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Invariant Cubics for Isogonal Transformation in the Geometry of the Triangle.

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Invariant Cubics for Isogonal Transformation in the Geometry of the Triangle,

BY

K. OGURA in Sendai.

1. Let x_1, x_2, x_3 be the normal coordinates of a point with respect to the fundamental triangle $A_1 A_2 A_3$; then the isogonal transformation with respect to this triangle is given by

$$x_1' : x_2' : x_3' = x_2 x_3 : x_3 x_1 : x_1 x_2 .$$

In order that the cubic curve

$$\sum a_{ijk} x_i x_j x_k = 0 \qquad (i, j, k = 1, 2, 3)$$

is invariant for this transformation, it must pass through either the three vertices of the fundamental triangle, or twice one vertex and once another; for, any cubic which satisfies neither one of these conditions should be transformed into a curve of higher degree. In this note we will confine ourselves to treat the former case only, which is of the most importance in the geometry of the triangle.

Now since the cubic passing through the three vertices of the fundamental triangle

$$a_{112} x_1^2 x_2 + a_{113} x_1^2 x_3 + a_{122} x_1 x_2^2 + a_{223} x_2^2 x_3$$
$$+ a_{133} x_1 x_3^2 + a_{233} x_2 x_3^2 + a_{123} x_1 x_2 x_3 = 0 \qquad (1)$$

is trasformed into

$$a_{112} x_2 x_3^2 + a_{113} x_2^2 x_3 + a_{122} x_1 x_3^2 + a_{223} x_1^2 x_3$$
$$+ a_{133} x_1 x_2^3 + a_{233} x_1^2 x_2 + a_{123} x_1 x_2 x_3 = 0 ,$$

it is necessary and sufficient that

$$\frac{a_{112}}{a_{233}} = \frac{a_{113}}{a_{223}} = \frac{a_{122}}{a_{133}} = \frac{a_{223}}{a_{113}} = \frac{a_{133}}{a_{122}} = \frac{a_{233}}{a_{112}} = \frac{a_{123}}{a_{123}} = k \qquad (k \gtrless 0) ,$$

in order to secure that the cubic (1) is invariant for the transformation. Hence it must be either

$$a_{112} = a_{233} , \qquad a_{113} = a_{223} , \qquad a_{122} = a_{133} ; \qquad \text{(A)}$$

or

$$a_{112} = -a_{233} , \quad a_{113} = -a_{223} , \quad a_{122} = -a_{133} , \quad a_{123} = 0 . \qquad \text{(B)}$$

2. We will begin with the second case (B). In this case, if we put

$$a_{133}=\lambda_1\,,\qquad a_{112}=\lambda_2\,,\qquad a_{223}=\lambda_3\,,$$

the invariant cubic becomes

$$\lambda_1 x_1 (x_2^2-x_3^2)+\lambda_2 x_2 (x_3^2-x_1^2)+\lambda_3 x_3 (x_1^2-x_2^2)=0\,,\text{(}^1\text{)} \qquad (2)$$

which passes through the seven points: the three vertices of the triangle $A_1 A_2 A_3$, the incentre I (1, 1, 1) and the three excentres I_1 (−1, 1, 1), I_2 (1, −1, 1), I_3 (1, 1, −1).

Conversely, it is evident that any cubic passing through these seven points belongs to the above net (2), where λ_1, λ_2, λ_3 are arbitrary constants.

It will be seen that the two curves in the net (2)

$$a_1 x_1 (x_2^2-x_3^2)+a_2 x_2 (x_3^2-x_1^2)+a_3 x_3 (x_1^2-x_2^2)=0$$

and

$$\frac{1}{a_1}x_1(x_2^2-x_2^2)+\frac{1}{a_2}x_2(x_3^2-x_1^2)+\frac{1}{a_3}x_3(x_1^2-x_2^2)=0$$

pass through the point $P(a_1, a_2, a_3)$ and its isogonal conjugate P' $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right)$. Hence all the cubics passing through the eight points A_1, A_2, A_3, I, I_1, I_2, I_3, P,

$$\sum\left(a_i+\frac{\mu}{a_i}\right)x_i\,(x_j^2-x_k^2)=0^{(2)} \qquad \left(\begin{matrix} i,\ j,\ k=1,\ 2,\ 3\,; \\ i\gtrless j\gtrless k\,, \end{matrix}\right)$$

μ being an arbitrary constant, will always pass through a fixed point isogonal conjugate to P.

