \frac{\partial A_u}{\partial v}=-\frac{\partial A_v}{\partial u};$$

and

$$\frac{\partial^2 A_u}{\partial u^2}+\frac{\partial^2 A_u}{\partial v^2}=0, \quad \frac{\partial^2 A_v}{\partial u^2}+\frac{\partial^2 A_v}{\partial v^2}=0.$$

Consequently, for the conservative field of force under which two orthogonal families of curves can be orbits, the two actions, corresponding to these two families of orbits respectively, are conjugate functions of a complex function of a point on this surface[1].

(ii) Conversely, let A, A' be any conjugate functions of a point on the surface

[1] Klein, Über Riemann's Theorie der algebraischen Funktion und ihre Integrale (1882), pp. 16–22; Picard, Traité d'analyse, t. 2, 2e éd. (1905), p. 8.

$$ds^2 = E\,du^2 + 2\,F du\,dv + G dv^2$$

referred to any coordinate system (u, v). Then A and A' satisfy the equation

$$\Delta_2 A = 0,$$

i.e.

$$\frac{\partial}{\partial u}\left(\frac{F\dfrac{\partial A}{\partial v} - G\dfrac{\partial A}{\partial u}}{\sqrt{EG-F^2}}\right) + \frac{\partial}{\partial v}\left(\frac{F\dfrac{\partial A}{\partial u} - E\dfrac{\partial A}{\partial v}}{\sqrt{EG-F^2}}\right) = 0 \ (^1).$$

But it is well known that in this case the linear element may be written

$$ds^2 = \lambda\,(dA^2 + dA'^2) \ (^2),$$

where

$$\lambda = \frac{1}{\Delta_1 A} = \frac{1}{\Delta_1 A'},$$

i.e.

$$\frac{1}{\lambda} = \frac{E\left(\dfrac{\partial A}{\partial v}\right)^2 - 2\,F\dfrac{\partial A}{\partial u}\dfrac{\partial A}{\partial v} + G\left(\dfrac{\partial A}{\partial u}\right)^2}{EG-F^2}$$

$$= \frac{E\left(\dfrac{\partial A'}{\partial v}\right)^2 - 2\,F\dfrac{\partial A'}{\partial u}\dfrac{\partial A'}{\partial v} + G\left(\dfrac{\partial A'}{\partial u}\right)^2}{EG-F^2}.$$

Hence the curves of equal action $A=$const. and $A'=$const. form an isothermal system. But since the orthogonal trajectories of these curves of equal action are

$$A' = \text{const. and } A = \text{const.}$$

respectively, they form an isothermal system. Consequently, by Art. 31, these two families can be the orbits under the force-function

$$U + h = \frac{k^2}{\lambda},$$

k being an arbitrary constant. Thus the equivalence of the conditions I and II has been proved.

(1) Klein, loc. cit.; Picard, loc. cit..
(2) Bianchi, Vorlesungen über Differentialgeometrie, 2. Aufl. (1910), p. 68.

(iii) Lastly, if A and A' be conjugate functions, then by (ii) the orbits corresponding to A are given by

$$A' = \text{const.},$$

and those corresponding to A' are

$$A = \text{const.};$$

whence the family of orbits and the curves of equal action are convertible.

Conversely, if one family of orbits and the curves of equal action be convertible, the two families of orbits form an orthogonal system; so that by (i) the corresponding actions must be conjugate. Hence the conditions II and III are equivalent.

Thus *Tait's analogy between the action in the conservative field and the velocity potential in steady fluid motion (or the electric potential in electric conduction) has been completely established* (1).

Dynamical interpretation of the condition that two families of curves on a surface can be orbits.

45. In Art. 30 we have proved that if the two families of parametric curves $u=$const., $v=$const. on a surface S be extremals for the function φ, it must be

$$(83)\qquad \begin{aligned} \frac{\partial \log \varphi}{\partial u} &= \frac{F}{E}\begin{Bmatrix}1\,1\\2\end{Bmatrix} + \frac{E}{G}\begin{Bmatrix}2\,2\\1\end{Bmatrix},\\ \frac{\partial \log \varphi}{\partial v} &= \frac{G}{E}\begin{Bmatrix}1\,1\\2\end{Bmatrix} + \frac{F}{G}\begin{Bmatrix}2\,2\\1\end{Bmatrix}.\end{aligned}$$

