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On the T—System on a Surface.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 9, Nos. 1, 2.,
edited by TSURUICHI HAYASHI, College of Science,
Tôhoku Imperial University, Sendai, Japan,
with the collaboration of Messrs.
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February 1916

On the T—System on a Surface,

by

KINNOSUKE OGURA in Sendai.

Among special systems, each of which consists of $2\infty^1$ curves on a surface, the following five are well known.

Asymptotic lines, lines of curvature, characteristic lines, lines of torsion, Occhipinti's lines([1]).

Besides these there exists a more general system called the conjugate system which contains the lines of curvature and the characteristic lines as particular cases. In this paper I will deal with another system of curves containing all the five systems above mentioned as particular cases.

Fundamental properties and some particular cases.

1. Let us consider a system of $2\infty^1$ curves on a surface. When the two curves of the system which pass through any point have the geodesic torsions $1:T$ which are different only in sign at this point, the system of curves is called the *T-system.*

If we take a T-system as the parametric curves u, v and denote the geodesic torsions of $u=$const. and $v=$const. by $1:T_u$ and $1:T_v$ respectively, then

$$\frac{1}{T_u}=\frac{FL-EM}{E\sqrt{EG-F^2}},\quad \frac{1}{T_v}=\frac{GM-FN}{G\sqrt{EG-F^2}},\quad \frac{1}{T_u}=-\frac{1}{T_v},$$

([1]) For the last three systems, see Ch. Dupin, Développements de géométrie (1813), p. 192; K. Peterson, Über Kurven und Flächen (1868), p. 35; R. Hoppe, Archiv d. Math. u. Physik, I, 69 (1883), p. 19; E. Pucci, Atti della R. Accad. dei Lincei, IV, 5 (1889), p. 501; V. Reina, Ibid., p. 881; R. Raffy, Bulletin de la Société Math. de France, 30 (1902), p. 226; R. v. Lilienthal, Math. Annalen, 62 (1906), p. 539; L. P. Eisenhart, Treatise on the differential geometry (1909), p. 130; R. v. Lilienthal, Vorlesungen über Differentialgeometrie, II, 1 (1913), p. 196; J. Knoblauch, Grundlagen der Differentialgeometrie (1913), pp. 595-6; R. Occhipinti, L'enseignement mathématique (1914), p. 38; R. D. Beetle, Annals of math., II, 15 (1914), p. 179; T. Hayashi, Science reports of the Tôhoku Imperial University, Series I, 3 (1914), p. 217; H. Hilton and R. H. Colomb, Messenger of math., 44 (1915), p. 167.

where E, F, G are the fundamental quantities of the first order and L, M, N those of the second order. Hence it follows that

$$(1)\qquad \frac{L}{E}F=\frac{N}{G}F.$$

Consequently we have either

$$(2)\qquad F=0,$$

or

$$(3)\qquad \frac{L}{E}=\frac{N}{G}.$$

Now we consider another system of $2\infty^1$ curves on the surface. When the two curves of the system which pass through any point have the equal normal curvatures $1:R$ at this point, the system of curves is called the *R-system.*

If we take an R-system as the parametric curves u, v and denote the normal curvatures of u=const. and v=const. by $1:R_u$ and $1:R_v$ respectively, then

$$\frac{1}{R_u}=\frac{L}{E},\quad \frac{1}{R_v}=\frac{N}{G},\quad \frac{1}{R_u}=\frac{1}{R_v}.$$

Hence

$$(3)\qquad \frac{L}{E}=\frac{N}{G}.$$

Therefore we get the theorem:

Any T-system is either an orthogonal system or an R-system; conversely, any orthogonal system([1]) *and any R-system belong to T-systems.*

2. In the following table the well known five particular cases of the T-system and their characteristic properties are arranged. The curves of the systems there mentioned respectively are taken for the parametrics. The total curvature K and the mean curvature H of the surface are connected by the well known formula:

$$(4)\qquad \frac{1}{T^2}+\frac{1}{R^2}=\frac{H}{R}-K(^2).$$

([1]) See Bianchi, Vorlesungen über Differentialgeometrie, 2. Aufl. p. 165.
([2]) J. Knoblauch, p. 197; Beetle.