Now according to a theorem due to E. Weyr,(³) all the cubics passing through the four vertices and the three diagonal points of a complete quadrilateral and also any other point S, pass through the

(¹) When $\lambda_1:\lambda_2:\lambda_3=\cos A_1+\cos A_2\cos A_3:\cos A_2+\cos A_3\cos A_1:\cos A_3+\cos A_1\cos A_2$ $=\sin A_1:\sin A_2:\sin A_3$, this cubic becomes the seventeen-point cubic. See J. Casey, A treatise on the analytical geometry, 2. ed. (1893), p. 460; H. Pfaff, Hoffmann's Zeitschrift, 39 (1908), p. 460; H. Pfaff, ibid, 44 (1913), p. 123. For some extensions of the seventeen-point cubic, see a paper of Mr. K. Yanagihara, this Journal, Vol. 4. p. 25.

When $\lambda_1:\lambda_2:\lambda_3=\cos A_1-\cos A_2\cos A_3:\cos A_2-\cos A_3\cos A_1:\cos A_3-\cos A_1\cos A_2$, it becomes Lucas' cubic. See Casey, ibid, p. 543; F. G.-M., Exercices de géométrie, 5. ed. (1912), p. 556.

(²) In this Journal, Vol. 4, p. 9, Prof. S. Nakagawa has stated the invariancy of this equation in the case where P is the circumcentre.

(³) E. Weyr, Wien Berichte, 58 (1868), p. 663; H. Durège, Die ebenen Curven dritter Ordnung (1871), p. 220.

nineth point S' which is the conjugate pole of S with respect to the pencil of conics passing through the four vertices. Now for the quadrilateral $I\,I_1\,I_2\,I_3$, the diagonal points are the vertices A_1, A_2, A_3; whence it is easy to prove that S' in this case is isogonal conjugate to S with respect to the fundamental triangle. Therefore our result is nothing but a concrete example of Weyr's theorem.

3. Next let us treat the first case (A). In this case, if we put

$$a_{122}=\beta_1\,,\qquad a_{233}=\beta_2\,,\qquad a_{113}=\beta_3\,,\qquad a_{123}=\beta_4\,,$$

the invariant cubic becomes

$$\beta_1 x_1(x_2^2+x_3^2)+\beta_2 x_2(x_3^2+x_1^2)+\beta_3 x_3(x_1^2+x_2^2)+\beta_4 x_1 x_2 x_3=0\,. \quad (3)$$

We can see that the curve (3) cuts the three sides of the fundamental triangle at the points M_1 (0, β_2, $-\beta_3$), M_2 ($-\beta_1$, 0, β_3), M_3 (β_1, $-\beta_2$, 0) respectively; and these three points are on the straight line

$$\frac{x_1}{\beta_1}+\frac{x_2}{\beta_2}+\frac{x_3}{\beta_3}=0\,, \qquad (4)$$

which is the trilinear polar of the point $Q(\beta_1, \beta_2, \beta_3)$, or the polar-line of Q with respect to the three sides of the fundamental triangle

$$x_1 x_2 x_3=0\,. \qquad (5)$$

Since it can be shewn that the cubic (3) and the conic

$$\frac{x_2 x_3}{\beta_1}+\frac{x_3 x_1}{\beta_2}+\frac{x_1 x_2}{\beta_3}=0 \qquad (6)$$

have the common tangents

$$\beta_2 x_2+\beta_3 x_3=0\,,\quad \beta_3 x_3+\beta_1 x_1=0\,,\quad \beta_1 x_1+\beta_2 x_2=0$$

at the vertices A_1, A_2, A_3 respectively, the cubic (3) touches the transformed conic of the straight line (4), or the polar-conic of Q with respect to the three sides of the fundamental triangle (5), at the vertices of the triangle; and the common tangents $A_1 T_1$, $A_2 T_2$, $A_3 T_3$ are isogonal conjugate to $A_1 M_1$, $A_2 M_2$, $A_3 M_3$ respectively.

