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by

KINNOSUKE OGURA in Sendai.

CHAPTER V.

Determination of the Function φ by the Families of Extremals in Space, especially by the Parametric Curves of Curvilinear Coordinates.

General Case.

35. Let any point (x, y, z) in space be represented by any curvilinear coordinates (u_1, u_2, u_3) such that

$$x=x(u_1, u_2, u_3), \quad y=y(u_1, u_2, u_3), \quad z=z(u_1, u_2, u_3).$$

Then the linear element has the expression

$$ds^2=dx^2+dy^2+dz^2=\Sigma a_{rs}\, du_r\, du_s \quad (r, s=1, 2, 3),$$

where

$$a_{rr}=\left(\frac{\partial x}{\partial u_r}\right)^2+\left(\frac{\partial y}{\partial u_r}\right)^2+\left(\frac{\partial z}{\partial u_r}\right)^2,$$

$$a_{rs}=a_{sr}=\frac{\partial x}{\partial u_r}\frac{\partial x}{\partial u_s}+\frac{\partial y}{\partial u_r}\frac{\partial y}{\partial u_s}+\frac{\partial z}{\partial u_r}\frac{\partial z}{\partial u_s}.$$

Next let us denote by a the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and by A_{rs} the algebraic complement of a_{rs} divided by a, and by $\left\{\begin{matrix}\lambda\,\mu\\ i\end{matrix}\right\}$ the expression

$$\frac{1}{2}\sum_r A_{ir}\left(\frac{\partial a_{\lambda r}}{\partial u_\mu}+\frac{\partial a_{\mu r}}{\partial u_\lambda}-\frac{\partial a_{\lambda\mu}}{\partial u_r}\right), \qquad (i, r, \lambda, \mu=1, 2, 3).$$

And also the corresponding quantities obtained from a, A_{rs} and $\left\{\begin{matrix}\lambda\,\mu\\ i\end{matrix}\right\}$ by substituting

$$\overline{a}_{rs}=\varphi^2(u_1, u_2, u_3)\, a_{rs} \quad \text{for} \quad a_{rs} \quad (r, s=1,\ 2,\ 3)$$

will be expressed by $\overline{a}$, $\overline{A}_{rs}$ and $\left\{\begin{matrix}\overline{\lambda\ \mu}\\ i\end{matrix}\right\}'$ respectively.

Now it is well known that the extremals of

$$\int ds=\int\sqrt{\Sigma\, a_{rs}\frac{du_r}{ds}\,\frac{du_s}{ds}}\;ds=\min.$$

are given by

$$\frac{d^2 u_i}{d s^2}+\sum_{\lambda, \mu}\left\{\begin{matrix}\lambda\ \mu\\ i\end{matrix}\right\}\frac{du_\lambda}{ds}\,\frac{du_\mu}{ds}=0\,(^1), \quad (i=1,\ 2,\ 3).$$

Hence if we put

$$d\overline{s}=\varphi\, ds=\sqrt{\Sigma\,\varphi^2 a_{rs}\frac{du_r}{ds}\,\frac{du_s}{ds}}\;ds,$$

it follows that the extremals of

$$\int d\overline{s}=\int\varphi\, ds=\min. \tag{87}$$

are given by

$$\frac{d^2 u_i}{ds^2}+\sum_{\lambda, \mu}\left\{\begin{matrix}\overline{\lambda\ \mu}\\ i\end{matrix}\right\}'\frac{du_\lambda}{ds}\,\frac{du_\mu}{ds}=0\,(^2), \quad (i=1,\ 2,\ 3). \tag{88}$$

From the last equations we can derive the following three equations

$$\begin{aligned}
&du_1\, d^2 u_2-du_2\, d^2 u_1+\left\{\begin{matrix}\overline{1\,1}\\ 2\end{matrix}\right\}'du_1^3-\left\{\begin{matrix}\overline{2\,2}\\ 1\end{matrix}\right\}'du_2^3+\left\{\begin{matrix}\overline{3\,3}\\ 2\end{matrix}\right\}'du_1\, du_3^2-\left\{\begin{matrix}\overline{3\,3}\\ 1\end{matrix}\right\}'du_2\, du_3^2\\
&\quad+\left(2\left\{\begin{matrix}\overline{1\,2}\\ 2\end{matrix}\right\}'-\left\{\begin{matrix}\overline{1\,1}\\ 1\end{matrix}\right\}'\right)du_1^2\, du_2-\left(2\left\{\begin{matrix}\overline{1\,2}\\ 1\end{matrix}\right\}'-\left\{\begin{matrix}\overline{2\,2}\\ 2\end{matrix}\right\}'\right)du_1\, du_2^2\\
&\quad+2\left\{\begin{matrix}\overline{1\,3}\\ 2\end{matrix}\right\}'du_1^2\, du_3-2\left\{\begin{matrix}\overline{2\,3}\\ 1\end{matrix}\right\}'du_2^2\, du_3+2\left(\left\{\begin{matrix}\overline{2\,3}\\ 2\end{matrix}\right\}'-\left\{\begin{matrix}\overline{1\,3}\\ 1\end{matrix}\right\}'\right)du_1\, du_2\, du_3=0,\\
&du_2\, d^2u_3-du_3\, d^2u_2+\left\{\begin{matrix}\overline{2\,2}\\ 3\end{matrix}\right\}'du_2^3-\left\{\begin{matrix}\overline{3\,3}\\ 2\end{matrix}\right\}'du_3^3+\left\{\begin{matrix}\overline{1\,1}\\ 3\end{matrix}\right\}'du_2\, du_1^2-\left\{\begin{matrix}\overline{1\,1}\\ 2\end{matrix}\right\}'du_3\, du_1^2\\
&\quad+\left(2\left\{\begin{matrix}\overline{2\,3}\\ 3\end{matrix}\right\}'-\left\{\begin{matrix}\overline{2\,2}\\ 2\end{matrix}\right\}'\right)du_2^2\, du_3-\left(2\left\{\begin{matrix}\overline{2\,3}\\ 1\end{matrix}\right\}'-\left\{\begin{matrix}\overline{3\,3}\\ 3\end{matrix}\right\}'\right)du_2\, du_3^2\\
&\quad+2\left\{\begin{matrix}\overline{1\,2}\\ 3\end{matrix}\right\}'du_1\, du_2^2-2\left\{\begin{matrix}\overline{1\,3}\\ 2\end{matrix}\right\}'du_3^2\, du_1+2\left(\left\{\begin{matrix}\overline{1\,3}\\ 3\end{matrix}\right\}'-\left\{\begin{matrix}\overline{1\,2}\\ 2\end{matrix}\right\}'\right)du_1\, du_2\, du_3=0,\\
&du_3\, d^2u_1-du_1\, d^2u_3+\left\{\begin{matrix}\overline{3\,3}\\ 1\end{matrix}\right\}'du_3^3-\left\{\begin{matrix}\overline{1\,1}\\ 3\end{matrix}\right\}'du_1^3+\left\{\begin{matrix}\overline{2\,2}\\ 1\end{matrix}\right\}'du_3\, du_2^2-\left\{\begin{matrix}\overline{2\,2}\\ 3\end{matrix}\right\}'du_2^2\, du_1\\
&\quad+\left(2\left\{\begin{matrix}\overline{1\,3}\\ 1\end{matrix}\right\}'-\left\{\begin{matrix}\overline{3\,3}\\ 3\end{matrix}\right\}'\right)du_3^2\, du_1-\left(2\left\{\begin{matrix}\overline{1\,3}\\ 3\end{matrix}\right\}'-\left\{\begin{matrix}\overline{1\,1}\\ 1\end{matrix}\right\}'\right)du_3\, du_1^2\\
&\quad+2\left\{\begin{matrix}\overline{2\,3}\\ 1\end{matrix}\right\}'du_2\, du_3^2-2\left\{\begin{matrix}\overline{1\,2}\\ 3\end{matrix}\right\}'du_1^2\, du_2+2\left(\left\{\begin{matrix}\overline{1\,2}\\ 1\end{matrix}\right\}'-\left\{\begin{matrix}\overline{2\,3}\\ 3\end{matrix}\right\}'\right)du_1 du_2 du_3=0\,(^1).
\end{aligned} \tag{89}$$