Ortho-gonal system.	Lines of curvature.			Lines of torsion.
	$F=0,\quad M=0.$			$F=0,\quad \frac{L}{E}=\frac{N}{G}.$
	$\frac{1}{R_u}\neq\frac{1}{R_v}$ (in general).			
	$\frac{1}{T_u}=-\frac{1}{T_v}=0.$			
R-system.	Asymptotic lines.	Characteristic lines.	Occhipinti's lines.	
	$L=0,\quad N=0.$	$M=0,\quad \frac{L}{E}=\frac{N}{G}.$	$\frac{L}{E}=-\frac{M}{F}=\frac{N}{G}.$	$\frac{1}{R_u}=\frac{1}{R_v}=\frac{H}{2}$ ([4]).
	$\frac{1}{R_u}=\frac{1}{R_v}=0.$	$\frac{1}{R_u}=\frac{1}{R_v}=\frac{2K}{H}$ ([2]).	$\frac{1}{R_u}=\frac{1}{R_v}=\sqrt{K}$ ([3]).	
	$\frac{1}{T_u}=-\frac{1}{T_v}=\sqrt{-K}$ ([1]).	$\frac{1}{T_u}=-\frac{1}{T_v}=\frac{1}{H}\sqrt{H^2-4K}.$	$\frac{1}{T_u}=-\frac{1}{T_v}=\sqrt[4]{K}\sqrt{H-2\sqrt{K}}.$	$\frac{1}{T_u}=-\frac{1}{T_v}=\frac{1}{2}\sqrt{H^2-4K}.$

The quantity $\frac{1}{T^2}+\frac{1}{R^2}$ is nothing but Gilbert's *flexion* (or *the first normal curvature* $\frac{1}{r_p}$ of Prof. Fr. Meyer). Hence *when* $R_u=R_v$, *we have* $(r_p)_u=(r_p)_v$; *and conversely.* See Ph. Gilbert, Bruxelles Mém., 37 (1868), p. 1; W. Fr. Meyer, Über die Theorie benachbarter Geraden (1911), p. 94, p. 141.

([1]) Enneper, Göttinger Nachrichten (1870), p. 493.
([2]) Reina; Beetle; Hayashi.
([3]) Occhipinti.
([4]) Hayashi.

From this table we can see that

I. *The lines of torsion are the only curves which belong to both an orthogonal system and an R-system.*

II. *The lines of curvature and the characteristic lines are the only curves which belong to both a conjugate system and a T-system.*

3. I. From Euler's formula for the normal curvature it follows that *the directions of the two curves of an R-system passing through a point (being no umbilical point) are symmetric with respect to the directions of the lines of curvature passing through that point, and conversely.*

In order to prove the first part of this theorem by another way, let us consider the equation of the lines of curvature referred to an R-system. In this case

$$\begin{vmatrix} dv^2 & E & L \\ -du\,dv & F & M \\ du^2 & G & N \end{vmatrix} = 0$$

becomes by (3)

$$E\left(\frac{M}{F}-\frac{L}{E}\right)du^2 - G\left(\frac{M}{F}-\frac{N}{G}\right)dv^2 = 0\,;$$

but since we have

$$\frac{L}{E} = \frac{N}{G} \neq \frac{M}{F}$$

at any point (being no umbilical point), the above equation becomes

$$(5) \qquad E\,du^2 - G\,dv^2 = 0,$$

which proves the first part.

As a consequence of this, *the directions of the two curves of an R-system passing through a point (being no umbilical point) are isogonal conjugate with respect to the directions of the two curves of any other R-system passing through that point.*

II. Now we see from (5) that *a necessary and sufficient condition that an isothermal system should be an R-system is that the surface is isothermic; and in this case the R-system consists of the lines of torsion.* Hence by Lie's theorem[1] *the lines of torsion on an isothermic surface can be found by quadratures only.*

(1) Bianchi, p. 73.

III. Lastly, when the surface is referred to the lines of curvature, the equations to the lines of torsion, the characteristic lines and Occhipinti's lines are respectively[1]

$$(6) \qquad E\,du^2 - G\,dv^2 = 0,$$

$$(7) \qquad \sqrt{EL}\,du^2 - \sqrt{GN}\,dv^2 = 0,$$

$$(8) \qquad L\,du^2 - N\,dv^2 = 0.$$

In what follows we will confine ourselves to consider the surfaces, exclusively the sphere and the developable surface. For, *when and only when the surface is either a sphere or a developable surface, two or three systems of the lines of torsion, the characteristic lines and Occhipinti's lines coincide with one another.*

Laplace's equation for the R-system.

