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Some Theorems in the Geometry of Oriented Circles in a Plane.

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Some Theorems in the Geometry of Oriented Circles in a Plane,

BY

K. OGURA in Sendai.

1. Let k be the vertex of a right isosceles cone of rotation standing on a circular base K in a plane Π, and let us suppose the circle K has a direction determined by the side of the plane Π on which the point k lies. The oriented circle K thus considered in the plane Π is the cyclographic representation of the point k in space. (1) The intersection of the above stated cone and the plane at infinity which is a fixed circle C, independent of the position of the vertex k, will be called the *fundamental circle at infinity with respect to the plane* Π.

By application of cyclography, Prof. E. Müller has obtained some theorems concerning such oriented circles in a plane. (2) In this note I will also give some theorems of the same nature.

2. If three straight lines l_1, l_2, l_3 in space form a triangle $k_{12}k_{23}k_{31}$, then the straight line l_4 joining any two points k_{14} and k_{24} on l_1 and l_2 respectively intersects the third straight line l_3 at a point k_{34}.

Transforming this theorem by cyclography, we have the following theorem (Fig. 1).

Let (L_1, L_1'), (L_2, L_2'), (L_3, L_3') *be three pairs of common (proper) tangents of three given oriented cercles* K_{12}, K_{23}, K_{31} *taken in pairs, and* K_{14} *and* K_{24} *be any two oriented circles touching the two pairs of oriented lines* (L_1, L_1'), (L_2, L_2') *respectively, and* (L_4, L_4') *be a pair of common tangents of the two oriented circles* K_{14} *and* K_{24}; *then the four oriented lines* L_3, L_3', L_4, L_4' *are touched by an oriented circle* K_{34}.

In the particular case where each of K_{14} and K_{24} reduces to a point, K_{34} will also reduce to a point. Hence we have the well-known theorem due to Monge.

3. There are always two transversals cutting four given straight lines

(1) W. Fiedler, Zyklographie, 1882; E. Müller, Jhrsb. Dtsch. Math.-Ver. 14 (1905), p. 574.

(2) E. Müller, Jhrsb. Dtsch. Math.-Ver. 20 (1911), p. 168.

in space. If more than two of such transversals exist, then an infinite number of such transversals must exist.

Transforming this theorem by cyclography, we get the following theorem (Fig. 2).

There are always two pairs of oriented lines (G_1, G_1') *cutting four given pairs of oriented lines* (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4') *in a plane* Π, *so that each of four pairs of four oriented lines* (G_1, G_1', L_1, L_1'), (G_1, G_1', L_2, L_2'), (G_1, G_1', L_3, L_3'), (G_1, G_1', L_4, L_4') *can be described to an oriented circle. If more than two of such pairs, then an infinite number of such pairs must exist.*

4. If $l_1, l_2, l_3, l_4, l_5, l_6$ be six sides of a space-hexagon s and k_{12}, $k_{23}, k_{34}, k_{45}, k_{53}, k_{61}$ be its vertices, and the opposite sides l_1, l_4; l_2, l_5; l_3, l_6 intesect at k_{14}, k_{25}, k_{36} respectively, then the three straight lines

$$k_{12}\,k_{45}=l_{36}, \quad k_{23}\,k_{56}=l_{14}, \quad k_{34}\,k_{61}=l_{25}$$

joining the opposite vertices of the hexagon will intersect at a point z, which is called the Brianchon point of the hexagon s. [1]

Conversely, if three pairs of two points (k_{12}, k_{45}), (k_{23}, k_{56}), (k_{34}, k_{61}) respectively lie on any three straight lines l_{36}, l_{14}, l_{25} passing through a point z, then each pair of two straight lines

$$k_{61}\,k_{12}=l_1,\ k_{34}\,k_{45}=l_4\,;\ k_{12}\,k_{23}=l_2,\ k_{45}\,k_{56}=l_5\,;\ k_{23}\,k_{34}=l_3,\ k_{53}\,k_{61}=l_6$$

intersect at k_{14}, k_{25}, k_{36} respectively.

Transforming this theorem by cyclography, we get the following theorem (Fig. 3).