Conversely, we will prove that any cubic, which passes through the three vertices of the fundamental triangle, and whose intersections M_1, M_2, M_3 with its three sides are collinear, and whose tangents $A_1 T_1$, $A_2 T_2$, $A_3 T_3$ at its vertices are isogonal conjugate to $A_1 M_1$, $A_2 M_2$, $A_3 M_3$ respectively, must take the form (3).

If the cubic passing through A_1, A_2, A_3 be

$$a_{112}x_1^2 x_2+a_{113}x_1^2 x_3+a_{122}x_1 x_2^2+a_{223}x_2^2 x_3$$
$$+a_{133}x_1 x_3^2+a_{233}\,a_2 x_3^2+a_{123}x_1 x_2 x_3=0\,,$$

then the coordinates of M_1, M_2, M_3 are

$$(0\,,\ a_{233}\,, -a_{223}),\quad (-a_{133}\,,\ 0\,,\ a_{113}),\quad (a_{122}\,, -a_{112}\,,\ 0)$$

respectively; and hence the condition for collinearity of these three points is

$$a_{122}\, a_{233}\, a_{311} = a_{133}\, a_{322}\, a_{211}\,. \qquad (7)$$

But since the tangents

$$A_1\, T_1\, (a_{211}\, x_2 + a_{311}\, x_3 = 0)\,,$$
$$A_2\, T_2\, (a_{322}\, x_3 + a_{122}\, x_1 = 0)\,,$$
$$A_3\, T_3\, (a_{133}\, x_1 + a_{233}\, x_2 = 0)$$

are isogonal conjugate to

$$A_1\, M_1\, (a_{223}\, x_2 + a_{233}\, x_3 = 0)\,,$$
$$A_2\, M_2\, (a_{331}\, x_3 + a_{311}\, x_1 = 0)\,,$$
$$A_3\, M_3\, (a_{112}\, x_1 + a_{122}\, x_2 = 0)$$

respectively, we must have the relations:

$$\frac{a_{211}}{a_{233}} = \frac{a_{311}}{a_{223}} = k_1\,,\qquad \frac{a_{322}}{a_{311}} = \frac{a_{122}}{a_{331}} = k_2\,,\qquad \frac{a_{133}}{a_{122}} = \frac{a_{233}}{a_{112}} = k_3\,,$$
$$(k_1 \gtreqless 0\,,\qquad k_2 \gtreqless 0\,,\qquad k_3 \gtreqless 0).$$

Hence

$$k_1\, a_{233}\,.\ a_{311}\,.\ a_{122} = a_{211}\,.\ k_1\, a_{223}\,.\ k_2\, a_{331}\,,$$
$$a_{122}\,.\ a_{233}\,.\ k_2\, a_{311} = k_2\, a_{331}\,.\ k_3\, a_{112}\,.\ a_{322}\,,$$
$$a_{311}\,.\ a_{233}\,.\ k_2\, a_{331} = k_1\, a_{223}\,.\ k_3\, a_{112}\,.\ a_{122}\,;$$

or, by (7)

$$k_2 = 1\,,\quad k_3 = 1\,,\quad k_2 = k_1\, k_3\,;$$

that is, $$k_1 = k_2 = k_3 = 1\,, \qquad (8)$$

which proves our proposition.