4. Gauss obtained the following fundamental formulae for the Cartesian coordinates x, y, z of any point of a surface:

$$\frac{\partial^2 x}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + LX, \qquad \ldots\ldots\ldots;$$

$$\frac{\partial^2 x}{\partial v^2} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \frac{\partial x}{\partial u} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \frac{\partial x}{\partial v} + NX, \qquad \ldots\ldots(^2),$$

where X, Y, Z are the direction-cosines of the normal to the surface and $\begin{Bmatrix} 11 \\ 1 \end{Bmatrix}$, ... Christoffel's symbols. Hence if an R-system be taken for the parametric curves, we must have by (3)

$$G\frac{\partial^2 x}{\partial u^2} - E\frac{\partial^2 x}{\partial v^2} = \left[G\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - E\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\right]\frac{\partial x}{\partial u} + \left[G\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} - E\begin{Bmatrix} 22 \\ 2 \end{Bmatrix}\right]\frac{\partial x}{\partial v},$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots$$

Thus we can state the fundamental theorem:

The Cartesian coordinates x, y, z of a movable point of a surface referred to an R-system are solutions of one and the same Laplace's equation of the form

(1) Occhipinti.

(2) Bianchi, p. 88.

$$a\frac{\partial^2\vartheta}{\partial u^2}-b\frac{\partial^2\vartheta}{\partial v^2}=\alpha\frac{\partial\vartheta}{\partial u}+\beta\frac{\partial\vartheta}{\partial v}, \tag{9}$$

$$\left(a=G,\quad b=E,\quad \alpha=G\left\{\begin{matrix}1\,1\\1\end{matrix}\right\}-E\left\{\begin{matrix}2\,2\\1\end{matrix}\right\},\quad \beta=G\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}-E\left\{\begin{matrix}2\,2\\2\end{matrix}\right\}\right).$$

Conversely, *if $x(u,v)$, $y(u,v)$, $z(u,v)$ be solutions of one and the same Laplace's equation* (9), *the curves $u=const.$ and $v=const.$ on the surface*

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v)$$

form an R-system.

In order to prove this converse, let us take the three equations

$$G\frac{\partial^2 x}{\partial u^2}-E\frac{\partial^2 x}{\partial v^2}=\alpha\frac{\partial x}{\partial u}+\beta\frac{\partial x}{\partial v},$$

$$G\frac{\partial^2 y}{\partial u^2}-E\frac{\partial^2 y}{\partial v^2}=\alpha\frac{\partial y}{\partial u}+\beta\frac{\partial y}{\partial v},$$

$$G\frac{\partial^2 z}{\partial u^2}-E\frac{\partial^2 z}{\partial v^2}=\alpha\frac{\partial z}{\partial u}+\beta\frac{\partial z}{\partial v}.$$

Multiplying these equations by X, Y, Z respectively and adding,

$$G.\sum X\frac{\partial^2 x}{\partial u^2}-E.\sum X\frac{\partial^2 x}{\partial v^2}=\alpha.\sum X\frac{\partial x}{\partial u}+\beta.\sum X\frac{\partial x}{\partial v};$$

so that by using the well known formulae

$$\sum X\frac{\partial^2 x}{\partial u^2}=L,\quad \sum X\frac{\partial^2 x}{\partial v^2}=N,\quad \sum X\frac{\partial x}{\partial u}=0,\quad \sum X\frac{\partial x}{\partial v}=0,$$

we have

$$GL-EN=0, \tag{3$'$}$$

which proves our converse theorem.