If (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4'), (L_5, L_5'), (L_3, L_3') *be six pairs of common tangents of six given oriented circles* $K_{12}, K_{23}, K_{34}, K_{45}, K_{56}, K_{61}$, *taken in pairs, in a plane* Π, *and* L_1, L_1', L_4, L_4'; L_2, L_2', L_5, L_5'; L_3, L_3', L_6, L_6' *be described to three oriented circles* K_{14}, K_{25}, K_{36} *respectively, then the three pairs of common tangentes of* K_{12}, K_{45}; K_{23}, K_{53}; K_{34}, K_{61} *are touched by an oriented circle* Z. This oriented circle Z will be called the *Brianchon circle* of the six given oriented circles $K_{12}, \ldots, K_{61}$.

Conversely, *if three pairs of two oriented circles* K_{12}, K_{45}; K_{23}, K_{56}; K_{34}, K_{61} *touch respectively each pair of tangents* (L_{36}, L'_{36}), (L_{14}, L'_{14}), (L_{25}, L'_{25}) *of an oriented circle and if* (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4'), (L_5, L_5'), (L_6, L_6') *be the pair of common tangents of* K_{61}, K_{12}; K_{12}, K_{23}; K_{23}, K_{34}; K_{24}, K_{45}; K_{45}, K_{56} *respectively, then the three*

[1] See O. Staude, Analytische Geometrie d. Punktepaares, d. Kegelschnittes und d. Fläche zweiter Ordnung (1910), p. 913.

pairs of four oriented lines (L_1, L_1', L_4, L_4'), (L_2, L_2', L_5, L_5'), (L_3, L_3', L_6, L_6') *are touched by the other oriented circles* K_{14}, K_{25}, K_{36} *respectively.*

By the change of order of the sides we get six space-hexagons:

$$s_1=l_1\,l_2\,l\,l_4\,l_5\,l_6, \qquad s_4=l_1\,l_4\,l_3\,l_2\,l_5\,l_6,$$
$$s_2=l_1\,l_4\,l\,l_6\,l_5\,l_2, \qquad s_5=l_1\,l_2\,l_3\,l_6\,l_5\,l_4,$$
$$s_3=l_1\,l_6\,l_3\,l_2\,l_5\,l_4, \qquad s_6=l_1\,l_6\,l_3\,l_4\,l_5\,l_2;$$

and we see that the three Brianchon points z_1, z_2, z_3 of the three space-hexagons s_1, s_2, s_3 lie on a Steiner line g_1, and the other three Brianchon points z_4, z_5, z_6 lie on another Steiner line g_2.

Hence, in the plane Π, by the change of order of the oriented circles we get the six Brianchon circles $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$. The three Brianchon circles Z_1, Z_2, Z_3 have a pair of common tangents (G_1, G_1') and the other three Brianchon circles Z_4, Z_5, Z_6 have another pair of common tangents (G_2, G_2').

In the particular case where each of the six straight line $l_1, l_2, l_3, l_4, l_5, l_6$ intersect the fundamental circle at infinity C with respect to the plane Π, we obtain the following theorem.

If in a plane six oriented circles $K_1, K_2, K_3, K_4, K_5, K_6$ be described so that each one K_i touches its two neighbours at $A_{i-1,i}, A_{i,i+1}$ respectively, and if there are three oriented circles K_{36}, K_{14}, K_{25} having contact with K_1, K_2, K_4, K_5; K_2, K_3, K_5, K_6; K_3, K_4, K_6, K_1 at the points $A_{12}, A_{12}, A_{45}, A_{45}$; $A_{23}, A_{23}, A_{56}, A_{56}$; $A_{34}, A_{34}, A_{61}, A_{61}$ respectively, then three pairs of common tangents of K_1, K_4; K_2, K_5; K_3, K_6 are touched by another oriented circle.

