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On a Certain System of a Doubly Infinite Curves on a Surface.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 8, No. 2,
edited by Tsuruichi Hayashi, College of Science,
Tôhoku Imperial University, Sendai, Japan,
with the collaboration of Messrs.
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December 1915

On a Certain System of a Doubly Infinite Curves on a Surface,

by

KINNOSUKE OGURA in Sendai.

The present note, which may be looked upon as a continuation of my previous paper "On the integral curves of ordinary differential equations of the second order of a certain type"(1), has for its main object to deal with the surface-curves defined by the differential equation

$$v''=(Av'-B)(1+v'^2)+C(1+v'^2)^{\frac{3}{2}},\quad \left(v'=\frac{dv}{du},\ v''=\frac{d^2v}{du^2}\right),$$

where u and v are isothermal parameters of a surface, and A, B, C are functions of u and v alone.

1. Let the linear element of a surface S be

$$(1)\qquad ds^2=\lambda\,(du^2+dv^2),$$

where the *isothermal parameters* u and v are taken for the parametric curves; and let us consider a system of ∞^2 curves Γ, on this surface S, one through each point in each direction.

If the centres of geodesic curvature of all curves Γ at any point P lie on a conic K having that point P as a focus, then the differential equation of these ∞^2 curves Γ is of the form

$$(2)\qquad v''=(Av'-B)(1+v'^2)+C(1+v'^2)^{\frac{3}{2}},$$

where A, B, C are arbitrary functions of u and v. The converse is valid also.

Particularly, when the conic K is a circle having the point P as its centre, the differential equation of Γ becomes

$$(3)\quad v''=\left(-\frac{1}{2}\frac{\partial}{\partial u}\log\lambda\cdot v'+\frac{1}{2}\frac{\partial}{\partial v}\log\lambda\right)(1+v'^2)+C(1+v'^2)^{\frac{3}{2}};$$

and conversely.

On the tangent plane to the surface S at the point $P(u, v)$, we

(1) The Tôhoku Math. Journal, Vol. 8 (1915), p. 93.

take the polar coordinates (r, θ) whose pole is P and whose initial line is the tangent to the parametric curve $v=$const. at P. Then the polar equation of any conic having the point P as a focus is

$$r=\frac{p}{2}\,\frac{1}{1+e\cos(\theta-\theta_0)}, \tag{4}$$

where

$$\operatorname{tg}\theta=v'. \tag{5}$$

Since the geodesic curvature $\frac{1}{\rho_g}$ of the surface S has the expression

$$\frac{1}{\rho_g}=\frac{\lambda\,du^3}{ds^3}\begin{vmatrix} 1 & \frac{1}{2\lambda}\frac{\partial\lambda}{\partial u}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial v}v'-\frac{1}{2\lambda}\frac{\partial\lambda}{\partial u}v'^2 \\ v' & -\frac{1}{2\lambda}\frac{\partial\lambda}{\partial v}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial u}v'+\frac{1}{2\lambda}\frac{\partial\lambda}{\partial v}v'^2+v'' \end{vmatrix},$$

we find

$$\begin{aligned}\frac{1}{\rho_g}&=\lambda^{-\frac{1}{2}}(1+v'^2)^{-\frac{3}{2}}v'' \\ &+\lambda^{-\frac{1}{2}}(1+v'^2)^{-\frac{1}{2}}\left(-\frac{1}{2}\frac{\partial}{\partial v}\log\lambda+\frac{1}{2}\frac{\partial}{\partial u}\log\lambda\,.\,v'\right).\end{aligned} \tag{6}$$

Now if the centre of geodesic curvature lie on the conic (4), it must be

$$r=\rho_g.$$

Hence (4) becomes

$$\frac{1}{\rho_g}=\frac{2}{p}+\frac{2}{p}e\,(1+v'^2)^{-\frac{1}{2}}(\cos\theta_0+\sin\theta_0\,.\,v'),$$

and therefore it follows from (6) that

$$\begin{aligned}v''=\frac{2}{p}\lambda^{\frac{1}{2}}(1+v'^2)^{\frac{3}{2}}+(1+v'^2)\Big\{&\left(\frac{2}{p}e\lambda^{\frac{1}{2}}\cos\theta_0+\frac{1}{2}\frac{\partial}{\partial v}\log\lambda\right) \\ &+\left(\frac{2}{p}e\lambda^{\frac{1}{2}}\sin\theta_0-\frac{1}{2}\frac{\partial}{\partial u}\log\lambda\right)v'\Big\}.\end{aligned}$$

If we put

$$\begin{aligned}A&=\frac{2}{p}e\sqrt{\lambda}\sin\theta_0-\frac{1}{2}\frac{\partial}{\partial u}\log\lambda, \\ B&=-\frac{2}{p}e\sqrt{\lambda}\cos\theta_0-\frac{1}{2}\frac{\partial}{\partial v}\log\lambda,\end{aligned}\qquad C=\frac{2}{p}\sqrt{\lambda}, \tag{7}$$

or

$$\begin{aligned}&p=\frac{2\sqrt{\lambda}}{C},\qquad \operatorname{tg}\theta_0=-\frac{A+\frac{1}{2}\frac{\partial}{\partial u}\log\lambda}{B+\frac{1}{2}\frac{\partial}{\partial v}\log\lambda}, \\ &e=\pm\frac{\sqrt{\left(A+\frac{1}{2}\frac{\partial}{\partial u}\log\lambda\right)^2+\left(B+\frac{1}{2}\frac{\partial}{\partial v}\log\lambda\right)^2}}{C},\end{aligned} \tag{8}$$

the last equation takes the form

$$v''=(Av'-B)(1+v'^2)+C(1+v'^2)^{\frac{3}{2}}. \tag{2}$$

The totality of the integral curves Γ is called *the V-system on the surface S*.