5. For an example, we consider the equation

$$\frac{\partial^2\vartheta}{\partial u^2}-\frac{\partial^2\vartheta}{\partial v^2}=0.$$

The general solution is the sum of two arbitrary functions, one of which depends upon $u+v$ only and the other upon $u-v$ only. Hence if we put

$$\bar u=u+v,\qquad \bar v=u-v,$$

and

$$x=f_1(\bar u)+\phi_1(\bar v),\quad y=f_2(\bar u)+\phi_2(\bar v),\quad z=f_3(\bar u)+\phi_3(\bar v), \tag{10}$$

the two families $\bar u+\bar v=$const., $\bar u-\bar v=$const. form an R-system on the

surface (10). But it is well known that equations (10) represent the general surface of translation and $\bar u$, $\bar v$ form a conjugate system. Therefore *on the surface of translation* (10), *the diagonal curves $\bar u+\bar v=const.$, $\bar u-\bar v=const.$ of the conjugate system $\bar u=const.$, $\bar v=const.$ form an R-system.*

Particularly, let $\bar u$ and $\bar v$ be the curve lengths of the two generating curves

$$\begin{cases} x=f_1(\bar u),\quad y=f_2(\bar u),\quad z=f_3(\bar u);\\ x=\phi_1(\bar v),\quad y=\phi_2(\bar v),\quad z=\phi_3(\bar v)\end{cases} \tag{11}$$

respectively. Then the linear element of the surface (10) takes the form

$$ds^2=d\bar u^2+2(f_1'\,\phi_1'+f_2'\,\phi_2'+f_3'\,\phi_3')\,d\bar u\,d\bar v+d\bar v^2,$$

where prime denotes differentiation with respect to the corresponding argument. Hence

$$ds^2=2(1+f_1'\,\phi_1'+f_2'\,\phi_2'+f_3'\,\phi_3')du^2+2(1-f_1'\,\phi_1'-f_2'\,\phi_2'-f_3'\,\phi_3')dv^2,$$

which shows us that u and v form an orthogonal system.

Consequently, by Art. 2, I, we infer that *if $\bar u$ and $\bar v$ be the curve lengths of any two curves respectively, the diagonal curves $\bar u+\bar v=const.$, $\bar u-\bar v=const.$ on the surface of translation generated by the two given curves are the lines of torsion.*

6. In exactly the same way as in Art. 4, *for the characteristic lines* we have the simultaneous Laplace's equations of the form

$$\begin{cases} a\dfrac{\partial^2\vartheta}{\partial u^2}-b\dfrac{\partial^2\vartheta}{\partial v^2}=\alpha\dfrac{\partial\vartheta}{\partial u}+\beta\dfrac{\partial\vartheta}{\partial v},\\ \dfrac{\partial^2\vartheta}{\partial u\partial v}=\gamma\dfrac{\partial\vartheta}{\partial u}+\delta\dfrac{\partial\vartheta}{\partial v},\end{cases} \tag{12}$$

$$\left(\gamma=\left\{\begin{matrix}1\,2\\1\end{matrix}\right\},\quad \delta=\left\{\begin{matrix}1\,2\\2\end{matrix}\right\}\right).$$

Also we have *for Occhipinti's lines*

$$\begin{cases} a\dfrac{\partial^2\vartheta}{\partial u^2}-b\dfrac{\partial^2\vartheta}{\partial v^2}=\alpha\dfrac{\partial\vartheta}{\partial u}+\beta\dfrac{\partial\vartheta}{\partial v},\\ a_1\dfrac{\partial^2\vartheta}{\partial u^2}+b_1\dfrac{\partial^2\vartheta}{\partial u\partial v}=\alpha_1\dfrac{\partial\vartheta}{\partial u}+\beta_1\dfrac{\partial\vartheta}{\partial v},\end{cases} \tag{13}$$

$$\left(a_1=F,\quad b_1=E,\quad \alpha_1=F\left\{\begin{matrix}1\,1\\1\end{matrix}\right\}+E\left\{\begin{matrix}1\,2\\1\end{matrix}\right\},\quad \beta_1=F\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}+E\left\{\begin{matrix}1\,2\\2\end{matrix}\right\}\right);$$

which are equivalent to

$$(13)' \qquad \begin{cases} a_1 \dfrac{\partial^2 \vartheta}{\partial u^2} + b_1 \dfrac{\partial^2 \vartheta}{\partial u \partial v} = \alpha_1 \dfrac{\partial \vartheta}{\partial u} + \beta_1 \dfrac{\partial \vartheta}{\partial v}, \\[2ex] b_1 \dfrac{\partial^2 \vartheta}{\partial u \partial v} + c_1 \dfrac{\partial^2 \vartheta}{\partial v^2} = \alpha_2 \dfrac{\partial \vartheta}{\partial u} + \beta_2 \dfrac{\partial \vartheta}{\partial v} \ (^1), \end{cases}$$

$$\left(c_1 = G, \quad \alpha_2 = F \left\{ \begin{matrix} 2\,2 \\ 1 \end{matrix} \right\} + G \left\{ \begin{matrix} 1\,2 \\ 1 \end{matrix} \right\}, \quad \beta_2 = F \left\{ \begin{matrix} 2\,2 \\ 2 \end{matrix} \right\} + G \left\{ \begin{matrix} 1\,2 \\ 2 \end{matrix} \right\} \right).$$

Inversion and collineation.