5. Let us state Pascal's theorem for a circle in the following form:

If $g_1, g_2, g_3, g_4, g_5, g_6$ be the six straight lines joining the six given points $A_{12}, A_{23}, A_{34}, A_{45}, A_{53}, A_{61}$ so that $A_{i-1,i}\,A_{i,i+1}=g_i$, and if A_{14}, A_{25}, A_{33} be the point of intersection of g_1, g_4; g_2, g_5; g_3, g_6 respectively, and if the six given points lie on a circle X, then A_{14}, A_{25} and A_{36} are collinear.

Then since the circle X may be considered as a pair of oriented circles X, X' having different orientations to each other, so if there exist two right isosceles cones of rotation symmetrically with respect to a plane P and passing through the six given points, then A_{14}, A_{25}, A_{36} are collinear; and both of these two cones contain the fundamental circle at infinity with respect to the plane P.

This theorem is transformed into the following by collineation.

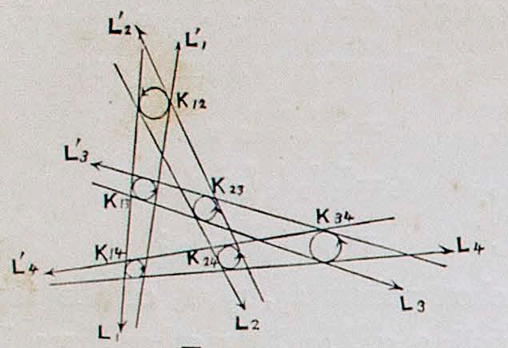

Fig. 1.

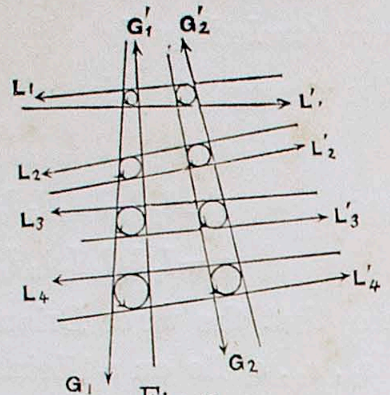

Fig. 2.

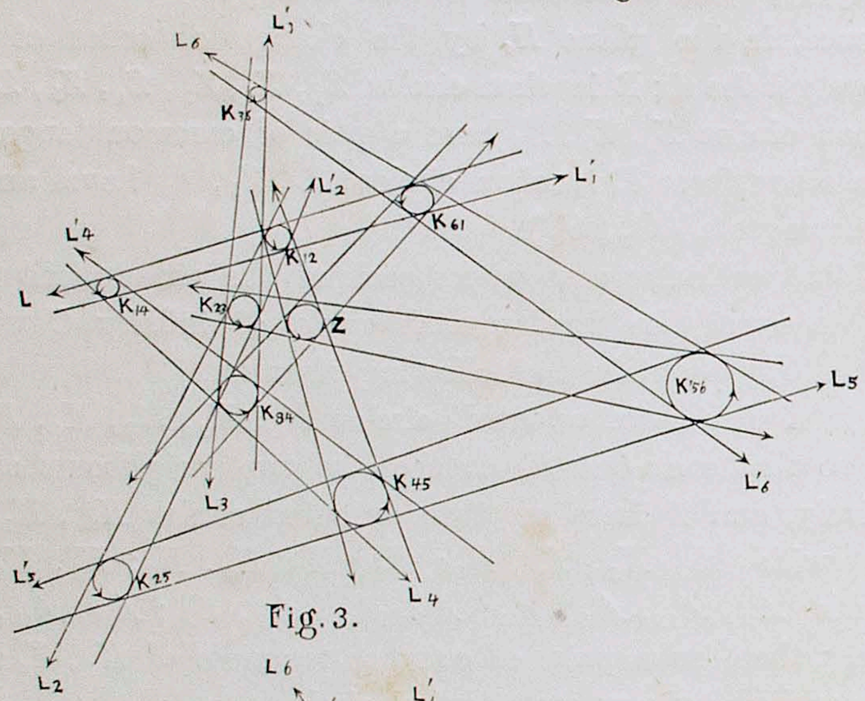

Fig. 3.