Let the radius of geodesic curvature ρ of $\phi(u, v)=$const. be taken as positive when the centre of geodesic curvature is in the direction such that the function $\phi(u, v)$ is increasing. Then we have

$$\frac{1}{\rho_u} = \frac{\sqrt{EG-F^2}}{G\sqrt{G}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}, \qquad \frac{1}{\rho_v} = \frac{\sqrt{EG-F^2}}{E\sqrt{E}}\begin{Bmatrix}1\,1\\2\end{Bmatrix} \ (^2).$$

And the angle Ω between the parametric curves is given by

$$\sin\Omega = \frac{\sqrt{EG-F^2}}{\sqrt{EG}}, \qquad \operatorname{tg}\Omega = \frac{\sqrt{EG-F^2}}{F}.$$

(1) Compare with Kirchhoff, Über die stationären elektrischen Strömungen in einer gekrümmten leitenden Fläche, Berliner Monatsberichte (1875), p. 487.
(2) Bianchi, loc. cit., p. 149.

Hence equations (83) may be written

$$\frac{1}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}=\operatorname{cosec}\Omega.\frac{1}{\rho_u}+\cot\Omega.\frac{1}{\rho_v},$$

$$\frac{1}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v}=\cot\Omega.\frac{1}{\rho_u}+\operatorname{cosec}\Omega.\frac{1}{\rho_v}.$$

Therefore

$$\frac{1}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}-\frac{1}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v}=\operatorname{tg}\frac{\Omega}{2}\cdot\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right),$$

so that

$$(115)\qquad \cos\frac{\Omega}{2}\frac{1}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v}+\sin\frac{\Omega}{2}\frac{1}{\rho_u}$$
$$=\cos\frac{\Omega}{2}\frac{1}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}+\sin\frac{\Omega}{2}\frac{1}{\rho_v}.$$

Similarly

$$(116)\qquad \sin\frac{\Omega}{2}\frac{1}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v}-\cos\frac{\Omega}{2}\frac{1}{\rho_u}$$
$$=-\sin\frac{\Omega}{2}\frac{1}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}+\cos\frac{\Omega}{2}\frac{1}{\rho_v}.$$

In the conservative field of force, let F_ϕ be the tangential component of force along the path $\phi=$const. and F_{ρ_ϕ} be the component of force along the tangent to the surface which is perpendicular to the path. Then

$$F_\phi=q\frac{\partial q}{\partial s_\phi}=q^2\frac{\partial \log q}{\partial s_\phi},\qquad F_{\rho_\phi}=\frac{q^2}{\rho_\phi};$$

but

$$q=\varphi,\qquad ds_u=\sqrt{G}\,dv,\qquad ds_v=\sqrt{E}\,du,$$

so that

$$F_u=\frac{\varphi^2}{\sqrt{G}}\frac{\partial \log \varphi}{\partial v},\qquad F_{\rho_u}=\frac{\varphi^2}{\rho_u};$$

$$F_v=\frac{\varphi^2}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u},\qquad F_{\rho_v}=\frac{\varphi^2}{\rho_v}.$$

Hence (115) and (116) become respectively

$$(117)\qquad F_u\cos\frac{\Omega}{2}+F_{\rho_u}\sin\frac{\Omega}{2}=F_v\cos\frac{\Omega}{2}+F_{\rho_v}\sin\frac{\Omega}{2},$$

$$(118)\qquad F_u\sin\frac{\Omega}{2}-F_{\rho_u}\cos\frac{\Omega}{2}=-F_v\sin\frac{\Omega}{2}+F_{\rho_v}\cos\frac{\Omega}{2}.$$

Consequently, the sum of the projections of the two vectors F_u and F_{ρ_u} upon the tangent to the surface which bisects the internal angle between the parametric curves is equal to the sum of the projections of the two vectors F_v and F_{ρ_v} upon that tangent; and the similar result is also true for the bisector corresponding to the external angle between the parametric curves.