7. Now recalling Art. 3, I and the invariantive property for the lines of curvature with respect to an inversion, we can state the theorem:

When a surface is transformed by an inversion into a second surface, any R-system of the former becomes an R-system of the latter.

Here we add an analytical proof (^{2}): An inversion, or a transformation by reciprocal radii, is given by

$$\bar{x} = \frac{c^2 x}{x^2+y^2+z^2}, \quad \bar{y} = \frac{c^2 y}{x^2+y^2+z^2}, \quad \bar{z} = \frac{c^2 z}{x^2+y^2+z^2},$$

where c denotes a constant. Now if x, y, z be solutions of equation (9), then $x^2+y^2+z^2$ is also a solution of it. For, if we put

$$\rho = x^2+y^2+z^2,$$

we have

$$G \frac{\partial^2 \rho}{\partial u^2} - E \frac{\partial^2 \rho}{\partial v^2} = 2G \left(x \frac{\partial^2 x}{\partial u^2} + y \frac{\partial^2 y}{\partial u^2} + z \frac{\partial^2 z}{\partial u^2} + E \right)$$
$$- 2E \left(x \frac{\partial^2 x}{\partial v^2} + y \frac{\partial^2 y}{\partial v^2} + z \frac{\partial^2 z}{\partial v^2} + G \right)$$

(^{1}) Occhipinti.

(^{2}) We can also prove this in the following manner: It is well known that for the inverse surface

$$\bar{E}\bar{N} - \bar{G}\bar{L} = - \frac{c^6}{(x^2+y^2+z^2)^3} (EN - GL).$$

(See A. R. Forsyth, Lectures on the differential geometry (1912), p. 106; R. Rothe, Math. Ann., 72 (1912), p. 57.) Hence an R-system $EN-GL=0$ on a surface becomes an R-system $\bar{E}\bar{N}-\bar{G}\bar{L}=0$ on the inverse surface. Compare with my paper "On the differential geometry of inversion" which will appear in this journal in the near future.

$$= 2x \left(G \frac{\partial^2 x}{\partial u^2} - E \frac{\partial^2 x}{\partial v^2} \right) + 2y \left(G \frac{\partial^2 y}{\partial u^2} - E \frac{\partial^2 y}{\partial v^2} \right)$$
$$+ 2z \left(G \frac{\partial^2 z}{\partial u^2} - E \frac{\partial^2 z}{\partial v^2} \right)$$
$$= 2x \left(\alpha \frac{\partial x}{\partial u} + \beta \frac{\partial x}{\partial v} \right) + 2y \left(\alpha \frac{\partial y}{\partial u} + \beta \frac{\partial y}{\partial v} \right)$$
$$+ 2z \left(\alpha \frac{\partial z}{\partial u} + \beta \frac{\partial z}{\partial v} \right)$$
$$= 2\alpha \left(x \frac{\partial x}{\partial u} + y \frac{\partial y}{\partial u} + z \frac{\partial z}{\partial u} \right)$$
$$+ 2\beta \left(x \frac{\partial x}{\partial v} + y \frac{\partial y}{\partial v} + z \frac{\partial z}{\partial v} \right)$$
$$= \alpha \frac{\partial \rho}{\partial u} + \beta \frac{\partial \rho}{\partial v}.$$