In the particular case where

$$A=-\frac{1}{2}\frac{\partial}{\partial u}\log\lambda,\qquad B=-\frac{1}{2}\frac{\partial}{\partial v}\log\lambda,$$

the eccentricity e vanishes, and hence the conic K becomes a circle. And the totality of the integral curves (3) is called *the E-system on the surface S*.

For example, the ∞^2 curves of constant geodesic curvature with equal magnitude form an E-system.

2. Suppose that the two functions A and B are fixed. If we decrease the absolute value of C monotonously, the direction of the major axis of the conic K is fixed, while the eccentricity e increases monotonously, so that the conic K approaches to the directrix D, corresponding to the focus P,

$$r\cos(\theta-\theta_0)=\pm\frac{\sqrt{\lambda}}{\sqrt{\left(A+\frac{1}{2}\frac{\partial}{\partial u}\log\lambda\right)^2+\left(B+\frac{1}{2}\frac{\partial}{\partial v}\log\lambda\right)^2}}$$

whose position is independent of C; and *in the limit $C=0$, the conic K reduces to the directrix D.* In this case the system of the ∞^2 integral curves of the equation

$$v''=(Av'-B)(1+v'^2) \tag{9}$$

is called *the velocity system* or *the V_0-system.* Hence we arrive at the theorem which is an extension of the well known theorem due to Profs. Scheffers and Kasner[1].

The centres of geodesic curvature of all curves of the V_0-system at any point lie on a straight line, and conversely.

3. Now it is easily seen that almost all the theorems for the V-system on a plane in the previous paper can be extended to the V-system on a surface. We, therefore, only think it necessary to select out a few of typical theorems.

A necessary and sufficient condition that

$$v''=\Phi(u, v, v') \qquad (10)$$

should be the differential equation of the V-system is that the expression

$$\frac{\partial \Phi}{\partial v'}-\frac{3\Phi}{1+v'^2} \qquad (11)$$

is a linear integral function of v'.

The only point transformations which convert every V-system on a surface into a V-system on the transformed surface are conformal transformations, and any conformal transformation actually converts any V-system on a surface into a V-system on the transformed surface.

4. When a function $f(u, v)$ exists such that

$$A=\frac{\partial f}{\partial u}, \qquad B=\frac{\partial f}{\partial v},$$

the system of ∞^2 curves Γ is called *the N-system.*

A necessary and sufficient condition that (10) should be the differential equation of the N-system is that the expression (11) is the total differential quotient of a function of u and v with respect to u.

Moreover, in the particular case where $C=0$, the N-system is called *the N_0-system* which is nothing but *the natural family on the surface*[2]. For, the differential equation of the extremals of

$$\int_{s_0}^{s_1}\varphi(u, v)\, ds=\min.$$

[1] Scheffers, "Über gewisse zweifach unendliche Curvenschaaren in der Ebene," Leipziger Berichte, Bd. 50 (1898), p. 261; Kasner, "Natural families of trajectories," Trans. Amer. Math. Soc., Vol. 10 (1909), p. 201. While I was reading the proof-sheets of this note, I found this theorem proved by a different method in Dr. J. Lipka's interesting paper "Geometric characterization of isogonal trajectories on a surface," Annals of Mathematics, Series 2, vol. 15 (1913–14), p. 71.

[2] See K. Ogura, "Trajectories in the conservative field of force," Part I, Chapt. IV, The Tôhoku Math. Journal, Vol. 7 (1915).

on a surface having the linear element

$$ds^2=\lambda(du^2+dv^2)$$

is equivalent to that of the geodesics on a surface having the linear element

$$ds^2=\varphi^2\lambda(du^2+dv^2),$$

and conversely. Hence it takes the form

$$v''=(Av'-B)(1+v'^2),$$

where

$$A=-\frac{\partial}{\partial u}\left(\log\sqrt{\lambda}\,\varphi\right), \qquad B=-\frac{\partial}{\partial v}\left(\log\sqrt{\lambda}\,\varphi\right)^{(1)}.$$

By any conformal transformation any N-system (N_0-system) converts into an N-system (N_0-system); conversely, the only V-systems (V_0-systems) which convert into N-systems (N_0-systems) by any conformal transformation are N-systems (N_0-systems).

[1] See Lipka, loc. cit; or Lipka, "Natural families of curves in a general curved space of n-dimensions," Trans. Amer. Math. Soc., Vol. 13 (1912), p. 93.

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