(¹) For example, see Bianchi, Vorlesungen über Differentialgeometrie, 1. Aufl. (1899), p. 569.

(²) J. Lipka, Trans. of Amer. Math. Soc., 13 (1912), p. 80.

Hence *a necessary and sufficient condition that the ∞^2 parametric curves*

$$u_2=const., \qquad u_3=const.$$

should be the extremals is

$$\left\{\begin{matrix}\overline{1\,1}\\ 2\end{matrix}\right\}'=0, \qquad \left\{\begin{matrix}\overline{1\,1}\\ 3\end{matrix}\right\}'=0,$$

which may be written

$$\begin{aligned}
&\left\{\begin{matrix}1\,1\\ 2\end{matrix}\right\}'-a_{11}\left(A_{21}\frac{\partial\log\varphi}{\partial u_1}+A_{22}\frac{\partial\log\varphi}{\partial u_2}+A_{23}\frac{\partial\log\varphi}{\partial u_3}\right)=0,\\
&\left\{\begin{matrix}1\,1\\ 3\end{matrix}\right\}'-a_{11}\left(A_{31}\frac{\partial\log\varphi}{\partial u_1}+A_{32}\frac{\partial\log\varphi}{\partial u_2}+A_{33}\frac{\partial\log\varphi}{\partial u_3}\right)=0\,(^2).
\end{aligned} \tag{90}$$

Similarly, such a condition for the parametric curves

$$u_3=\text{const.}, \qquad u_1=\text{const.},$$

is

$$\left\{\begin{matrix}\overline{2\,2}\\ 1\end{matrix}\right\}'=0, \qquad \left\{\begin{matrix}\overline{2\,2}\\ 3\end{matrix}\right\}'=0,$$

or

$$\begin{aligned}
&\left\{\begin{matrix}2\,2\\ 1\end{matrix}\right\}'-a_{22}\left(A_{11}\frac{\partial\log\varphi}{\partial u_1}+A_{12}\frac{\partial\log\varphi}{\partial u_2}+A_{13}\frac{\partial\log\varphi}{\partial u_3}\right)=0,\\
&\left\{\begin{matrix}2\,2\\ 3\end{matrix}\right\}'-a_{22}\left(A_{31}\frac{\partial\log\varphi}{\partial u_1}+A_{32}\frac{\partial\log\varphi}{\partial u_2}+A_{33}\frac{\partial\log\varphi}{\partial u_3}\right)=0;
\end{aligned} \tag{91}$$

and that for the parametric curves

$$u_1=\text{const.}, \qquad u_2=\text{const.}$$

is

(¹) For another form of these equations, see O. Staude, Leipziger Berichte, 45 (1893), p. 513.

(²) Compare with equations (31) and (80) in Part I of this paper, this Journal, 7 (1915).

$$\begin{Bmatrix}3\,3\\1\end{Bmatrix}'=0, \qquad \begin{Bmatrix}3\,3\\2\end{Bmatrix}'=0,$$

or

$$(92)\qquad \begin{aligned} &\begin{Bmatrix}3\,3\\1\end{Bmatrix}'-a_{33}\left(A_{11}\frac{\partial \log\varphi}{\partial u_1}+A_{12}\frac{\partial \log\varphi}{\partial u_2}+A_{13}\frac{\partial \log\varphi}{\partial u_3}\right)=0,\\ &\begin{Bmatrix}3\,3\\2\end{Bmatrix}'-a_{33}\left(A_{21}\frac{\partial \log\varphi}{\partial u_1}+A_{22}\frac{\partial \log\varphi}{\partial u_2}+A_{23}\frac{\partial \log\varphi}{\partial u_3}\right)=0.\end{aligned}$$

36. For an example, take a triply pseudospherical orthogonal system. Let $u_3=$const. be a family of pseudospherical surfaces having their radii R which depend upon u_3 only, and let $2\,\omega$ be the angle between the two asymptotic curves $u_1=$const. and $u_2=$const. passing through any point (u_1, u_2, u_3) on the surface $u_3=$const.. Then the linear element has the expression

$$ds^2=du_1^2+2\cos 2\,\omega\, du_1\, du_2+du_2^2+R^2\left(\frac{\partial\omega}{\partial u_3}\right)^2 du_3^2\ (^1),$$

so that the ∞^2 curves $u_2=$const., $u_3=$const. form a system of asymptotic curves on the ∞^1 surfaces $u_3=$const., and also form a system of geodesics on the other ∞^1 surfaces $u_2=$const..