Hence if the substitution

$$\bar{\vartheta} = \frac{\vartheta}{\rho}$$

be effected upon the equation (9),

$$a \frac{\partial^2 \bar{\vartheta}}{\partial u^2} - b \frac{\partial^2 \bar{\vartheta}}{\partial v^2} = \frac{1}{\rho^3} \left\{ \rho^2 \left(a \frac{\partial^2 \vartheta}{\partial u^2} - b \frac{\partial^2 \vartheta}{\partial v^2} \right) - \rho \vartheta \left(a \frac{\partial^2 \rho}{\partial u^2} - b \frac{\partial^2 \rho}{\partial v^2} \right) \right.$$
$$- 2\rho \left(a \frac{\partial \rho}{\partial u} \frac{\partial \vartheta}{\partial u} - b \frac{\partial \rho}{\partial v} \frac{\partial \vartheta}{\partial v} \right)$$
$$\left. + 2\vartheta \left[a \left(\frac{\partial \rho}{\partial u} \right)^2 - b \left(\frac{\partial \rho}{\partial v} \right)^2 \right] \right\}$$
$$= \frac{1}{\rho^3} \left\{ \rho^2 \left(\alpha \frac{\partial \vartheta}{\partial u} + \beta \frac{\partial \vartheta}{\partial v} \right) - \rho \vartheta \left(\alpha \frac{\partial \rho}{\partial u} + \beta \frac{\partial \rho}{\partial v} \right) \right.$$
$$- 2\rho \left(a \frac{\partial \rho}{\partial u} \frac{\partial \vartheta}{\partial u} - b \frac{\partial \rho}{\partial v} \frac{\partial \vartheta}{\partial v} \right)$$
$$\left. + 2\vartheta \left[a \left(\frac{\partial \rho}{\partial u} \right)^2 - b \left(\frac{\partial \rho}{\partial v} \right)^2 \right] \right\}$$
$$= \frac{1}{\rho^3} \left\{ \rho^3 \left(\alpha \frac{\partial \bar{\vartheta}}{\partial u} + \beta \frac{\partial \bar{\vartheta}}{\partial v} \right) - 2a \rho^2 \frac{\partial \rho}{\partial u} \frac{\partial \bar{\vartheta}}{\partial u} \right.$$
$$\left. + 2b \rho^2 \frac{\partial \rho}{\partial v} \frac{\partial \bar{\vartheta}}{\partial v} \right\}$$

$$=\left(\alpha-\frac{2a}{\rho}\frac{\partial\rho}{\partial u}\right)\frac{\partial\bar\vartheta}{\partial u}+\left(\beta+\frac{2b}{\rho}\frac{\partial\rho}{\partial v}\right)\frac{\partial\bar\vartheta}{\partial v},$$

so that the resulting equation becomes

$$\bar a\frac{\partial^2\bar\vartheta}{\partial u^2}-\bar b\frac{\partial^2\bar\vartheta}{\partial v^2}=\bar\alpha\frac{\partial\bar\vartheta}{\partial u}+\bar\beta\frac{\partial\bar\vartheta}{\partial v},$$

$$\left(\bar a=\bar G,\ \bar b=\bar E,\ \bar\alpha=\frac{c^4}{\rho^2}\left(\alpha-\frac{2a}{\rho}\frac{\partial\rho}{\partial u}\right),\quad \bar\beta=\frac{c^4}{\rho^2}\left(\beta+\frac{2b}{\rho}\frac{\partial\rho}{\partial v}\right)\right),$$

which is the same form as (9). Therefore u and v must be an R-system on the transformed surface

$$\bar x=\bar x(u,v),\qquad \bar y=\bar y(u,v),\qquad \bar z=\bar z(u,v),$$

which proves the theorem.

Now by an inversion any orthogonal system on a surface is transformed into an orthogonal system on the transformed surface. We also infer from this, in combination with the above theorem, the following result :

When a surface is transformed by an inversion into a second surface, any T-system of the former becomes a T-system of the latter.

Particularly we can infer the following result from Art. 2, I and Art. 7 :

When a surface is transformed by an inversion into another surface, the lines of torsion of the former become the lines of torsion of the latter.

This may also be seen from the facts that the lines of torsion are bisectors of the angle between the lines of curvature and that by an inversion the lines of curvature on a surface become the lines of curvature on the transformed surface.

8. On the contrary, for a collineation we have the following theorem :

When a surface is transformed by any collineation into a second surface, the asymptotic lines of the former constitute the only R-system which becomes an R-system of the latter.