Squaring equations (117) and (118) respectively and adding

$$(119)\qquad F_u^2+F_{\rho_u}^2=F_v^2+F_{\rho_v}^2.$$

Next, dividing (117) by (118) side by side and after a short calculation

$$(120)\qquad \operatorname{tg}\Omega=\frac{\dfrac{F_{\rho_u}}{F_u}+\dfrac{F_{\rho_v}}{F_v}}{1-\dfrac{F_{\rho_u}}{F_u}\dfrac{F_{\rho_v}}{F_v}},$$

that is,

$$(120)'\qquad \Omega=\operatorname{arc\,tg}\frac{F_{\rho_u}}{F_u}+\operatorname{arc\,tg}\frac{F_{\rho_v}}{F_v}.$$

It follows from (119) and (120)′ that a necessary and sufficient condition that $u=$const., and $v=$const. can be orbits in the conservative field of force is that the resultant of the two vectors F_u and F_{ρ_u} is equal to the resultant of F_v and F_{ρ_v}.

But since this result is almost self-evident from dynamical considerations and our processes are all reversible, we may consider that another proof for equations (83) has been obtained by an elementary method from the standpoint of dynamics.

46. In the particular case where u and v form an orthogonal system, we have $\Omega=\frac{\pi}{2}$, so that (117) and (118) become respectively

$$F_u+F_{\rho_u}=F_v+F_{\rho_v},\qquad F_u-F_{\rho_u}=-F_v+F_{\rho_v};$$

whence

$$(121) \qquad F_u = F_{\rho_v}, \qquad F_v = F_{\rho_u};$$

and conversely, when equations (121) hold good, we see from (120) that u, v must form an orthogonal system.

Thus the dynamical condition for an orthogonal system to be the orbits has been obtained.

Lastly, in the general case of natural families we can state the theorem: *A necessary and sufficient condition that u=const., v=const. can be the extremals for the function φ is that the resultant of the two vectors $(grad\ \varphi)_u$ and $(grad\ \varphi)_{\rho_u}$ is equal to the resultant of the two vectors $(grad\ \varphi)_v$ and $(grad\ \varphi)_{\rho_v}$.*

Free paths on a surface under the force whose equipotential surfaces are parallel.

47. Suppose that there exists a conservative force under which the $2\infty^1$ curves u=const., v=const. are the free paths on a surface S. Then, by Art. 9, these curves must be the asymptotic lines on the surface. Now, let PT_u and PT_v be the tangents to u=const. and v=const. at the point P respectively and let PN_u be the vector with the magnitude equal to the geodesic curvature of u=const. at P, directed toward the centre of geodesic curvature, and PN_v be the corresponding vector for v=const.. Also let Q be the point such that QN_u and QN_v are parallel to T_uP and T_vP respectively. Since the vector PQ has the direction of the resultant of the force-components F_u and F_{ρ_u} (or F_v and F_{ρ_v}), and the force-vector itself touches the surface S at P (Art. 9), the direction of the vector PQ is that of the force-vector at P.

Hence, if we assume that *the equipotential surfaces U=const. are parallel* in this field of force, the straight line PQ for every point P of the surface is always normal to the given family U=const..

Conversely, if the straight line PQ, determined by the above method, for the asymptotic curves u, v at every point P on the surface be normal to a family of parallel surfaces U=const., then the curves u and v are the free paths on the surface under a certain force whose equipotential surfaces are U=const..

48. Particularly, we consider the *minimal surface* S having the linear element

$$(122) \qquad ds^2 = E\,(du^2 + dv^2),$$

where the asymptotic lines are taken for the parametric curves. We have already shown that the force-function, under which the two families u and v may be orbits, always exists and is given by

$$U + h = \frac{k^2}{E},$$

k being an arbitrary constant (Art. 14). But in order that these families should be the free paths under this force, the force-function U must satisfy

$$(20) \qquad \frac{\partial U}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial z}{\partial y} - \frac{\partial U}{\partial z} = 0,$$

or

$$(21) \qquad \frac{\partial U}{\partial u}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} + \frac{\partial U}{\partial v}\begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0$$

at every point on the surface (Art. 9). As a matter of fact, however, this condition is not convenient for practical work; and hence we will give an entirely different condition for the particular case where the equipotential surfaces are parallel.