Now the condition (90) for these ∞^2 curves to be extremals is given by the differential equations

$$\cos 2\,\omega\frac{\partial \log\varphi}{\partial u_1}-\frac{\partial \log\varphi}{\partial u_2}-2\sin 2\,\omega\frac{\partial\omega}{\partial u_1}=0, \qquad \frac{\partial \log\varphi}{\partial u_3}=0,$$

which have always common integrals. Since we have $\dfrac{\partial\varphi}{\partial u_3}=0$ from the second equation, the gradient vector of φ at any point (u_1, u_2, u_3) touches the surface $u_3=$const. at this point, while $u_2=$const., $u_3=$const. constitute a family of asymptotic curves on $u_3=$const.. Therefore from Art. 9, we can infer the theorem:

In a triply pseudospherical orthogonal system, the ∞^2 curves $u_2=$const., $u_3=$const. form a system of the free paths (described by a particle in the conservative field of force) on the ∞^1 pseudospherical surfaces $u_3=$cons$'$..

In exactly the same way it may be proved that the ∞^2 curves $u_1=$const., $u_3=$const. form the other system of the free paths on $u_3=$const., and the function φ for these trajectories is determined by

$$-\frac{\partial \log\varphi}{\partial u_1}+\cos 2\,\omega\frac{\partial \log\varphi}{\partial u_2}-2\sin 2\,\omega\frac{\partial\omega}{\partial u_2}=0, \qquad \frac{\partial \log\varphi}{\partial u_3}=0.$$

(1) Bianchi, loc. cit., p. 530.

Systems of parametric curves as extremals.

37. In Art. 3, we have proved that the function φ can be determined if we know $3\,\infty^2$ out of the totality of ∞^4 extremals, each of three systems of ∞^2 curves passing through each point of space. Here we will give a detailed discussion with the case where the three systems of parametric curves are the extremals.

It will be seen from Art. 35 that a necessary and sufficient condition for the $3\,\infty^2$ curves

$$u_2,\ u_3=\text{const.};\quad u_3,\ u_1=\text{const.};\quad u_1,\ u_2=\text{const.}$$

to be the extremals is that the six equations (90), (91), (92) have a common integral. Now in order that these equations should be consistent, we must have

$$(93)\qquad \frac{1}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'=\frac{1}{a_{33}}\begin{Bmatrix}3\,3\\1\end{Bmatrix}',\quad \frac{1}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}'=\frac{1}{a_{11}}\begin{Bmatrix}1\,1\\2\end{Bmatrix}',\quad \frac{1}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'=\frac{1}{a_{22}}\begin{Bmatrix}2\,2\\3\end{Bmatrix}';$$

and if the condition be satisfied, it will be sufficient to consider only the three equations

$$(94)\qquad \begin{aligned} A_{11}\frac{\partial \log\varphi}{\partial u_1}+A_{12}\frac{\partial \log\varphi}{\partial u_2}+A_{13}\frac{\partial \log\varphi}{\partial u_3}&=\frac{1}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}',\\ A_{21}\frac{\partial \log\varphi}{\partial u_1}+A_{22}\frac{\partial \log\varphi}{\partial u_2}+A_{23}\frac{\partial \log\varphi}{\partial u_3}&=\frac{1}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}',\\ A_{31}\frac{\partial \log\varphi}{\partial u_1}+A_{32}\frac{\partial \log\varphi}{\partial u_2}+A_{33}\frac{\partial \log\varphi}{\partial u_3}&=\frac{1}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'.\end{aligned}$$

If we introduce a new dependent variable f such that

$$f=f(v, u_1, u_2, u_3), \quad \text{where} \quad v=\log\varphi,$$

and transform the equations (94) by means of the relations

$$(95)\qquad \frac{\partial f}{\partial v}\frac{\partial v}{\partial u_i}+\frac{\partial f}{\partial u_i}=0 \qquad (i=1, 2, 3),$$

then the integral will be given by the equation

$$f(v, u_1, u_2, u_3)=\text{const.}.$$

But since

$$\begin{vmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{vmatrix}=\frac{a^2}{a^3}=\frac{1}{a},$$

we find from (94) and (95),

$$(96)\qquad \frac{\partial f}{\partial u_1}+\left\{1\right\}\frac{\partial f}{\partial v}=0,\quad \frac{\partial f}{\partial u_2}+\left\{2\right\}\frac{\partial f}{\partial v}=0,\quad \frac{\partial f}{\partial u_3}+\left\{3\right\}\frac{\partial f}{\partial v}=0,$$

where

$$(97)\qquad \begin{aligned}\left\{1\right\}&=\frac{a_{11}}{a_{22}}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}'+\frac{a_{12}}{a_{33}}\left\{\begin{matrix}3\,3\\2\end{matrix}\right\}'+\frac{a_{13}}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}',\\ \left\{2\right\}&=\frac{a_{21}}{a_{22}}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}'+\frac{a_{22}}{a_{33}}\left\{\begin{matrix}3\,3\\2\end{matrix}\right\}'+\frac{a_{23}}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}',\\ \left\{3\right\}&=\frac{a_{31}}{a_{22}}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}'+\frac{a_{32}}{a_{33}}\left\{\begin{matrix}3\,3\\2\end{matrix}\right\}'+\frac{a_{33}}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}'.\end{aligned}$$

But equations (96) have a common integral when and only when they form a complete system; and the condition of completeness of the system is found to be

$$(98)\qquad \frac{\partial\{2\}}{\partial u_1}=\frac{\partial\{1\}}{\partial u_2},\quad \frac{\partial\{3\}}{\partial u_2}=\frac{\partial\{2\}}{\partial u_3},\quad \frac{\partial\{1\}}{\partial u_3}=\frac{\partial\{3\}}{\partial u_1}.$$

If the condition (98) be satisfied, the system (96) has one common integral which is given by

$$f=v-\int\{1\}du_1+\{2\}du_2+\{3\}du_3=\text{const.},$$

and hence

$$(99)\qquad \varphi=k\,e^{\int\{1\}du_1+\{2\}du_2+\{3\}du_3},$$

where k denotes an arbitrary constant. Thus we have arrived at the theorem:

A necessary and sufficient condition that the three systems of parametric curves

$$u_2,\ u_3=const.;\quad u_3,\ u_1=const.;\quad u_1,\ u_2=const.$$

should be the extremals of the integral

$$\int\varphi(u_1,\ u_2,\ u_3)\,ds=min.$$

is that

$$\frac{1}{a_{22}}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}'=\frac{1}{a_{33}}\left\{\begin{matrix}3\,3\\1\end{matrix}\right\}',\quad \frac{1}{a_{33}}\left\{\begin{matrix}3\,3\\2\end{matrix}\right\}'=\frac{1}{a_{11}}\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}',\quad \frac{1}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}'=\frac{1}{a_{33}}\left\{\begin{matrix}2\,2\\3\end{matrix}\right\}'$$

and

$$\frac{\partial\{2\}}{\partial u_1}=\frac{\partial\{1\}}{\partial u_2},\quad \frac{\partial\{3\}}{\partial u_2}=\frac{\partial\{2\}}{\partial u_3},\quad \frac{\partial\{1\}}{\partial u_3}=\frac{\partial\{3\}}{\partial u_1};$$

and if the condition be fulfilled, the function φ is given by

$$\varphi=k\,e^{\int\{1\}du_1+\{2\}du_2+\{3\}du_3},$$

where k denotes an arbitrary constant.

Now we add a remark that *any two systems of parametric curves as extremals are sufficient to determine the function φ.*

For, if the $2\,\infty^2$ parametric curves

$$u_2,\ u_3=\text{const.};\quad u_3,\ u_1=\text{const.}$$

should be the extremals we must have from (90) and (91)

$$\frac{1}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}'=\frac{1}{a_{22}}\left\{\begin{matrix}2\,2\\3\end{matrix}\right\}';$$

and if the condition be satisfied, we have

$$\begin{aligned}A_{11}\frac{\partial\log\varphi}{\partial u_1}+A_{12}\frac{\partial\log\varphi}{\partial u_2}+A_{13}\frac{\partial\log\varphi}{\partial u_3}&=\frac{1}{a_{22}}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}',\\ A_{21}\frac{\partial\log\varphi}{\partial u_1}+A_{22}\frac{\partial\log\varphi}{\partial u_2}+A_{23}\frac{\partial\log\varphi}{\partial u_3}&=\frac{1}{a_{11}}\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}',\\ A_{31}\frac{\partial\log\varphi}{\partial u_1}+A_{32}\frac{\partial\log\varphi}{\partial u_2}+A_{33}\frac{\partial\log\varphi}{\partial u_3}&=\frac{1}{a_{11}}\left\{\begin{matrix}1\,1\\3\end{matrix}\right\}'.\end{aligned}$$

Hence by a similar reasoning as before, we can determine the function φ, if it exist, up to a constant multiple.

The differential equation of all extremals.

38. Suppose that the function φ has been found, under which the $3\,\infty^2$ parametric curves

$$u_2,\ u_3=\text{const.};\quad u_3,\ u_1=\text{const.};\quad u_1,\ u_2=\text{const.}$$

are extremals. Then we can find the differential equations of all extremals for this function φ.

For, in consequence of (99), we find

$$\begin{aligned}\left\{\begin{matrix}\lambda\,\mu\\i\end{matrix}\right\}'=\left\{\begin{matrix}\lambda\,\mu\\i\end{matrix}\right\}'&+\{\mu\}(A_{i1}a_{\lambda 1}+A_{i2}a_{\lambda 2}+A_{i3}a_{\lambda 3})\\ &+\{\lambda\}(A_{i1}a_{\mu 1}+A_{i2}a_{\mu 2}+A_{i3}a_{\mu 3})-a_{\lambda\mu}(A_{i1}\{1\}+A_{i2}\{2\}+A_{i3}\{3\}),\end{aligned}$$

so that

$$
(100)\quad
\begin{aligned}
&\begin{Bmatrix}\overline{1\,1}\\1\end{Bmatrix}'=\begin{Bmatrix}1\,1\\1\end{Bmatrix}'+2\{1\}-\frac{a_{11}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}', &&\begin{Bmatrix}\overline{1\,2}\\1\end{Bmatrix}'=\begin{Bmatrix}1\,2\\1\end{Bmatrix}'+\{2\}-\frac{a_{12}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}',\\
&\begin{Bmatrix}\overline{1\,3}\\1\end{Bmatrix}'=\begin{Bmatrix}1\,3\\1\end{Bmatrix}'+\ \{3\}-\frac{a_{13}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}', &&\begin{Bmatrix}\overline{2\,3}\\1\end{Bmatrix}'=\begin{Bmatrix}2\,3\\1\end{Bmatrix}'-\frac{a_{23}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}',\\
&\begin{Bmatrix}\overline{2\,2}\\2\end{Bmatrix}'=\begin{Bmatrix}2\,2\\2\end{Bmatrix}'+2\{2\}-\frac{a_{22}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}', &&\begin{Bmatrix}\overline{2\,3}\\2\end{Bmatrix}'=\begin{Bmatrix}2\,3\\2\end{Bmatrix}'+\{3\}-\frac{a_{23}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}',\\
&\begin{Bmatrix}\overline{1\,2}\\2\end{Bmatrix}'=\begin{Bmatrix}1\,2\\2\end{Bmatrix}'+\ \{1\}-\frac{a_{12}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}', &&\begin{Bmatrix}\overline{1\,3}\\2\end{Bmatrix}'=\begin{Bmatrix}1\,3\\2\end{Bmatrix}'-\frac{a_{13}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}',\\
&\begin{Bmatrix}\overline{3\,3}\\3\end{Bmatrix}'=\begin{Bmatrix}3\,3\\3\end{Bmatrix}'+2\{3\}-\frac{a_{33}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}', &&\begin{Bmatrix}\overline{1\,3}\\3\end{Bmatrix}'=\begin{Bmatrix}1\,3\\3\end{Bmatrix}'+\{1\}-\frac{a_{13}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}',\\
&\begin{Bmatrix}\overline{2\,3}\\3\end{Bmatrix}'=\begin{Bmatrix}2\,3\\3\end{Bmatrix}'+\ \{2\}-\frac{a_{23}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}', &&\begin{Bmatrix}\overline{1\,2}\\3\end{Bmatrix}'=\begin{Bmatrix}1\,2\\3\end{Bmatrix}'-\frac{a_{12}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'.
\end{aligned}
$$