For, it is seen that for the surface transformed by a collineation we have

$$\bar L=\lambda L,\quad \bar M=\lambda M,\quad \bar N=\lambda N\,(^1),$$

where λ is a function of x, y, z, $\frac{\partial x}{\partial u}$, ...$\frac{\partial z}{\partial v}$, which does not

(1) Lilienthal, Differentialgeometrie, II, 1, p. 201.

vanish identically. Hence in order that

$$G\,L-E\,N=0$$

should be transformed into

$$\bar G\,\bar L-\bar E\,\bar N=0,$$

it is necessary and sufficient that either

$$L=0,\quad N=0,$$

or

$$\frac{\bar E}{E}=\frac{\bar G}{G}.$$

But the latter case does not hold in general. For example, if we transform the helicoid

$$x=u\cos v,\quad y=u\sin v,\quad z=v,$$

by the collineation

$$\bar x=\frac{1}{2}x+\frac{1}{3}y,\quad \bar y=\frac{\sqrt{3}}{2}x+\frac{\sqrt{8}}{3}y,\quad \bar z=z,$$

we have

$$E=1,\qquad G=1+u^2,$$

$$\bar E=1+\frac{1+2\sqrt{6}}{6}\sin 2v,\qquad \bar G=1+u^2\left(1-\frac{1+2\sqrt{6}}{6}\sin 2v\right);$$

so that

$$\frac{\bar E}{E}\neq\frac{\bar G}{G}.$$

Deformation.

9. Let us suppose that there exists a deformation such that an R-system on a surface S corresponds to an R-system on the other surface $\bar S$, and let

$$ds^2=E\,du^2+2\,F\,du\,dv+G\,dv^2,\qquad d\bar s^2=\bar E\,d\bar u^2+2\bar F\,d\bar u\,d\bar v+\bar G\,d\bar v^2$$

be the linear elements of S and $\bar S$ referred to these corresponding systems respectively. Then for $u=\bar u$, $v=\bar v$, we have

$$E=\bar E,\qquad F=\bar F,\qquad G=\bar G;$$

so that the lines of curvature on S

$$E\,du^2-G\,dv^2=0 \tag{5}$$

correspond to

$$\bar{E}\,d\bar{u}^2-\bar{G}\,d\bar{v}^2=0.$$

Combining this with Art. 3, I, we can infer the theorem :

A necessary and sufficient condition that by a deformation an R–system should be preserved is that the lines of curvature are also preserved.

Also, *a necessary and sufficient condition that by a deformation the lines of torsion should be preserved is that the lines of curvature are also preserved.*

In order to prove this, it is sufficient to show that any deformation with preservation of the lines of curvature will also preserve the lines of torsion. Let

$$ds^2=E\,du^2+G\,dv^2,\qquad d\bar{s}^2=\bar{E}\,d\bar{u}^2+\bar{G}\,d\bar{v}^2$$

be the linear elements of S, $\bar{S}$ referred to the lines of curvature respectively. Then for $u=\bar{u}$, $v=\bar{v}$ we have

$$E=\bar{E},\qquad G=\bar{G};$$

whence by this deformation the equation to the lines of torsion on S

$$(6)\qquad E\,du^2-G\,dv^2=0$$

is transformed into

$$\bar{E}\,d\bar{u}^2-\bar{G}\,d\bar{v}^2=0,$$

which denotes the equation to the lines of torsion on $\bar{S}$.

Thus the so-called Codazzi-Bonnet's problem ([1]) is equivalent to the following: To find all the surfaces which can be deformed with preservation of their lines of torsion.

10. On the other hand, no surface can be deformed to a distinct surface, such that Occhipinti's lines on the two surfaces correspond. For, let any two surfaces S and $\bar{S}$ be referred to Occhipinti's lines respectively ; then

$$\frac{L}{E}=-\frac{M}{F}=\frac{N}{G}=\sqrt{K},\qquad \frac{\bar{L}}{\bar{E}}=-\frac{\bar{M}}{\bar{F}}=\frac{\bar{N}}{\bar{G}}=\sqrt{\bar{K}}.$$

If these two surfaces be deformable to each other in such a way that

$$u=\bar{u},\qquad v=\bar{v},$$

we must have

([1]) Codazzi, Annali di mat., I, 7 (1857), p. 410; Bonnet, Journal de l'Ecole Polyt., 42 (1867), p. 58.

$$E=\bar{E},\qquad F=\bar{F},\qquad G=\bar{G},\qquad K=\bar{K};$$

so

$$L=\bar{L},\qquad M=\bar{M},\qquad N=\bar{N};$$

whence by a fundamental theorem due to Bonnet, S and $\bar{S}$ must be congruent.