In this case the angle T_vPT_u is right and the points N_u and N_v lie on PT_v and PT_u respectively, and PN_uQN_v becomes a rectangle. If x, y, z be the Cartesian coordinates of $P(u, v)$, the direction-cosines α_u, β_u, γ_u of the tangent PT_u are given by

$$\alpha_u = \frac{dx}{ds_u} = \frac{1}{\sqrt{E}}\frac{\partial x}{\partial v}, \qquad \beta_u = \frac{dy}{ds_u} = \frac{1}{\sqrt{E}}\frac{\partial y}{\partial v},$$

$$\gamma_u = \frac{dz}{ds_u} = \frac{1}{\sqrt{E}}\frac{\partial z}{\partial v}.$$

Also if $\frac{1}{R_u}$ be the curvature of u=const. and λ_u, μ_u, ν_u the direction-cosines of the principal normal to u=const., we have by Frenet-Serret's formulae

$$\frac{\lambda_u}{R_u}=\frac{d\alpha}{ds_u}=\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial v}\right),$$

$$\frac{\mu_u}{R_u}=\frac{d\beta}{ds_u}=\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial y}{\partial v}\right),$$

$$\frac{\nu_u}{R_u}=\frac{d\gamma}{ds_u}=\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial z}{\partial v}\right).$$

But since $u=$const. are asymptotic curves, the curvature $\frac{1}{R_u}$ becomes the geodesic curvature $\frac{1}{\rho_u}$ and the principal normal to $u=$const. becomes PT_v which is perpendicular to PT_u. Hence the coordinates of N_u are

$$x+\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial v}\right),\qquad y+\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial y}{\partial v}\right),$$

$$z+\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial z}{\partial v}\right).$$

In exactly the same way the coordinates of N_v are found to be

$$x+\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right),\qquad y+\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial y}{\partial u}\right),$$

$$z+\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial z}{\partial u}\right).$$

Therefore the coordinates of Q are given by

$$x+\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right)+\frac{1}{\sqrt{E}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial x}{\partial v}\right),\quad \cdots\cdots\cdots,$$

that is,

$$x+\frac{1}{E}\left(\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}\right)-\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right),\quad \cdots\cdots\cdots.$$

But since the asymptotic lines are taken for the parametric curves, we have the well known formulae [1]

$$\frac{\partial^2 x}{\partial u^2}=\left\{\begin{matrix}1\,1\\1\end{matrix}\right\}\frac{\partial x}{\partial u}+\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}\frac{\partial x}{\partial v},\qquad \frac{\partial^2 x}{\partial v^2}=\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}\frac{\partial x}{\partial u}+\left\{\begin{matrix}2\,2\\2\end{matrix}\right\}\frac{\partial x}{\partial v};$$

(1) Bianchi, loc. cit., p. 109.

which become in our case

$$\frac{\partial^2 x}{\partial u^2}=-\frac{\partial^2 x}{\partial v^2}=\frac{1}{2E}\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}-\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right).$$

Hence the coordinates of Q have the expressions:

$$x-\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right),$$

$$y-\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial y}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial y}{\partial v}\right),$$

$$z-\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial z}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial z}{\partial v}\right).$$

Consequently the direction-cosines l, m, n of PQ are

$$l=-\frac{\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right)}{\sqrt{\sum\left\{-\frac{1}{2E^2}\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right)\right\}^2}},\ \cdots\cdots;$$

and recalling that

$$\sum\left(\frac{\partial x}{\partial u}\right)^2=E,\qquad \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}=0,\qquad \sum\left(\frac{\partial x}{\partial v}\right)^2=E,$$

we obtain

$$l=-\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right):\sqrt{E}\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2},$$

(123) $$m=-\left(\frac{\partial E}{\partial u}\frac{\partial y}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial y}{\partial v}\right):\sqrt{E}\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2},$$

$$n=-\left(\frac{\partial E}{\partial u}\frac{\partial z}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial z}{\partial v}\right):\sqrt{E}\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2};$$

whence the equations of PQ are

(124) $$\xi=x+l\tau,\qquad \eta=y+m\tau,\qquad \zeta=z+n\tau,$$

$(\xi,\ \eta,\ \zeta)$ being current coordinates and τ a parameter.