Hence it follows, from the first and the third of equations (89), that *the differential equations of the extremals for* φ *are*

$$
(101)\quad
\begin{aligned}
&\frac{d^2u_2}{du_1^2}-2\left(\begin{Bmatrix}2\,3\\1\end{Bmatrix}'-\frac{a_{23}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'\right)\left(\frac{du_2}{du_1}\right)^2\frac{du_3}{du_1}\\
&\qquad-\left(2\begin{Bmatrix}1\,2\\1\end{Bmatrix}'-\begin{Bmatrix}2\,2\\2\end{Bmatrix}'+\frac{a_{22}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}'-2\,\frac{a_{12}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'\right)\left(\frac{du_2}{du_1}\right)^2\\
&\qquad+2\left(\begin{Bmatrix}2\,3\\2\end{Bmatrix}'-\begin{Bmatrix}1\,3\\1\end{Bmatrix}'+\frac{a_{13}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'-\frac{a_{23}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}'\right)\frac{du_2}{du_1}\,\frac{du_3}{du}\\
&\qquad+\left(2\begin{Bmatrix}1\,2\\2\end{Bmatrix}'-\begin{Bmatrix}1\,1\\1\end{Bmatrix}'+\frac{a_{11}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'-2\,\frac{a_{12}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}'\right)\frac{du_2}{du_1}\\
&\qquad+2\left(\begin{Bmatrix}1\,3\\2\end{Bmatrix}'-\frac{a_{13}}{a_{33}}\begin{Bmatrix}3\,3\\2\end{Bmatrix}'\right)\frac{du_3}{du_1}=0,\\
&\frac{d^2u_3}{du_1^2}-2\left(\begin{Bmatrix}2\,3\\1\end{Bmatrix}'-\frac{a_{23}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'\right)\frac{du_2}{du_1}\left(\frac{du_3}{du_1}\right)^2\\
&\qquad-\left(2\begin{Bmatrix}1\,3\\1\end{Bmatrix}'-\begin{Bmatrix}3\,3\\3\end{Bmatrix}'+\frac{a_{33}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'-2\,\frac{a_{13}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'\right)\left(\frac{du_3}{du_1}\right)^2\\
&\qquad+2\left(\begin{Bmatrix}1\,2\\1\end{Bmatrix}'-\begin{Bmatrix}2\,3\\3\end{Bmatrix}'+\frac{a_{23}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'-\frac{a_{12}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'\right)\frac{du_2}{du_1}\,\frac{du_3}{du_1}\\
&\qquad+\left(2\begin{Bmatrix}1\,3\\3\end{Bmatrix}'-\begin{Bmatrix}1\,1\\1\end{Bmatrix}'+\frac{a_{11}}{a_{22}}\begin{Bmatrix}2\,2\\1\end{Bmatrix}'-2\,\frac{a_{13}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'\right)\frac{du_3}{du_1}\\
&\qquad+2\left(\begin{Bmatrix}1\,2\\3\end{Bmatrix}'-\frac{a_{12}}{a_{11}}\begin{Bmatrix}1\,1\\3\end{Bmatrix}'\right)\frac{du_2}{du_1}=0.
\end{aligned}
$$

A triply orthogonal system.

39. In the following lines we will deal with the particular case where the parametric surfaces

$$u_1=\text{const.},\quad u_2=\text{const.},\quad u_3=\text{const.}$$

form a triply orthogonal system.

In this case the linear element takes the form

$$ds^2=H_1^2\,du_1^2+H_2^2\,du_2^2+H_3^2\,du_3^2,$$

and therefore

$$
\begin{aligned}
a_{ii}&=H_i^2, & a_{ij}&=0, & &\\
A_{ii}&=\frac{1}{H_i}, & A_{ij}&=0, & &(i,j=1,2,3;\ i\neq j);\\
\begin{Bmatrix}i\,i\\i\end{Bmatrix}'&=\frac{1}{H_i}\,\frac{\partial H_i}{\partial u_i}, & \begin{Bmatrix}i\,i\\j\end{Bmatrix}'&=-\frac{H_i}{H_j^2}\,\frac{\partial H_i}{\partial u_j}, & &\\
\begin{Bmatrix}i\,j\\j\end{Bmatrix}'&=\frac{1}{H_j}\,\frac{\partial H_j}{\partial u_i}, & \begin{Bmatrix}i\,j\\k\end{Bmatrix}'&=0, & &\left(\begin{matrix}i,j,k=1,2,3\\ i\neq j,\ j\neq k,\ k\neq i\end{matrix}\right).
\end{aligned}
$$

Now since equations (90) become

$$\frac{\partial H_1}{\partial u_2}+H_1\,\frac{\partial \log\varphi}{\partial u_2}=0,\qquad \frac{\partial H_1}{\partial u_3}+H_1\,\frac{\partial \log\varphi}{\partial u_3}=0,$$

we see that *the condition for the* ∞^2 *parametric curves* $u_2=const.$, $u_3=const.$ *to be the extremals for the function* φ *is given by*

$$(102)\qquad H_1\,\varphi=\Psi_1(u_1),$$

where $\Psi_1(u_1)$ *is an arbitrary function of* u_1 *only.* Similarly, such conditions for $u_3=$const., $u_1=$const. and $u_1=$const., $u_2=$const. are

$$(103)\qquad H_2\,\varphi=\Psi_2(u_2)$$

and

$$(104)\qquad H_3\,\varphi=\Psi_3(u_3)$$

respectively, where $\Psi_2(u_2)$ is an arbitrary function of u_2 only and $\Psi_3(u_3)$ of u_3 only.