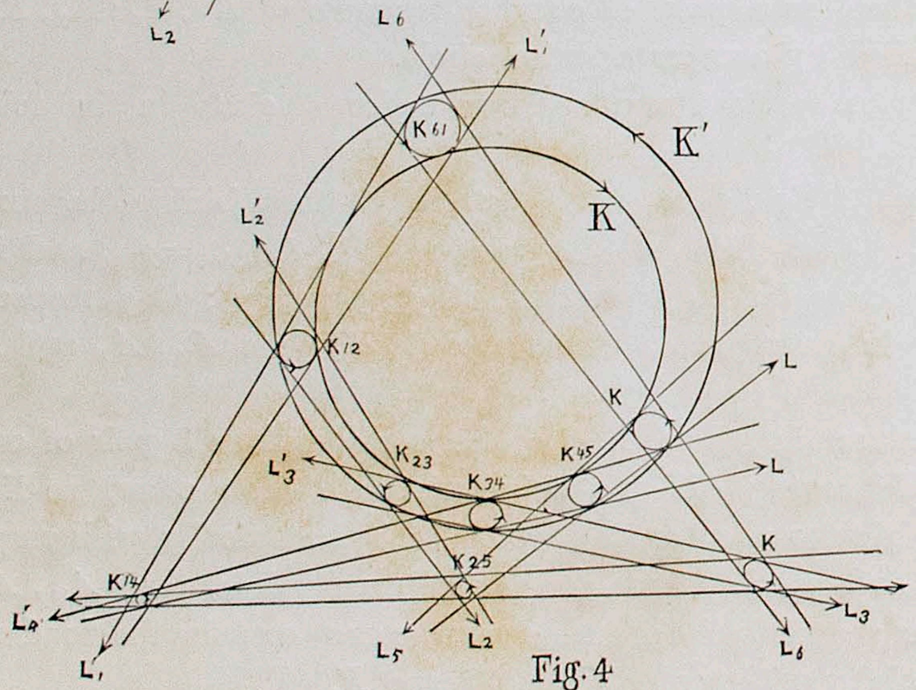

Fig. 4

If $l_1, l_2, l_3, l_4, l_5, l_6$ be the six straight lines joining six given points $k_{12}, k_{23}, k_{34}, k_{45}, k_{56}, k_{61}$ so that $k_{i-1,i}\, k_{i,i+1}=l_i$, and if l_1, l_4; l_2, l_5; l_3, l_6 intersect to each other at k_{14}, k_{25}, k_{36} respectively, and if there exist two cones of the second degree containing the given six points and a conic, then the three points k_{14}, k_{25}, k_{36} are collinear.

In the particular case, where the conic and the two cones are the fundamental circle at infinity C and the two right isosceles cones of rotation with respect to a plane Π respectively, we get by cyclographic transformation the following theorem which is an extension of Pascal's theorem for a circle (Fig. 4).

If (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4'), (L_5, L_5'), (L_6, L_6') *be six pairs of common tangents of six given oriented circles* $K_{12}, K_{23}, K_{34}, K_{45}, K_{56}, K_{61}$ *in a plane, so that the common tangents of* $K_{i-1,i}$, $K_{i,i+1}$ *are* (L_i, L_i'), *and if the three pairs of four oriented lines* $(L_1, L_1', L_4\, L_4')$, (L_2, L'_2, L_5, L'_5), (L_3, L'_3, L_6, L'_6) *be touched by other three oriented circles* K_{14}, K_{25}, K_{36} *respectively, and if the six given oriented circles touch two oriented circles* K *and* K', *then the three oriented circles* K_{14}, K_{25}, K_{36} *have a pair of common tangents.*

Similarly we can apply the cyclographic transformation to the Pascal configuration, and we can substitute some oriented circles and some pairs of common tangents of oriented circles for points, Steiner's, Kirkman's, Salmon's, etc., and lines, Pascal's, Cayley-Salmon's, Plücker's, etc.