Now the necessary and sufficient condition that PQ for every point $P(x,\ y,\ z)$ should be normal to a surface (and hence a family of parallel surfaces) is that

$$l\,dx+m\,dy+n\,dz$$

is the total differential with respect to u and v. But since

$$l\,dx+m\,dy+n\,dz=-\frac{\sum\left\{\left(\frac{\partial E}{\partial u}\frac{\partial x}{\partial u}+\frac{\partial E}{\partial v}\frac{\partial x}{\partial v}\right)\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)\right\}}{\sqrt{E}\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2}}$$

$$=-\sqrt{E}\left(\frac{\partial E}{\partial u}du+\frac{\partial E}{\partial v}dv\right):\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2},$$

the above stated condition becomes

$$\frac{\partial}{\partial v}\left\{\frac{\sqrt{E}\frac{\partial E}{\partial u}}{\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2}}\right\}=\frac{\partial}{\partial u}\left\{\frac{\sqrt{E}\frac{\partial E}{\partial v}}{\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2}}\right\},$$

from which we obtain

$$(125)\qquad \frac{\partial E}{\partial u}\frac{\partial E}{\partial v}\left(\frac{\partial^2 E}{\partial u^2}-\frac{\partial^2 E}{\partial v^2}\right)=\left\{\left(\frac{\partial E}{\partial u}\right)^2-\left(\frac{\partial E}{\partial v}\right)^2\right\}\frac{\partial^2 E}{\partial u\,\partial v}.$$

And if the condition be satisfied, the surfaces orthogonal to every straight line PQ are given by

$$(126)\qquad \begin{aligned}\xi&=x(u,v)+l(u,v)\{f(u,v)+c\},\\ \eta&=y(u,v)+m(u,v)\{f(u,v)+c\},\\ \zeta&=z(u,v)+n(u,v)\{f(u,v)+c\},\end{aligned}$$

where c is an arbitrary constant and

$$(127)\qquad f(u,v)=\int\frac{\sqrt{E}\left(\frac{\partial E}{\partial u}du+\frac{\partial E}{\partial v}dv\right)}{\sqrt{\left(\frac{\partial E}{\partial u}\right)^2+\left(\frac{\partial E}{\partial v}\right)^2}}.$$

Thus we have arrived at the theorem:

A necessary and sufficient condition that there should exist the two families of the free paths on the minimal surface (122) *under a field of force in which the totality of the force-vectors forms a normal congruence, is that the relation* (125) *is satisfied; and if the condition be fulfilled, the equipotential surfaces are given by* (126).

49. I. We can see from (126) that *the equipotential curves on the minimal surface* (122) are

$$f(u,v)+c=0.$$

Differentiating and dividing by the factor which does not vanish,

$$\frac{\partial E}{\partial u}du+\frac{\partial E}{\partial v}dv=0,$$

and hence by integration

$$(128)\qquad E=\text{const.},$$

which coincides with the result obtained from the equation

$$U+h=\frac{k^2}{E}=\text{constant}.$$

II. Now the differential equation (125) is satisfied by

$$E=\Phi\left(\frac{u+av+b}{\sqrt{1+a^2}}\right),$$

where a and b are arbitrary constants and Φ is an arbitrary positive function.

For an example, let us take *the ordinary helicoid* (Art. 18)

$$x=\sinh u\cos v,\qquad y=\sinh u\sin v,\qquad z=v.$$

In this case, since

$$E=\cosh^2 u,$$

the condition (125) is satisfied, and the equipotential surfaces are

$$\xi=-c\cos v,\qquad \eta=-c\sin v,\qquad \zeta=v,$$

that is, the coaxial circular cylinders whose common axis is the axis of the generating helices; the equipotential lines on the helicoid are u=const., so that the direction of the force coincides with the generating line.

For the second example, take *the catenoid* (Art. 14)

$$x=\cosh(u+v)\cos(u-v),\qquad y=\cosh(u+v)\sin(u-v),\qquad z=u+v.$$

Since

$$E=2\cosh^2(u+v),$$

the condition (125) is satisfied, and the equipotential surfaces are

$$\begin{aligned}\xi&=[\cosh(u+v)-\text{tgh}(u+v)\{\sin(u+v)+c\}]\cos(u-v),\\ \eta&=[\cosh(u+v)-\text{tgh}(u+v)\{\sin(u+v)+c\}]\sin(u-v),\\ \zeta&=(u+v)-\text{sech}(u+v)\{\sin(u+v)+c\}.\end{aligned}$$

The equipotential lines on the catenoid are $u+v=$const., so that the force touches the curve $u-v=$const., i.e. the meridian; whence the force-vectors intersect with the axis of the catenoid, and the equipotential surfaces form a family parallel to the pseudosphere having the same axis as the catenoid.

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