For example, if we take the elliptic coordinates (u_1, u_2, u_3) such that

$$x^2=\frac{(a^2+u_1)(a^2+u_2)(a^2+u_3)}{(a^2-b^2)(a^2-c^2)},\quad y^2=\frac{(b^2+u_1)(b^2+u_2)(b^2+u_3)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2=\frac{(c^2+u_1)(c^2+u_2)(c^2+u_3)}{(c^2-a^2)(c^2-b^2)}, \quad (a,\ b,\ c,\ \text{arbitrary constants}),$$

the linear element has the expression

$$ds^2=\frac{1}{4}\left\{\frac{(u_1-u_2)(u_1-u_3)}{(a^2+u_1)(b^2+u_1)(c^2+u_1)}du_1^2+\frac{(u_1-u_3)(u_2-u_1)}{(a^2+u_2)(b^2+u_2)(c^2+u_2)}du_2^2\right.$$
$$\left.+\frac{(u_3-u_1)(u_3-u_2)}{(a^2+u_3)(b^2+u_3)(c^2+u_3)}du_3^2\right\}.$$

Hence the function φ under which the parametric curves $u_2=$const., $u_3=$const. can be the extremals is given by

$$\varphi=\frac{\Psi_1(u_1)}{\sqrt{(u_1-u_2)(u_1-u_3)}}.$$

These ∞^2 curves which are orthogonal trajectories of the confoca quadrics $u_1=$const. form a family of lines of curvature on the other confocal quadrics $u_3=$const. (or $u_2=$const.)(¹).

40. For the orbits in the conservative field of force, the results in the last article can also be obtained by using Prof. Joukovsky's method (Art. 22).

The kinetic energy T of a particle with unit mass is given by

$$T=\frac{1}{2}(H_1^2\dot{u}_1^2+H_2^2\dot{u}_2^2+H_3^2\dot{u}_3^2),$$

where dots denote differentiations with respect to the time t; and Lagrange's equations of motion become therefore

$$(105)\qquad \frac{d}{dt}(H_1^2\dot{u}_1)-\frac{1}{2}\frac{\partial H_1^2}{\partial u_1}\dot{u}_1^2=\frac{\partial U}{\partial u_1},$$

$$(106)\qquad -\frac{1}{2}\frac{\partial H_1^2}{\partial u_2}\dot{u}_1^2=\frac{\partial U}{\partial u_2},$$

$$(107)\qquad -\frac{1}{2}\frac{\partial H_1^2}{\partial u_3}\dot{u}_1^2=\frac{\partial U}{\partial u_3},$$

where U denotes the force-function under which the ∞^2 parametric curves

$$u_2=\text{const.},\qquad u_3=\text{const.}$$

can be described by the particle. But equation (105) may be written, as in Art. 22,

(¹) Compare with Art. 25, III.

$$(108)\qquad \frac{1}{2}\frac{\partial}{\partial u_1}(H_1^2\dot{u}_1^2)=\frac{\partial U}{\partial u_1}.$$

I. Now in the case where

$$\frac{\partial H_1}{\partial u_2}\neq 0 \quad\text{and}\quad \frac{\partial H_1}{\partial u_3}\neq 0,$$

eliminating $\dot{u}_1^2$ among (106), (107) and (108), we have

$$\frac{\partial}{\partial u_1}\left(\frac{H_1^2\frac{\partial U}{\partial u_2}}{\frac{\partial H_1^2}{\partial u_2}}\right)+\frac{\partial U}{\partial u_1}=0,\qquad \frac{\partial}{\partial u_1}\left(\frac{H_1^2\frac{\partial U}{\partial u_3}}{\frac{\partial H_1^2}{\partial u_3}}\right)+\frac{\partial U}{\partial u_1}=0;$$

from which we obtain as in Art. 22

$$H_1^2\frac{\partial U}{\partial u_2}+U\frac{\partial H_1^2}{\partial u_2}=-h\frac{\partial H_1^2}{\partial u_2},\qquad H_1^2\frac{\partial U}{\partial u_3}+U\frac{\partial H_1^2}{\partial u_3}=-h\frac{\partial H_1^2}{\partial u_3},$$

h being the total energy which is constant. Integrating the last two equations we get

$$(102)'\qquad H_1^2(U+h)=\Phi(u_1),$$

where $\Phi(u_1)$ is an arbitrary positive function of u_1 only.

II. Next, in the case where

$$\frac{\partial H_1}{\partial u_2}=0 \quad\text{and}\quad \frac{\partial H_1}{\partial u_3}\neq 0,$$

it follows from (106) that H_1 and U are independent of u_2. Eliminating $\dot{u}_1^2$ from (107) and (108), and integrating with respect to u_1, we see that $H_1^2(U+h)$ is independent of u_3(¹). Hence we have also the same result as before.

III. Lastly, in the case where

$$\frac{\partial H_1}{\partial u_2}=0 \quad\text{and}\quad \frac{\partial H_1}{\partial u_3}=0,$$

it follows, from (106) and (107), that H_1 and U are independent of u_2 and u_3(²). Hence we have also the same result as before.

In this case the parametric surfaces $u_1=$const. must be a family of parallel surfaces, provided H_1 is a function of u_1 only. Conversely, if

(¹), (²) In these cases equation (108) expresses the conservation of energy which is trivial; hence $\Phi(u_1)$ can be taken as an arbitrary function.

$u_1=$const. be a family of parallel surfaces belonging to a triply orthogonal system, H_1 and hence φ (or U) becomes a function of u_1 only. But it is well known that any family of parallel surfaces belongs to a triply orthogonal system. Thus *a necessary and sufficient condition that the* ∞^2 *normals of a family of* ∞^1 *parallel surfaces should be the extremals of the form*

$$\int \varphi\, ds = min.$$

is that the ∞^1 *(equipotential) surfaces*

$$\varphi = const.$$

coincide with the given family of parallel surfaces.

For example, if we put

$$x = \frac{a}{b} u_1 + \frac{a^2-b^2}{b} \frac{b\cos u_2 - u_1}{a - b\cos u_2 \cos u_3},$$

$$y = \frac{\sqrt{a^2-b^2}\,(a - u_1 \cos u_3)}{a - b\cos u_2 \cos u_3} \sin u_2,$$

$$z = \frac{\sqrt{a^2-b^2}\,(b\cos u_2 - u_1)}{a - b\cos u_2 \cos u_3} \sin u_3,$$

a and b being constants, then we have $H_1=1$; and therefore $u_1=$const. form a family of Dupin's cyclides which are parallel surfaces, and so the function φ, for which all normals of the family of cyclides are extremals, is an arbitrary function of u_1 only.