In practical drawing of figures, a point, a straight line and a circle may be considered as a circle, two straight lines and two circles respectively in strict sense. Hence this theorem may be taken as Pascal's theorem for a circle in approximate mathematics. [1]

6. Let us state Brianchon's theorem for a circle in the following form :

If $g_1, g_2, g_3, g_4, g_5, g_6, g_{14}, g_{25}, g_{36}$ be the straight lines joining six given points $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$, so that $A_{i-1,i}\, A_{i,i+1}=g_i$, $A_{i-2,i-1}\, A_{j+1,j+2}=g_{i,j}$, and if g_{36}, g_{14}; g_{14}, g_{25}; g_{25}, g_{36} intersect to each other at A_{25}, A_{36}, A_{14} respectively, and if the six straight lines $g_1, g_2, g_3, g_4, g_5, g_6$ touch a circle X, then the three points A_{25}, A_{36}, A_{14} will coincide with a point z.

Then since a circle and its tangent may be considered as a pair

([1]) For another treatment of Pascal's theorem (for a conic) in approximate mathematics, see F. Klein, Anwendung d. Differential- u. Integralrechnung auf Geometrie (1902), p. 360 ; P. Böhmer, Ueber geometrische Approximation, Göttinger Diss. (1904) p. 29

of oriented circles and a pair of tangents having different orientations to each other respectively, by a treatment similar to that in the last article, we obtain the following theorem.

If (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4'), (L_5, L_5'), (L_6, L_6'), (L_{14}, L'_{14}), (L_{25}, L'_{25}), (L_{36}, L'_{36}) *be the pair of common tangents of six given oriented circles* $K_{12}, K_{23}, K_{34}, K_{45}, K_{56}, K_{61}$ *in a plane* Π, *so that the common tangents of* $K_{i,i-1}$, $K_{i,i+1}$; $K_{i-2,i-1}$, $K_{j+1,j+2}$ *are* (L_i, L_i'), $(L_{i,j}, L'_{i,j})$ *respectiverely, and if* $(L_{36}, L'_{36}, L_{14}, L'_{14})$, $(L_{14}, L'_{14}, L_{25}, L'_{25})$, $(L_{25}, L'_{25}, L_{36}, L'_{36})$ *touch other three oriented circles* K_{25}, K_{36}, K_{14} *respectively and if the six oriented lines* $L_1, L_2, L_3, L_4, L_5, L_6$ *and the other six* $L_1', L_2', L_3', L_4', L_5', L_6'$ *are touched by two oriented circles* K *and* K′ *respectively, then the three oriented circles* K_{25}, K_{36}, K_{14} *will coincide with an oriented circle* Z ; *that is, the three pairs of oriented lines* (L_{14}, L'_{14}), (L_{25}, L'_{25}), (L_{36}, L'_{36}) *will be touched by the oriented circle* Z.

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of oriented circles and a pair of tangents having different orientations to each other respectively, by a treatment similar to that in the last article, we obtain the following theorem.

If (L_1, L_1'), (L_2, L_2'), (L_3, L_3'), (L_4, L_4'), (L_5, L_5'), (L_6, L_6'), (L_{14}, L'_{14}), (L_{25}, L'_{25}), (L_{36}, L'_{36}) *be the pair of common tangents of six given oriented circles* K_{12}, K_{23}, K_{34}, K_{45}, K_{56}, K_{61} *in a plane* Π, *so that the common tangents of* $K_{i,i-1}$, $K_{i,i+1}$; $K_{i-2,i-1}$, $K_{j+1,j+2}$ *are* (L_i, L_i'), $(L_{i,j}, L'_{i,j})$ *respectiverely, and if* $(L_{26}, L'_{36}, L_{14}, L'_{14})$, $(L_{14}, L'_{14}, L_{25}, L'_{25})$, $(L_{25}, L'_{25}, L_{36}, L'_{36})$ *touch other three oriented circles* K_{25}, K_{36}, K_{14} *respectively and if the six oriented lines* L_1, L_2, L_3, L_4, L_5, L_6 *and the other six* L_1', L_2', L_3', L_4', L_5', L_6' *are touched by two oriented circles* K *and* K' *respectively, then the three oriented circles* K_{25}, K_{36}, K_{14} *will coincide with an oriented circle* Z; *that is, the three pairs of oriented lines* (L_{14}, L'_{14}), (L_{25}, L'_{25}), (L_{36}, L'_{36}) *will be touched by the oriented circle* Z.

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