Thus we have arrived at the following theorem:

The invariant cubics passing through the three vertices of the fundamental triangle $A_1\, A_2\, A_3$ are divided into two classes:

(i) the cubics passing through the incentre and the three excentres, and

(ii) the cubics whose intersections M_1, M_2, M_3 with the three sides of the triangle are collinear and whose tangents at its vertices are isogonal conjugate to $A_1\, M_1$, $A_2\, M_2$, $A_3\, M_3$.

Conversely, the cubics belonging to these two classes only are the invariant cubics passing through the three vertices.

4. In the particular case where A_2 and A_3 are the two imaginary points at infinity, we may take

$$x_1 = 1\,,\quad x_2 = x + y\sqrt{-1}\,,\quad x_3 = x - y\sqrt{-1}\,,$$

x, y being the coordinates of a point in the rectangular coordinates; and then the tranformation formulae become

$$x' = \frac{x}{x^2 + y^2}\,,\qquad y' = \frac{y}{x^2 + y^2}\,, \qquad (9)$$

which represent the inversion having the positive power 1 with respect to

$$x^2 + y^2 = 1\,.$$

Since the invariant cubic with respect to the inversion, that is, the anallagmatic cubic passes through at least an imaginary point at infinity (Art. 1), the real anallagmatic cubic must be a circular cubic passing through the centre of inversion.

Now let us put

$$f(x_1\,,\ x_2\,,\ x_3) = \Sigma\, a_i\, x_i\, (k_j^2 - x_k^2) \qquad \begin{pmatrix} i, j, k = 1, 2, 3; \\ i \gtreqless j \gtreqless k\,. \end{pmatrix}$$
$$= f(x, y)\,;$$

Then $$f(x, y) = (A\,x + B\,y)\,(x^2 + y^2) + C\,x\,y - (A\,x + B\,y)\,.$$

Hence in order that

$$f(x',\ y') = \frac{1}{(x^2 + y^2)^2}\, f(x, y)\,,$$

it must be $$A = 0,\quad B = 0\,;$$

that is, $$f(x, y) = C\,x\,y\,.$$

On the contrary, if we put

$$\varphi(x_1\,,\ x_2\,,\ x_3) = \Sigma\, \beta_i\, x_i (x_j^2 + x_k^2) + \beta_4\, x_1\, x_2\, x_3$$
$$= \varphi(x, y),$$

we have

$$\varphi(x, y) = (D\,x + E\,y)\,(x^2 + y^2) + (F\,x^2 + G\,y^2) + (D\,x + E\,y)\,.$$

Hence $$\varphi(x,' y') = \frac{1}{(x^2 + y^2)^2}\, \varphi(x, y)\,.$$

Therefore any real anallagmatic cubic will take the form

$$(D\,x + E\,y)\,(x^2 + y^2) + F\,x^2 + C\,x\,y + G\,y^2 + (D\,x + E\,y) = 0\,, \quad (10)$$

which is easily verified by direct calculation.

Let us avail this opportunity to add a remark on the inversion having the negative power -1

$$x' = -\frac{x}{x^2 + y^2}\,,\qquad y' = -\frac{y}{x^2 + y^2}\,, \qquad (11)$$

with respect to $$x^2 + y^2 = -1\,.$$

In this case, we have

$$f(x', y') = \frac{1}{(x^2+y^2)^2} f(x, y) \, ;$$

but in order that

$$\varphi(x', y') = \frac{1}{(x^2+y^2)^2} \varphi(x, y) \, ,$$

it must be $$D=0, \quad E=0 \, ;$$

that is, $$\varphi(x, y) = F x^2 + G y^2 \, .$$

From this we can infer that any real anallagmatic cubic with respect to the inversion having the negative power -1 shall be

$$(A x + B y)(x^2+y^2) + F x^2 + C x y + G y^2 - (A x + B y) = 0 \, . \quad (12)$$

It may be said that $f(x, y) = 0$ and $\varphi(x, y) = 0$ are the proper anallagmatic cubics with respect to the inversions (9), (11) having the negative power -1 and the positive power 1, respectively.

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