41. A necessary and sufficient condition that the $2\infty^2$ curves

$$u_2=\text{const.},\ u_3=\text{const.};\quad \text{and}\quad u_3=\text{const.},\ u_1=\text{const.}$$

should be the extremals is, from (102) and (103), that

$$\frac{H_1}{\Psi_1(u_1)} = \frac{H_2}{\Psi_2(u_2)};$$

and if the condition be fulfilled, φ is given by

$$\varphi = \frac{H_1}{\Psi_1(u_1)} = \frac{H_2}{\Psi_2(u_2)}. \tag{109}$$

Now in this case the linear element on the surface $u_3=$const. is of the form

$$ds^2 = \frac{1}{\varphi^2}\left\{\Psi_1^2(u_1)\, du_1^2 + \Psi_2^2(u_2)\, du_2^2\right\}$$

so that $u_1=$const. and $u_2=$const. form an isothermal system on the surface $u_3=$const.. But by Dupin-Darboux's theorem these curves are lines of curvature on this surface. Therefore it follows that the ∞^1 surfaces $u_3=$const. form a family of isothermal surfaces.

Next, let r_{31} and r_{32} be the principal radii of curvature of the surface $u_3=$const. along the curves of intersection with the surfaces $u_1=$const. and $u_2=$const. respectively. Then by the well known formulae [1]

$$\frac{1}{r_{31}} = \frac{1}{H_3 H_1}\frac{\partial H_1}{\partial u_3} = \frac{1}{H_3\varphi . \Psi_1(u_1)} . \Psi_1(u_1) \frac{\partial \varphi}{\partial u_3} = \frac{1}{H_3\varphi}\frac{\partial \varphi}{\partial u_3},$$

$$\frac{1}{r_{32}} = \frac{1}{H_3 H_2}\frac{\partial H_2}{\partial u_3} = \frac{1}{H_3\varphi . \Psi_2(u_2)} . \Psi_2(u_2) \frac{\partial \varphi}{\partial u_3} = \frac{1}{H_3\varphi}\frac{\partial \varphi}{\partial u_3},$$

so that

$$\frac{1}{r_{31}} = \frac{1}{r_{32}}.$$

Hence the ∞^1 surfaces $u_3=$const. must be a family of spheres (or planes).

Conversely, if the surfaces $u_3=$const. be spheres (or planes), it must be

$$\frac{1}{r_{31}} = \frac{1}{r_{32}},$$

or

$$\frac{1}{H_3 H_1}\frac{\partial H_1}{\partial u_3} = \frac{1}{H_3 H_2}\frac{\partial H_2}{\partial u_3},$$

from which we get

$$H_1 = H_2 f(u_1, u_2),$$

where $f(u_1, u_2)$ denotes an arbitrary function of u_1 and u_2 only. Hence the linear element on the surface $u_3=$const. takes the form

$$ds^2 = H_2^2 \{ f^2(u_1, u_2)\, du_1^2 + du_2^2 \}.$$

Moreover, if $u_1=$const. and $u_2=$const. form an isothermal system on the surface $u_3=$const., the function f must take the form

$$f(u_1, u_2) = \frac{\Psi_1(u_1)}{\Psi_2(u_2)},$$

so that

(1) Bianchi, loc. cit, p. 489.

$$\frac{H_1}{\Psi_1(u_1)}=\frac{H_2}{\Psi_2(u_2)}.$$

Hence we can infer the theorem:

A necessary and sufficient condition that in the triply orthogonal system $u_1=const.$, $u_2=const.$, $u_3=const.$, the two families of parametric curves

$$u_2=const.,\ u_3=const.;\quad and \quad u_3=const.,\ u_1=const.$$

should be the extremals is that the parametric surfaces $u_3=const.$ are spheres (or planes), and the curves $u_1=const.$ and $u_2=const.$ on these spheres (or planes) form isothermal systems.

For example, in the polar coordinates (u_1, u_2, u_3) where

$$x=u_3\cos u_1\sin u_2,\quad y=u_3\sin u_1\sin u_2,\quad z=u_3\cos u_2,$$

or

$$\operatorname{tg} u_1=\frac{y}{x},\quad \operatorname{tg}^2 u_2=\frac{x^2+y^2}{z^2},\quad u_3^2=x^2+y^2+z^2,$$

the linear element is

$$ds^2=u_3^2\sin^2 u_2\,du_1^2+u_3^2\,du_2^2+du_3^2,$$

so that

$$\Psi_1(u_1)=\frac{1}{k},\quad \Psi_2(u_2)=\frac{1}{k\sin^2 u_2},$$

where k denotes an arbitrary constant. Hence the function φ, under which all meridians u_3, $u_1=\text{const.}$ and parallels u_3, $u_2=\text{const.}$ of concentric spheres $u_3=\text{const.}$ are the extremals, is given by

$$\varphi=\frac{k}{u_3\sin u_2}=\frac{k}{\sqrt{x^2+y^2}}.$$

42. Now a necessary and sufficient condition that the $3\,\infty^2$ curves

$$u_2,\ u_3=\text{const.};\quad u_3,\ u_1=\text{const.};\quad u_1,\ u_2=\text{const.}$$

should be the extremals is, from (102), (103) and (104), that

$$\frac{H_1}{\Psi_1(u_1)}=\frac{H_2}{\Psi_2(u_2)}=\frac{H_3}{\Psi_3(u_3)};$$

and if the condition be satisfied, φ is given by

$$(110)\qquad \frac{1}{\varphi}=\frac{H_1}{\Psi_1(u_1)}=\frac{H_2}{\Psi_2(u_2)}=\frac{H_3}{\Psi_3(u_3)},$$

so that the linear element takes the form

$$ds^2=\frac{1}{\varphi^2}\left\{\Psi_1^2(u_1)\,du_1^2+\Psi_2^2(u_2)\,du_2^2+\Psi_3^2(u_3)\,du_3^2\right\}.$$

Therefore if we put

$$\xi_i=\int\Psi_i(u_i)\,du_i\quad (i=1, 2, 3),\quad \varphi_1(\xi_1, \xi_2, \xi_3)=\varphi(u_1, u_2, u_3),$$

we have

$$(111)\qquad dx^2+dy^2+dz^2=\frac{1}{\varphi_1^2(\xi_1, \xi_2, \xi_3)}(d\,\xi_1^2+d\,\xi_2^2+d\,\xi_3^2),$$

which shows us that the space (ξ_1, ξ_2, ξ_3) is nothing but the conformal representation of the space (x, y, z).