Such a result is also true for the characteristic lines. For, these curves are determined by

$$M=0,\quad \frac{L}{E}=\frac{N}{G}=\sqrt{\frac{(EG-F^2)K}{EG}}\ \text{([1])}.$$

Now we will solve the problem :

To find the most general group of the surfaces of revolution of variable curvature which are deformable to each other such that the characteristic lines of one surface correspond to Occhipinti's lines of the other.

Let S and $\bar{S}$ be the most general surfaces of revolution of variable curvature deformable to each other by putting

$$u=\bar{u},\qquad v=\bar{v}.$$

Then the equations to these surfaces are given, up to the rotation about their axes and the symmetry with respect to their meridian planes, by the following respectively([2]) :

$$(14)\qquad x=U(u)\cos v,\qquad y=U(u)\sin v,\qquad z=\int\sqrt{1-\left(\frac{dU}{du}\right)^2}\,du;$$

$$(15)\qquad \bar{x}=a\,U(\bar{u})\cos\frac{\bar{v}}{a},\quad \bar{y}=a\,U(\bar{u})\sin\frac{\bar{v}}{a},\quad \bar{z}=\int\sqrt{1-a^2\left(\frac{dU}{d\bar{u}}\right)^2}\,d\bar{u},$$

where U denotes an arbitrary function of u (or $\bar{u}$) only and a is an arbitrary constant. The parametric curves u, v and $\bar{u}$, $\bar{v}$ are the lines of curvature of these surfaces respectively.

The equation to Occhipinti's lines of S is

$$(7)\qquad \sqrt{EL}\,du^2-\sqrt{NG}\,dv^2=0,$$

or

$$\left(\frac{dv}{du}\right)^2=\sqrt{\frac{\dfrac{d^2U}{du^2}}{U^3\cdot\left\{1-\left(\dfrac{dU}{du}\right)^2\right\}}},$$

([1]) Hilton and Colomb.

([2]) Minding, Journ. f. d. r. u. ang. Math., 18 (1838), p. 367; Scheffers, Theorie der Flächen, 1. Aufl. (1902), p. 296.

and the equation to the characteristic lines of $\bar{S}$ is

$$(8) \qquad \bar{L}\, d\bar{u}^2 - \bar{N}\, d\bar{v}^2 = 0,$$

or

$$\left(\frac{d\bar{v}}{d\bar{u}}\right)^2 = -\frac{a^2 \dfrac{d^2 U}{d\bar{u}^2}}{U.\left\{1 - a^2\left(\dfrac{dU}{d\bar{u}}\right)^2\right\}}.$$

In order that these two curves correspond when $u=\bar{u}$, $v=\bar{v}$, the function U must satisfy the differential equation

$$(16) \qquad a^4 U.\left\{1-\left(\frac{dU}{du}\right)^2\right\}\frac{d^2U}{du^2} + \left\{1 - a^2\left(\frac{dU}{du}\right)^2\right\}^2 = 0,$$

provided that $\dfrac{d^2 U}{du^2} \neq 0$ (1). Consequently the differential equation to the meridian curves

$$(17) \qquad x = U(u), \qquad z = \int\sqrt{1-\left(\frac{dU}{du}\right)^2}\, du$$

of the surface S is

$$a^4 x \frac{d^2 x}{dz^2} + \left\{1 + (1-a^2)\left(\frac{dx}{dz}\right)^2\right\}^2 = 0,$$

which gives

$$(18) \qquad \int\sqrt{\frac{a^4}{2(1-a)\log bx} - 1}\; dx \pm \frac{z}{\sqrt{1-a^2}} = c,$$

where b and c are constants of integration.

November 1915.

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(1) If $\dfrac{d^2U}{du^2}=0$, then $L=0$ and hence the surface must be developable which contradicts our assumption.

THE TÔHOKU MATHEMATICAL JOURNAL.

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Price per volume (consisting of four numbers) payable in advance: 3 yen = 6 shillings = 6 Mark = 7,50 francs = 1.50 dollars. Postage inclusive.