Now according to a theorem due to Liouville(¹), we have either

$$(112)\qquad \varphi_1=c,$$

or

$$(113)\qquad \varphi_1=\frac{1}{k}\left\{(\xi_1-a_1)^2+(\xi_2-a_2)^2+(\xi_3-a_3)^2\right\},$$

where c, k, a_1, a_2, a_3 are arbitrary constants.

I. In the first case, the conformal transformation is any similar transformation, up to any motion in space. Hence the triply orthogonal system ξ_1, ξ_2, ξ_3 consists of three families of parallel planes, any two planes of different families cutting at right angles to each other; so that the given $3\,\infty^2$ extremals form three families of straight lines which are parallel to three straight lines which are at right angles to each other. And also since the function φ becomes a constant, all the extremals are ∞^4 straight lines.

II. In the second case, the conformal transformation is any transformation of reciprocal radii, up to any motion in space. Hence by applying a proper motion, we find

$$\xi_1-a_1=\frac{k(x-b_1)}{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2},\quad \xi_2-a_2=\frac{k(y-b_2)}{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2},$$

$$\xi_3-a_3=\frac{k(z-b_3)}{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2},\quad (b_1, b_2, b_3,\ \text{constants});$$

so that

(¹) See Bianchi, loc. cit., p. 487 and Encyklopädie der mathematischen Wissenschaften, Bd. III 3, p. 59 and p. 370.

$$\varphi=\frac{k}{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2};$$

and the triply orthogonal system ξ_1, ξ_2, ξ_3 consists of three families of spheres touching each other at the same point O, any two spheres of different families cutting at right angles to each other. But the equations (101) of all the extremals become

$$\frac{d^2\xi_2}{d\xi_1^2}=0, \quad \frac{d^2\xi_3}{d\xi_1^2}=0;$$

so that

$$\xi_2=c_1\xi_1+c_2, \quad \xi_3=c_3\xi_1+c_4,$$

where c_1, c_2, c_3, c_4 denote arbitrary constants; whence all the extremals are the ∞^4 circles passing through the same point O.

Thus we have arrived at the theorem:

In order that the three families of parametric curves of a triply orthogonal system should be the extremals, it is necessary and sufficient that either (i) *each family of parametric surfaces consists of all parallel planes, or* (ii) *each family of parametric surfaces consists of spheres touching each other at the same point* O. *In the first case, the function* φ *is constant, and all the extremals are all straight lines in space; and in the second case, the function* φ *is given by*

$$\varphi=\frac{k}{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2},$$

k, b_1, b_2, b_3 *being constants, and all the extremals are the* ∞^4 *circles passing through the same point* O (1).

(To be continued)

(1) Compare with Kasner, Transactions of the American Mathematical Society, 12 (1911), p. 70.

CORRIGENDA

on my note "Trajectories in the conservative field of force, Part I," Vol. 7 (1915), pp. 123–186.

P. 134, l. 16; read "(11)′" for "(12)."

P. 136, l. 23: read "(11)′" for "(12)."

P. 146, l. 5: read "$-\dfrac{2E}{\dfrac{\partial E}{\partial v}}\cdots$" for "$-\dfrac{2G}{\dfrac{\partial E}{\partial v}}\cdots$."

P. 147, l. 23: read "$\log(U+h)=-\int\cdots$" for "$\log(U+h)=-\frac{1}{2}\int\cdots$."

P. 155, l. 11: read "$\frac{dz}{dv}$" for "$\frac{dz}{du}$."

P. 160, l. 9: read "$f(x, y, z, p, q)\,s$" for "$f(x, y, z, p, q)\,t$."

P. 160, ls. 13, 19: read "(56)′" for "(52)."

P. 162, l. 2: read "$2pqt(-s\pm\cdots)$" for "$2pqt(s\pm\cdots)$."

P. 165, l. 7: read "$\left\{E\left(\frac{\partial u_1}{\partial v}\right)^2-\cdots\right\}$" for "$\left\{E\left(\frac{\partial u_1}{\partial v}\right)-\cdots\right\}$."

P. 165, l. 14: read "$\cdots\frac{\partial E_1}{\partial v_1}\dot{u}_1^2+\cdots$" for "$\cdots\frac{\partial E_1}{\partial v_1}\dot{v}_1^2+\cdots$."

P. 167, l. 9: read "$U_{u_1}+h$" for "$U_{v_1}+h$."

P. 168, l. 3: read "that there exists a" for "that the."

P. 168, l. 22: read "of Liouville's type" for "Liouville's type."

P. 169, l. 24: read "if" for "fi."

P. 170, l. 10: read "$(-c^2>u>-b^2>v>-a^2)$" for "$(-c^2>u^2>-b^2>v^2>-a^2)$."

P. 171, l. 6: read "S" for "$\overline{S}$."

P. 172, l. 1: read "$-2F\frac{\partial v_1}{\partial v}\frac{\partial v_1}{\partial u}+G\left(\frac{\partial v_1}{\partial u}\right)^2$" for "$-2F\frac{\partial v_1}{\partial v}\frac{\partial v_1}{\partial u}G+\left(\frac{\partial v_1}{\partial u}\right)^2$."

P. 173, l. 23: read "that there exists a" for "that the."

P. 174, l. 19: read "of Liouville's type" for "Liouville's type."

P. 178, l. 29: read "$\left\{\begin{matrix}\overline{1\,1}\\2\end{matrix}\right\}$" for "$\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}$."

P. 180, l. 16: read "$\frac{\lambda(u, v)}{k^2}$" for "$\lambda(u, v)$."

P. 180, l. 20: read "$\sqrt{\lambda(u, v)}$" for "$\lambda(u, v)$."

P. 181, l. 16: read "$\sqrt{\sin\omega}$" for "$\sin\omega$."

P. 182, l. 24: read "$\sqrt[4]{}$" for "$\sqrt{}$."

P. 183, l. 1: read "$\sqrt[4]{}$" for "$\sqrt{}$."

P. 183, l. 2 of the foot-note: read "$\cdots+F\frac{\partial E}{\partial v}-2F\frac{\partial F}{\partial u}$" for "$\cdots+F\frac{\partial E}{\partial v}+2F\frac{\partial F}{\partial u}$."

P. 185, ls. 6, 7 of the foot-note: read "(86)′" for "(86)."

Kinnosuke Ogura.

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